

The Design of Monetary and Fiscal Policy: A Global Perspective*

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Abstract

We study the emergence of multiple equilibria in models with capital and bonds under various monetary and fiscal policies. We show that the presence of capital is indeed another independent source of local and global multiplicities, even under active policies that yield local determinacy. We also show how a very similar mechanism generates multiplicities in models with bonds and distortionary taxation. We then explore the design of monetary policies that avoid multiple equilibria. We show that interest rate policies that respond to the output gap, while potentially a source of significant inefficiencies, may be effective in preventing multiple equilibria and costly oscillatory equilibrium dynamics.

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1 Introduction

Recent papers have analyzed the role and the effectiveness of monetary policy in models of the economy where capital is one of the assets¹. These studies, some of which use higher-order approximations around the steady state, analyze the time series properties of calibrated models and explore the effectiveness of monetary and fiscal policy rules in terms of welfare and stabilization. At the same time, another related strand of this literature² uncovered the possibility of local and global indeterminacies in dynamic equilibrium models of monetary policy for a wide range of model specifications and monetary policy rules. In particular, in sticky price models allowing the interest rate to affect the marginal cost and output through its effect on real balances, or to affect fiscal policy through the government budget constraint and distortionary taxes, provided plausible mechanisms for the emergence of multiple equilibria. In this paper we show that the presence of capital is indeed another independent source of local and global multiplicities, even under standard calibrations and active monetary policies that yield local determinacy. We also show that there is a common mechanism that generates multiple equilibria in the models with money, models with distortionary taxes arising from the interaction of monetary and fiscal policies, and models with capital.

We then explore the design of monetary policies that avoid multiple equilibria³. We show that interest rate policies that respond to the output gap, while potentially a source of significant inefficiencies as shown by Schmitt-Grohe and Uribe (2004) and Woodford (2003), may be effective in preventing multiple equilibria and costly oscillatory equilibrium dynamics.

In the next section we spell out the general model with capital and bonds. The following section gives the analysis of equilibria under various fiscal and monetary rules. We start with the analysis of the model with capital only, and then we provide the economic intuition for local and global multiplicities. Next we turn to the model without capital, but with bonds and distortionary taxes under various fiscal policy rules and we study the emergence of multiple equilibria. In the subsequent section we discuss the formal equivalence of the model with capital and the model with bonds. Finally, in the last section we explore the role of monetary policies that respond to the output gap in addition to inflation.

¹(See for example Carlstrom and Fuerst (2004), Christiano, Eichenbaum and Evans (2003), Schmitt-Grohe and Uribe(2004), Sveen and Weinke(2004))

²See Carlstrom and Fuerst (2004), Dupor (2001), Meng(2000), Benhabib,Schmitt-Grohe and Uribe (2001a, 2001b, 2002, 2002b), Eusepi (2002), Hong (2002).

³For an alternative approach to the design of policies to select good equilibria when there are potentially many local and/or global equilibria see Benhabib, Schmitt-Grohé, and Uribe (2002), Christiano and Rostagno (2001) and Christiano and Harrison (1999) .

2 The model

We consider a simple monetary model with explicit microfoundations and nominal rigidity.

2.1 Private Sector

The economy is populated by a continuum of identical utility maximizing agents taking consumption and production decisions. Each agent consumes a composite good made of a continuum of differentiated goods, and produces only one differentiated good. Producers have market power and therefore set their price to maximize profits. They face convex adjustment costs of changing prices, the only source of nominal rigidity in the model.

Each agent j maximizes the intertemporal utility function

$$U^j = \sum_{s=t}^{\infty} \beta^{s-t} \left[\frac{C_{js}^{1-\sigma}}{1-\sigma} - h_{js} - \frac{\psi}{2} \left(\frac{P_{js}}{P_{js-1}} - 1 \right)^2 \right] \quad (1)$$

where $C_{j,t}$ denotes the composite good, $h_{j,t}$ denotes hours worked and where the last term measures the utility cost of changing prices⁴. The composite good is defined as

$$C_t^j = \left[\int_0^1 (Y_{jt})^{\frac{\eta}{1-\eta}} dj \right]^{\frac{1-\eta}{\eta}} \quad (2)$$

where $\eta > 1$ and $Y_{j,t}$ is the differentiated good. Given (2), the consumers' demand for each differentiated good is

$$Y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\eta} C_t^j$$

where P_t defines the following price index

$$P_t = \left[\int_0^1 (P_{jt})^{1-\eta} dj \right]^{\frac{1}{1-\eta}}.$$

Each producer uses labor and capital as factors of production

$$Y_{jt} = K_{jt}^{\theta} h_{jt}^{1-\theta} \quad (3)$$

where K_{jt} denotes the amount of capital used by producer j . The production function is Cobb-Douglas and displays constant returns to scale. Capital is subject to depreciation and evolves as

$$K_{jt+1} = (1 - \delta)K_{jt} + I_{jt} \quad (4)$$

⁴In reduced form our model is almost identical to one with Calvo pricing, but see the discussion in section 2.4.

where I_{jt} is the investment good. It is assumed to be of the same form as the composite consumption good, so that the producers have a demand of $Y_{jt} = \left(\frac{P_{jt}}{P_t}\right)^{-\eta} I_{jt}$ for each differentiated good. Producers face the constraint that aggregate demand for their good needs to be satisfied at the posted prices

$$Y_{j,t} = K_{jt}^\theta h_{jt}^{1-\theta} = \left(\frac{P_{jt}}{P_t}\right)^{-\eta} a_t \quad (5)$$

where $a_t = \int (C_{jt} + I_{jt}) dj$ denotes absorption.

Each consumer-producer's wealth evolves according to

$$\frac{B_{jt}}{P_t} = \frac{R_{t-1}}{\pi_t} \frac{B_{jt-1}}{P_{t-1}} + (1 - \tau_t) \frac{P_{jt}}{P_t} Y_{jt} - C_{jt} - I_{jt} \quad (6)$$

where B_{jt} is a one period bond and R_t denotes the gross nominal interest rate. Each agent j receives income from selling her output. Income is taxed in proportion τ_t . In order to simplify the analysis we assume a cashless economy so that the only two assets available are bonds and capital.

Summing up, given R_t, τ_t, P_t, a_t and K_0, B_0 , the problem of agent j consists of choosing sequences of $C_{jt}, I_{jt}, B_{jt}, P_{jt}$ and h_{jt} in order to maximize (1) under the constraints (6), (5) and (4).

2.2 Government

The monetary authority sets the nominal interest rate by following the Taylor-type policy rule

$$R_t = \bar{R} \left(\frac{P_t}{P_{t-1}}\right)^{\phi_\pi} \left(\frac{Y_t}{\bar{Y}}\right)^{\phi_y} \quad (7)$$

where the inflation target is set equal to zero and $\bar{R} = \beta^{-1}$. The central bank responds to deviations of output from its steady state value, not from the efficient level of output. We focus on the case where $\phi_\pi > 1$, so that the Taylor principle is satisfied. We chose a specification where the central bank responds to current rather than to future inflation because forward looking policy rules have been shown to be destabilizing. (See for example Carlstrom and Fuerst (2004), Eusepi (2002,2003)).

The fiscal authority's liabilities in real terms evolve according to

$$\frac{B_t}{P_t} = \frac{R_{t-1}}{\pi_t} \frac{B_{t-1}}{P_{t-1}} + \left(\bar{g} - \tau_t \int \frac{P_{jt}}{P_t} Y_{jt} di\right) \quad (8)$$

where \bar{g} denotes (constant) government purchases. For simplicity we assume $\bar{g} = 0$. The government fiscal rule is:

$$\tau_t \int \frac{P_{jt}}{P_t} Y_{jt} di = \phi_0 + \phi_1 R_{t-1} \frac{B_{t-1}}{P_{t-1}} + \phi_2 \left[\bar{g} + \left(\frac{R_{t-1}}{\pi_t} - 1\right) \frac{B_{t-1}}{P_{t-1}} \right]. \quad (9)$$

Following Schmitt-Grohe and Uribe (2004) we consider two different fiscal policies. The first is a balanced budget rule that keeps the total amount of real debt *constant*. It corresponds to the case where $\phi_0 = \phi_1 = 0$ and $\phi_2 = 1$. The second is a fiscal rule requiring taxes to respond to deviations of real bonds from a target, here normalized to zero. In this case $\phi_2 = 0$ and $\phi_1 > 0$. As well known in the literature, this fiscal policy rule can be “passive” or “active”. In the passive case, to a first approximation, we set $|(1 - \phi_1)| < 1$ so that the growth rate of government debt is lower than the real interest rate. This implies that the government sets fiscal policy to satisfy its intertemporal budget constraint. In the active case the government conducts fiscal policy disregarding the effects on its intertemporal budget constraint so that other variables such as the price level need to adjust to guarantee the solvency of the fiscal authority.

2.3 Equilibrium

Each period the goods markets clear, that is

$$\int Y_{j,t} dj = h_t \left(\frac{K_t}{h_t} \right)^\theta = \int \left(\frac{P_{jt}}{P_t} \right)^{-\eta} a_t \quad (10)$$

where we use the fact that given our production function, the capital/labor ratio is the same for every producer. Walras’ law, (10), implies that the bonds market also clears. We impose a symmetric equilibrium and we further assume that each agent begins with identical quantities of bonds and capital, so that $C_{jt} = C_t, K_{jt} = K_t, B_{jt} = B_t, P_{jt} = P_t$ and $h_{jt} = h_t$. The first order conditions give three behavioral equations for the private sector. First we have the IS curve

$$C_t^{-\sigma} = \frac{\beta R_t C_{t+1}^{-\sigma}}{\pi_{t+1}} \quad (11)$$

which defines the “demand channel” of monetary policy. Second, the arbitrage condition equates real return on bonds with real return of capital, net of the depreciation rate:

$$\frac{R_t}{\pi_{t+1}} = \left[\left(\frac{s_{t+1}^{\frac{1}{1-\theta}} (1-\theta)}{C_{t+1}^\sigma} \right)^{\frac{1-\theta}{\theta}} + 1 - \delta \right] \quad (12)$$

The real marginal cost s_t is given by

$$s_t = \frac{C_t^\sigma K_t^{-\theta} h_t^\theta}{(1-\theta)(1-\tau_t)} = \frac{C_t^\sigma}{MPL_t(1-\theta)} \quad (13)$$

and can be expressed in terms of the marginal product of labor net of taxes. Finally, we have the Phillips curve describing the behavior of the inflation rate

$$\pi_t (\pi_t - 1) = \beta \pi_{t+1} (\pi_{t+1} - 1) + \frac{C_t^{-\sigma} (K_t^{\theta-1} h_t^{1-\theta}) K_t \eta (1-\tau_t)}{\psi} \left(s_t - \frac{\eta-1}{\eta} \right) \quad (14)$$

where $\frac{\eta-1}{\eta}$ defines the inverse of the mark up. The model is closed with the private sector resource constraint

$$b_t + K_{t+1} = R_{t-1}b_{t-1} + (1 - \delta)K_t + (1 - \tau_t) (K_t^{\theta-1}h_t^{1-\theta}) K_t - C_t, \quad (15)$$

where b_t denotes real bonds, the government budget constraint (8) and the monetary and fiscal rules (7), (9). We define an equilibrium a sequence $\{\pi_t, C_t, K_t, h_t, s_t, \tau_t, R_t, b_t\}_{t=0}^{\infty}$ such that (11)-(15) are satisfied and the transversality conditions hold.

2.4 Model Calibration

In order to perform the nonlinear analysis of the model we must calibrate some of its parameters. We fix the numerical values of those parameters that are less controversial in the literature. We set $\theta = 0.3$, consistent with the cost share of capital. Following Schmitt-Grohe and Uribe (2004), we set the steady state tax rate $\bar{\tau} = 0.2$, consistent with the ratio of tax revenues over GDP for the US over the years 1997-2001. We set the discount factor β equal to 0.99, implying an annual discount rate of approximately 4%, and the depreciation of capital, δ , to 0.02. Finally, we set the price elasticity of demand $\eta = 5$, which imply a mark up of 25%, consistent with the empirical findings of Basu and Fernald. Our choice for η implies a somewhat higher mark-up with respect to the rest of the literature, where η can take a value as high as 11⁵. Nevertheless, our results below are invariant to alternative values of η .

