Optimal conventional and unconventional monetary policy in the presence of collateral constraints and the zero bound

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Abstract

We take a sticky price business cycle model with collateral constrained entrepreneurs and investigate how optimal monetary policy is affected by the presence of the zero bound to central bank interest rates; we also consider the advantages of using a second unconventional monetary policy instrument that loosely approximates 'credit easing' undertaken by some central banks recently. We compare commitment with discretion, and explore how the cost of not being able to commit interacts with the zero bound and the potential to use the second instrument. Solving under discretion, and with the 3 state variables our model entails, requires us to make a small innovation to previous algorithms which we suggest may be of more general use to others.

1 Introduction[1]

This paper studies questions of monetary policy design arising from several features of the recent financial crisis. The first such feature is that the severe downturn in activity may be considered to have had its origin in financial markets, via a sudden reduction in the quantity of lending advanced to firms and a tightening of the terms on which such lending was made. Second, whatever its origin, difficulties in the financial sector have served to amplify and propagate the shock over time. Third, central bank interest rates were driven to the zero lower bound (ZLB) as the monetary authorities in many countries sought to counter the effects that the shock would otherwise have had on inflation and the real economy. Fourth, and as a result of finding interest rates driven to their natural floor, central banks were forced to consider unconventional means of stimulating spending. One such means was the possibility, advocated in work by Eggertson and Woodford (2003) and others, that central

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[1] We thank Richard Harrisson for talking us through his algorithm for solving the model under commitment at the zero bound. We thank seminar participants in the Monetary Assessment and Strategy Division for comments. Remaining errors are ours alone. Views expressed here are not those of the Bank of England nor the Monetary Policy Committee.

banks should commit themselves to maintaining interest rates low some time into the future. Another was the purchase of certain financial assets including private securities - a practice dubbed by some 'credit easing' - and long term government securities - known as 'quantitative easing'.

These features of the financial crisis lead naturally to the following questions: how should optimal monetary policy respond to shocks that have their origin in the financial sector, or are amplified by it? How does this response depend on the ability of the authorities to commit to future actions? How does the ZLB to the conventional interest rate tool affect optimal interest rate policy? Of what benefit is the purchase of assets held by the private sector, and how should these purchases be orchestrated alongside the conventional monetary policy tool, and in the light of the ZLB constraint on that tool?

The model laboratory we use to answer these questions is a version of Iacoviello (2005). In this model patient consumers lend to impatient entrepreneurs, who combine the labour of the consumers with commercial real estate to produce. They finance production by borrowing against the expected net present value of their commercial real estate, and, as in Iacoviello (2005) and Kiyotaki and Moore (1997) are assumed to be able to borrow only up to a fraction of that value. To capture the notion that the financial crisis may have had its origin in financial markets, we consider exogenous disturbances to the loan to value ratio.

Leverage for conventional monetary policy in this model exists due to two nominal rigidities: Calvo stickiness in the price of the final consumption good; and nominal debt contracts. With this leverage over the real economy also comes the familiar requirement that the central bank seek to stabilise inflation so as to minimise unplanned relative price changes that inflation implies when some firms are unable to change their price (and minimise undesirable fluctuations in the real burden of debt). Against this the authorities trade off the distortions induced by the credit friction. In the face of a shock that would drive down demand and tighten credit, on the one hand a monetary policy loosening, can, by stimulating aggregate demand and pushing up on collateral values counteract this tightening of credit constraints, but the price for doing so will be inflation which is itself very costly. In our model, the zero lower bound interacts with these borrowing constraint via two channels. First, debt is nominal and falls in the price level that may occur at the zero bound increase the real debt burden and lead to further feedback effects. Second, increases in the real interest rates that may occur at the zero bound reduce the borrowing capacity of entrepreneurs further because the borrowing constraint is specified in present value terms.

We introduce a new instrument that the authorities - either the government or the central bank - can wield: purchases of securitised loans made by banks to entrepreneurs. To induce banks to sell on claims to the loans they have made to entrepreneurs, the authorities offer to buy these at a subsidy. This subsidy is then passed onto entrepreneurs themselves. These purchases look most like the actions dubbed 'credit easing' by the Fed and some other central banks. Asset purchases are financed from a distortionary tax on labour income.

The model has no role for what has been dubbed 'quantitative easing', just as in the canonical New Keynesian model. At the ZLB, expansions of the money supply to buy government securities that are not accompanied by changes in expected future interest rates will not affect allocations in this model, i.e. will not change demand or inflation. As has been noted by others, this was prefigured in the 'irrelevance proposition' of Wallace (1981). One way of seeing why this irrelevance holds at the ZLB is to note that an operation to buy government bonds with reserves is an exchange of one zero interest, default-risk free asset (central bank reserves) for another (government bonds). Since private portfolios are not materially affected, there is no need for agents to adjust any other aspect of their behaviour. But note that we deploy a model that embodies this irrelevance for simplicity, and not to take any stance on the efficacy of quantitative easing, an issue we rule beyond the scope of this paper. There is a relatively consistent seam of empirical evidence now showing that purchases of long-dated government bonds lowered yields during the crisis - see Gagnon *et al.* (2010) and Joyce *et al.* (2010) - and that relative bond supplies affect relative yields in general - see, for example Greenwood and Vayanos (2010). Harrison (2010) works through the consequences for optimal policy of assuming that long term government bonds confer some special non-pecuniary value on their holders, and such a mechanism could readily be grafted on to the framework we have here.

In our model, Wallace's irrelevance proposition fails to hold regarding public purchases of *private* debt because the subsidies that these purchases entail transfer resources from unconstrained agents - patient consumers - to those who are constrained - impatient, borrowing entrepreneurs. In addition, these subsidies are financed by distortionary taxation.

The goal of the paper is to compute jointly optimal conventional and unconventional policy in the presence of the ZLB, and in response to a financial disturbance, modelled as a shock to the pledgability ratio. Along the way we discuss three dimensions of optimal monetary policy: the presence or absence of the ZLB to central bank interest rates; the availability or otherwise of the instrument of private sector asset purchases; and the benefits of commitment versus discretion. There are 3 reasons why considering discretion is useful, aside from the usual concerns one might have about the lack of any technology to enforce commitment in stabilisation policy. First, for most central banks operating unconventional policy, it was the first time they had done it consciously, or on a significant scale, so it might be optimistic to think that commitment equilibria could be sustained. Second, the crisis constituted a large shock, which might be thought to put commitment strategies under most strain. Third, at the ZLB, commitment strategies prescribe a prolonged period of very low interest rates and a corresponding 'overshooting' of desired long run inflation rates. Many commentators have expressed scepticism that such a commitment would be believed.

Previewing our main results: Optimal monetary policy with one instrument works by generating an income effect for entrepreneurs - a boom in demand for their goods - to offset the substitution effect of the tightening in the borrowing constraint encouraging them to deleverage and scale back on their commercial real estate holdings, at the expense of a little inflation on impact. Using the second instrument rather unsurprisingly improve matters, because this second instrument - which takes the form of a subsidy to the cost of borrowing - relaxes the borrowing constraint directly. We find that under discretion, the ZLB can have a dramatic effect on welfare relative to the case where it is absent causing large falls in output and inflation. Part of the reason for this is the effect of deflation induced by insufficiently stabilising policy on the real burden of entrepreneurial debt. In findings closely related to those of Eggertson (2010), we find that the expansionary policy that would be pursued in the absence of a zero lower may be rendered impossible in its presence, because the excess capacity that that policy implies would reduce inflation expectations to such an extent that the real interest rate rises - contracting aggregate demand rather than expanding it The potential for such large effects was not evident from previous studies which did not consider the interaction of the ZLB with credit frictions. The benefit of the second instrument is magnified by the presence of the zero lower bound (an extra instrument is even more helpful when you can't move the first) or when commitment is not possible. Our experiments allow us to investigate the determinants of the welfare cost of not being able to commit. These are increased by the presence of the zero lower bound: this eliminates the possibility of substituting cutting the short rate for managing expected future short rates. And these costs are reduced by having a second instrument, with our without the ZLB. In part, our unconventional tool makes up for not being able to manage expected future short rates.

2 Related literature and our contribution

2.1 Building blocks

We give here a brief account of some of the papers on which we build organised along two themes: work that serves as an analytical building block point for this paper; and other work studying conventional and unconventional monetary policy.

The starting point, and the laboratory for our experiments, is Iacoviello (2005). We make two modifications to Iacoviello's model. One is to introduce costs of converting real estate from commercial to residential use or vice versa. These costs are a function of the rate of change of the portion devoted to one use or the other. This naturally has the effect of slowing down the speed at which real estate is converted from housing to commercial use or vice versa. This modification appealed to us on grounds of realism. The second modification we make is to introduce a second policy instrument: central bank purchases of loans to entrepreneurs that are originated by banks. At the same time we introduce a rule for distortionary labour taxes used to finance these purchases, so that the authorities have some discipline on their use of the second instrument. Iacoviello (2005) focused on positive questions: on the ability of collateral constraints to propagate technology shocks and to contribute to accounting for business cycle dynamics. Our focus is normative: on the implications of his model for monetary policy design. We are assisted in our task by Andres *et al.* (2009) who studied study optimal monetary policy (under commitment, with 1 instrument, and in the absence of the ZLB constraint) in their version of the Iacoviello model, modified to include a banking sector with endogenously time varying spreads. The derivation of the criterion function for policymakers that approximates the welfare of the agents in our model follows that in Andres *et al.* (2009), modified only to account for the costs of adjusting real estate use. Our analysis builds on theirs in a few directions: we incorporate the zero bound to nominal interest rates; we consider how the availability of a second unconventional instrument affects the policy problem; we also study both commitment and discretion.

Our paper exploits but also contributes to the literature on how to compute rational expectations equilibria under optimal policy in the presence of the zero lower bound. For the commitment case, we use the 'piecewise linear' algorithm set out in Eggertson and Woodford (2003). For the case of discretion, we develop our own algorithm. This algorithm has the advantage that it can handle multiple state variables, of which there are three endogenous states in our model and the shocks. A natural comparison is with Adam and Billi (2007). They use projection methods on the non-linear system formed by the zero bound constraint and the other, linearised equilibrium conditions of the model. Their method has the advantage of allowing for a solution in which agents rationally expect the possibility of further shocks (we restrict ourselves to the situation in which agents have the perfect foresight that there will be no further shocks) but would not be practical in our model of multiple state variables. A realistic model of credit frictions generates multiple state variables and that is what makes our solution algorithm useful. Our algorithm works by solving for a terminal Markov-stationary equilibrium that will obtain once inequality constraints have ceased to bind, then iterates backward through time on the value function problem associated with this to obtain the time-varying policy rule in each period prior to this terminal regime. An appendix goes into more detail.

2.2 Other work on the ZLB and unconventional monetary policy

Our study of the impact of the zero bound constraint on optimal monetary policy follows and exploits methods and insights from a line of work that has included Eggertson and Woodford (2003), Adam and Billi (2007), Jung *et al.* (2005), Nakov (2008) and others. To this work has recently been added a few studies which consider policy with an additional unconventional monetary policy instrument at the ZLB, which we recount briefly below.

Harrison (2010) assumes that a representative agent has a desire to keep the ratio of one period government bonds to consuls (perpetuities) close to a target level. This gives the authorities leverage over aggregate demand even when the short rate is at the ZLB, through purchases of long duration bonds, though these purchases likewise entail costs (pushing agents away from their target level). This paper can be thought of as characterising the optimal quantiative easing policy, starting from the assumption that this policy has traction. Eggertson *et al.* (2009) study a sticky-price version of Kiyotaki and Moore (2008). Their model has a borrowing constraint like the one we inherit from Iacoviello (2005) and a constraint that only a certain portion of illiquid assets (shares) can be sold each period. The financial crisis is modelled as a disturbance to this resaleability constraint, holding the (parameterization of the) borrowing constraint constant. Their Fed has as a second instrument purchases of illiquid assets held by the private sector (interpreted as referring to shares, mortgage backed securities...). They document that, with interest rates following a Taylor Rule, in the absence of Fed purchases of private assets, the US economy would have suffered a Great Depression style recession, and thus that the second instrument facilitated the 'great escape' in the title of their paper. In Gertler and Karadi (2009) there are two agency problems, one between firms and banks, and another between banks and their depositors. They assume that the authorities have as a second instrument the facility to engage in purchases of private bank equity. They simulate with interest rates and bank equity purchases following simple feedback rules. They shock bank capital and demonstrate that the active rule for bank equity purchases substantially mutes the response of the output gap and inflation. The closest paper to ours in purpose is Curdia and Woodford (2010). In their model, only a fraction of loans made by financial intermediaries are repaid, and they introduce a financial disturbance by peturbing this fraction. Just as in our model, their central bank can purchase the loans banks make to impatient consumer-borrowers. They compute jointly optimal interest rate policy and central bank lending under commitment, finding that there is a role for substantial central bank lending to the private sector, particularly, but not exclusively, if interest rate setting is not optimal but instead follows a Taylor rule. It is also worth noting the work of Eggertson and Woodford (2004). They also look at jointly optimal, two-instrument policy in the presence of the ZLB. Their second instrument is a consumption tax. Our second instrument is also loosely interpreted as a tax/subsidy, but it is levied on a factor of production (borrowing to finance production) rather than the final good. Conventional taxes are levied in the background, in order to finance the unconventional policy that subsidises, but these follow a simple rule to stabilise the stock of government debt. Aside from the details of the instruments deployed or the frictions in the models, our contribution relative to these two papers by Woodford and his co-authors is to contrast commitment with discretion.

3 Model details

As noted, the model we use is a variant upon the basic model of Iacoviello (2005)/Andres *et al.* (2009), modified to include costs of converting real estate from one use to another, and to allow the policymaker to make use of an alternative instrument. Six classes of agent feature: households, entrepreneurs, final goods firms, construction firms, banks and the policymaker. We present their choice problems in turn.

3.1 Households

The model works with a measure $\omega \in [0,1]$ of households and $(1-\omega)$ of entrepreneurs. At time t households maximise the objective function:

$$U_t^h = E_t \sum_{s=0}^{\infty} \beta^s \left\{ \frac{c_{t+s}^{1-\sigma}-1}{1-\sigma} - \frac{(l_t^s)^{1+\varphi}}{1+\varphi} + \vartheta \ln(h_t) \right\}, \tag{1}$$

where l_t^s is the household's labour supply, h_t the quantity of housing it owns and c_t is the usual Dixit-Stiglitz sub-utility function across the unit-measure continuum of differential goods produced:

$$c_t = \left[\int_0^1 c_t(j)^{\frac{\varepsilon_t - 1}{\varepsilon_t}} dj \right]^{\frac{\varepsilon_t}{\varepsilon_t - 1}}, \tag{2}$$

where we allow the possibility that the elasticity of substitution across goods, ε_t, is time-varying. If the money price of good j is $p_t(j)$, the minimum expenditure required to obtain a unit of c_t, P_t, is given by:

$$P_t = \left[\int_0^1 p_t(j)^{1-\varepsilon_t}\, dj\right]^{\frac{1}{1-\varepsilon_t}}. \tag{3}$$

The gross rate of consumer price inflation, $\frac{P_t}{P_{t-1}}$, is denoted π_t in what follows.

Households optimise subject to the period-by-period budget constraint (expressed in real terms):

$$w_t(1-t_t)\, l_t^s + T_t + \frac{R_{t-1}}{\pi_t} s_{t-1} = c_t + p_t^h\left[\left(1+\tau^h\right) h_t - h_{t-1}\right] + s_t, \tag{4}$$

with w_t the real wage, t_t the rate of labour income taxation, p_t^h the real price of housing, s_t the real quantity of saving (in nominal bonds, paying gross interest R_t), T_t a collection of lump-sum transfers to and from profit-making firms and the government and τ^h a housing tax introduced to ensure steady-state efficiency. This constraint is coupled with a usual transversality/'no-Ponzi' restriction.

3.2 Entrepreneurs

Entrepreneurs employ workers and make use of commercial real estate to produce intermediate goods, y_t, which are sold in a perfectly competitive market at price p^I to final goods firms. These entrepreneurs maximise a utility function expressed over consumption goods alone:

$$U_t^e = E_t \sum_{s=0}^{\infty} (\beta^e)^s \frac{\left(c_{t+s}^e\right)^{1-\sigma} - 1}{1-\sigma}, \tag{5}$$

where the superscript e distinguishes their consumption of final goods from the household's, and $\beta^e < \beta$ holds. This is subject to the period-by-period budget constraint:

$$b_t + (1-\tau^e)\left(p_t^I y_t - w_t l_t^d\right) = c_t^e + p_t^{h^e}\left(h_t^e - h_{t-1}^e\right) + \frac{Q_{t-1}}{\pi_t} b_{t-1}, \tag{6}$$

where l_t^d denotes labour demand, h_t^e commercial real estate (whose real price is $p_t^{h^e}$) and b_t real borrowing, for which entrepreneurs are charged gross nominal rate Q_t. τ^e is a tax on the proceeds of investment, also introduced to ensure steady-state efficiency. This is combined with an associated transversality/'no-Ponzi' condition, along with the collateral constraint:

$$b_t \leqslant m_t E_t \frac{\pi_{t+1}}{Q_t} p_{t+1}^{h^e} h_t^e, \tag{7}$$

with m_t the fraction of the monetary value of next period's commercial real estate that the entrepreneur is permitted to commit to the repayment of loans (this is subject to random fluctuations about a steady-state value denoted m), and the production function:

$$y_t = a_t \left(l_t^d\right)^{1-v} \left(h_{t-1}^e\right)^v, \tag{8}$$

where the level of TFP, a_t, may likewise contain a stochastic component. So long as the expected returns available to entrepreneurs from holding an extra unit of commercial real estate exceed the borrowing rate, the collateral constraint must hold with equality.[2] In this event entrepreneurs make their intertemporal choices as if faced with a single 'composite' asset, obtained by purchasing a unit of commercial real estate that they then leverage to the maximum possible extent. Thus they face an effective ex-post real rate of return on their savings, say RR_{t+1}^e, given by:

$$RR_{t+1}^e = \frac{(1-\tau^e)\, v p_{t+1}^I \frac{y_{t+1}}{h_t^e} + p_{t+1}^{h^e} - \frac{m_t E_t \pi_{t+1} p_{t+1}^{h^e}}{\pi_{t+1}}}{\left[p_t^{h^e} - m_t E_t \left(\frac{\pi_{t+1}}{Q_t} p_{t+1}^{h^e}\right)\right]}. \tag{9}$$

Many of the equilibrium consequences of fluctuations in the permitted leverage ratio m_t are best understood via their effects on this effective rate of return. In steady state it must equal the inverse of the entrepreneurial discount factor β^e. Notice that it is only by barring households from investing in commercial real estate that we can provide the distinct rates of return that are necessary to guarantee stationary equilibrium consumption profiles for both entrepreneurs and households – despite the relative impatience of the former.

3.3 Final goods firms

Final goods producers are monopolistically-competitive price setters owned by households, free to reset their prices only at stochastically-determined intervals – as in Calvo (1983). Each firm has access to a linear technology, converting intermediate goods one-for-one into final goods. The period-t profit level of firm j, $\Pi_t(j)$, thus satisfies:

$$\Pi_t(j) = \left((1+\tau)\, p_t(j) - P_t p_t^I\right) y_t(j), \tag{10}$$

where τ is a production subsidy used to eliminate steady-state underemployment due to market power. Prices are then chosen to maximise the net present value (to households) of the firm's future stream of profits, assuming a fixed probability of resetting prices equal to θ each period.

3.4 Construction firms

There exists in addition a construction sector, whose role is to convert real estate between commercial and residential uses. This is done at the start of each time period, in order to satisfy the pattern of relative demand

[2] If it did not then the entrepreneur could always take on an extra ε units of commercial real estate, at real price $p_t^{h^e}$, and borrow an extra $\varepsilon p_t^{h^e}$ (for ε sufficiently small). Given the rate of return differential this would deliver an expected welfare gain at time $t+1$.

from households and entrepreneurs. Construction firms are assumed to be perfectly competitive, with each firm that chooses to convert housing into CRE making use of the Leontief production function:

$$(h_t^e)^c = \min\left\{h_{t-1}^c, \max\left\{\frac{C_t^c}{\psi^e\left(h_t^e - h_{t-1}^e\right)}, 0\right\}\right\}, \tag{11}$$

where $(h_t^e)^c$ denotes the number of units of real estate that the firm in question converts to commercial use, having purchased h_{t-1}^c units of property previously used for residential purposes and made use of $C_t^c \geq 0$ units of the 'final' good as a productive input. ψ^e is a parameter that indexes the size of construction costs, and $h_t^e - h_{t-1}^e$ is the *aggregate* change in the level of commercial real estate per entrepreneur – which our small firm treats as given.[3] If the firm instead wishes to convert CRE into housing, a symmetric production function applies:

$$h_t^c = \min\left\{\left(h_{t-1}^e\right)^c, \max\left\{\frac{C_t^c}{\psi^e\left(h_{t-1}^e - h_t^e\right)}, 0\right\}\right\}.^4 \tag{12}$$

Profit maximisation on the part of construction firms then requires that a wedge must exist in equilibrium between the real price of CRE and that of housing, satisfying the equation:

$$p_t^{h^e} = p_t^h + \psi^e\left(h_t^e - h_{t-1}^e\right). \tag{13}$$

It is perhaps useful to add a few words on the appropriate interpretation of these firms' production functions. The terms in $\left(h_t^e - h_{t-1}^e\right)$ imply that an externality effect is present, linking the aggregate evolution of real estate to the marginal cost of converting it between uses. The more real estate is converted from one use to the other, the greater the real unit cost of that type of conversion for all firms – though no construction firm considers how its actions worsen this 'congestion' effect when deciding on optimal behaviour. As justification for this approach, one could suppose that each conversion project demands the services of professionals (not modelled formally) who are sufficiently few in number to attract a premium for their work when it is in particularly high demand. From a more pragmatic perspective, it is a simple way to introduce frictions that prevent unrealistic volatility in the use of real estate from one period to the next.

3.5 Banks

As in standard New Keynesian models, the monetary authority controls the gross interest rate R_t that is paid on money, and by arbitrage this must equal the nominal interest rate faced by households (who are net holders of financial assets in equilibrium, and always have the option of holding money). Entrepreneurs do not have the right to issue money, and must instead borrow from banks when leveraging their commercial real estate purchases. In normal times the monetary authority ensures that banks must borrow at rate R_t, and for

[3] In the event that $h_t^e = h_{t-1}^e$ the production function given above is not well defined. In this case we assume it takes the form $(h_t^e)^c = h_{t-1}^c$.

simplicity we assume that these banks then merely apply a time-invariant markup Γ when setting the rate charged to entrepreneurs, so we have:

$$Q_t = \Gamma R_t. \tag{14}$$

3.6 Policy

In addition to setting the nominal interest rate, the monetary authority (in conjunction with the government) also has the capacity to carry out 'unconventional policy', in the form of purchases of securitised loans from commercial banks. The purpose of these purchases is to reduce the borrowing rate Q_t directly: this may be the only policy option available in the event that the conventional instrument, the nominal savings rate, has come up against the zero lower bound. Specifically, we assume that banks are still able to apply the same markup of Γ on the cost of their funds, but now can sell (risk-free) one-period nominal bonds to the monetary authority at a price, S_t, that may exceed their market value, R_t^{-1} – so long as these bond sales are backed by equivalent holdings of one-period debt issued by entrepreneurs. This is equivalent to presenting banks with a marginal cost of funds equal to S_t^{-1}, to which they apply the spread Γ as before; hence:

$$Q_t = \Gamma S_t^{-1}. \tag{15}$$

Given this, in what follows we neglect the variable S_t and assume simply that the policymaker controls Q_t directly. Since banks are always able to raise funds from depositors at rate R_t, there is an upper bound on the value of Q_t that policy can implement: $Q_t \leq \Gamma R_t$. This states that the central bank can bid *up* the market price for entrepreneurial loans (the limiting factor on this being only the necessity to finance such purchases using distortionary taxation), but it cannot bid it down: if the state offers a lower than market price for a loan, the private bank will simply refuse to sell. We also consider the usual lower bound on the gross nominal interest rate, $R_t \geq 1$. This has no formal justification in a purely 'cashless' economy such as the one treated here, but if we were to make the additional assumption that households always have the right to convert one-period nominal bonds issued by the central bank into zero-coupon perpetuities (at an exchange rate that is fixed over time) interest-bearing money is dominated by the non-interest-bearing variant ('cash') as soon as $R_t \geq 1$ ceases to hold.

Central bank asset purchases are an indirect way of subsidising borrowing, and the funding of this requires us to consider the public sector balance sheet. Defining D_t as the stock of one-period nominal central bank/government bonds issued at time t (equivalent to money in our New Keynesian environment), and B_t^G the total quantity of entrepreneurial obligations that have effectively been purchased by the government at time t (paying interest rate Q_t at the start of time $t+1$), the debt stock evolves according to:

$$D_t = R_{t-1}D_{t-1} - Q_{t-1}B_{t-1}^G + \Gamma B_t^G - \omega P_t w_t t_t l_t^s. \tag{16}$$

Defining real net debt outstanding at the *start* of time t, d_t, by $\frac{1}{P_t}\left[R_{t-1}D_{t-1} - Q_{t-1}B^G_{t-1}\right]$, and noting that all entrepreneurial debt will be sold to the policymaker in the event that $Q_t < \Gamma R_t$, this implies:

$$d_t = \frac{R_{t-1}}{\pi_t}\left(d_{t-1} - \omega w_{t-1} t_{t-1} l^s_{t-1}\right) + \left(\frac{\Gamma R_{t-1} - Q_{t-1}}{\pi_t}\right)(1-\omega)\, b_{t-1}. \tag{17}$$

The purpose of articulating the profile of conventional fiscal instruments here is to provide some discipline on use of the second instrument that we think accords with the real policy dilemma: policies that deliver effective subsidies to the private sector have to be financed, and real financing options for governments impose other costs. Our distortionary labour income tax captures that. That said, we do not study the optimal setting of these conventional fiscal instruments in this paper: first, the two instrument problem in the presence of the zero bound offers enough complications. Second, our study mirrors the institutional separation of the wielding of the conventional fiscal tools - done by the finance ministry - and the monetary policy tools - done usually by the central bank. We use these terms 'fiscal' and 'monetary' somewhat cautiously, acknowledging that the unconventional monetary policy instrument is isomorphic to most economist's definition of a fiscal tool.

We assume labour income taxes are set via a simple feedback rule to ensure debt sustainability:

$$t_t = \alpha d_t. \tag{18}$$

Policy is chosen to maximise the objective function W_t:

$$W_t = E_t \sum_{s=0}^{\infty} \beta^s \left\{ \omega \left[\frac{c_{t+s}^{1-\sigma} - 1}{1-\sigma} - \frac{\left(l^s_t\right)^{1+\varphi}}{1+\varphi} + \vartheta \ln\left(h_t\right) \right] + (1-\omega)\frac{\left(c^e_{t+s}\right)^{1-\sigma} - 1}{1-\sigma} \right\}. \tag{19}$$

Notice that this is not a 'welfarist' objective, in the sense that it is not equivalent to a function that takes as its inputs the t-dated subjective utility levels of households and entrepreneurs alone. The policymaker discounts the value of future entrepreneurial consumption at a lower rate than entrepreneurs themselves, so prefers them to be more patient than they themselves wish. This has the advantage of ensuring the steady state is stationary. If the policymaker instead applied entrepreneurs' own discount factor to their consumption utility the optimal policy from the perspective of time t would see a negative trend in entrepreneurial wealth (and hence consumption) through time – with households gradually becoming richer. The disadvantage or our approach is that optimal policies under the objective W_t will not generally be Pareto efficient. Given the option, entrepreneurs would always be willing to trade away some of their future consumption in return for some of that currently enjoyed by households – who could in turn benefit from the trade. The segmented market structure – in particular the leverage constraints restricting entrepreneurial borrowing – rules out such trades in equilibrium (borrowing is a direct way for entrepreneurs to sell on the future returns from investment projects to households). But in general it remains the case that the policymaker may at times prefer one equilibrium outcome to another despite both types of agent having the opposite preference ranking.