This leaves five other parameters to be calibrated. Two of them are related to the structure of the economy : σ and ψ . Different values of σ are found in the literature, ranging from 3 to 1/3. There is also disagreement about an exact measure of price rigidity in the economy. Most of the literature assumes a Calvo type nominal rigidity, allowing for a more direct comparison with the data. Estimated models of nominal rigidities set the probability of not changing prices between 0.66 (Sbordone, 2002) and 0.83 (Gali and Gertler, 1999). Our choice of ψ , a measure of the cost of changing prices in our model, is consistent with this interval. From the linearized Phillips curve, derived in the Appendix, we set $\psi = \frac{\bar{Y}\bar{C}^{-\sigma}\eta\alpha\bar{s}}{(1-\alpha\beta)(1-\alpha)}$, where α is the probability of not changing the price in a Calvo type model of price rigidity. This choice of ψ implies that the *linearized* Phillips curve is *identical* under our Rotemberg specification and the Calvo pricing model, independently of our assumption about price rigidity. In both cases, the linearized Phillips curve becomes

$$\pi_t = \beta\pi_{t+1} + \xi s_t$$

where the parameter ξ measures how changes in real marginal cost impact on inflation. The value of ξ clearly depends on the degree of price rigidity. In the case of Calvo pricing we have

$$\xi = \frac{(1 - \alpha)(1 - \alpha\beta)}{\alpha} \quad (16)$$

⁵We thank the referee for pointing this out.

where α is the probability of *not* changing the price, while in the case of Rotemberg pricing

$$\xi = \frac{\bar{Y}\bar{C}^{-\sigma}\eta\bar{s}}{\psi} \quad (17)$$

where ψ is the adjustment cost parameter. As a consequence of that any local result in the Propositions is valid for both modelling assumption. Moreover, in order to be consistent with most of the literature we are going to define and calibrate price rigidity in terms of α and set ψ so that (16) and (17) are the same.

Notice that this version of the model is equivalent to assuming homogeneous labor and capital markets. As shown in Woodford (2003) and Sveen and Weinke (2004), assuming firm-specific labor and capital markets increases inflation persistence, for a given level of price rigidity. That is

$$\xi = \frac{(1 - \alpha)(1 - \alpha\beta)}{\alpha} A$$

where $A < 1$. Sveen and Weinke's paper shows that the local results in this paper would also hold in the case of firm specific factor markets, if we allow for a different interpretation of the parameters. More precisely, the same results are obtained for *lower* values of α and therefore for a lesser degree of price rigidity.

The Calvo and Rotemberg pricing models are different in their nonlinear components. Our choice of Rotemberg pricing simplifies the nonlinear analysis considerably. In fact, the fully nonlinear capital model under Rotemberg pricing is four dimensional, including inflation, consumption, marginal cost and the capital stock, while under Calvo pricing it is six dimensional, as shown in Schmitt-Grohe and Uribe (2004). The same holds true for the bonds model in section 3.2. Given that the higher order terms are different, the two models might have different implications for global indeterminacy. In other words, the global results discussed in the Propositions below may depend on the assumptions of Rotemberg pricing. We leave the analysis of the nonlinear Calvo model for further research.

The remaining three parameters describe monetary and fiscal policy. In the sections below we discuss how the choice of these parameters affects the existence of multiple equilibria.

3 Active Monetary Policy Rules and Multiple Equilibria

This section discusses the main result of the paper. An active policy rule might not be sufficient to achieve the inflation target and stabilize the economic system. In fact, we show that multiple equilibria arise once we consider the global dynamics of the model. In order to simplify the analysis and the exposition of the results, we consider two cases. First, we discuss the model with capital, abstracting from the fiscal authority, i.e. no government liabilities and no taxation. Second, we consider the model with the government and without capital

accumulation. In later sections we point out that these two models share a very similar source of multiple equilibria.

3.1 The Model with Capital

Let us assume that there is no fiscal authority in the model. Then any of the model's perfect foresight solutions takes the following form

$$Z_{t+1} = F(Z_t)$$

where $Z_{t+1} = [C_{t+1}, s_{t+1}, \pi_{t+1}, K_{t+1}]$, given the initial value for K_0 . The following proposition characterizes the equilibria of the model. In this section we consider a monetary authority that responds only to the inflation rate. This has been advocated to be the (constrained) optimal policy by many authors; see for example Rotemberg and Woodford (1997), Gali and Gertler(1999), Schmitt-Grohe and Uribe (2004,a, b).

Proposition 1 *Consider the model with capital only under the benchmark calibration. For each $\xi(\alpha) \in (\xi(0.77), \xi(0.84))$ there exists a $\bar{\phi}_{\pi-}^{\xi}$ and $\bar{\phi}_{\pi+}^{\xi}$ such that for $1 < \phi_{\pi} < \bar{\phi}_{\pi-}^{\xi}$ and $\phi_{\pi} > \bar{\phi}_{\pi+}^{\xi}$ the equilibrium is locally determinate, and for $\phi_{\pi} \in (\bar{\phi}_{\pi-}^{\xi}, \bar{\phi}_{\pi+}^{\xi})$ the equilibrium is locally indeterminate. Furthermore, for an interval $S_+^1 = (\bar{\phi}_{\pi+}^{\xi}, \bar{\phi}_{\pi+}^{\xi} + \varepsilon)$, $\varepsilon > 0$, there exists a closed invariant curve which bifurcates from the steady state as ϕ_{π} crosses $\bar{\phi}_{\pi+}^{\xi}$ from below, and for initial conditions of capital k_0 close to this invariant curve, there exists a continuum of initial values $\{c_0, x_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve⁶. In addition, for an interval $S_+^2 = (\bar{\phi}_{\pi-}^{\xi}, \bar{\phi}_{\pi-}^{\xi} + \varepsilon)$, $\varepsilon > 0$ there is a "determinate" invariant curve bifurcating from the steady state as ϕ_{π} crosses $\bar{\phi}_{\pi-}^{\xi}$, so that given an initial condition for capital k_0 close to the invariant curve, there exists initial values of $\{c_0, x_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve⁷.*

Proof. See Appendix. ■

Proposition 1 shows that multiple equilibria exist in the case of an active policy rule that achieves local uniqueness or the local determinacy of the equilibrium (cf. Carlstrom and Fuerst (2004)). This clearly indicates the importance of considering the full nonlinear solution of the model.

⁶On the invariant curve the dynamics of the variables may be periodic if the ratio of the angle of rotation to pi is rational. Otherwise the dynamics remain on the curve but will not be exactly periodic. For example in the simple degenerate case of a two dimensional linear system with complex roots $a \pm bi$ of unit modulus, the dynamics of (x_1, x_2) is given by the map $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \cos \theta t + x_2 \sin \theta t \\ x_1 \sin \theta t - x_2 \cos \theta t \end{pmatrix}$ with period $2\pi/\theta$ where $\theta = \tan^{-1}(b/a)$, but $2\pi/\theta$ may be irrational.

⁷Determinate invariant curves, like locally determinate steady states, are particularly interesting from the perspective of learnability. We leave this for further research.

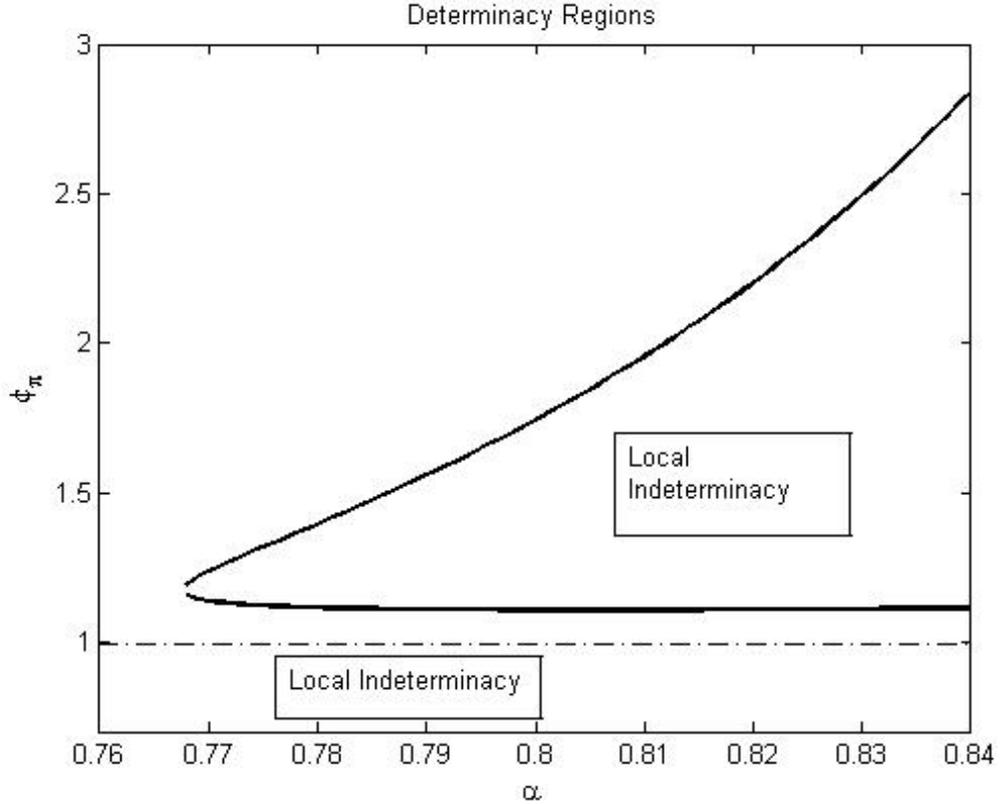


Figure 1

Figure (1) shows the combination of α , the probability of not resetting the price in a Calvo model, and ϕ_π that delivers local determinacy and indeterminacy. Notice first that the relation between α and ϕ_π is a correspondence with two ‘branches’. Also, above the dotted line the Taylor Principle is satisfied. For a given level of price rigidity, local determinacy can be achieved by choosing a sufficiently aggressive response to inflation or by a very mild response very close to one. For intermediate values of ϕ_π indeterminacy occurs, even though the Taylor Principle is satisfied. Proposition 1 states that under the benchmark calibration, for parameter values in a neighborhood above both the lower and the upper branches, there are equilibrium trajectories that converge to and cycle on an invariant curve that bifurcates from the steady state.

Choosing a value of ϕ_π which is close to the lower branch does not seem a realistic option. First, the stability corridor is rather small, since ϕ_π must be higher than one. Second, given the uncertainty about ‘true’ parameter values it likely that the choice of ϕ_π falls in the local indeterminacy region. For this reason, from a policy perspective the upper branch is the most realistic

parameter combination on which to focus the analysis.

One possibility for the central bank however may be to choose extreme values of ϕ_π so the chosen parameter is sufficiently distant from the bifurcation value, as shown in Figure (1). But this could lead to severe instability. In fact, a small mistake in controlling inflation would induce extreme volatility in the policy instrument and therefore output and inflation⁸.

As is clear by Proposition (1), achieving local determinacy might not be sufficient to avoid global indeterminacy. Figure (2) shows a possible equilibrium for the benchmark calibration, where inflation, output and the interest rate converge to a cycle. Note however that the stable manifold of the invariant curve (including the direction along the curve), is of one less dimension than the dimension of the system. Since only one variable, k , is predetermined, the system is globally indeterminate. Theoretically in our proofs, the existence of an invariant curve with a stable manifold of dimension three is established by computing the "Lyapunov" exponent⁹. If this exponent is negative, the dimension of stable manifold corresponds to the number of roots inside the unit circle other than the two complex roots crossing the unit circle at the bifurcation point, plus two. (See the proof of Proposition (1) in the appendix). In simulations however, approximation errors can never make it possible for us to stay on the stable manifold that eliminates divergent trajectories. Thus, as Figure (2) makes clear, we can converge towards and stay arbitrarily close to the invariant curve for a very long time, but eventually, due to approximation errors, one of the variables, notably capital, will start to diverge, pulling along the other variables as well. In the next section, where we analyze the model with bonds but without capital, the stable manifold of the invariant curve will be of full dimension, thereby avoiding this computational problem with simulations.

⁸For further comments on this, see Svensson and Woodford (2003).

⁹Lyapunov exponents, just like characteristic roots in the case of steady states, determine the dimension of the local stable and unstable manifolds of the invariant curve bifurcating from the steady state.

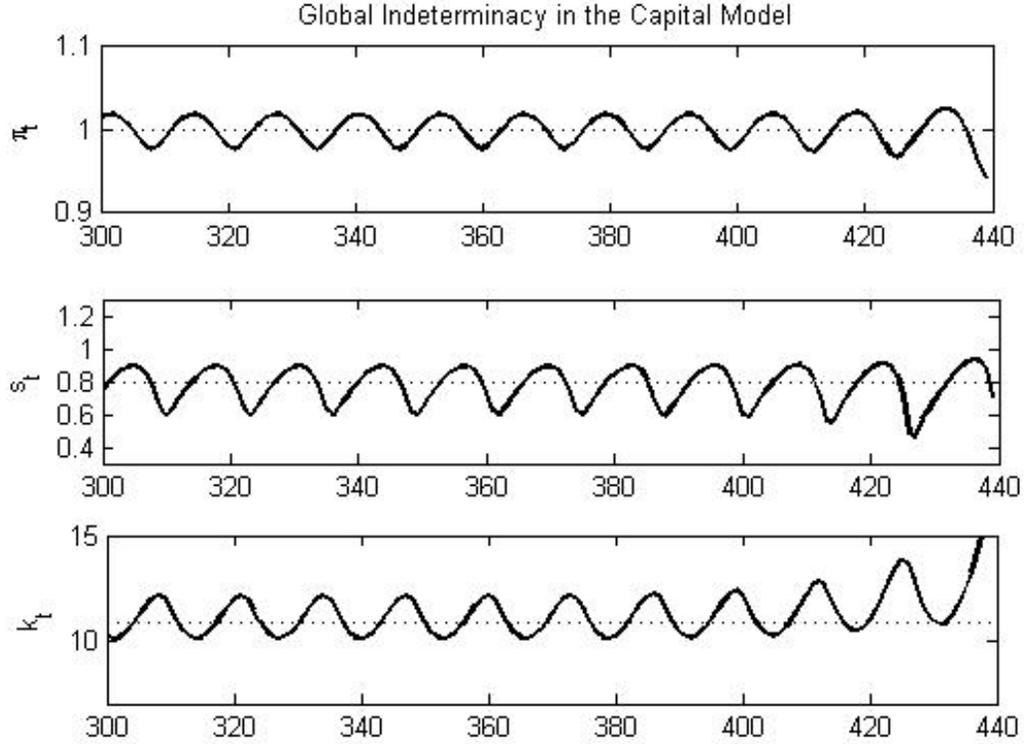


Figure 2

3.1.1 The economic intuition of the local and global indeterminacies

We now turn to an intuitive economic explanation of Proposition 1. We begin, for purposes of exposition, with a much simpler model of the Phillips curve. We assume that marginal costs depend positively on the interest rate, maybe because wages are paid in advance of production, or because real balances held by firms affect output by reducing the real cost of transactions (see Benhabib, Schmitt-Grohe and Uribe (2000)). Under this assumption monetary policy aimed at controlling the nominal interest rate also operates directly on the marginal costs, through the so called "cost channel." In such a model, the Phillips curve is

$$\pi_t = \beta\pi_{t+1} + \xi s(y_t, R(\pi_t)) \quad s_y, s_\pi > 0 \quad (18)$$

Here β is the discount factor, ξ is the parameter positively related to the rigidity of prices and $s(\cdot)$ represents marginal costs that depend, through the

interest rate and the monetary policy rule, on inflation. Let us first consider the case where $s_\pi = 0$ and no cost channel exists. An increase in expected inflation¹⁰ next period increases current inflation and, because of an active Taylor rule, increases the interest rate. Aggregate demand falls, decreasing the marginal cost of production and putting downward pressure on current inflation. From (18), current inflation increases by less than expected inflation. For the initial increase in expected inflation to be self-fulfilling, a further rise in expected inflation in the subsequent period is needed, which, as we iterate forward, would result in the divergence of inflation, violating transversality conditions. In other words, the existence of a cost channel effect is necessary to have indeterminacy in this model.

If $s_\pi > 0$, the initial increase in inflation expectations can lead to an increase in the marginal cost and, if the cost channel is sufficiently strong, to a larger increase in actual inflation. In fact, the increase in the interest rate puts upward pressure on the marginal cost, contributing to a further increase in the current inflation rate. In this case inflation reverts back to the steady state. Expectations are self-fulfilling and the the inflation path is an equilibrium, giving the indeterminacy result¹¹.

While the process generating local and global indeterminacies in a model with capital but without the "cost channel" is related to the mechanism described above, it is more complicated. Consider instead a similar thought experiment, where we start with an increase in expected inflation, and trace the dynamic effects through the Phillips curve of equation 14. Note that the second term on the right of the Phillips curve consists of ψ^{-1} , measuring price rigidity, a composite term $C_t^{-\sigma} (K_t^{\theta-1} h_t^{1-\theta}) K_t \eta (1 - \tau_t)$ that captures output or absorption, and the marginal cost term given in equation 13. The impact effect of a rise in expected inflation is the sum of two effects: the first effect is a rise in expected inflation which raises current inflation by a factor of $\beta < 1$, a direct effect. The second effect under an active Taylor rule is a rise in expected inflation that would raise the real rate, which would then generate a drop in current consumption, as well as a rise in the rate of growth of consumption via the IS curve, or the Euler equation. A higher interest rate would require a higher marginal value product of capital next period, which in part would come about through a decline in the capital stock for the next period, implying a lower investment level today. Thus output today would decline, and, given the current stock of capital, so would the level of employment h_t . The result would be a fall in current consumption, employment and marginal cost s_t , all of which dampen the rise in current inflation, so that on account of both the direct and indirect effects of expected inflation, current inflation rises by less than expected inflation. Following the same logic as in the example based on the the simpler "cost channel" model above, if the expected inflation were to be

¹⁰Since there is no uncertainty in our example we are in a perfect foresight world, and expected inflation corresponds to realized inflation.

¹¹It is also possible for $s_\pi (\pi)$ to be so large that a rise in expected inflation causes a decline in current inflation giving rise to either convergent or divergent oscillations, but we will not pursue this further since our focus is to shed light on the model with capital.

self-fulfilling, a rising inflation rate would require expected inflation to rise by even more in each subsequent period, resulting a divergent trend in inflation. However the fall in investment and capital, together with the recovery of growing consumption, would then begin to raise employment and the marginal cost s in the subsequent periods, eventually reversing the direction of inflation, and resulting in oscillatory dynamics. Such oscillations can either be divergent or convergent. The strength or the amplitude of oscillations will depend, among other things, on ψ^{-1} , which measures the rigidity of prices. If prices are very flexible and ψ is very low, the effect of the second term, embodying output and marginal cost responses, is significantly magnified, and we get divergent oscillations. If ψ is very high and prices are very rigid, then output and marginal cost responses become insignificant for current inflation, so that only the direct effects of expected inflation are operational, and again, they lead to divergence since $\beta < 1$. In some intermediate range for ψ we obtain convergent oscillations and local indeterminacy.

So far the above analysis is local, in the sense that the dynamic effects considered correspond to the dynamics of a system linearized around a steady state. Our particular interest is in the more global analysis of the locally determinate case, where the dynamics of inflation and other variables are characterized by locally divergent oscillations, so that higher order terms start to dominate the dynamics as we move further away from the steady state. These higher order terms may either reinforce the divergence for the range of the parameters that corresponds to local determinacy, or they may contain it, resulting in convergence to a cycle (or more precisely to a circle), so that local determinacy translates into global indeterminacy.

Our analysis of the Lyapunov exponents in the appendix is a formal investigation of this, and we find, for the plausible parameter ranges where we have local determinacy, that we also have a continuum of initial conditions, arbitrarily close to the steady state, giving rise to trajectories converging to a circle. Since dynamics converging to a circle satisfy transversality conditions, we have global indeterminacy. The forces that contain the local divergence are in fact related to the second and higher order terms in the expansion of the Phillips curve¹²: divergence of capital, consumption and labor away from the steady state triggers the containment response in marginal costs, and results in convergence to equilibrium cycles.

For further intuition of the global indeterminacy result, consider a sudden increase in inflation expectations. If the economy were at the steady state, we know from the linearized model that the the only path for inflation which is consistent with the initial increase in expectations is an increasing inflation rate. But as inflation moves sufficiently far from the steady state, the nonlinearities in the Phillips curve start to operate. The increase in the marginal cost has a more pronounced effect on actual inflation, which now becomes *higher* than inflation in the subsequent periods, following the same intuition as in the case of

¹²In fact our simulations confirm that if we artificially eliminate higher order terms from the Phillips curve equation only, the system remains divergent.

local indeterminacy. Therefore, inflation reverses course back toward the steady state. But as inflation is close enough to the steady state, repelling forces drive it away the cycle emerges.

3.2 The Model with Bonds

In this section we discuss the model with only bonds, and we show how most of the results obtained for the capital model also hold in this case.

3.2.1 Constant real bonds rule.

Let us consider the model with the constant real bonds rule. This is the case where $\phi_1 = \phi_0 = 0$ and $\phi_2 = 1$ in equation (9). The solutions of models with bonds only can be expressed as

$$Z_{t+1} = \tilde{F}(Z_t)$$

where $Z_{t+1} = [y_{t+1}, \pi_{t+1}, \pi_t]$. The following proposition discusses the existence of multiple equilibria.

Proposition 2 *Consider the model with bonds and the constant bonds rule under the benchmark calibration. For each $\xi(\alpha) \in (\xi(0.8385), \xi(0.95))$ there exists a $\bar{\phi}_{\pi-}^{\xi}$ and $\bar{\phi}_{\pi+}^{\xi}$ such that for $1 < \phi_{\pi} < \bar{\phi}_{\pi-}^{\xi}$ and $\phi_{\pi} > \bar{\phi}_{\pi+}^{\xi}$ the equilibrium is locally determinate, and for $\phi_{\pi} \in (\bar{\phi}_{\pi-}^{\xi}, \bar{\phi}_{\pi+}^{\xi})$ the equilibrium is locally indeterminate. Furthermore, for an interval $S_+^1 = (\hat{\phi}_{\pi+}^{\xi} - \epsilon, \hat{\phi}_{\pi+}^{\xi})$, $\epsilon > 0$, "determinate" invariant curve bifurcating from the steady state as ϕ_{π} crosses $\hat{\phi}_{\pi+}^{\xi}$, so that given an initial condition for π_{-1} close to the invariant curve, there is a set of initial values of $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve. In addition, if $\xi(\alpha) \in (\xi(0.8385), \xi(0.89))$ for an interval $S_+^2 = (\hat{\phi}_{\pi-}^{\xi}, \hat{\phi}_{\pi-}^{\xi} + \epsilon)$, $\epsilon > 0$ there is a "determinate" invariant curve bifurcating from the steady state as ϕ_{π} crosses $\hat{\phi}_{\pi-}^{\xi}$, so that given an initial condition for π_{-1} close to the invariant curve, there is a set of initial values of $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve. If $\xi(\alpha) \in (\xi(0.89), \xi(0.95))$ for an interval $S_+^1 = (\hat{\phi}_{\pi-}^{\xi} - \epsilon, \hat{\phi}_{\pi-}^{\xi})$, $\epsilon > 0$, there exists a closed invariant curve which bifurcates from the steady state as ϕ_{π} crosses $\hat{\phi}_{\pi-}^{\xi}$ from above, and for initial conditions for π_{-1} close to this invariant curve, there exists a continuum of initial values $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve.*

Proof. See Appendix. ■

The model with balanced budget gives qualitative results identical to the model with capital. Figure (3) shows the determinacy correspondence, as in the case of capital.

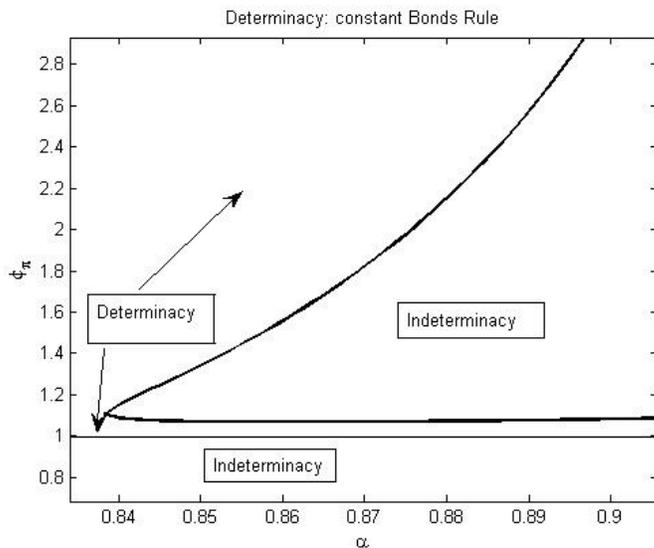


Figure 3

The intuition for the result follows the same logic as for the capital model. Consider an increase in inflationary expectations. The central bank reacts by increasing the real interest rate. This increases the cost of servicing the debt and thus triggers an increase in taxes. Since taxes are distortionary, from (13) the marginal productivity of labor decreases and thus marginal cost goes up. Once again the decrease in aggregate demand and the increase in the marginal cost have opposite effects on the inflation rate, opening the door for multiple equilibria, as described for the capital model.