4 Model solution

4.1 Non-linear equations

Solving in the usual way, we proceed to list the basic model's key equations. First, households' holdings of nominal assets must satisfy a usual Euler equation:

$$c_t^{-\sigma} = \beta E_t c_{t+1}^{-\sigma} \frac{R_t}{\pi_{t+1}}, \tag{20}$$

whilst optimal housing purchases imply a similar condition:

$$\left(1+\tau^h\right) c_t^{-\sigma} p_t^h = \beta E_t c_{t+1}^{-\sigma} p_{t+1}^h + \vartheta h_t^{-1}. \tag{21}$$

Assuming that entrepreneurs' borrowing constraint always binds – that is, that the (risk-adjusted) expected returns from commercial real estate always exceed the cost of borrowing – their optimal choices must satisfy an Euler equation, given by:

$$\left[p_t^{h^e} - m_t E_t \left(\frac{\pi_{t+1}}{Q_t} p_{t+1}^{h^e}\right)\right] \left(c_t^e\right)^{-\sigma} = \beta^e E_t \left\{\left(c_{t+1}^e\right)^{-\sigma} \left[\left(1-\tau^e\right) \upsilon p_{t+1}^I \frac{y_{t+1}}{h_t^e} + p_{t+1}^{h^e} - \frac{Q_t b_t}{\pi_{t+1} h_t^e}\right]\right\}. \tag{22}$$

The entrepreneurial borrowing constraint binds with equality:

$$b_t = m_t E_t \frac{\pi_{t+1}}{Q_t} p_{t+1}^{h^e} h_t^e, \tag{23}$$

and entrepreneurs additionally satisfy the budget equation already specified:

$$b_t + \left(1-\tau^e\right) \upsilon p_t^I y_t = c_t^e + p_t^{h^e}\left(h_t^e - h_{t-1}^e\right) + \frac{Q_{t-1}}{\pi_t} b_{t-1}. \tag{24}$$

Optimal price-setting for Calvo-constrained firms (where $\widetilde{P}_t$ is the price chosen by firms resetting at t) gives:

$$E_t \sum_{s=0}^{\infty} (\beta\theta)^t \frac{c_{t+s}^{-\sigma}}{c_t^{-\sigma}} \left\{(1+\tau) \frac{\widetilde{P}_t}{P_{t+s}} - \frac{\varepsilon_t}{\varepsilon_t - 1} p_{t+s}^I\right\} P_{t+s}^{\varepsilon_t} y_{t+s}^f = 0, \tag{25}$$

with the aggregate quantity of the final (composite) consumption good demanded, y_t^f, equal to its supply – which in turn is equal to the aggregate quantity of intermediate goods produced, corrected to allow for price dispersion across the final goods:

$$\Delta_t y_t^f = (1-\omega) y_t, \tag{26}$$

where the price dispersion index Δ_t is defined by:

$$\Delta_t = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_t} dj. \tag{27}$$

The consumer price index then evolves in accordance with the Calvo pricing structure:

$$P_t = \left[\theta P_{t-1}^{1-\varepsilon_t} + (1-\theta)\,\widetilde{P}_t^{1-\varepsilon_t}\right]^{\frac{1}{1-\varepsilon_t}}. \tag{28}$$

Allowing for labour market clearing, the production function is:

$$y_t = a_t\left(\frac{\omega}{(1-\omega)} l_t^s\right)^{1-v} \left(h_{t-1}^e\right)^v, \tag{29}$$

whilst equilibrium in the labour market in turn requires that the marginal revenue product of an extra worker should be equal to the marginal rate of substitution between leisure and consumption:

$$\left(l_t^s\right)^{\varphi} c_t^{\sigma} = (1-t_t)(1-v)\frac{(1-\omega)}{\omega} p_t^I y_t \left(l_t^s\right)^{-1} \tag{30}$$

Goods market clearing (making use of 26) gives:

$$\omega c_t + (1-\omega)\,c_t^e + \psi^e\left(h_t^e - h_{t-1}^e\right)^2 = (1-\omega)\,y_t \Delta_t^{-1}, \tag{31}$$

where we note that the technology assumed for the construction sector renders aggregate costs quadratic in the change in quantity of commercial real estate.

Market clearing in the real estate sector (with a fixed supply $\overline{h}$) requires:

$$\omega h_t + (1-\omega)\,h_t^e = \overline{h}. \tag{32}$$

As already noted, equilibrium in the construction sector demands the following relationship between the prices of commercial real estate and housing:

$$p_t^{h^e} = p_t^h + \psi^e\left(h_t^e - h_{t-1}^e\right). \tag{33}$$

The nominal interest rate charged to borrowers is a fixed markup on the cost to banks of obtaining funding for the associated loans:

$$Q_t = \min\left\{\Gamma R_t, \Gamma S_t^{-1}\right\}, \tag{34}$$

which incorporates the restriction that policy cannot *raise* the borrowing rate above ΓR_t.

Subsidising entrepreneurial borrowing results in public-sector debt accumulation, which we have already noted follows the evolution equation:

$$d_t = \frac{R_{t-1}}{\pi_t}\left(d_{t-1} - \omega w_{t-1} t_{t-1} l_{t-1}^s\right) + \left(\frac{\Gamma R_{t-1} - Q_{t-1}}{\pi_t}\right)(1-\omega)\,b_{t-1}, \tag{35}$$

with labour income taxes set according to the rule:

$$t_t = \alpha d_t \tag{36}$$

Our simple rule for taxes reflects that we are not analyzing jointly optimal fiscal and monetary policy.

4.2 Linearised model

The linearised versions of these equations follow (eliminating $\widetilde{P}_t$, Δ_t and y_t^f prior to linearising, effectively condensing equations 26, 25, 27 and 28 to one – which is the familiar New Keynesian Phillips Curve). We use 'hats' to denote log deviations from steady state, with the exception of the variables $\widehat{d}_t \equiv d_t$ and $\widehat{t}_t \equiv t_t$. We assume that debt and taxes are zero in the steady state. This simplifying assumption is not innocuous because zero steady state government debt implies that inflation drops out of the log-linear accumulation equation for public debt and inflation cannot be used as a shock absorber. We leave the analysis of nonzero steady state government debt for future work.

$$-\sigma\widehat{c}_t = -\sigma E_t \widehat{c}_{t+1} + \widehat{R}_t - E_t \widehat{\pi}_{t+1} \tag{37}$$

$$\left(1+\tau^h\right) p_{ss}^h c_{ss}^{-\sigma} \left(\widehat{p}_t^h - \sigma\widehat{c}_t\right) = \beta c_{ss}^{-\sigma} p_{ss}^h E_t \left(\widehat{p}_{t+1}^h - \sigma\widehat{c}_{t+1}\right) - \vartheta h_{ss}^{-1} \widehat{h}_t \tag{38}$$

$$\begin{aligned} & p_{ss}^h \left[\widehat{p}_t^{h^e} - \beta\Gamma^{-1} m E_t \left(m_t + \widehat{\pi}_{t+1} - \widehat{Q}_t + \widehat{p}_{t+1}^{h^e}\right)\right] - \sigma p_{ss}^h \left(1 - m\beta\Gamma^{-1}\right) \widehat{c}_t^e \\ = \; & -\sigma\beta^e \left[\left(1-\tau^e\right) v \frac{y_{ss}}{h_{ss}^e} + p_{ss}^h \left(1-m\right)\right] E_t \widehat{c}_{t+1}^e \\ & +\beta^e [\left(1-\tau^e\right) v \frac{y_{ss}}{h_{ss}^e} E_t \left(\widehat{p}_{t+1}^I + \widehat{y}_{t+1} - \widehat{h}_t^e\right) + p_{ss}^h E_t \widehat{p}_{t+1}^{h^e} \\ & -\beta^{-1}\Gamma \frac{b_{ss}}{h_{ss}^e} E_t \left(\widehat{Q}_t - \widehat{\pi}_{t+1} + \widehat{b}_t - \widehat{h}_t^e\right)] \end{aligned} \tag{39}$$

$$\widehat{b}_t = \widehat{m}_t + E_t \widehat{\pi}_{t+1} - \widehat{Q}_t + E_t \widehat{p}_{t+1}^{h^e} + \widehat{h}_t^e \tag{40}$$

$$b_{ss}\widehat{b}_t + \left(1-\tau^e\right) v y_{ss} \left(\widehat{y}_t + \widehat{p}_t^I\right) = c_{ss}\widehat{c}_t^e + p_{ss}^h h_{ss}^e \left(\widehat{h}_t^e - \widehat{h}_{t-1}^e\right) + b_{ss}\beta^{-1}\Gamma \left(\widehat{Q}_{t-1} - \widehat{\pi}_t + \widehat{b}_{t-1}\right) \tag{41}$$

$$\widehat{\pi}_t = \frac{\left(1-\theta\right)\left(1-\beta\theta\right)}{\theta} \left(\widehat{p}_t^I + \mu_t\right) + \beta E_t \widehat{\pi}_{t+1} \tag{42}$$

$$\widehat{y}_t = \widehat{a}_t + \left(1-v\right) \widehat{l}_t^s + v\widehat{h}_{t-1}^e \tag{43}$$

$$(1+\varphi)\,\widehat{l}_t^s + \sigma\widehat{c}_t = \widehat{p}_t^I + \widehat{y}_t - \widehat{t}_t \tag{44}$$

$$\omega\widehat{c}_t + (1-\omega)\,\widehat{c}_t^e = \widehat{y}_t \tag{45}$$

$$\widehat{h}_t = -\frac{(1-\omega)}{\omega}\frac{h_{ss}^e}{h_{ss}}\widehat{h}_t^e \tag{46}$$

$$\widehat{p}_t^{h^e} = \widehat{p}_t^h + \psi^e \frac{h_{ss}^e}{p_{ss}^h}\left(\widehat{h}_t^e - \widehat{h}_{t-1}^e\right) \tag{47}$$

$$\widehat{d}_t = \beta^{-1}\left[\widehat{d}_{t-1} - \omega c_{ss}^{\sigma}\left(l_{ss}^s\right)^{1+\varphi}\widehat{t}_{t-1} + (1-\omega)\,b_{ss}\Gamma\left(\widehat{R}_{t-1} - \widehat{Q}_{t-1}\right)\right] \tag{48}$$

$$\widehat{t}_t = \alpha\widehat{d}_t \tag{49}$$

The system is closed by adding a policy rule, assessed by minimising (either under commitment or discretion) a loss function obtained as a second-order log approximation to a weighted average of household and entrepreneur utility, discounted at the household rate of time preference. This is given by:

$$\begin{aligned} L_t \quad = \quad & E_t \frac{1}{2}\sum_{s=0}^{\infty}\beta^s c_{ss}^{1-\sigma}\{\frac{1+\varphi}{1-v}\left(\widehat{y}_{t+s} - \widehat{y}_{t+s}^{fl}\right)^2 + \omega(1-\omega)\left(\widehat{c}_{t+s} - \widehat{c}_{t+s}^e\right)^2 \\ & + (\sigma-1)\left(\omega\widehat{c}_{t+s}^2 + (1-\omega)\left(\widehat{c}_{t+s}^e\right)^2\right) + \frac{2\psi^e\left(h_{ss}^e\right)^2}{c_{ss}}\left(\widehat{h}_{t+s}^e - \widehat{h}_{t+s-1}^e\right)^2 \\ & + \frac{\varepsilon\theta}{(1-\theta)(1-\beta\theta)}\widehat{\pi}_{t+s}^2 + \beta v\frac{\omega\vartheta + \beta v c_{ss}^{1-\sigma}}{\omega\vartheta}\left(\widehat{h}_{t+s}^e\right)^2\}, \end{aligned} \tag{50}$$

where $\widehat{y}_t^{fl} \equiv \widehat{a}_{t+s} + v\widehat{h}_{t+s-1}^e$ is defined as the log deviation in intermediate goods production that would be observed if labour supply were fixed at its steady-state level. The derivation of this objective is explained in detail below.

4.3 Steady-state solutions

It remains for us to solve for the steady state of the economy. As in much of the literature, we focus on a zero inflation steady state. Such a zero inflation steady state is often consistent with optimal policy under commitment. Under discretion, it is likely that the long run inflation rate will deviate from zero, possibly substantially. We follow Adam and Billi (2007) and many others and abstract from this issue. That is, we keep the quadratic objective and the linear constraints identical across the analysis for commitment and discretion.

Our strategy is to obtain an equation that implicitly defines h^e_{ss}, with the other steady state values solving recursively from this. Market clearing in the real estate sector, together with the optimal allocation of real estate between uses, implies:

$$\vartheta\omega c^{\sigma}_{ss} = \beta v y_{ss}\left(\frac{\overline{h}}{h^e_{ss}} - (1-\omega)\right). \tag{51}$$

Together with goods market clearing, this solves for c_{ss} as a function of h_{ss}:

$$c^{\sigma-1}_{ss} = \frac{\beta v}{\vartheta\omega\,(1-\omega)}\left(\frac{\overline{h}}{h^e_{ss}} - (1-\omega)\right). \tag{52}$$

Optimal labour supply together with goods market clearing gives:

$$l^s_{ss} = \left(\frac{1-v}{\omega}\right)^{\frac{1}{\varphi+1}} c_{ss}^{\frac{1-\sigma}{1+\varphi}}. \tag{53}$$

Plugging this into the production function, again making use of goods market clearing, and solving for $c^{\sigma-1}_{ss}$ we obtain:

$$c^{\sigma-1}_{ss} = (1-\omega)^{v(1+\varphi)\eta}\,\omega^{(1-v)\varphi\eta}\,(1-v)^{(1-v)\eta}\,(h^e_{ss})^{v(1+\varphi)\eta}, \tag{54}$$

where we define $\eta \equiv \frac{\sigma-1}{\varphi+v+\sigma(1-v)}$. We have two expressions for $c^{\sigma-1}_{ss}$ in terms of h^e_{ss} and constants. Provided $\sigma > 1$,[5] the the first expression ranges (monotonically) from $+\infty$ to $\frac{-(1-\omega)\beta v}{\vartheta\omega(1-\omega)}$ as h^e_{ss} goes from 0 to $+\infty$, whilst the second ranges (monotonically) from 0 to $+\infty$; thus there is a unique solution for steady-state commercial real estate in the desired domain, and we use numerical methods to solve for it in the model simulations.