But the nonlinear implications of the model are different. In fact, if the central bank manages to choose ϕ_π to be in the locally determinate region, we can conjecture that no other equilibria exist, i.e. the steady state is 'globally' unique. The only exception is for values of ϕ_π close to the lower branch bifurcation values, which are not that interesting from a policy point of view, as discussed above. Global uniqueness is a promising feature of this fiscal rule. As shown in the next section, however, a liability targeting rule does induce global indeterminacy.

Summing up, the key to the local result in the Proposition is the link between the nominal interest rate and its effects on the marginal cost of production. In the case of capital the increase in the nominal rate increases the return on capital, which in turns decreases the capital stock, affecting labor productivity. In the bonds case, the increase in the nominal rate increases the return on bonds and thus the cost of servicing the debt. This affects the marginal product of labor via the tax increase necessitated by the tax rule.

3.2.2 Targeting rules for bonds

The case of debt targeting corresponds to setting $\phi_2 = 0$ in equation (9). Debt targeting introduces another policy parameter, ϕ_1 . In this section we discuss the interplay between monetary and fiscal policy and its consequences for multiple equilibria. To begin we consider different values for ϕ_1 and how this choice affects the role of monetary policy in generating multiple equilibria. We consider two cases. First, a low value for ϕ_1 : the fiscal authority is assumed to adjust taxes gradually to changes in total liabilities. Second, a higher value of ϕ_1 which denotes a more aggressive stabilization behavior.

Proposition 3 *Consider the model with bonds and targeting rule under the benchmark calibration.*

(i), *Let the fiscal stance be mild: $\phi_1 = 0.4$. For each $\xi(\alpha) \in (\xi(0.649), \xi(0.95))$ there exist a $\bar{\phi}_{\pi-}^{\xi}$ and $\bar{\phi}_{\pi+}^{\xi}$ such that for $1 < \phi_{\pi} < \bar{\phi}_{\pi-}^{\xi}$ and $\phi_{\pi} > \bar{\phi}_{\pi+}^{\xi}$ the equilibrium is locally determinate, and for $\phi_{\pi} \in (\bar{\phi}_{\pi-}^{\xi}, \bar{\phi}_{\pi+}^{\xi})$ the equilibrium is locally indeterminate. Furthermore, for intervals $S_+^2 = (\hat{\phi}_{\pi+}^{\xi} - \epsilon, \hat{\phi}_{\pi+}^{\xi})$, and $S_+^2 = (\hat{\phi}_{\pi-}^{\xi} + \epsilon, \hat{\phi}_{\pi-}^{\xi})$, $\epsilon > 0$ there is a "determinate" invariant curve bifurcating from the steady state as ϕ_{π} crosses $\hat{\phi}_{\pi+}^{\xi}$ and $\hat{\phi}_{\pi-}^{\xi}$, so that given an initial condition for L_0 close to the invariant curve, there is a set of initial values of $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve.*

(ii), *Let the fiscal stance be aggressive: $\phi_1 = 1.7$. For each $\xi(\alpha) \in (\xi(0.45), \xi(0.95))$ there exists a $\bar{\phi}_{\pi-}^{\xi}$ and $\bar{\phi}_{\pi+}^{\xi}$ such that for $1 < \phi_{\pi} < \bar{\phi}_{\pi-}^{\xi}$ and $\phi_{\pi} > \bar{\phi}_{\pi+}^{\xi}$ the equilibrium is locally determinate, and for $\phi_{\pi} \in (\bar{\phi}_{\pi-}^{\xi}, \bar{\phi}_{\pi+}^{\xi})$ the equilibrium is locally indeterminate. Furthermore, for an interval $S_+^1 = (\hat{\phi}_{\pi+}^{\xi}, \hat{\phi}_{\pi+}^{\xi} + \epsilon)$, $\epsilon > 0$, there exists a closed invariant curve which bifurcates from the steady state as ϕ_{π} crosses $\hat{\phi}_{\pi+}^{\xi}$ from below, and for initial conditions of L_0 close to this invariant curve, there exists a continuum of initial values $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve. In addition, for an interval $S_+^2 = (\hat{\phi}_{\pi-}^{\xi} + \epsilon, \hat{\phi}_{\pi-}^{\xi})$, $\epsilon > 0$ there is a "determinate" invariant curve bifurcating from the steady state as ϕ_{π} crosses $\hat{\phi}_{\pi-}^{\xi}$, so that given an initial condition for π_{-1} close to the invariant curve, there is a set of initial values of $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve.*

Proof. See Appendix. ■

As the Proposition shows, there is an important difference in the two fiscal approaches which would not be detected if we restricted the analysis to the linearized model. In fact, a mild response to variations in total government liabilities may guarantee a globally unique equilibrium, provided monetary policy is chosen to guarantee local determinacy. An active monetary policy may not be enough to stabilize the economy, if fiscal policy is aggressive. In fact, the

Proposition shows that multiple equilibria exist even if the equilibrium is locally determinate. Figure (4) shows a possible equilibrium when monetary policy is active and fiscal policy is aggressive. In the Appendix we show the determinacy correspondences for both case (a) and (b).

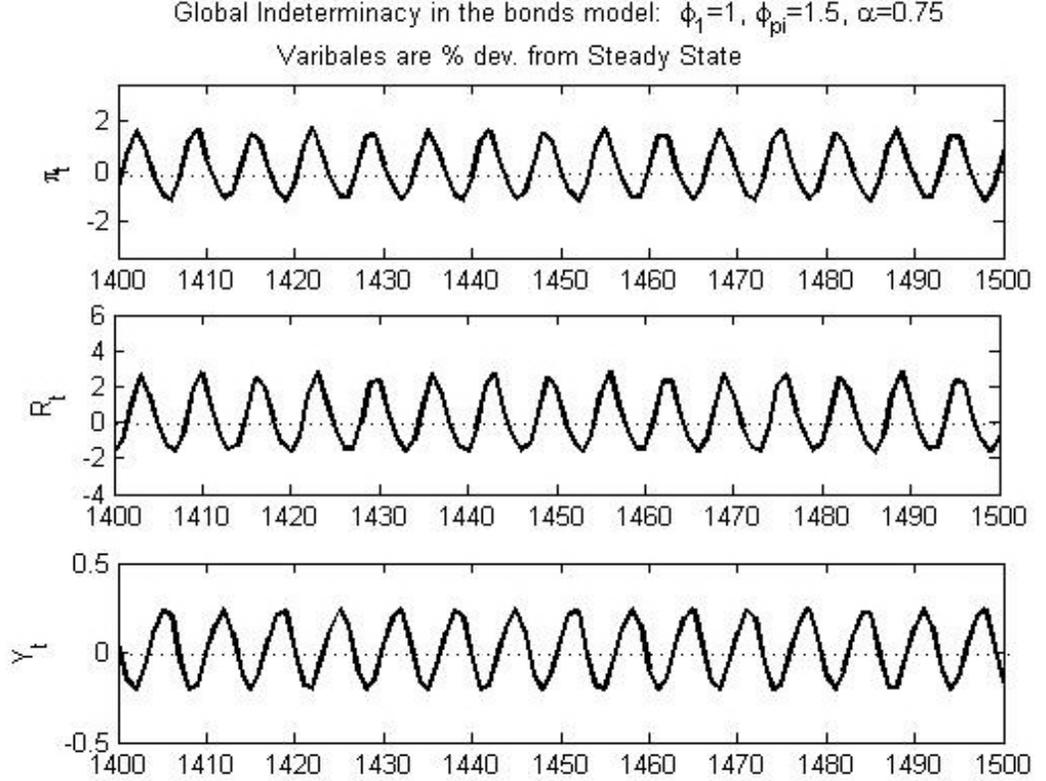


Figure 4

We now fix the coefficient of the monetary policy rule ϕ_π and explore the existence of multiple equilibria as we vary the fiscal parameter.

Proposition 4 Consider the model under benchmark calibration and $\phi_\pi = 1.5$. For each $\xi(\alpha) \in (\xi(0.745), \xi(0.95))$ there exist $1 - \beta < \phi_1^\xi$ such that for $1 - \beta < \phi_1 < \phi_1^\xi$ the equilibrium is locally determinate, and for $\phi_1 > \phi_{1+}^\xi$ the equilibrium is indeterminate. Moreover,

(a) If $\xi \in (\xi(0.7487), \xi(0.777))$, for an interval $S_+^1 = (\hat{\phi}_1^\xi - \epsilon, \hat{\phi}_1^\xi)$, $\epsilon > 0$, there exists a closed invariant curve which bifurcates from the steady state as ϕ_1 crosses $\hat{\phi}_1^\xi$ from above, and for initial conditions of b_0 close to this invariant curve, there exists a continuum of initial values $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve.

(b) if $\xi \in (\xi(0.745), \xi(0.7486)) \cup (\xi(0.778), \xi(0.95))$, or an interval $S_+^2 = (\hat{\phi}_1^\xi, \hat{\phi}_1^\xi + \epsilon)$, $\epsilon > 0$ there is a "determinate" invariant curve bifurcating from the steady state as ϕ_π crosses $\hat{\phi}_1^\xi$, so that given an initial condition for b_0 close to the invariant curve, there exists initial values of $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve.

Proof. See Appendix. ■

For values of ϕ_1 which are less than $-1 + \beta$ no equilibrium exists. As we increase ϕ_1 indeterminacy arises. Figure 5 shows the determinacy area as a function of the fiscal parameter and the measure of price rigidity. The function $\bar{\phi}_1$ is defined in the Appendix.

Considering global indeterminacy, if ϕ_1 is chosen to be very aggressive multiple equilibria may disappear. Notice that the result is obtained for values of ϕ_1 that make fiscal policy active, in the sense of Leeper (1991). Choosing both fiscal and monetary policy active may achieve a unique equilibrium.

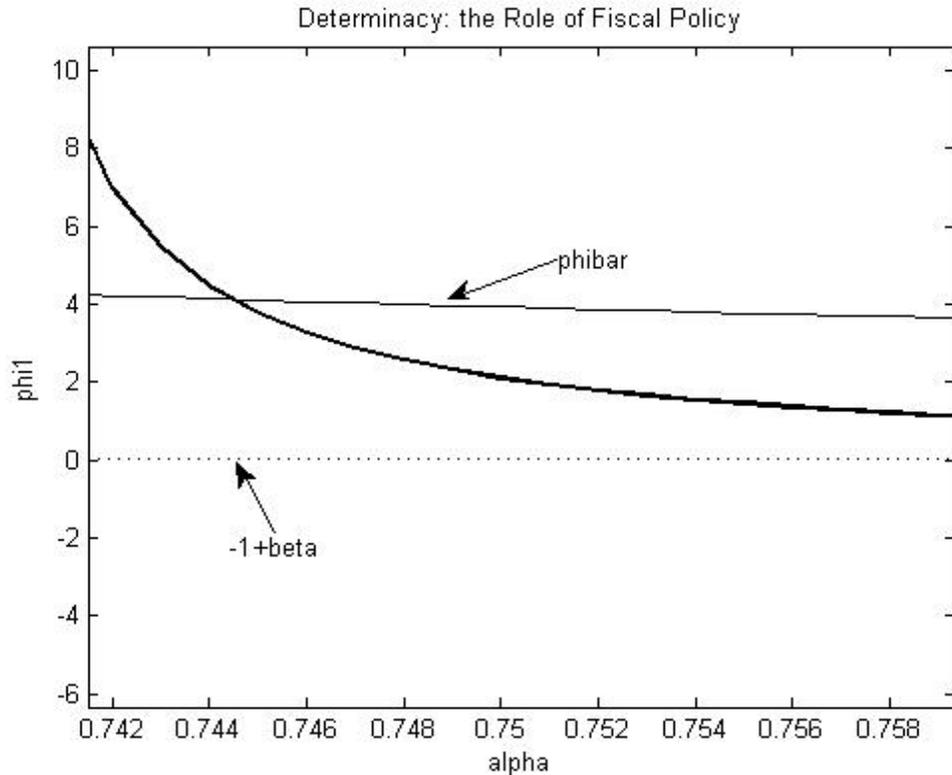


Figure 5

3.3 Sensitivity Analysis

The results in the Propositions above refer to a benchmark calibration, which is broadly consistent with the literature. Nevertheless there is uncertainty about some of the calibrated parameters, especially σ . It is therefore worth investigating how sensitive our results are with respect to this parameter. For the case of capital, choosing σ away from the benchmark value of $\phi_\pi = 1.5$ does not affect the global results. Similarly, as pointed out earlier, our results and analysis are invariant to alternative choices of η . In particular, the multiple equilibria continue exist for the parameter space that yields local determinacy.