5 Calibration

The starting point for our calibration is the values used in Iacoviello (2005). Relative to the Iacoviello calibration, we make the following changes. First, we move away from what we consider to be an unrealistic case of log utility and set the intertemporal elasticity of substitution a little lower. Second, we set Γ, the markup applied by banks, equal to 0.01. In annualized terms this amounts to banks charging a spread of 4 percentage points. Third, we set ψ^e, the parameter that determines the cost of adjusting real estate use, such that the cost associated with a within-period change in h^e equal to 5% of its steady-state value would amount to 1% of aggregate (not per-entrepreneur) output.

[5] Solving for the log-utility case is straightforward: $h^e_{ss} = \frac{\beta v\overline{h}}{(1-\omega)(\vartheta\omega+\beta v)}$ follows directly from earlier results. We restrict attention to values of $\sigma \geq 1$ in the analysis.

β	household discount factor	0.993
β^e	entreprener discount factor	0.95
σ	inverse elasticity of intertemporal substitution	2
ϑ	weight on housing utility	0.11
φ	inverse Frisch elasticity of labour supply	2
ε	elasticity of substitution across final goods	6
ω	measure of household sector	0.979
v	elasticity of output with respect to CRE	0.05
θ	Calvo hazard rate	0.67
ψ^e	real estate adjustment cost parameter	5.7×10^{-3}
m	steady state permitted collateral ratio	0.85
Γ	banking sector markup	1.01
α	fiscal feedback parameter	0.21

Table 1: Calibration

6 Deriving the policymakers' loss function

This derivation closely follows Andres *et al.* (2009), the minor differences relating to the introduction of costs of adjusting commercial real estate. The details are provided in an appendix. The approximate loss function that maximises the welfare of the agents in our model is given by:

$$
\begin{aligned}
W_t = & -\frac{1}{2} c_{ss}^{1-\sigma} E_t \sum_{s=0}^{\infty} \beta^s \{ \omega (1-\omega) \left(\widehat{c}_{t+s} - \widehat{c}^e_{t+s} \right)^2 \\
& + (\sigma - 1) \left(\omega \widehat{c}^2_{t+s} + (1-\omega) \left(\widehat{c}^e_{t+s} \right)^2 \right) + \frac{2\psi^\varepsilon \left(h^e_{ss} \right)^2}{(1-\omega) y_{ss}} \left(\widehat{h}^e_{t+s} - \widehat{h}^e_{t+s-1} \right)^2 \\
& + \frac{1+\varphi}{1-v} \left(\widehat{y}_{t+s} - \widehat{y}^{fl}_{t+s} \right)^2 + \frac{\varepsilon\theta}{(1-\theta)(1-\beta\theta)} \widehat{\pi}^2_{t+s} + \beta v \frac{\omega\vartheta + \beta v c_{ss}^{1-\sigma}}{\omega\vartheta} \left(\widehat{h}^e_{t+s} \right)^2 \} \\
& + t.i.p. + O^3,
\end{aligned} \tag{55}
$$

where the terms in a_{t+s} and the initial states $\widehat{\Delta}_{t-1}$ and $\widehat{h}^e_{t-1}$ are noted to be independent of policy.

Given the calibrated values for our model parameters, the weights in our loss function can be shown to be given by:

$$
\begin{aligned}
L_t = E_t \frac{1}{2} \sum_{s=0}^{\infty} \beta^s \{ & 3.3689 \left(\widehat{y}_{t+s} - \widehat{y}^{fl}_{t+s} \right)^2 + 0.0219 \left(\widehat{c}_{t+s} - \widehat{c}^e_{t+s} \right)^2 \\
& + 1.0444 \widehat{c}^2_{t+s} + 0.0224 \left(\widehat{c}^e_{t+s} \right)^2 + 3.205 \left(\widehat{h}^e_{t+s} - \widehat{h}^e_{t+s-1} \right)^2 \\
& + 38.8294 \widehat{\pi}^2_{t+s} + 0.0790 \left(\widehat{h}^e_{t+s} \right)^2 \},
\end{aligned} \tag{56}
$$

This reveals that despite the complications introduced into the basic New Keynesian model to characterise credit frictions, the dominant motive for monetary policy is still price stability, since the inflation term carries a weight of almost 40, one or more orders of magnitude greater than the weights on other arguments.

7 Solving for equilibrium with inequality constraints

We consider two cases, discretion and commitment. For commitment, we apply the method of Harrison (2010), which supplies a recipe for the case of multiple instruments at the zero bound, a generalisation of the analysis in Eggertson and Woodford (2003) which presents the case for one instrument. For our analysis under discretion, we present a new method, which combines the piecewise linear approach to inequality constraints of Eggertson and Woodford (2003) with iterations on the policymaker's value function under equilibrium-consistent assumptions about future policy. The algorithm works by solving for a terminal Markov-stationary equilibrium that will obtain once inequality constraints have ceased to bind, then iterates 'backwards through time' on the value function problem associated with this to obtain the (time-varying) policy rule in each period prior to this terminal regime.

This innovation is necessary in order to cope with the multiple state variables in the model of credit frictions that we are using. An alternative method described in Adam and Billi (2007) uses projection methods on the non-linear system formed from combining the zero bound constraint with the other linearized equilibrium conditions of the model. That method has an advantage over ours in that it allows one to solve for the equilibrium under which agents expect there to be some chance of future shocks. By contrast, we preserve the assumption of perfect foresight (that there will be no future shocks) inherited from Eggertson and Woodford (2003). But the payoff from doing so is that we have the ability to cope with multiple state variables, something that would not be practical to attempt using the method of Adam and Billi (2007). To study the recent crisis it seemed to us that we needed a model that had both the ZLB and credit frictions, and that therefore a realistic scrutiny of what policymakers did would require handling multiple state variables, and for that reason we hope our methodological contribution is of some practical use. We present the logic in an appendix, as applies to a general linear-quadratic problem.

8 Results

We hit the model with a negative shock of 5 percentage points to m_t,the portion of commercial real estate that can be pledged as collateral for loans from the banks, relative to its steady state value of 0.85. This shock is our attempt to proxy for the reduction in the preparedness to lend of financial intermediaries that characterised the early stages of the crisis. In order to illustrate the effects of the zero lower bound constraint and the second instrument, we begin by showing the behaviour of the model without either.

Chart 1 shows the model with one interest rate instrument, no ZLB constraint on interest rates, and compares the behaviour of the model under optimal monetary policy and in the case where interest rates follow a Taylor rule. The y axis in Chart 1 and subsequent charts is in terms of log deviations from log steady state. In this simulation the policy rate is the 'saving rate', and is equal to the rate entrepreneurs borrow at. When the second instrument is in play, these two rates differ, and the 'borrowing rate' indicates the path of this unconventional instrument.

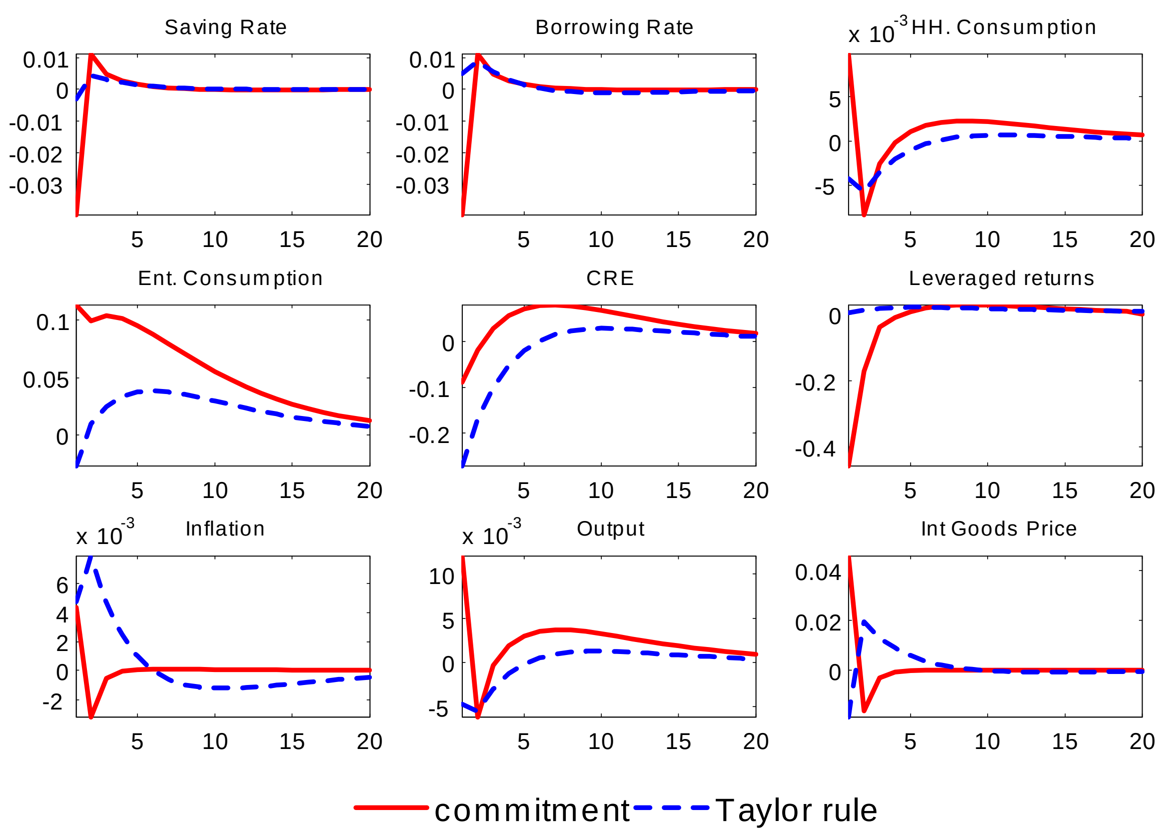

Chart 1: optimal monetary policy with 1 instrument (commitment, no zlb)

We can see how, relative to the Taylor Rule policy, optimal monetary policy is to cut rates sharply so as to engineer a consumption boom, which in turn generates an increase in the prices of the intermediate goods that entrepreneurs produce, and thereby discourages them from deleveraging and shrinking the commercial real estate they use. Optimal policy can be seen in part as an attempt to generate an income boost (in the form of increased prices for the goods they make) to offset the substitution away from real estate holdings that occurs because of the fall in the pledgeability ratio m_t.

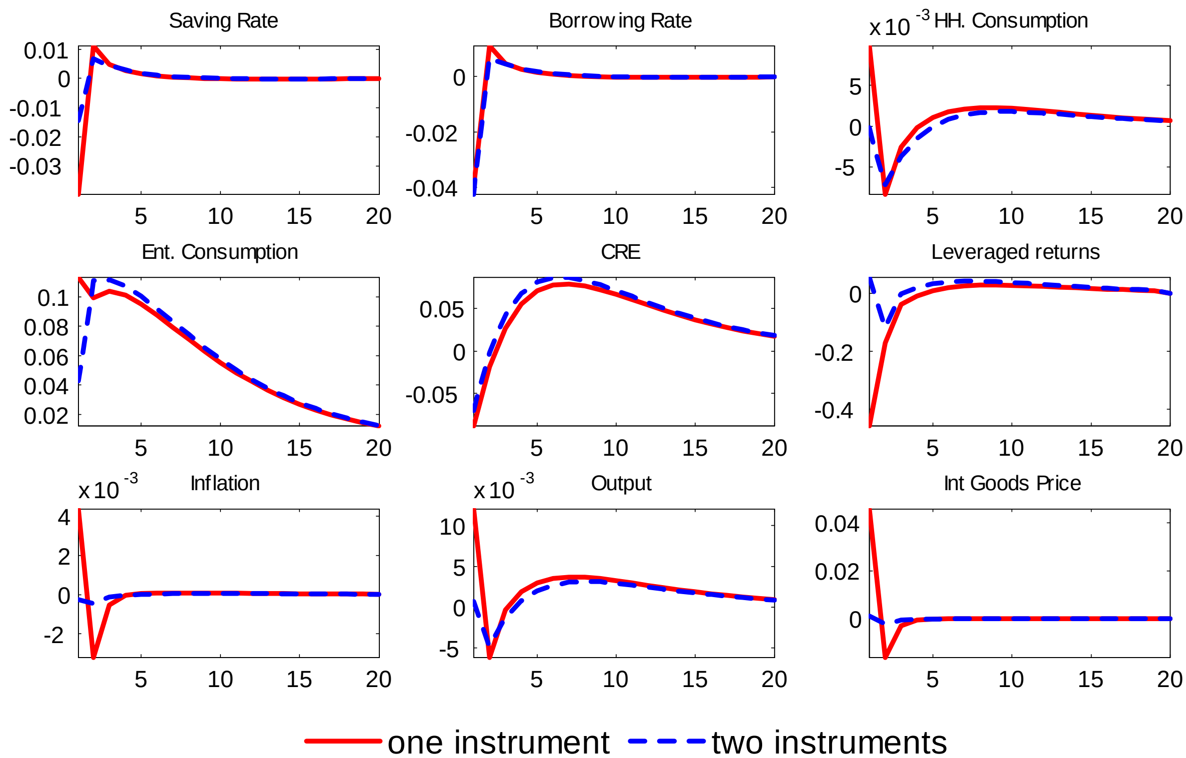

Chart 2: Commitment: one instrument vs. two instruments (no ZLB)

Chart 2 shows the effect of adding the option to use the second unconventional instrument. The ability to use the second instrument allows the policymaker to spread the burden of accommodating the shock to the pledgeability ratio; the saving rate now falls by less, and the borrowing rate now fails to track completely the path of the saving rate, indicating that assets are being purchased from the banks. However, note that the paths are still not much different, so not much use is being made of unconventional policy. Despite that, its use has a noticeable impact. Whereas before optimal policy was about generating an income boost to entrepreneurs by forcing up the price of the goods they sell, now the central bank has an instrument that can more directly offset the effect of the reduction in the pledgeability ratio. The borrowing constraint that entrepreneurs face involves terms in the borrowing rate. Recall that $b_t = m_t E_t \frac{\pi_{t+1}}{Q_t} p^{h^e}_{t+1} h^e_t$. Here the borrowing rate refers to Q_t. So the effect on b_t of the fall in m_t the pledgeability ratio is offset by the fall in Q_t the borrowing rate. There is no longer a need for such a large income effect, and therefore no longer a need for an inflation-inducing consumption boom, since the borrowing constraint can be relaxed more directly. Table 2 below records the welfare outcomes for different experiments we have conducted. Each cell in this table records the per period consumption loss which agents would regard as equivalent to the utility loss incurred by the fluctuations caused by the shock to the pledgeability ratio. We compute these losses as follows. We obtain welfare losses W_t from our simulations setting initial states to the deterministic steady state. We then solve the following equation for the consumption equivalent $\bar{C}$

$$\frac{1}{1-\beta}\left[\frac{\bar{C}^{1-\sigma}-1}{1-\sigma}+\omega\left(-\frac{\bar{l}^{1+\varphi}}{1+\varphi}+\vartheta\ln\left(\bar{h}\right)\right)\right]=W_t+\bar{W}$$

Turning to the effects of introducing the second instrument when we have commitment and no zero lower bound constraint, (illustrated in Chart 2), we see that this second instrument reduces the welfare loss from 0.0113% of per period consumption (required with one instrument) to 0.0101%.

	1 Instrument	**2 Instruments**
	No zero lower bound	
Commitment	0.0113%	0.0101%
Discretion	0.46%	0.46%
	Zero lower bound	
Commitment	0.0138%	0.0102%
Discretion	31.98%	2.09%

Table 2: Welfare losses: per period consumption equivalents as per cent of quarterly consumption

Chart 3 shows optimal policy under discretion compared with commitment. A word of caution at the outset: in some regions of our parameter space there exists more than one equilbrium under discretion and we pick a particular one. Section 9 discusses this issue in detail.

The broad outline of the discretionary strategy is the same as in the commitment case: cut rates sharply in order to engineer a consumption boom that persuades entrepreneurs not to deleverage and shrink their real estate holdings. But the policy is much less effective. The cut in the savings rate is much larger than in the commitment case shown in Chart 1, far lower than the zero lower bound would permit; despite this large cut the economy suffers a substantial deflation, which follows from the excess capacity created by the continued high levels of commercial real estate maintained by entrepreneurs. The welfare loss of the shock is equivalent to paying 0.4653% of per period consumption, compared with 0.0113% under commitment.

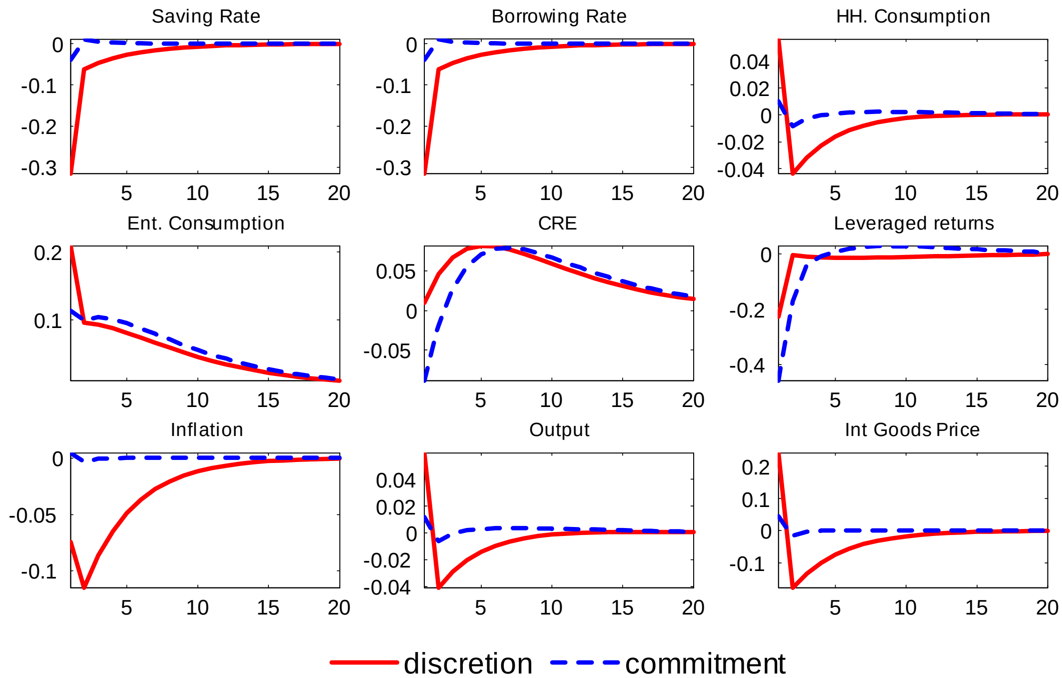

Chart 3: discretion vs commitment: 1 instrument, no zero lower bound

We next introduce the ZLB and compare commitment with discretion, in Chart 4. The outcome under discretion is now much worse than that under commitment - the lines for commitment appear almost flat in comparison with those under discretion. The welfare loss expressed as before rises from 0.4653% with no ZLB to about 32% with the ZLB constraint. By contrast, under commitment, the ZLB is much less deleterious in its effects: the welfare loss of the shock rises from 0.0113% to 0.0138%. Note that under discretion we are getting falls in output of more than 30%, and this far away from the steady state we have to take our first order approximation with a pinch of salt. Chart 4 raises a few other questions. First, why is the welfare loss imposed by the ZLB so much worse under discretion than commitment? Second, what is driving the result that the outcomes are so bad under discretion? Third, why are interest rates actually *rising* under discretion with the ZLB in place?

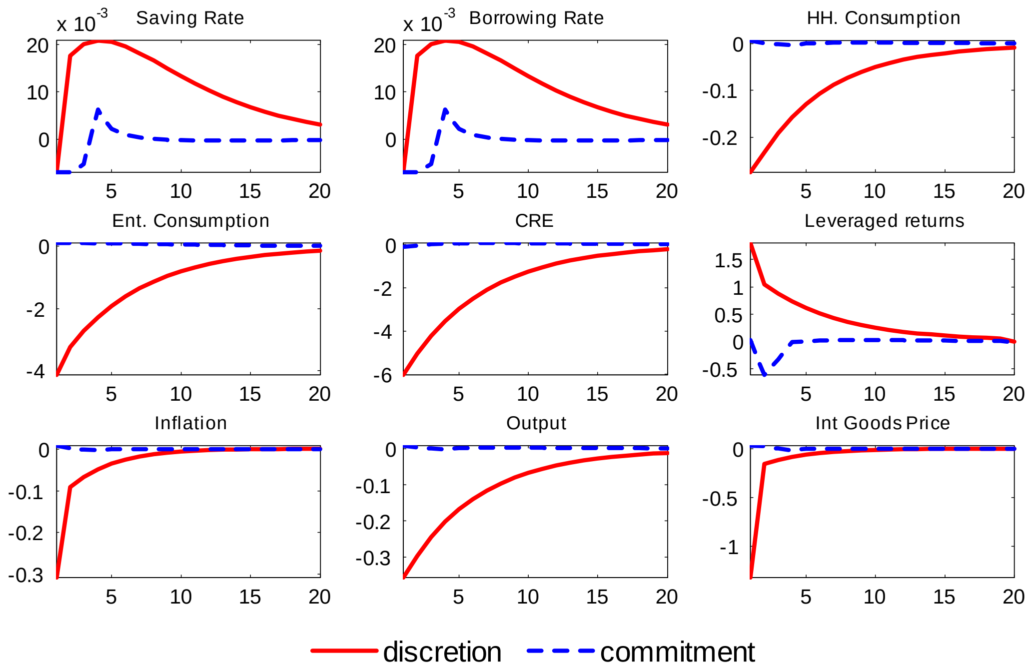

Chart 4: Discretion vs commitment at the ZLB with 1 instrument

One way to understand the difference between the commitment and discretion is to recall the mechanism by which policy operated in the absence of the zero lower bound. There, the reduction in permitted leverage reduced the returns to 'fully leveraged' commercial real estate investment, causing entrepreneurial investment to drop in the absence of a countervailing policy strategy. That strategy was to cut the main policy rate when the shock occurred, so as to induce a consumption boom that would raise entrepreneurs' profits – and thereby generate a positive income effect boosting entrepreneurs' saving levels (mitigating the negative substitution effect associated with the initial reduction leveraged returns). With a zero lower bound this strategy becomes impossible – a 'paradox of stimulus' that has close parallels with the 'paradox of toil' highlighted by Eggertson 2010. The reasoning is as follows. Suppose the policymaker were able to engineer a boom by reducing real interest rates. This should increase entrepreneurial wealth, and thus raise their CRE holdings over the medium term, as in Chart 3. But higher CRE implies that the economy would have a high level of productive capacity, which in turn implies that intermediate goods prices would be relatively low over the medium term – since commercial real estate, a productive input, would be relatively abundant. By pulling down final goods firms' marginal costs, this then implies that inflation would be relatively muted in future periods. Note that in this Calvo pricing setup inflation can be expressed as a discounted sum of future intermediate goods prices.

But low future inflation implies low initial inflation expectations, which in turn implies a higher initial real interest rate. This serves to undermine the initial attempt to cut real rates and stimulate consumption – and if the zero lower bound features, it simply proves impossible to cut nominal rates by enough to ensure the required initial consumption boom, given the subsequent deflationary consequences. In short, stimulus implies extra capacity, which induces disinflationary (or deflationary) pressure inconsistent with stimulus.

There is a slight complication that the above logic skirts. If a consumption boom were successfully engineered, unit labour costs would presumably rise (reflecting a reduced willingness on the part of workers to accept any given wage, given that their marginal utility of consumption is lower in a boom). This is helpful to the policymaker, putting upward pressure on inflation expectations. For our 'paradox of stimulus' to obtain we need the disinflationary consequences of increased CRE investment to exceed the inflationary consequences of a reduced willingness of household members to work.

So like Eggertson, we find that policymakers at the zero lower bound cannot be content with trying to engineer higher potential output in future time periods, since this may actively preclude the intertemporal prices required for demand to rise to meet this potential. Unlike Eggertson, we allow potential output to depend on the (endogenous) level of commercial real estate investment in the economy, and hence to be influenced by policy choices rather than being the product of consumers' exogenous desire to work. Thus our problem is more a 'paradox of stimulus' than a paradox of toil: if policymakers act under discretion, and so cannot commit to higher inflation at future horizons, they may find it impossible to inject stimulus via conventional monetary policy instruments without destroying the very mechanism that boosts demand.

A prima facie reason for why the ZLB worsens outcomes so much more under discretion is that the 'missing stimulus' imposed by the ZLB is so much larger. This can be seen from Chart 3 where it is clear that the optimal policy in the absence of the ZLB is for a larger cut on impact (thus a larger breach of the ZLB) than under discretion. As to why outcomes are so bad, a key factor here is the interaction of the deflation that insufficiently stabilising interest rate policy has with the borrowing constraint. Recall once more that this constraint has $b_t = m_t E_t \frac{\pi_{t+1}}{Q_t} p^{h^e}_{t+1} h^e_t$. The failure to cut the savings rate induces a deflation which exacerbates the fall in borrowing b_t necessitated by the fall in the pledgeability ratio m_t, which in turn induces even greater deflationary pressure, and so on. Similarly, debt deflation increases the real value of the debt burden and reduces the borrowing capacity of entrepreneurs further. The impact of this debt deflation can be seen by simulating a version of the model in which private debt is indexed. Chart 5 does this. The impulses of all key variables entering policymakers' loss function are substantially more benign when debt is indexed.

Finally, why are policy rates *rising* after the initial fall, especially when inflation and output are so depressed? The reason seems to be as follows. Partly because of the debt-deflation mechanism described above, and its anticipation, commercial real estate drops hugely on impact. Keeping interest rates low in the periods following the shock would stimulate demand and push up on inflation and output, but commercial real estate would move back to steady state quickly following the large initial fall. Our quadratic loss function penalizes variations in the growth rate of real estate and thus policymakers prefer to bring real estate back to target in a succession of small steps following a large initial fall.

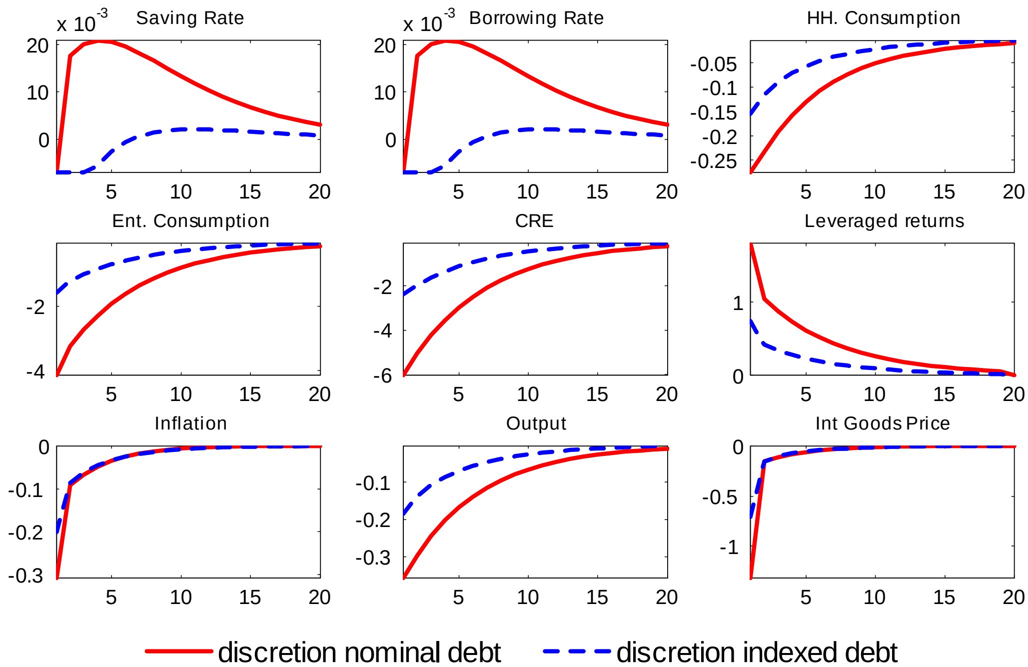

Chart 5: Discretion with 1 instrument and the ZLB, nominal vs indexed debt

Past analyses of discretionary policy in the New Keynesian model without credit frictions pointed to a fairly modest influence of the zero lower bound on the optimal response to technology shocks. In our model, outcomes under discretion at the zero lower bound can be quite bad and very different than those under commitment.