In the case of bonds, σ affects global indeterminacy. For the model with the targeting rule, low values of σ induce global indeterminacy also in the case where $\phi_1 = 0.4$. Hence, the possibility that a gradual adjustment to changes in bonds can rule out global indeterminacy is not robust to different choices of σ . Also, in the model with the constant bond rule, high values of σ (*i.e.* $\sigma = 3$) imply that for a large portion of the parameter space global indeterminacy can arise, thus qualifying the results in Proposition (2).

Summing up, the results are somewhat sensitive to different choices of σ , but also indicate that for a broad set of parameter values global indeterminacy is an issue in this type of models.

3.4 Equivalence between the bond only and capital only models

In the previous sections we showed that the mechanism generating multiple equilibria is the same in the two models, and arises from the direct effects of monetary policy on the marginal cost. In effect, the result holds whenever the model displays a sufficient link between asset accumulation and marginal cost. Changes in the monetary instrument affect the nominal interest rate and, via arbitrage conditions, changes the path of asset accumulation. In both models the decline in the asset increases the marginal cost. In the case of capital this occurs because the decline in capital is followed by the recovery of consumption and employment. In the case of bonds this happens as the fiscal authority increases taxes to respond for an increased cost in servicing the debt.

The equivalence between the two models is best understood by comparing the linearized equation for the marginal cost for the model with capital and the model with constant bonds. Consider the model with capital and set δ (the depreciation) equal to zero. We can rewrite the log linearized arbitrage equation for capital as:

$$\hat{s}_{t+1} = \frac{\theta}{1-\beta} \phi_\pi \pi_t - \frac{\theta}{1-\beta} \pi_{t+1} + (1-\theta) \sigma \hat{c}_{t+1}$$

The Euler equation for consumption and the Phillips curve are exactly the same as for the case with bonds. Consider now the equation that we obtain combining the marginal cost with the evolution of taxes for the case of balanced budget and bonds. We get

$$\hat{s}_t = \frac{\tau}{1-\beta} \phi_\pi \pi_{t-1} - \frac{\tau}{1-\beta} \pi_t + \left[(1-\tau)\sigma + \frac{(\tau-\theta)}{1-\theta} \right] \hat{c}_t$$

Note that τ measures how marginal cost is affected by changes in distortionary taxation, while θ in the model with capital measures the share of capital, or how marginal cost is affected by changes in the return to capital. If $\tau = \theta$, then the two equations are exactly the same.

3.5 The Role of the Output Gap

Previous results indicate that active monetary policy might generate policy induced fluctuations that are welfare reducing. Evaluating the performance of monetary policy rules by restricting the attention to the locally unique equilibrium, even if the analysis is conducted on a nonlinear approximation of the model's steady state, might lead to misleading results. As mentioned above, many authors conclude that the optimal policy rule (within the class of simple rules) puts a zero coefficient on the output gap.

In contrast, our results indicate that, unless extreme monetary and fiscal policies are adopted, a policy rule that responds only to actual inflation can lead to welfare reducing outcomes. Moreover, as Figure (6) shows for the model with bonds, analysis of the full nonlinear model shows some benefit may arise from responding to output. In fact, a sufficient response to the output gap helps eliminate *local* indeterminacy: a sufficiently aggressive response shifts the 'indeterminacy' frontier southward. A higher degree of price rigidity is now required for local indeterminacy to occur, for a given combination of fiscal and monetary policy.

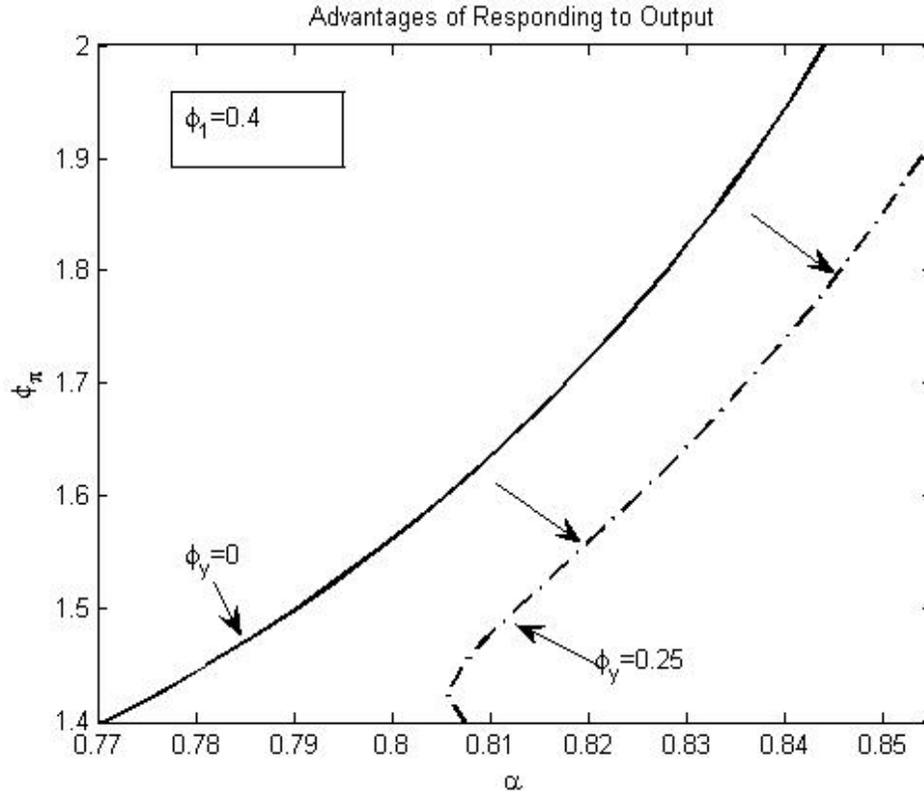


Figure 6

By reacting sufficiently strongly to output, the indeterminacy region shrinks to a parameter region that implies far too much rigidity than observed in actual economies¹³. As a caveat, the response to output must be non-negligible. As Figure (7) shows if the monetary authority does not respond sufficiently to output, the situation can actually get worse! In fact the frontier shift inwards.

¹³In a model with money, an excessive response to the output gap might lead to local indeterminacy, as shown by Eusepi (2003) and SU (2004). Nevertheless, numerical simulations (under the benchmark calibration) show that with a coefficient as low as 0.3 produces a sufficient shift in the local indeterminacy frontier to make indeterminacy unplausible .

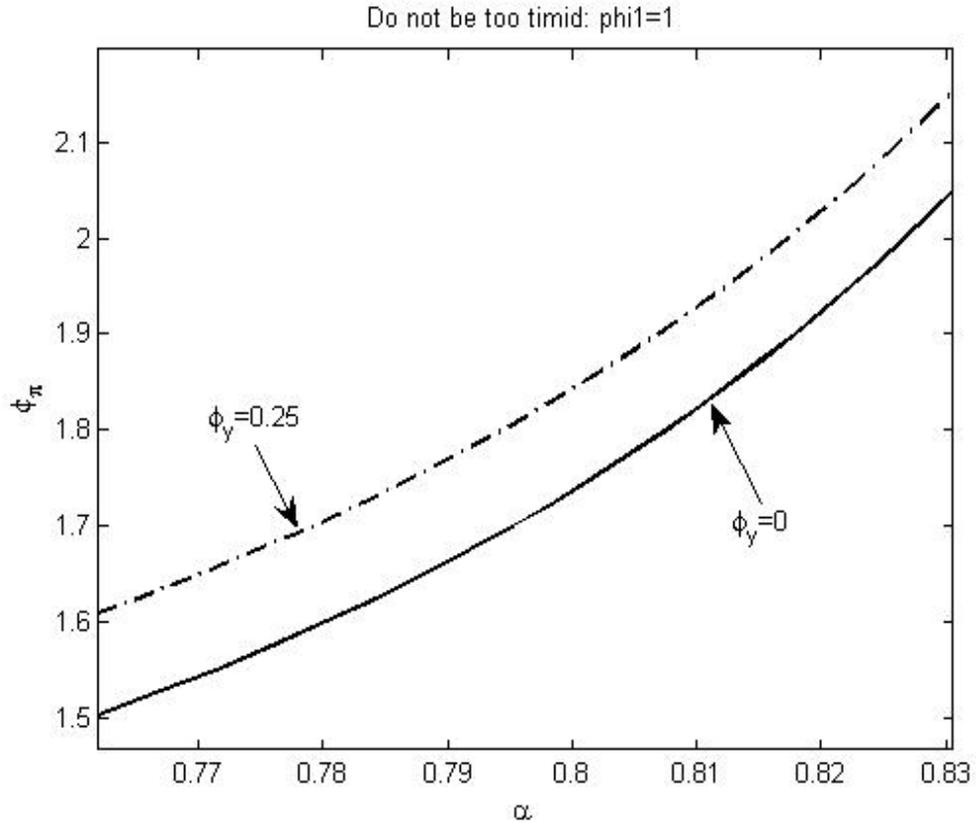


Figure 7

In the model with capital, numerical simulations show that a very small response to output shift widely the indeterminacy frontier. Depending on the calibration, a response as small as 0.1 is indeed sufficient to guarantee a locally unique equilibrium. Naturally, different and more complex models environment may require stronger responses.

Notice again that the above results have only *local* validity. Global equilibria still exists for parameter combinations that guarantee *local determinacy* north of the frontier, threatening economic instability. In fact, it can be shown that the global results discussed in the previous sections for both the model with capital and bonds remain valid, despite a positive response to output.

Summing up, adopting a monetary policy rule that responds to output may reduce welfare if the economy is at the locally unique equilibrium induced by the minimum state variable solution, but if we take into account the possibility of other welfare-reducing equilibria, a response to output may turn out to be stabilizing, and therefore welfare improving. Nevertheless, the caveat is that this policy rule does not necessary guarantee a globally unique equilibrium.

4 Appendix

4.1 Model Solution

The consumer-producer problem is described as follows

$$\max_{B_{jt}, K_{jt+1}, h_{jt}, P_{jt}} \sum_{s=t}^{\infty} \beta^{s-t} \left\{ \frac{\left[-\frac{B_{js}}{P_s} + \frac{R_{s-1}}{\pi_s} \frac{B_{js-1}}{P_{s-1}} + (1 - \tau_s) \left(\frac{P_{st}}{P_s} \right)^{1-\eta} a_s - K_{js+1} + (1 - \delta) K_{js} \right]^{1-\sigma}}{1 - \sigma} - h_{js} - \frac{\psi}{2} \left(\frac{P_{js}}{P_{js-1}} - 1 \right)^2 + \lambda_s \left[K_{js}^\theta h_{js}^{1-\theta} - \left(\frac{P_{js}}{P_s} \right)^{-\eta} a_s \right] \right\}$$

given $B_{j,t-1}, K_{j,t}$ and the transversality condition. The first order condition for $B_{j,t}$ is equation (11). The first order condition for $K_{j,t+1}$ is the equation

$$C_{jt}^{-\sigma} = \beta C_{jt+1}^{-\sigma} (\lambda_{t+1} C_{jt+1}^\sigma \theta K_{jt+1}^{\theta-1} h_{jt+1}^{1-\theta} + 1 - \delta).$$

The first order condition with respect to hours worked is

$$s_t = \frac{\lambda_t}{C_{jt}^{-\sigma} (1 - \tau_t)} = \frac{C_{jt}^\sigma (1 - \theta) K_t^\theta h_t^{-\theta}}{(1 - \tau_t)}$$

where s_t is the real marginal cost. Finally, the first order condition with respect to the price gives the Phillips curve (14). We assume a symmetric equilibrium where all agents choose the same consumption/production paths.