Chart 6 reproduces Chart 4 but when policymakers can use the second, unconventional instrument, asset purchases targeted at lowering the borrowing rate.

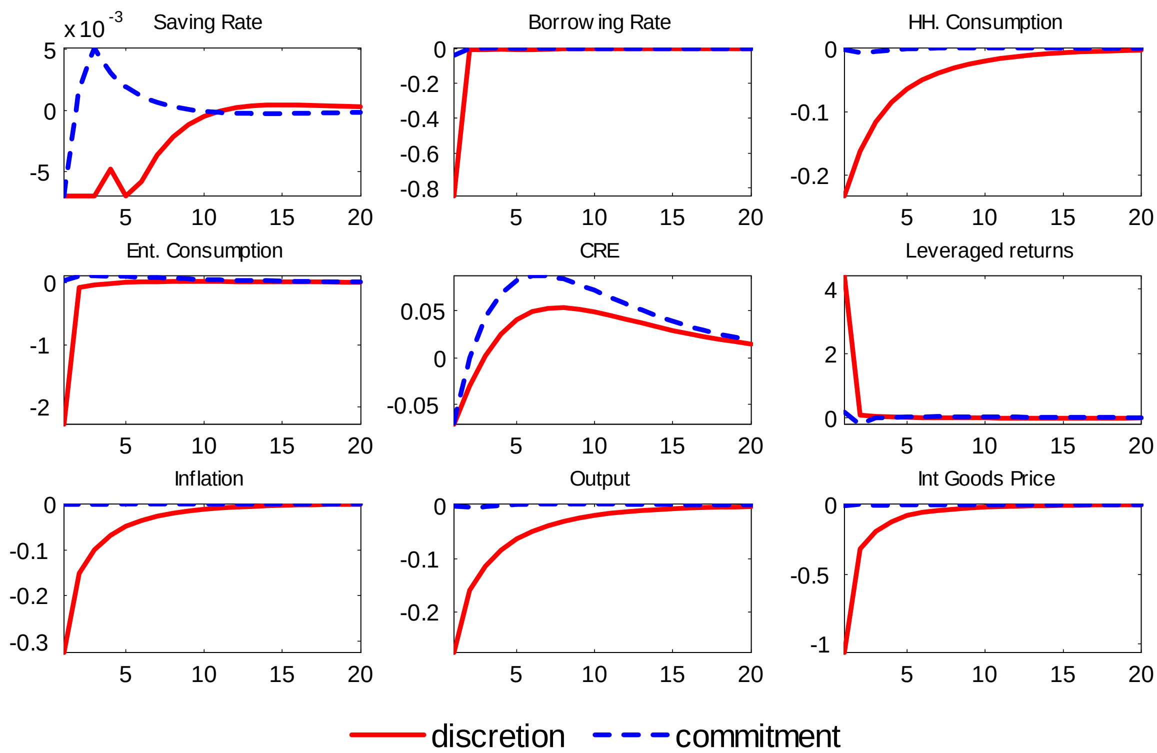

Chart 6: optimal policy with 2 instruments and the ZLB

There are several points to note. Under commitment, the second instrument is used for stabililsation purposes, but much less so than under discretion. This extra instrument improves the welfare loss associated with the shock from 0.0138% to 0.0102% of quarterly consumption (Table 2). In the case of discretion there is much greater exploitation of the second instrument (the borrowing rate falls dramatically). Outcomes for inflation and output are still of the same order of magnitude as with 1 instrument, but the stock of commercial real estate falls by much less. This translates to a reduction in the welfare loss due to the shock of more than a factor of 10, from about 32% to 2.1%of quarterly consumption. Once again, the precise numbers for the discretion case are probably questionable, on account of the inaccuracy of our first order approximation so far away from the steady state, but the qualitative point is clear. Having a second instrument is especially useful when you can't commit to managing expectations with the first.

The second point to take away from the experiment with 2 instruments and the ZLB (Chart 6) is that the benefits of having the second instrument are magnified by the presence of the zero lower bound. This is true for discretion and commitment. Looking back to Table 2: under commitment, the welfare gain from using the second instrument (quantified as before) amounts to a reduction in the compensating payments of 0.0012 percentage points without the ZLB, but with the ZLB this the second instrument reduces these payments by

0.0036 percentage points. For discretion, we record actually no benefit of having the second instrument without the ZLB, but a reduction in the payments required to compensate for the shock by a factor of more than 10 with the ZLB.

A final observation from our results concerns the benefits of commitment, or, as previous work termed it, the size of the 'stabilisation bias'. Using Table 2 we can deduce the following. With the ZLB, the benefit of commitment is greater when only 1 instrument is available than if 2 are. Without it, the benefit of commitment is not much affected by the second instrument, but if anything the benefit increases. The gain from commitment is also much greater when the ZLB binds, regardless of whether the policymaker has access to 1 or 2 instruments.

9 Notes on multiplicity

For our baseline calibration, we find two Markov-stationary equilibria under discretion. The possibility that there are multiple equilibria is a general property of rational expectations models with endogenous state variables when policy is set optimally under discretion, as explained in Blake and Kirsanova (2010) or King and Wolman (2004). In our model, the endogenous states are commercial real estate, entrepreneurial debt and the nominal interest rate. Chart 7 below shows the dynamics for the two equilibria model under discretion when no zero lower bound is imposed. Note that the second equilibrium displays a negative root in the decision rules giving rise to oscillating dynamics.

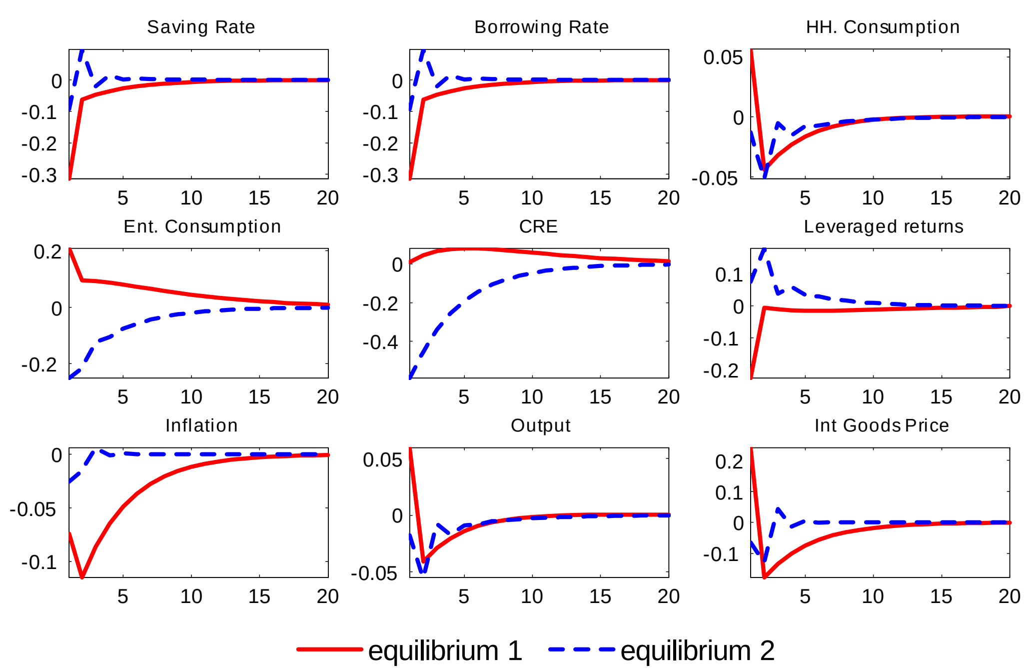

Chart 7: Discretionary Policy: Dynamics under two different equilibria (no ZLB, one instrument)

We neglect the equilibrium with oscillatory dynamics. We made this choice for 3 reasons. First, on grounds of realism: if this economy were to be estimated, the oscillatory equilibrium would probably have a lower likelihood. Second, on practical grounds: the oscillatory dynamics make it more cumbersome to implement algorithm for solving for optimal policy in the presence of the zero bound. That process involves guessing in which periods the lower bound is binding and when it is not, which is difficult when interest rates oscillates around zero. A third reason concerns what we find regarding the existence or otherwise of multiplicity on a quick exploration of the parameter space defined by our model. The oscillatory equilibrium is the only one that survives for values of our adjustment cost below the value we chose in our calibration. For values much above our baseline value, only the non-oscillatory equilibrium survives. At intermediate values for adjustment costs, both equilibria exist. If one assumes that the baseline calibration for adjustments is conservative and adjustment costs could plausibly be higher, then focusing on the equilibrium that survives for higher adjustment costs makes sense.

10 Conclusions

The contribution of this paper is to illustrate optimal monetary policy when there is access to a second unconventional instrument that affects spreads on private borrowing rates, a zero lower bound constraint on moving the conventional interest rate, when use of this second instrument has to be financed using distortionary taxation, and when there is a shock to financial conditions that tightens borrowing constraints on firms. Many of these issues are dealt with in separate papers, but we bring them all together in one place, and by doing so have a framework that can start to address what happened in the financial crisis.

The model we chose was a modification of Iacoviello (2005) and Andres *et al.* (2009). The credit friction in this model is that entrepreneurs,who are responsible for production, can only pledge a certain fraction of their commercial real estate holdings as collateral against loans they need to finance that production. We modified the model to incorporate costs of converting real estate from commercial to residential use and vice-versa. We simulated the model in the face of a downward shock to this portion - the pledgeability ratio - which reduces their borrowing capability. The unconventional policy instrument works directly against this friction. This instrument involves central bank purchases of claims issued by banks on the income streams flowing from loans made by them to entrepreneurs: essentially central bank purchases of securitised loans to firms. These purchases drive prices above (yields below) what would otherwise have prevailed.

Optimal monetary policy with one instrument aims at trying to generate an income effect for entrepreneurs to offset the substitution effect of the tightening in the borrowing constraint encouraging them to deleverage and scale back on their commercial real estate holdings. Using the second instrument rather unsurprisingly improves matters, because this second instrument helps relax the borrowing constraint directly. The zero lower bound constraint obviously has a deleterious effect on welfare, relative to the case where it is absent, and a disastrous effect when policy is set with discretion, using 1 instrument only. The potential for such large effects was not evident from previous studies which did not consider the interaction of the zero bound with credit frictions. The benefit of the second instrument is magnified by the presence of the zero lower bound (an extra instrument is even more helpful when you can't move the first) or when commitment is not possible. Our experiments allow us to investigate the determinants of the welfare cost of not being able to commit. These are increased by the presence of the zero lower bound: this eliminates the possibility of substituting cutting the short rate for managing expected future short rates. And these costs of discretion are reduced by having a second instrument, with our without the zero lower bound. In part, our unconventional tool makes up for not being able to manage expected future short rates.

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11 Appendix: deriving the policymaker's loss function

This derivation follows closely Andres et al (2010), which applies the logic set out in Woodford (2003) and elsewhere to derive an approximation to a function that maximises the welfare of the agents inhabiting our model economy. The loss function is derived from a second-order log approximation to the policymaker's objective function in the region of a zero-inflation, optimal steady state.[6] We first present the derivation of that approximation, and then turn to explain the derivation of the tax processes required to ensure that the steady state is distortion free.

First, we take a second-order log approximation to W_t directly:

$$\begin{aligned} W_t &= E_t \sum_{s=0}^{\infty} \beta^s \{ \omega c_{ss}^{1-\sigma} \left(\widehat{c}_{t+s} + \frac{1}{2}(1-\sigma)\widehat{c}_{t+s}^2 \right) \\ &\quad + (1-\omega) c_{ss}^{1-\sigma} \left(\widehat{c}_{t+s}^e + \frac{1}{2}(1-\sigma)\left(\widehat{c}_{t+s}^e\right)^2 \right) \\ &\quad - \omega l_{ss}^{1+\varphi} \left(\widehat{l}_{t+s}^s + \frac{1}{2}(1+\varphi)\left(\widehat{l}_{t+s}^s\right)^2 \right) + \omega\vartheta \widehat{h}_{t+s} \} + O^3. \end{aligned} \tag{57}$$

Next, we do likewise to the goods market equilibrium condition, making use of the fact that small deviations in prices do not affect the price dispersion index in a zero-inflation steady state, so linear deviations in the log of this index are already reflecting 'second-order' fluctuations:

[6] Approximating the model with log deviations makes the algebra more tractable, but requires us to neglect cases where $\sigma < 1$ if we are to use standard linear-quadratic techniques subsequently, since in this event consumption utility is a convex function of its log deviation from steady state, and the objective weight matrix may no longer be positive-semi-definite.

$$\begin{aligned}(1-\omega)\,y_{ss}\left(\widehat{y}_t+\frac{1}{2}\widehat{y}_t^2-\widehat{\Delta}_t\right) &= \omega c_{ss}\left(\widehat{c}_t+\frac{1}{2}\widehat{c}_t^2\right)\\ &\quad +(1-\omega)\,c_{ss}\left(\widehat{c}_t^e+\frac{1}{2}\left(\widehat{c}_t^e\right)^2\right)\\ &\quad +\psi^{\varepsilon}\left(h_{ss}^e\right)^2\left(\widehat{h}_t^e-\widehat{h}_{t-1}^e\right)^2+O^3, \end{aligned} \tag{58}$$

where we have also used the fact that $c_t = c_t^e$ must hold in any optimal steady state. Squaring both sides of this last equation and rearranging then gives:

$$\widehat{y}_t^2=\left(\frac{\omega}{(1-\omega)}\frac{c_{ss}}{y_{ss}}\right)^2\widehat{c}_t^2+\left(\frac{c_{ss}}{y_{ss}}\right)^2\left(\widehat{c}_t^e\right)^2+2\frac{\omega}{(1-\omega)}\left(\frac{c_{ss}}{y_{ss}}\right)^2\widehat{c}_t\widehat{c}_t^e+O^3, \tag{59}$$

which, together with the condition $\frac{c_{ss}}{y_{ss}}=(1-\omega)$ (which follows from goods market clearing when $c_t = c_t^e$) and some manipulation, gives us:

$$\widehat{y}_t=\omega\widehat{c}_t+(1-\omega)\,\widehat{c}_t^e+\widehat{\Delta}_t+\frac{1}{2}\omega\,(1-\omega)\left(\widehat{c}_t-\widehat{c}_t^e\right)^2+\frac{\psi^{\varepsilon}\left(h_{ss}^e\right)^2}{(1-\omega)\,y_{ss}}\left(\widehat{h}_t^e-\widehat{h}_{t-1}^e\right)^2+O^3. \tag{60}$$