4.2 Local Determinacy: The Linearized Models

4.2.1 Model with Capital

By log-linearizing the solution, we get the following equations for consumption, marginal cost and inflation:

$$\hat{c}_t = -\frac{1}{\sigma} \phi_\pi \hat{\pi}_t + \frac{1}{\sigma} \hat{\pi}_{t+1} + \hat{c}_{t+1} \quad (19)$$

$$\hat{s}_{t+1} = -\frac{(\pi_{t+1} - \phi_\pi \pi_t) \theta \beta^{-1}}{(\beta^{-1} - 1 + \delta)} + \sigma (1 - \theta) \hat{c}_{t+1}$$

$$\pi_t = \beta \pi_{t+1} + \xi \hat{s}_t$$

where

$$\xi = \frac{\bar{Y} \bar{C}^{-\sigma} \eta \bar{s}}{\psi}$$

measures the degree of nominal rigidity.

As shown by Carlstrom and Fuerst (2004) the capital equation is decoupled from the remaining equations in the system. They also show that the coefficient on the differential $dK_{t+1}/dK_t > 1$ for every parameter configuration. Hence, the local determinacy of the system is decided by the local stability properties of the following sub system

$$A_0^K \begin{bmatrix} c_{t+1} \\ \pi_{t+1} \\ s_{t+1} \end{bmatrix} = A_1^K \begin{bmatrix} c_t \\ \pi_t \\ s_t \end{bmatrix}$$

where

$$A_0^K = \begin{bmatrix} 1 & \frac{1}{\sigma} & 0 \\ 0 & \beta & 0 \\ -\sigma(1-\theta) & \frac{\beta^{-1}\theta}{(\beta^{-1}-1+\delta)} & 1 \end{bmatrix}$$

$$A_1^K = \begin{bmatrix} 1 & \frac{1}{\sigma}\phi_\pi & 0 \\ 0 & 1 & -\xi \\ 0 & \frac{\phi_\pi\beta^{-1}\theta}{(\beta^{-1}-1+\delta)} & 0 \end{bmatrix}.$$

Therefore, the Jacobian becomes

$$J_K = (A_0^K)^{-1} A_1^K \quad (20)$$

Given that capital is predetermined, local determinacy requires that two eigenvalues of (20) to be outside the unit circle and one inside.

Proposition 1 *Consider the model with capital only under the benchmark calibration. For each $\xi(\alpha) \in (\xi(0.77), \xi(0.84))$ there exists a $\bar{\phi}_{\pi-}^\xi$ and $\bar{\phi}_{\pi+}^\xi$ such that for $1 < \phi_\pi < \bar{\phi}_{\pi-}^\xi$ and $\phi_\pi > \bar{\phi}_{\pi+}^\xi$ the equilibrium is locally determinate, and for $\phi_\pi \in (\bar{\phi}_{\pi-}^\xi, \bar{\phi}_{\pi+}^\xi)$ the equilibrium is locally indeterminate. Furthermore, for an interval $S_+^1 = (\bar{\phi}_{\pi+}^\xi, \bar{\phi}_{\pi+}^\xi + \varepsilon)$, $\varepsilon > 0$, there exists a closed invariant curve which bifurcates from the steady state as ϕ_π crosses $\bar{\phi}_{\pi+}^\xi$ from below, and for initial conditions of capital k_0 close to this invariant curve, there exists a continuum of initial values $\{c_0, x_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve. In addition, for an interval $S_+^2 = (\bar{\phi}_{\pi-}^\xi, \bar{\phi}_{\pi-}^\xi + \varepsilon)$, $\varepsilon > 0$ there is a "determinate" invariant curve bifurcating from the steady state as ϕ_π crosses $\bar{\phi}_{\pi-}^\xi$, so that given an initial condition for capital k_0 close to the invariant curve, there is a set of initial values of $\{c_0, x_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve.*

Proof. The Jacobian J^K can be computed as:

$$J^K = \begin{bmatrix} 1 & \frac{1}{\sigma}\phi_\pi - \frac{1}{\sigma\beta} & \frac{1}{\sigma\beta}\xi \\ 0 & \frac{1}{\beta} & -\frac{1}{\beta}\xi \\ \sigma(1-\theta) & \phi_\pi \left(1-\theta + \theta \frac{1}{1-\beta(1-\delta)}\right) + -c & \xi c \end{bmatrix}$$

where $c = \frac{(\beta^{-1}-1+\theta+\delta-\theta\delta)}{(1-\beta(1-\delta))}$. The characteristic equation is

$$P(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \quad (21)$$

We will start by using the necessary and sufficient conditions provided by Woodford (2003) for $P(\lambda)$ to have two roots outside and one root inside the unit circle. The three mutually exclusive conditions are, either:

$$1. \quad P(-1) < 0 \text{ and } P(1) > 0 \quad (22)$$

or

$$2. \quad P(-1) > 0 \text{ and } P(1) < 0 \quad \text{and} \quad (a_0)^2 - a_0a_2 + a_2 - 1 > 0 \quad (23)$$

or

$$3. \quad P(-1) > 0 \text{ and } P(1) < 0 \quad \text{and} \quad (a_0)^2 - a_0a_2 + a_2 - 1 < 0 \quad \text{and} \quad |a_2| > 3 \quad (24)$$

We can show, by evaluating the characteristic equation 21 of J^K in our model, that $P(-1) > 0$, and if $\phi_\pi > 1$, that $P(1) < 0$. So we have to consider only cases 2 and 3. For our benchmark parametrization, the values of ϕ_π and ξ such that $(a_0)^2 - a_0a_2 + a_2 - 1 = 0$ must satisfy

$$b_2(\xi)\phi_\pi^2 + b_1(\xi)\phi_\pi + b_0(\xi) = 0 \quad (25)$$

where

$$b_2(\xi) = \frac{\theta^2\xi}{\beta(1-\beta+\beta\delta)}$$

$$b_1(\xi) = -\frac{\theta(\gamma_2 + \xi\gamma_1)}{\beta(1-\beta+\beta\delta)} + \gamma_1$$

$$\gamma_1 = 1 - \beta + \theta\beta + \theta\delta - \theta\beta\delta > 0$$

$$\gamma_2 = 1 + \delta\beta - \beta^2 + \beta^2\delta > 0$$

$$b_0(\xi) = \theta + \frac{1}{\xi} [(1-\beta)(1-\beta+\beta\delta)]$$

Consider the region of $\xi(\alpha) \in (\xi(0.77), \xi(0.84)) = S$. Given $\xi \in S$, we can solve for ϕ_π that satisfies 25. The two solution branches are given by

$$\hat{\phi}_{\pi_1}^\xi = \frac{-b_1(\xi) - \sqrt{b_1(\xi)^2 - 4b_2(\xi)b_0(\xi)}}{2b_2(\xi)} > 1 \quad (26)$$

$$\hat{\phi}_{\pi_2}^\xi = \frac{-b_1(\xi) + \sqrt{b_1(\xi)^2 - 4b_2(\xi)b_0(\xi)}}{2b_2(\xi)} > 1 \quad (27)$$

(see Figure 1). For our benchmark parametrization, we also compute that for $\xi \in S$, $|a_2| < 3$. Thus, focusing on the first branch, for small $\varepsilon > 0$ and a pair $(\hat{\phi}_{\pi_+}^\xi + \varepsilon, \xi)$, $\xi \in S$, 22 holds and 21 has two roots outside and one root inside the unit curve. For $(\hat{\phi}_{\pi_-}^\xi - \varepsilon, \xi)$ on the other hand, neither 22, nor 23, nor 24 holds: in particular we have $P(-1) > 0$, $P(1) < 0$, $(a_0)^2 - a_0 a_2 + a_2 - 1 < 0$ and $|a_2| < 3$. Thus as $\phi_{\pi_1}^\xi$ crosses from $S_-^1 = (\hat{\phi}_{\pi_1}^\xi - \varepsilon, \hat{\phi}_{\pi_1}^\xi)$ into $S_+^1 = (\hat{\phi}_{\pi_1}^\xi, \hat{\phi}_{\pi_1}^\xi + \varepsilon)$, we must have a change in stability, and the modulus of pair of complex roots must cross unity from below, since $P(-1) > 0$ and $P(1) < 0$ for $\phi_{\pi_1} > 1$. This is the standard case of a discrete time Hopf bifurcation, provided certain additional conditions hold (see Kuznetsov (1998), chapter 4, pages 125-137 and 183-186, chapter 5).

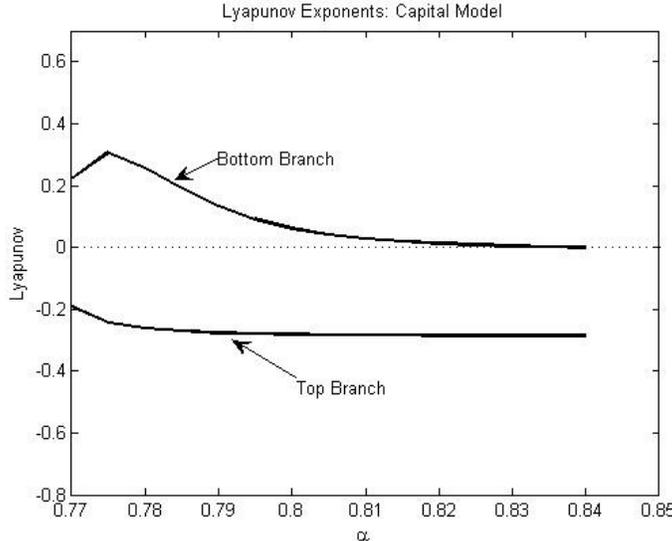


Figure 8

To check these conditions we must compute the related Lyapunov exponent for each $\xi \in S$ (see Figure 8) for the full four dimensional non-linear system. Since the Lyapunov exponent is negative, the invariant curve for $\xi \in S_+^1$ has a three dimensional stable manifold. It inherits one of the stable real roots of of the linearized system, and the two unstable complex roots of the linearized system induce an additional a two dimensional stable manifold for the bifurcating invariant curve. Thus the invariant curve has a locally stable manifold of dimension 3, including the dimension along the curve¹⁴. We note however that

¹⁴As it is clear from Carstrom and Fuest (2003), the parameter σ does not play any role for local determinacy. But it could still affect the criticality of the bifurcation through the higher order terms. Given our uncertainty about this parameter, it is interesting to check the robustness of the result for different values of σ .

the full system is four dimensional and includes capital, but the linearization at the steady state decouples into a separate three dimensional system, and an additional equation for the local dynamics of capital. Therefore when we simulate the four dimensional non-linear system, the invariant curve bifurcates with a three dimensional stable manifold in a four dimensional space.

For the second branch, the argument for the Hopf bifurcation is identical, except for one difference. As $\phi_{\pi_-}^\xi$ crosses from $S_-^2 = (\hat{\phi}_{\pi_2}^\xi - \varepsilon, \hat{\phi}_{\pi_2}^\xi)$ into $S_+^2 = (\hat{\phi}_{\pi_2}^\xi, \hat{\phi}_{\pi_2}^\xi + \varepsilon)$, we have a change in stability, and the modulus of pair of complex roots must cross unity this time from *above*: increasing $\phi_{\pi_-}^\xi$ moves us from the locally determinate to the locally indeterminate region (see Figure 1). Furthermore the associated Lyapunov exponent is now positive (see Figure 8). This implies that there is another invariant curve that lies in the locally indeterminate region $S_+^2 = (\hat{\phi}_{\pi_2}^\xi, \hat{\phi}_{\pi_2}^\xi + \varepsilon)$. This curve is now repelling, or determinate, in the sense that within the four dimensional space, it has a one dimensional stable manifold, but it surrounds a locally indeterminate steady state. ■

4.2.2 Model with Bonds: Constant Liability Rule

In this model government liabilities (i.e. government bonds) are constant in equilibrium. The dynamics of the economy is described by the tax rule, the consumption equation (19) and by the Phillips curve. The linearized tax rule is

$$\hat{\tau}_t = -\hat{y}_t + \frac{1}{(1-\beta)} (\phi_\pi \pi_{t-1} - \pi_t) \quad (28)$$

which expresses a link between past inflation (via Taylor rule) and the current amount of distortionary taxes. The Phillips curve includes now taxes, because they affect the marginal cost. We get

$$\pi_t = \beta \pi_{t+1} + \xi \left[\sigma (1 - \bar{\tau}) + \frac{\gamma (1 - \bar{\tau})}{(1 - \theta)} - (1 - \bar{\tau}) \right] \hat{y}_t + \frac{\xi \bar{\tau}}{(1 - \bar{\tau})} \hat{\tau}_t. \quad (29)$$

By inserting (28) in (29) we get the inflation equation

$$\pi_t = \tilde{\beta} \pi_{t+1} + \xi \rho \hat{y}_t + \frac{\xi \bar{\tau} \phi_\pi}{1 - \beta + \xi \bar{\tau}} \pi_{t-1}$$

where

$$\begin{aligned} \tilde{\beta} &= \left[\frac{\beta (1 - \beta)}{(1 + \xi \bar{\tau}) - \beta} \right] \\ \rho &= \left[\sigma (1 - \bar{\tau}) + \frac{\gamma (1 - \bar{\tau})}{(1 - \theta)} - 1 \right]. \end{aligned}$$

Local determinacy depends on the stability properties of the system

$$A_0^C \begin{bmatrix} c_{t+1} \\ \pi_{t+1} \\ \pi_t \end{bmatrix} = A_1^C \begin{bmatrix} c_t \\ \pi_t \\ \pi_{t-1} \end{bmatrix}$$

where

$$A_0^C = \begin{bmatrix} 1 & \frac{1}{\sigma} & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_1^C = \begin{bmatrix} 1 & \frac{1}{\sigma}\phi_\pi & 0 \\ -\xi\rho & 1 & -\frac{\xi\bar{\tau}\phi_\pi}{1-\beta+\xi\bar{\tau}} \\ 0 & 1 & 0 \end{bmatrix}$$

Local determinacy requires that the Jacobian

$$J_C = (A_0^C)^{-1} A_1^C$$

has two eigenvalues outside the unit circle and one eigenvalue inside.

Proposition 2 *Consider the model with bonds and constant bonds rule under the benchmark calibration. For each $\xi(\alpha) \in (\xi(0.8385), \xi(0.95))$ there exists a $\bar{\phi}_{\pi-}^\xi$ and $\bar{\phi}_{\pi+}^\xi$ such that for $1 < \phi_\pi < \bar{\phi}_{\pi-}^\xi$ and $\phi_\pi > \bar{\phi}_{\pi+}^\xi$ the equilibrium is locally determinate, and for $\phi_\pi \in (\bar{\phi}_{\pi-}^\xi, \bar{\phi}_{\pi+}^\xi)$ the equilibrium is locally indeterminate. Furthermore, for an interval $S_+^1 = (\hat{\phi}_{\pi+}^\xi - \epsilon, \hat{\phi}_{\pi+}^\xi)$, $\epsilon > 0$, "determinate" invariant curve bifurcating from the steady state as ϕ_π crosses $\hat{\phi}_{\pi+}^\xi$, so that given an initial condition for π_{-1} close to the invariant curve, there is a set of initial values of $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve. In addition, if $\xi(\alpha) \in (\xi(0.8385), \xi(0.89))$ for an interval $S_+^2 = (\hat{\phi}_{\pi-}^\xi, \hat{\phi}_{\pi-}^\xi + \epsilon)$, $\epsilon > 0$ there is a "determinate" invariant curve bifurcating from the steady state as ϕ_π crosses $\hat{\phi}_{\pi-}^\xi$, so that given an initial condition for π_{-1} close to the invariant curve, there is a set of initial values of $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve. If $\xi(\alpha) \in (\xi(0.89), \xi(0.95))$ for an interval $S_+^1 = (\hat{\phi}_{\pi-}^\xi - \epsilon, \hat{\phi}_{\pi-}^\xi)$, $\epsilon > 0$, there exists a closed invariant curve which bifurcates from the steady state as ϕ_π crosses $\hat{\phi}_{\pi-}^\xi$ from above, and for initial conditions for π_{-1} close to this invariant curve, there exists a continuum of initial values $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve.*

Proof. The Jacobian J^C can be computed as:

$$J^C = \begin{bmatrix} \frac{(\sigma\beta + \xi\rho)}{\beta\sigma} & \frac{(\tilde{\beta}\phi_\pi - 1)}{\tilde{\beta}\sigma} & \frac{\phi_\pi\xi\tau}{\sigma\tilde{\beta}(\tau\xi - \beta + 1)} \\ -\frac{\rho\xi}{\tilde{\beta}} & \tilde{\beta}^{-1} & \frac{\phi_\pi\xi\tau}{\tilde{\beta}(\tau\xi - \beta + 1)} \\ 0 & 1 & 0 \end{bmatrix}.$$

The characteristic equation is

$$P(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \quad (30)$$

and, as in Proposition (1), we make use of Woodford (2003). By evaluating the characteristic equation we obtain

$$P(1) = \frac{(\phi_\pi - 1)\xi\rho}{\beta\sigma} > 0 \text{ if } \phi_\pi > 0$$

and

$$P(-1) = -\frac{(2\sigma + \xi\rho + 2\sigma\tau\xi - \beta\xi\rho + \xi\rho\phi_\pi + 2\sigma\tau\xi\phi_\pi - \beta\xi\rho\phi_\pi - 2\sigma\beta^2)}{\sigma\beta(1 - \beta)} < 0$$

As in the capital model, we have to consider only cases 2 and 3 in Proposition (1). The values of ϕ_π and ξ such that $(a_0)^2 - a_0a_2 + a_2 - 1 = 0$ must satisfy

$$\phi_\pi^2 + b_1(\xi)\phi_\pi + b_0(\xi) = 0 \quad (31)$$

where

$$b_1(\xi) = \frac{(\beta\rho - \sigma\tau + \sigma\beta\tau - \tau\xi\rho - 2\beta^2\rho + \beta^3\rho + \beta\tau\xi\rho - \sigma\tau^2\xi)}{\tau^2\xi\sigma}$$

$$b_0(\xi) = \frac{\sigma\beta(\beta - 1)^2(1 - \beta) + (1 + \tau\xi)\beta(1 - \beta)}{\sigma\tau^2\xi^2}$$

Notice that imposing $\tau = \theta, \sigma = 1$, (31) takes the same *form* as in the case of the capital model with $\delta = 0$ (even though ξ has different values). Consider the region of $\xi(\alpha) \in (\xi(0.872), \xi(0.95)) = S$. Given $\xi \in S$, we can solve for ϕ_π that satisfies 31. The two solution branches are given by

$$\hat{\phi}_{\pi-}^\xi = \frac{-b_1(\xi) - \sqrt{b_1(\xi)^2 - 4b_0(\xi)}}{2} > 1 \quad (32)$$

$$\hat{\phi}_{\pi+}^\xi = \frac{-b_1(\xi) + \sqrt{b_1(\xi)^2 - 4b_0(\xi)}}{2} > 1 \quad (33)$$

(see Figure 3). For our benchmark parametrization, we also compute that for $\xi \in S$, $|a_2| < 3$. The rest of the proof follows Proposition (1). From Figure 9, we know that the Lyapunov exponent is positive at bifurcation values of ϕ_π corresponding to the upper branch. The Lyapunov exponent is positive for the lower branch for $\xi(\alpha) < \xi(0.89)$ but then it becomes negative.

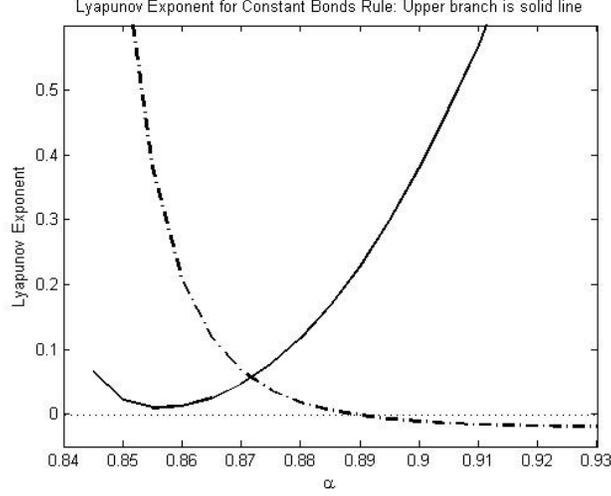


Figure 9

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4.2.3 Model with Bonds: Liability Targeting Rule

In this version of the model the linearized tax rule becomes

$$\hat{\tau}_t = -\hat{y}_t + \frac{\phi_1}{(1-\beta)} \hat{l}_{t-1} \quad (34)$$

where \hat{l}_t denotes the log-deviation of total real liabilities $\frac{R_t B_t}{P_t}$. Inserting the tax rule (34) in the Phillips curve (29) we get the following equation for inflation

$$\pi_t = \beta \pi_{t+1} + \xi \rho \hat{y}_t + \frac{\xi \bar{\tau} \phi_1}{(1-\beta)} \hat{l}_{t-1}.$$

The last equation describes the evolution of government liabilities

$$\hat{l}_t = \beta^{-1} (\phi_\pi - 1) \pi_t - \frac{\phi_1 (\beta^{-1} - 1)}{(1-\beta)} \hat{l}_{t-1}.$$

Again, local determinacy depends on the local stability of the system

$$A_0^T \begin{bmatrix} c_{t+1} \\ \pi_{t+1} \\ \hat{l}_t \end{bmatrix} = A_1^T \begin{bmatrix} c_t \\ \pi_t \\ \hat{l}_{t-1} \end{bmatrix}$$

where

$$A_0^T = \begin{bmatrix} 1 & \frac{1}{\sigma} & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_1^C = \begin{bmatrix} 1 & \frac{1}{\sigma}\phi_\pi & 0 \\ -\xi\rho & 1 & -\frac{\xi\bar{\tau}\phi_1}{(1-\beta)} \\ 0 & \beta^{-1}(\phi_\pi - 1) & -\frac{\phi_1(\beta^{-1}-1)}{(1-\beta)} \end{bmatrix}$$

If the Jacobian

$$J_C = (A_0^C)^{-1} A_1^C$$

has two eigenvalues outside the unit circle and one inside, we have a locally determinate equilibrium.

Proposition 3 *Consider the model with bonds and targeting rule under the benchmark calibration.*

(i), *Let the fiscal stance be mild: $\phi_1 = 0.4$. For each $\xi(\alpha) \in (\xi(0.649), \xi(0.95))$ there exist a $\bar{\phi}_{\pi-}^\xi$ and $\bar{\phi}_{\pi+}^\xi$ such that for $1 < \phi_\pi < \bar{\phi}_{\pi-}^\xi$ and $\phi_\pi > \bar{\phi}_{\pi+}^\xi$ the equilibrium is locally determinate, and for $\phi_\pi \in (\bar{\phi}_{\pi-}^\xi, \bar{\phi}_{\pi+}^\xi)$ the equilibrium is locally indeterminate. Furthermore, for each interval $S_+^2 = (\hat{\phi}_{\pi+}^\xi - \epsilon, \hat{\phi}_{\pi+}^\xi)$, and $S_+^2 = (\hat{\phi}_{\pi-}^\xi + \epsilon, \hat{\phi}_{\pi-}^\xi)$, $\epsilon > 0$, there is a "determinate" invariant curve bifurcating from the steady state as ϕ_π crosses $\hat{\phi}_{\pi+}^\xi$ for the first interval and $\hat{\phi}_{\pi-}^\xi$ for the second interval, so that given an initial condition for L_0 close to the invariant curve, there exists initial values $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve.*

(ii), *Let the fiscal stance be aggressive: $\phi_1 = 1.7$. For each $\xi(\alpha) \in (\xi(0.45), \xi(0.95))$ there exists a $\bar{\phi}_{\pi-}^\xi$ and $\bar{\phi}_{\pi+}^\xi$ such that for $1 < \phi_\pi < \bar{\phi}_{\pi-}^\xi$ and $\phi_\pi > \bar{\phi}_{\pi+}^\xi$ the equilibrium is locally determinate, and for $\phi_\pi \in (\bar{\phi}_{\pi-}^\xi, \bar{\phi}_{\pi+}^\xi)$ the equilibrium is locally indeterminate. Furthermore, for an interval $S_+^1 = (\hat{\phi}_{\pi+}^\xi, \hat{\phi}_{\pi+}^\xi + \epsilon)$, $\epsilon > 0$, there exists a closed invariant curve which bifurcates from the steady state as ϕ_π crosses $\hat{\phi}_{\pi+}^\xi$ from below, and for initial conditions of L_0 close to this invariant curve, there exists a continuum of initial values $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve. In addition, for an interval $S_+^2 = (\hat{\phi}_{\pi-}^\xi, \hat{\phi}_{\pi-}^\xi + \epsilon)$, $\epsilon > 0$ there is a "determinate" invariant curve bifurcating from the steady state as ϕ_π crosses $\hat{\phi}_{\pi-}^\xi$ from below, so that given an initial condition for π_{-1} close to the invariant curve, there exists initial values $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve.*