The production function is log-linear, and can be used to replace terms in labour supply; we have:

$$\widehat{y}_t=\widehat{a}_t+(1-v)\,\widehat{l}_t^s+v\widehat{h}_{t-1}^e. \tag{61}$$

Collecting these expressions together, the objective can be written as:

$$\begin{aligned} W_t &= E_t\sum_{s=0}^{\infty}\beta^s\{c_{ss}^{1-\sigma}[\widehat{y}_{t+s}-\widehat{\Delta}_{t+s}-\frac{1}{2}\omega\,(1-\omega)\left(\widehat{c}_{t+s}-\widehat{c}_{t+s}^e\right)^2\\ &\quad -\frac{\psi^{\varepsilon}\left(h_{ss}^e\right)^2}{(1-\omega)\,y_{ss}}\left(\widehat{h}_{t+s}^e-\widehat{h}_{t+s-1}^e\right)-\frac{1}{2}(\sigma-1)\left(\omega\widehat{c}_{t+s}^2+(1-\omega)\left(\widehat{c}_{t+s}^e\right)^2\right)\\ &\quad -(1-v)\left(\frac{\widehat{y}_{t+s}-a_{t+s}-v\widehat{h}_{t+s-1}^e}{(1-v)}+\frac{1}{2}\frac{(1+\varphi)}{(1-v)^2}\left(\widehat{y}_{t+s}-\widehat{a}_{t+s}-v\widehat{h}_{t+s-1}^e\right)^2\right)]\\ &\quad +\omega\vartheta\widehat{h}_{t+s}\}+O^3. \end{aligned} \tag{62}$$

(This uses the fact that $\omega\left(l_{ss}^s\right)^{1+\varphi}=(1-v)\,c_{ss}^{1-\sigma}$ follows jointly from steady-state equilibrium in the goods and labour markets – assuming that the distortionary impact of final goods firms' market power has been offset, as outlined below.)

Next we exploit a second-order approximation to the real-estate market clearing condition, giving:

$$\omega h_{ss}\left(\widehat{h}_t+\frac{1}{2}\widehat{h}_t^2\right)+(1-\omega)\,h_{ss}^e\left(\widehat{h}_t^e+\frac{1}{2}\left(\widehat{h}_t^e\right)^2\right)=0. \tag{63}$$

An optimal steady state must involve the marginal (social) value product of a unit of real estate used for commercial purposes being equal to the marginal (social) value of the residential services it could deliver. Together with goods market clearing, that implies:

$$\vartheta h_{ss}^{-1} = \frac{\beta v}{(1-\omega)} \frac{c_{ss}^{1-\sigma}}{h_{ss}^{e}}. \tag{64}$$

Inserting this in the previous equation and simplifying:

$$\omega\vartheta\widehat{h}_t + \beta v c_{ss}^{1-\sigma}\widehat{h}_t^e = -\frac{1}{2}\left[\omega\vartheta\widehat{h}_t^2 + \beta v c_{ss}^{1-\sigma}\left(\widehat{h}_t^e\right)^2\right]. \tag{65}$$

Some protracted but ultimately straightforward algebra (see, for instance, Woodford, 2003) allows the price dispersion terms near a zero-inflation steady state to be expressed as squared deviations of the inflation rate:

$$\sum_{s=0}^{\infty}\beta^s\widehat{\Delta}_{t+s} = \frac{\theta}{1-\beta\theta}\widehat{\Delta}_{t-1} + \frac{1}{2}\frac{\varepsilon\theta}{(1-\theta)(1-\beta\theta)}\sum_{s=0}^{\infty}\beta^s\widehat{\pi}_{t+s}^2. \tag{66}$$

So that we finally have:

$$\begin{aligned} W_t \quad = \quad & -\frac{1}{2}c_{ss}^{1-\sigma}E_t\sum_{s=0}^{\infty}\beta^s\{\omega(1-\omega)\left(\widehat{c}_{t+s}-\widehat{c}_{t+s}^e\right)^2 \\ & +(\sigma-1)\left(\omega\widehat{c}_{t+s}^2+(1-\omega)\left(\widehat{c}_{t+s}^e\right)^2\right)+\frac{2\psi^{\varepsilon}\left(h_{ss}^e\right)^2}{(1-\omega)\,y_{ss}}\left(\widehat{h}_{t+s}^e-\widehat{h}_{t+s-1}^e\right)^2 \\ & +\frac{1+\varphi}{1-v}\left(\widehat{y}_{t+s}-\widehat{y}_{t+s}^{fl}\right)^2+\frac{\varepsilon\theta}{(1-\theta)(1-\beta\theta)}\widehat{\pi}_{t+s}^2+\beta v\frac{\omega\vartheta+\beta v c_{ss}^{1-\sigma}}{\omega\vartheta}\left(\widehat{h}_{t+s}^e\right)^2\} \\ & +t.i.p.+O^3, \end{aligned} \tag{67}$$

where the terms in a_{t+s} and the initial states $\widehat{\Delta}_{t-1}$ and $\widehat{h}_{t-1}^e$ are noted to be independent of policy.

12 Appendix: steady-state taxes

A 'simple' linear-quadratic approach to the optimal policy problem – in which a second-order approximation to the policymaker's objective criterion is maximised subject to first-order approximations to the economy's equilibrium conditions – will in general only be appropriate to finding the best path in the neighbourhood of a steady state that is itself optimal from the policymaker's perspective. This steady-state optimality implies the first-order effects of fluctuations on the value of the objective function are zero (given the equilibrium restrictions constraining their impact) – a fact that can be used to eliminate linear terms from the second-order expansion of the welfare criterion, as we do above. Given a purely quadratic approximation to the objective, the second-order effects of fluctuations on the policymaker's constraints can be neglected – since the impact on welfare of *first*-order relaxations of these constraints is already of second order at best.[7] To ensure this approximation is

[7] Benigno and Woodford (2008) present general methods for obtaining a linear-quadratic formulation of the optimal policy problem without introducing steady-state taxes. But their method relies on the 'timeless perspective' criterion for ranking stationary

possible we follow much of the literature and assume certain taxes and subsidies are in place that correct the tendency of market variables to be located away from the policymaker's preferred steady state.

As noted in passing in the derivation above, we require the following three conditions in particular to be satisfied in an optimal steady state:[8]

$$c_{ss} = c_{ss}^{e} \tag{68}$$

$$\omega\vartheta h_{ss}^{-1} = \frac{\omega}{(1-\omega)}(1-\omega)\beta v y_{ss}\left(h_{ss}^{e}\right)^{-1} c_{ss}^{-\sigma} \tag{69}$$

$$\omega\left(l_{ss}^{s}\right)^{\varphi} = (1-v)(1-\omega)\, y_{ss}\left(l_{ss}^{s}\right)^{-1} c_{ss}^{-\sigma} \tag{70}$$

The first of these follows from the fact that the policymaker places relative weights on the consumption utilities of households and entrepreneurs that are proportional to their respective population shares. The second states that the value of giving households an extra unit of residential housing must equal the value of reallocating that real estate to entrepreneurs and allowing them to consume the marginal product of it one period hence. The third is the usual intratemporal requirement that the marginal disutility of labour supply should equal the marginal value of consuming its product.

To guarantee that these conditions are met, three time-invariant market interventions are introduced: a proportional tax on the profits of entrepreneurs in each period, τ^{e}; a proportional tax on the value of residential property held by households at the end of each period, denoted τ^{h}; and a proportional revenue subsidy paid to final goods producers, τ These are set as follows:

$$\left(1-\tau^{e}\right) = \frac{1-\omega}{v\left(1-\beta^{e}\right)}\frac{\Gamma\left(1-\beta^{e}\right) - m\left(\beta - \Gamma\beta^{e}\right)}{\Gamma - m\beta} \tag{71}$$

$$\left(1+\tau^{h}\right) = \beta + \frac{\beta}{\beta^{e}}\frac{v\left(\Gamma - m\beta\right)\left(1-\beta^{e}\right)}{\Gamma\left(1-\omega\right)} \tag{72}$$

$$(1+\tau) = \frac{\varepsilon}{\varepsilon - 1} \tag{73}$$

The first two of these taxes together ensure that entrepreneurs are content with receiving a stable consumption profile when they hold the unique level of commercial real estate consistent with an optimal allocation,[9] *and* that in this event the implied relative wealth shares of entrepreneurs and households are equal – so they both consume the same quantities. To show this, we first substitute the expression for $\left(1+\tau^{h}\right)$ into a steady-state

policy plans, and the normative basis for this is unclear.

[8]The logic here again closely follows that of Andres et al (2010).

[9]Put differently, the returns that entrepreneurs can obtain at the margin from holding an extra unit of commercial real estate *and* leveraging this purchase to the maximum extent permitted will now equal $\left(\beta^{e}\right)^{-1}$ at precisely the optimal level of commercial real estate.

version of the household optimality condition for residential property holdings, obtaining:

$$c_{ss}^{-\sigma} p_{ss}^{h} \frac{\beta}{\beta^{e}} \frac{v\left(\Gamma - m\beta\right)\left(1-\beta^{e}\right)}{\Gamma\left(1-\omega\right)} = \vartheta h_{ss}^{-1}. \tag{74}$$

Similarly, inserting the expression for $(1-\tau^{e})$ into a steady-state version of the entrepreneur's Euler condition and simplifying gives:

$$p_{ss}^{h} = \beta^{e} \frac{\left(1-\omega\right)\Gamma}{v\left(\Gamma - m\beta\right)\left(1-\beta^{e}\right)} v \frac{y_{ss}}{h_{ss}^{e}}. \tag{75}$$

Using this equation in the previous confirms that the allocation of real estate is optimal. To show that entrepreneurial and household consumption levels are equal, we use the entrepreneurial budget constraint together with the expression for $(1-\tau^{e})$ and the solution for p_{ss}^{h} just derived, simplifying down to obtain:

$$c_{ss}^{e} = \left(1-\omega\right) y_{ss}. \tag{76}$$

Steady-state goods market clearing gives us $\omega c_{ss} + \left(1-\omega\right) c_{ss}^{e} = \left(1-\omega\right) y_{ss}$, and equal consumption immediately follows.

Finally, the first-order condition for optimal price-setting on the part of final goods firms in steady state gives:

$$\left(1+\tau\right) = \frac{\varepsilon}{\varepsilon - 1} p_{ss}^{I}. \tag{77}$$

Intermediate goods firms set the (real) marginal revenue product of hiring an additional worker equal to the real wage:

$$p_{ss}^{I} \left(1-v\right) \frac{y_{ss}}{l_{ss}^{d}} = w_{ss}. \tag{78}$$

Workers allow their marginal rate of substitution to equal the same:

$$\left(l_{ss}^{s}\right)^{\varphi} c_{ss}^{\sigma} = w_{ss}. \tag{79}$$

Labour market clearing ($\omega l_{ss}^{s} = \left(1-\omega\right) l_{ss}^{d}$), together with the value $p_{ss}^{I} = 1$ implied by τ, then confirms the final optimality condition.

13 Appendix: a new algorithm for solving for optimal monetary policy under discretion with inequality constraints

The economy is described by a set of linear rational-expectations equations, stacked in matrix form to give:

$$A_0 y_t = A_1 y_{t-1} + A_2 E_t y_{t+1} + A_3 x_t + A_4 x_{t-1} + A_5 v_t, \qquad (80)$$

where y_t is a vector of endogenous variables, x_t a vector of controls and v_t a vector of mean-zero, iid disturbances. Additionally, a set of inequality restrictions constrain the behaviour of the control variables:

$$S x_t \geq s, \qquad (81)$$

where s is a vector of constants and S a coefficient matrix. The policymaker sets these controls in order to minimise the discounted sum of loss L_t:

$$L_t = E_t \sum_{s=0}^{\infty} \beta^s y'_{t+s} W y_{t+s}, \qquad (82)$$

where W is a symmetric, positive-semi-definite weighting matrix and $\beta \in [0, 1)$ the rate of pure time preference.[10]

The policymaker at time t is unable to commit to a policy strategy at later dates, and must therefore take future policy responses to the state of the economy as given – effectively playing the role of Stackelberg leader in a non-cooperative game with its own future incarnations. We initially assume that in the equilibrium under study the policymaker at time $t + 1$ behaves in a way that ensures the following state-contingent evolution for the endogenous variables:

$$y_{t+1} = H_{1,t+1} y_t + H_{2,t+1} x_t + H_{3,t+1} v_{t+1} + H_{4,t+1}, \qquad (83)$$

where the transition matrices are indexed by the time period to which they apply. $H_{4,t+1}$ is a vector of constants that will enter into equilibrium behaviour in the event that certain policy variables are constrained to equal non-zero scalar values.