Proof. (i) Given the additional parameter ϕ_1 analytical expressions become complicated. We therefore focus on the benchmark calibration. Following the same steps as for the Propositions above we get

$$\begin{aligned} C1 &= 1 + a_2 + a_1 + a_0 \\ &= 0.3468\xi\phi_\pi - 0.34675\xi - 0.00008 \end{aligned}$$

$$\begin{aligned}
C2 &= -1 + a_2 - a_1 + a_0 \\
&= 14.911\xi - 17.575\xi\phi_\pi - 6.4567
\end{aligned}$$

where $C1 > 0$ and $C2 < 0$ if $\phi_\pi > 1$. Consider the region of $\xi(\alpha) \in (\xi(0.872), \xi(0.95)) = S$. Given $\xi \in S$, we can solve for ϕ_π that satisfies a quadratic equation equivalent to 31. The two solution branches are given by (see Figure 10):

$$\hat{\phi}_{\pi_-}^\xi, \hat{\phi}_{\pi_+}^\xi = \frac{1.3476 \times 10^{-2}}{\xi^2} \left(1.5143\xi + 74.105\xi^2 \mp \frac{1}{2} \sqrt{8.7334\xi^2 - 50.021\xi^3 + 57.625\xi^4} \right)$$

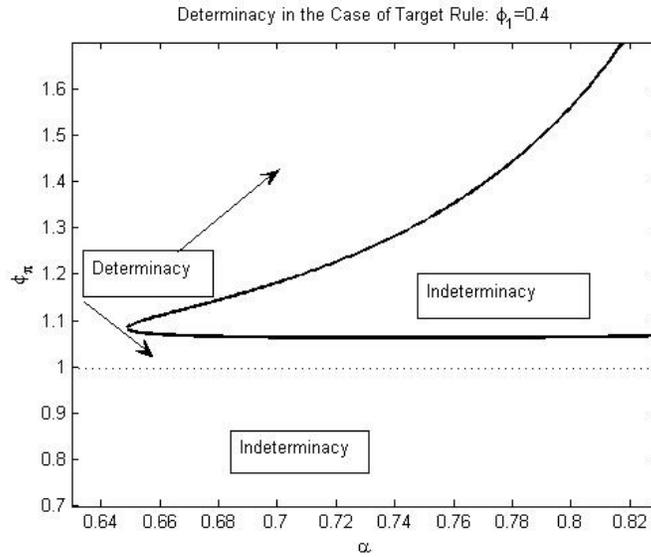


Figure 10

For our benchmark parametrization, we also compute that for $\xi \in S$, $|a_2| < 3$. Finally, Figure 11 shows that the Lyapunov exponent is positive for both branches.

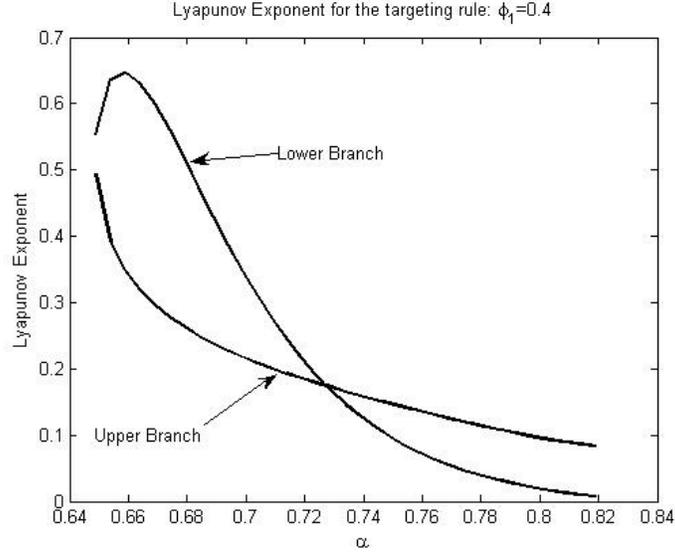


Figure 11

(ii) Again, we have

$$\begin{aligned}
 C1 &= 1 + a_2 + a_1 + a_0 \\
 &= 1.503\xi\phi_\pi - 1.5033\xi + 0.00003
 \end{aligned}$$

$$\begin{aligned}
 C2 &= -1 + a_2 - a_1 + a_0 \\
 &= 69.123\xi - 68.945\xi\phi_\pi - 1.1776
 \end{aligned}$$

where $C1 > 0$ and $C2 < 0$ if $\phi_\pi > 1$. Consider the region of $\xi(\alpha) \in (\xi(0.872), \xi(0.95)) = S$. Given $\xi \in S$, we can solve for ϕ_π that satisfies a quadratic equation equivalent to 31. The two solution branches are given by

$$\hat{\phi}_{\pi-}^\xi, \hat{\phi}_{\pi+}^\xi = \frac{8.7943 \times 10^{-4}}{\xi^2} \left(28.441\xi + 1184.6\xi^2 \mp \frac{1}{2} \sqrt{3101.3\xi^2 - 3688.2\xi^3 + 837.88\xi^4} \right)$$

see Figure 12 .

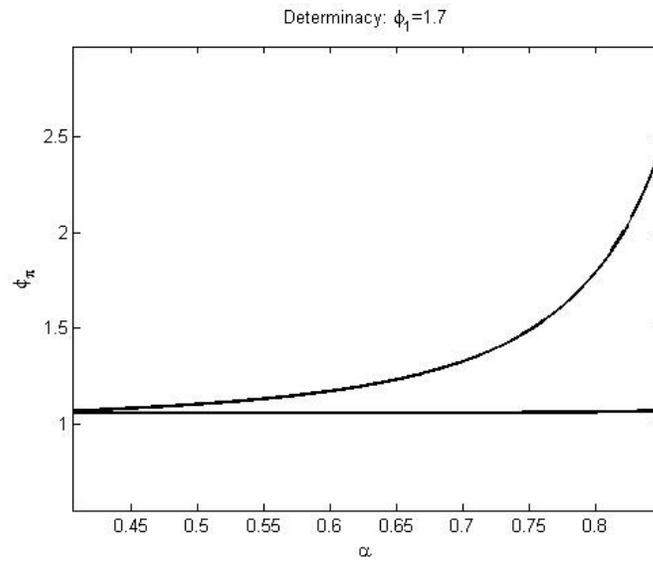


Figure 12

For our benchmark parametrization, we also compute that for $\xi \in S$, $|a_2| < 3$. Figure 13 shows that the Lyapunov exponent is negative for the upper branch and positive for the lower branch.

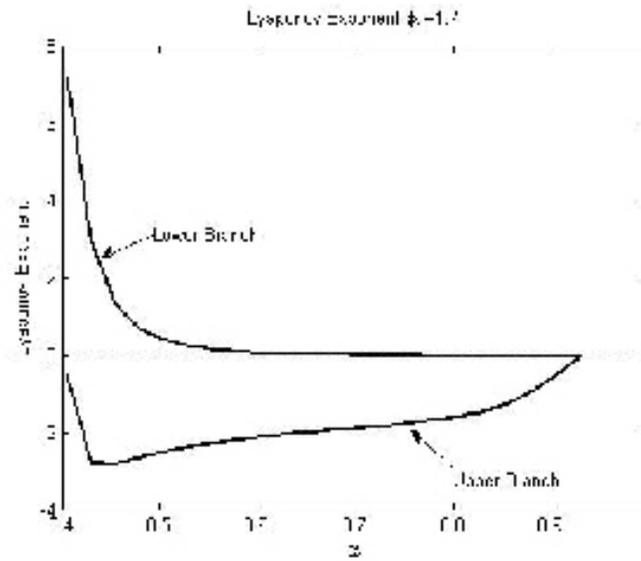


Figure 13

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Proposition 4 Consider the model under benchmark calibration and $\phi_\pi = 1.5$. For each $\xi(\alpha) \in (\xi(0.745), \xi(0.95))$ there exist $1 - \beta < \phi_1^\xi$ such that for $1 - \beta < \phi_1 < \phi_1^\xi$ the equilibrium is locally determinate, and for $\phi_1 > \phi_1^\xi$ the equilibrium is indeterminate. Moreover,

(a) If $\xi \in (\xi(0.7487), \xi(0.777))$, for an interval $S_+^1 = (\hat{\phi}_1^\xi - \epsilon, \hat{\phi}_1^\xi)$, $\epsilon > 0$, there exists a closed invariant curve which bifurcates from the steady state as ϕ_1 crosses $\hat{\phi}_1^\xi$ from above, and for initial conditions of b_0 close to this invariant curve, there exists a continuum of initial values $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve.

(b) if $\xi \in (\xi(0.745), \xi(0.7486)) \cup (\xi(0.778), \xi(0.95))$, or an interval $S_+^2 = (\hat{\phi}_1^\xi, \hat{\phi}_1^\xi + \epsilon)$, $\epsilon > 0$ there is a "determinate" invariant curve bifurcating from the steady state as ϕ_π crosses $\hat{\phi}_1^\xi$, so that given an initial condition for b_0 close to the invariant curve, there exists initial values of $\{c_0, \pi_0\}$ for which the equilibrium trajectories converge to the invariant curve.

Proof. In this case we have that

$$C_1 = \beta^{-2} \sigma^{-1} (\phi_\pi - 1) (\beta + \phi_1 - 1) \xi \rho$$

so that if $\phi_\pi > 1$, $C_1 > 0$ provided $\phi_1 > -1 + \beta$, and

$$C_2 = \phi_\pi \xi (\rho - \rho \phi_1 + \beta \rho \phi_1 + 2\sigma \beta \tau \phi_1 - \beta^2 \rho)$$

so that $C_1 < 0$ provided ϕ_1 satisfies

$$\bar{\phi}_1^\xi < \frac{(\beta^2 - 1) \rho}{(2\sigma \beta \tau - \rho(1 - \beta)) \phi_\pi \xi}. \quad (35)$$

Consider the region of $\xi(\alpha) \in (\xi(0.745), \xi(0.95)) = S$. Given $\xi \in S$, we can solve for ϕ_π that satisfies a quadratic equation equivalent to 31. The solution is given by

$$\phi_1^\xi = \frac{c_1(\xi) - \frac{1}{2} \sqrt{c_2(\xi)} + 1.5455 \times 10^{-4}}{-8.8244\xi + 73.328\xi^2 + 1.0407 \times 10^{-2}} \quad (36)$$

$$\begin{aligned} c_1(\xi) &= -7.1515 \times 10^{-2} \xi - 7.6519 \xi^2 \\ c_2(\xi) &= -2.2026 \times 10^{-4} \xi + 0.15778 \xi^2 + 24.291 \xi^3 + 56.818 \xi^4 + 1.1031 \times 10^{-8} \end{aligned}$$

see Figure 5. For our benchmark parametrization, we also compute that for $\xi \in S$, $|a_2| < 3$. It is also possible to show that for $\xi \in S$ $\bar{\phi}_1^\xi > \phi_1^\xi$ so that condition (35) is satisfied provided (36) is satisfied. Finally, Figure 14 shows that the Lyapunov exponent is negative for $\xi \in (\xi(0.7487), \xi(0.777))$ and positive for $\xi \in (\xi(0.745), \xi(0.7486)) \cup (\xi(0.778), \xi(0.95))$.

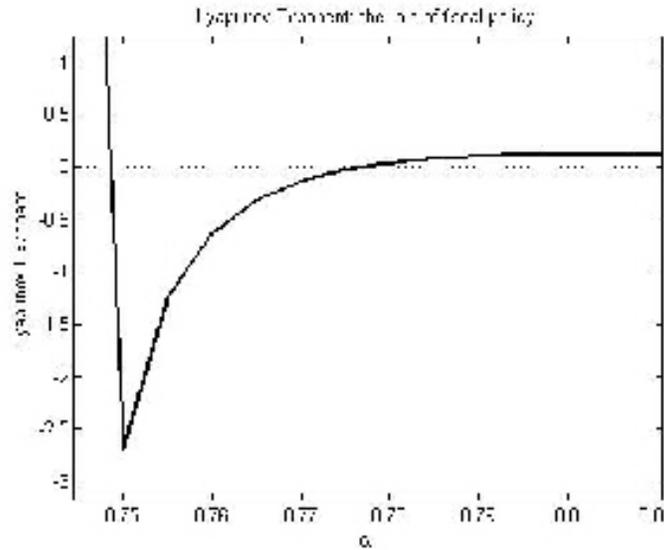


Figure 14

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