This implies that the policymaker acting at time t faces the following consolidated constraint, alongside the inequality restrictions:

$$\left(A_0 - A_2 H_{1,t+1}\right) y_t = A_1 y_{t-1} + \left(A_3 + A_2 H_{2,t+1}\right) x_t + A_4 x_{t-1} + A_5 v_t + A_2 H_{4,t+1}. \qquad (84)$$

We require for what follows that $(A_0 - A_2 H_{1,t+1})$ is invertible. The piecewise approach to solving the model under inequality constraints assumes perfect foresight in the aftermath of a shock, allowing one to specify the precise timing of transition from one 'regime' to another along the stabilisation path (where each regime is characterised by a distinct subset of the inequality constraints binding). This is an extremely useful simplification for our purposes, as it allows us to find a discretionary equilibrium by working backwards – first solving for a stationary Markov equilibrium in the regime consistent with the control variables being close to

[10] It would be straightforward to generalise to cases in which the policymaker also faces loss directly from deviations in the control variables x_t, and indeed to cases in which expected future values of the controls feature in the model's structural equations.

steady state, then finding the optimal response to this future behaviour in earlier periods, that are subject to different binding policy constraints. To this end we set up a value function problem with the advantage that it can be solved iteratively for a stationary solution in the terminal regime whilst still characterising the optimal policy problem along the prior, non-stationary path. Given the linear-quadratic nature of the problem, it makes sense to conjecture that the value function in the arbitrary time period $t+1$ satisfies:

$$V_{t+1}\left(y_t, x_t\right) = y_t' V_{t+1}^{yy} y_t + 2 y_t' V_{t+1}^{yx} x_t + x_t' V_{t+1}^{xx} x_t + 2 y_t' V_{t+1}^{yc} + 2 x_t' V_{t+1}^{xc} + \Sigma_{t+1}, \tag{85}$$

using superscript notation in the obvious way (c indicates constant terms), and collecting in Σ_{t+1} the sum of all terms that involve the shock vector, together with a constant (these are not relevant to the analysis that follows). The linear terms in the lagged endogenous and control variables will be non-zero in the event that policy is constrained by a non-zero scalar. We assume that the combined value matrix conformable with the vector $z_t \equiv [y_t', x_t']'$, obtained by collecting the relevant sub-components above, is positive semi-definite. Given this, by the definition of the value function the policymaker at time t faces the following problem (written in Lagrangian form):

$$\begin{aligned} \min_{x_t, y_t} \max_{\lambda_t, \mu_t \geqq 0} & \{ y_t' W y_t + \beta V_{t+1}\left(y_t, x_t\right) - 2\lambda_t' [\left(A_0 - A_2 H_{1,t+1}\right) y_t \\ & - A_1 y_{t-1} - \left(A_3 + A_2 H_{2,t+1}\right) x_t - A_4 x_{t-1} - A_5 v_t - A_2 H_{4,t+1}] \\ & - 2\mu_t' \left(S x_t - s\right) \}. \end{aligned} \tag{86}$$

Reasoning in the usual way, four necessary conditions for an optimum are:

$$\left(W + \beta V_{t+1}^{yy}\right) y_t + \beta V_{t+1}^{yx} x_t + \beta V_{t+1}^{yc} - \left(A_0' - H_{1,t+1}' A_2'\right) \lambda_t = 0, \tag{87}$$

$$\beta \left(V_{t+1}^{yx}\right)' y_t + \beta V_{t+1}^{xx} x_t + \beta V_{t+1}^{xc} + \left(A_3' + H_{2,t+1}' A_2'\right) \lambda_t - S' \mu_t = 0, \tag{88}$$

$$\left(A_0 - A_2 H_{1,t+1}\right) y_t - A_1 y_{t-1} - \left(A_3 + A_2 H_{2,t+1}\right) x_t - A_4 x_{t-1} - A_5 v_t - A_2 H_{4,t+1} = 0, \tag{89}$$

$$\mu_t' \left(S x_t - s\right) = 0. \tag{90}$$

Manipulating the first of these:

$$\lambda_t = \left(A_0' - H_{1,t+1}' A_2'\right)^{-1} \left[\left(W + \beta V_{t+1}^{yy}\right) y_t + \beta V_{t+1}^{yx} x_t + \beta V_{t+1}^{yc}\right], \tag{91}$$

which implies:

$$\begin{aligned} & \beta\left(V_{t+1}^{yx}\right)' y_t + \beta V_{t+1}^{xx} x_t + \beta V_{t+1}^{xc} \\ & + \left(A_3' + H_{2,t+1}' A_2'\right)\left(A_0' - H_{1,t+1}' A_2'\right)^{-1}\left[\left(W + \beta V_{t+1}^{yy}\right) y_t + \beta V_{t+1}^{yx} x_t + \beta V_{t+1}^{yc}\right] \\ = \; & S' \mu_t. \end{aligned} \tag{92}$$

The third condition solves for y_t:

$$y_t = \left(A_0 - A_2 H_{1,t+1}\right)^{-1}\left[A_1 y_{t-1} + \left(A_3 + A_2 H_{2,t+1}\right) x_t + A_4 x_{t-1} + A_5 v_t + A_2 H_{4,t+1}\right]. \tag{93}$$

Combining the previous two equations, we have a relationship describing the optimal values for the t-dated controls, of the form:

$$\Delta_1 x_t + \Delta_2 y_{t-1} + \Delta_3 x_{t-1} + \Delta_4 v_t + \Delta_5 - S' \mu_t = 0, \tag{94}$$

where:

$$\begin{aligned} \Delta_1 = \; & \left[\beta\left(V_{t+1}^{yx}\right)' + \left(A_3' + H_{2,t+1}' A_2'\right)\left(A_0' - H_{1,t+1}' A_2'\right)^{-1}\left(W + \beta V_{t+1}^{yy}\right)\right] \\ & \cdot \left(A_0 - A_2 H_{1,t+1}\right)^{-1}\left(A_3 + A_2 H_{2,t+1}\right) + \beta V_{t+1}^{xx} + \left(A_3' + H_{2,t+1}' A_2'\right)\left(A_0' - H_{1,t+1}' A_2'\right)^{-1} \beta V_{t+1}^{yx}, \end{aligned} \tag{95}$$

$$\Delta_2 = \left[\beta\left(V_{t+1}^{yx}\right)' + \left(A_3' + H_{2,t+1}' A_2'\right)\left(A_0' - H_{1,t+1}' A_2'\right)^{-1}\left(W + \beta V_{t+1}^{yy}\right)\right]\left(A_0 - A_2 H_{1,t+1}\right)^{-1} A_1, \tag{96}$$

$$\Delta_3 = \left[\beta\left(V_{t+1}^{yx}\right)' + \left(A_3' + H_{2,t+1}' A_2'\right)\left(A_0' - H_{1,t+1}' A_2'\right)^{-1}\left(W + \beta V_{t+1}^{yy}\right)\right]\left(A_0 - A_2 H_{1,t+1}\right)^{-1} A_6, \tag{97}$$

$$\Delta_4 = \left[\beta\left(V_{t+1}^{yx}\right)' + \left(A_3' + H_{2,t+1}' A_2'\right)\left(A_0' - H_{1,t+1}' A_2'\right)^{-1}\left(W + \beta V_{t+1}^{yy}\right)\right]\left(A_0 - A_2 H_{1,t+1}\right)^{-1} A_5, \tag{98}$$

$$\begin{aligned} \Delta_5 = \; & \left[\beta\left(V_{t+1}^{yx}\right)' + \left(A_3' + H_{2,t+1}' A_2'\right)\left(A_0' - H_{1,t+1}' A_2'\right)^{-1}\left(W + \beta V_{t+1}^{yy}\right)\right] \\ & \cdot \left(A_0 - A_2 H_{1,t+1}\right)^{-1} A_2 H_{4,t+1} + \beta V_{t+1}^{xc} + \left(A_3' + H_{2,t+1}' A_2'\right)\left(A_0' - H_{1,t+1}' A_2'\right)^{-1} \beta V_{t+1}^{yc}. \end{aligned} \tag{99}$$

Provided Δ_1 is non-singular, this relationship will allow us to tie down the value of the controls at time t in precisely as many dimensions as are left free by the complementary slackness condition on the inequality constraints (where an inequality constraint binds, the above condition tells us the value of the multiplier on it). Solving for x_t is therefore a matter of combining the inequality constraints with the previous expression in a way that will always permit a solution for x_t. Stacking the two, we have:

$$\begin{bmatrix} I - J_k & J_k S \\ -S' J_k & \Delta_1 \end{bmatrix} \begin{bmatrix} \mu_t \\ x_t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\Delta_2 & -\Delta_3 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & J_k S \\ -\Delta_4 & -\Delta_5 \end{bmatrix} \begin{bmatrix} v_t \\ 1 \end{bmatrix}, \tag{100}$$

where J_k is an indicator matrix that has unit entries on the diagonal at positions corresponding to non-zero elements of μ_t, and zeros elsewhere. The coefficient matrix on the left-hand side here is then invertible by construction (again, provided Δ_1 is non-singular). Inverting it allows us to extract a solution for x_t in the form:

$$x_t = F_{1,t} y_{t-1} + F_{2,t} x_{t-1} + F_{3,t} v_t + F_{4,t}, \tag{101}$$

where the coefficient matrices here are read off from the relevant blocks in the matrices

$$\begin{bmatrix} I - J_k & J_k S \\ -S' J_k & \Delta_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ -\Delta_2 & -\Delta_3 \end{bmatrix} \text{and} \begin{bmatrix} I - J_k & J_k S \\ -S' J_k & \Delta_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & J_k S \\ -\Delta_4 & -\Delta_5 \end{bmatrix}$$

.

Using this to replace the contemporaneous controls in the earlier solution for y_t, we have an evolution equation for the endogenous variables in the desired form:

$$y_t = H_{1,t} y_{t-1} + H_{2,t} x_{t-1} + H_{3,t} v_t + H_{4,t}, \tag{102}$$

where:

$$H_{1,t} = (A_0 - A_2 H_{1,t+1})^{-1} \left[A_1 + (A_3 + A_2 H_{2,t+1}) F_{1,t}\right], \tag{103}$$

$$H_{2,t} = (A_0 - A_2 H_{1,t+1})^{-1} \left[A_4 + (A_3 + A_2 H_{2,t+1}) F_{2,t}\right], \tag{104}$$

$$H_{3,t} = (A_0 - A_2 H_{1,t+1})^{-1} \left[A_5 + (A_3 + A_2 H_{2,t+1}) F_{3,t}\right], \tag{105}$$

$$H_{4,t} = (A_0 - A_2 H_{1,t+1})^{-1} \left[A_2 H_{4,t+1} + (A_3 + A_2 H_{2,t+1}) F_{4,t}\right]. \tag{106}$$

This therefore gives us an updating procedure for the policymaker's optimal law of motion for the endogenous variables, given the behaviour of future policymakers. Using these solutions for x_t and y_t, we are able to substitute in to the t-dated objective function directly, and express the minimum value of the policymaker's loss function in terms of y_{t-1}, x_{t-1}, v_t and a constant. Doing so yields the following solutions for the value matrices specified above:

$$V_t^{yy} = H_{1,t}' \left[W + \beta V_{t+1}^{yy} \right] H_{1,t} + \beta H_{1,t}' V_{t+1}^{yx} F_{1,t} + \beta F_{1,t}' \left(V_{t+1}^{yx} \right)' H_{1,t} + \beta F_{1,t}' V_{t+1}^{xx} F_{1,t}, \tag{107}$$

$$V_t^{yx} = H_{1,t}' \left[W + \beta V_{t+1}^{yy} \right] F_{2,t} + \beta H_{1,t}' V_{t+1}^{yx} F_{2,t} + \beta F_{1,t}' \left(V_{t+1}^{yx} \right)' H_{2,t} + \beta F_{1,t}' V_{t+1}^{xx} F_{2,t}, \tag{108}$$

$$V_t^{xx} = H_{2,t}' \left[W + \beta V_{t+1}^{yy} \right] H_{2,t} + \beta H_{2,t}' V_{t+1}^{yx} F_{2,t} + \beta F_{2,t}' \left(V_{t+1}^{yx} \right)' H_{2,t} + \beta F_{2,t}' V_{t+1}^{xx} F_{2,t}, \tag{109}$$

$$\begin{aligned} V_t^{yc} &= H_{1,t}' \left[W + \beta V_{t+1}^{yy} \right] H_{4,t} + \beta H_{1,t}' V_{t+1}^{yx} F_{4,t} + \beta F_{1,t}' \left(V_{t+1}^{yx} \right)' H_{4,t} \\ &\quad + \beta F_{1,t}' V_{t+1}^{xx} F_{4,t} + \beta H_{1,t}' V_{t+1}^{yc} + \beta F_{1,t}' V_{t+1}^{xc}, \end{aligned} \tag{110}$$

$$\begin{aligned} V_t^{xc} &= H_{2,t}' \left[W + \beta V_{t+1}^{yy} \right] H_{4,t} + \beta H_{2,t}' V_{t+1}^{yx} F_{4,t} + \beta F_{2,t}' \left(V_{t+1}^{yx} \right)' H_{4,t} \\ &\quad + \beta F_{2,t}' V_{t+1}^{xx} F_{4,t} + \beta H_{2,t}' V_{t+1}^{yc} + \beta F_{2,t}' V_{t+1}^{xc}. \end{aligned} \tag{111}$$

Given a particular 'steady state' regime – characterised by the relevant indicator matrix J_k (denoting the inequality constraints that bind in the region of steady state) – we can use the results above to obtain a stationary Markov equilibrium, which is assumed to obtain in perpetuity in our perfect foresight equilibrium once the impact of initial shocks has faded sufficiently. To be clear, the algorithm for finding this equilibrium is as follows:

1. Initialise values for the H and V matrices that are assumed to apply at some terminal date T. In applications we have found that null matrices usually suffice, though if multiple stationary Markov equilibria exist, different initialisations for the terminal value matrices can be used to find these.[11]

2. Iterate 'backwards through time', solving for the H and V matrices at each horizon given their next-period value, using the method outlined above. Continue until all matrices have converged (with $H_{1,t} = H_{1,t+1}$, $V_t^{yy} = V_{t+1}^{yy}$ and so on).

The 'terminal' stationary equilibrium found, we know that the policymaker choosing in the last period of the regime that is immediately *prior* to the terminal one faces a next-period value function and transition equation consistent with that terminal equilibrium, but must optimise with a different sub-set of the inequality constraints binding (that is, applying a different value of the indicator matrix J_k). Optimal dynamics in that period will thus satisfy H and F matrices that are obtained from a single application of the solution procedure above,

[11] One might conjecture that a set of null terminal value matrices should always deliver convergence on the stationary equilibrium that delivers lowest loss, though this may not be true for all initialisations of the H matrices.

inputting next-period value and transition matrices associated with the terminal equilibrium but constraints consistent with the prior regime. This in turn implies a distinct set of value matrices associated with that particular time period. One can work backwards in precisely the same way for as many periods (and through as many regime shifts) as required, retaining the value and transition matrices (both H and F) at each horizon, so as to be able to chart the non-stationary evolution of the variables of interest proceeding from any initial shocks.

The precise sequencing of regimes unfortunately requires some guesswork. In general a solution must be characterised by the inequality constraints being satisfied, and the multipliers associated with these inequality constraints being non-negative, at all horizons. But these conditions can only be checked *after* solving the model for a hypothetical sequencing: if satisfied, the associated dynamics constitute a discretionary equilibrium; if not, one must start again with a new conjecture for the timing of the different regimes.