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On Variance Bounds for Asset Price Changes*

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Abstract

This paper considers variance bounds for stock price changes in a general setting that allows for ex-dividend stock prices, risk averse investors, and exponentially-growing dividends. I show that providing investors with more information about future dividends can either increase or decrease the variance of stock price changes, depending on key parameters, namely, those governing the properties of dividends and the stochastic discount factor. This finding contrasts with the results of Engel (2005) who shows that news about future dividends will always decrease the variance of stock price changes in a specialized setting with cum-dividend stock prices and risk neutral investors.

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JEL Classification: E44, G12, G14.

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1 Introduction

In theory, the price of a stock represents the market's consensus forecast of the discounted sum of future dividends that will accrue to the owner. If dividends are stationary, the theory says that the variance of observed stock prices (the forecast) should be lower than the variance of discounted realized dividends (the object being forecasted). Shiller (1981) and LeRoy and Porter (1981) argued that this rationality principle appears to be violated in the case of U.S. stock prices.

West (1988) extended the analysis to allow for nonstationary dividends. He showed that the variance of *unexpected* changes in the stock price will decline if risk neutral investors are given more information about future dividends. This is because rational investors will use any new information to improve the precision of their dividend forecasts, thereby reducing the variance of the forecast errors.[1]

Engel (2005) extends the analysis of West (1988) to consider the variance of *actual* changes in the stock price, i.e., $\Delta p_t \equiv p_t - p_{t-1}$, as opposed to unexpected changes. Engel's analysis allows the level of real dividends to evolve as an arithmetic random walk or a stationary stochastic process. Assuming that stock prices are "cum-dividend," he shows that the variance of Δp_t will decline if risk neutral investors are given more information about future dividends. In particular, he proves analytically that $Var(\Delta p_t) \geq Var(\Delta p_t^*)$, where p_t^* is the stock price computed using perfect foresight about future dividends. Also assuming risk neutral investors, LeRoy (1984) had previously demonstrated the result $Var(\Delta p_t) > Var(\Delta p_t^*)$ in a calibrated model where stock prices are "ex-dividend." LeRoy's analysis assumes that dividends are stationary but highly persistent.[2] The perfect foresight case is the same benchmark used by Shiller (1981) to argue that theory predicts the opposite variance ordering when it comes to the price level, i.e., $Var(p_t) \leq Var(p_t^*)$.

The foregoing results have been interpreted to imply that news about future cash flows must decrease the volatility of asset price changes. For example, Engel (2014, p. 464) states "...the variance of changes in the asset price falls with more information...[N]ews can account for a high variance in the real exchange rate, but not for a high variance in the change in the real exchange rate."

[1] However, on page 41, West (1988) includes the caveat that his result "may not extend immediately if logarithms or logarithmic differences are required to induce stationarity [of the dividend process]."

[2] The result in LeRoy (1984) is further discussed by Gilles and LeRoy (1991), p. 771.

This paper expands the modeling framework of Engel (2005) to consider the more-standard setup of "ex-dividend" stock prices, risk averse investors, and exponentially-growing dividends. I consider three different information sets labeled I_t, I_t^o, and I_t^* that contain progressively increasing amounts of information, i.e., $I_t \subseteq I_t^o \subseteq I_t^*$. Under set I_t, the investor can observe current and past dividend realizations and thereby identify the law of motion for dividends. Set I_t^o, denoted by the superscript "o," allows investors to have one-period foresight about dividends. This setup captures the possibility that investors may have some news that allows them to accurately forecast dividends over the near-term.[3] Finally, set I_t^*, denoted by the superscript "$*$," provides the maximum amount of investor information, corresponding to perfect knowledge about the entire stream of past and future dividends. I use the symbols p_t, p_t^o, and p_t^* to represent the equilibrium prices under the three information sets.

I show that providing investors with more information about future dividends can either increase or decrease the variance of stock price changes. In particular, I show that the direction of the price-change variance inequality can be reversed, depending on the values assigned to some key parameters of the model. These include a dividend persistence parameter ρ, the investor's subjective time discount factor β, and the coefficient of relative risk aversion α.

Following Engel (2005), I initially consider an economy where the representative investor is risk neutral ($\alpha = 0$) and dividends follow an arithmetic AR(1) process that allows for a unit root as a special case. When observed stock prices are cum-dividend, I recover a variance ordering consistent with Engel's theoretical propositions, namely, $Var\left(\Delta p_t\right) \geq Var\left(\Delta p_t^o\right) \geq Var\left(\Delta p_t^*\right)$. However when observed stock prices are ex-dividend, I show that $Var\left(\Delta p_t\right)$ can be greater or less than $Var\left(\Delta p_t^*\right)$, depending on the values of ρ and β. The two variance statistics are exactly equal when the parameters satisfy the condition $\rho\left(1+\beta\right) = 1$. For a typical model calibration where dividends are a close to a random walk and the discount factor is close to unity, we have $\rho\left(1+\beta\right) > 1$ which in turn yields $Var\left(\Delta p_t\right) > Var\left(\Delta p_t^*\right)$, thus confirming the numerical results obtained by LeRoy (1984). LeRoy's model calibration satisfies the condition $\rho\left(1+\beta\right) > 1$. Engel's cum-dividend model can be interpreted as imposing the parameter restriction $\rho\beta \simeq 1$ such that the condition $\rho\left(1+\beta\right) > 1$ is once again satisfied. However, if dividends are less persistent or the future is more heavily discounted such that $\rho\left(1+\beta\right) < 1$, then the variance inequality will be reversed, yielding $Var\left(\Delta p_t\right) < Var\left(\Delta p_t^*\right)$. Similarly, I show that variance ordering for Δp_t and Δp_t^o can be reversed if stock prices are

[3] Information set I_t^o connects to recent research on business cycles that focuses on "news shocks" as an important quantitative source of economic fluctuations. See, for example, Barsky and Sims (2011).

ex dividend and $\rho < 1.$

The explanation for the variance ordering reversals is linked to the discounting mechanism. The parameters ρ and β both affect the degree to which future dividend innovations influence the perfect foresight price p_t^* via discounting from the future to the present. The future dividend innovations have no effect on p_t because the expected value of future innovations is zero. When dividends are highly persistent and the investor's discount factor is close to unity such that $\rho\,(1+\beta) > 1$, the discounting weights applied to successive future dividend innovations decay gradually. By taking the first-difference of the p_t^* series, the terms involving future dividend innovations tend to cancel out, thus shrinking the magnitude of $Var\,(\Delta p_t^*)$ relative to $Var\,(\Delta p_t)$. In contrast, when $\rho\,(1+\beta) < 1$, the discounting weights applied to successive future innovations decay rapidly, so these terms do not tend to difference out, resulting in a higher value for $Var\,(\Delta p_t^*)$ relative to $Var\,(\Delta p_t)$. Similar logic applies to the relationship between between $Var\,(\Delta p_t^o)$ and $Var\,(\Delta p_t)$.

In the model with risk aversion and exponentially-growing dividends, I specify the growth rate dividends as an ARMA$(1,1)$ process. Within this framework, I derive approximate analytical expressions for the variance of the log price-change under the three information sets, i.e., $Var\,[\Delta \log{(p_t)}]$, $Var\,[\Delta \log{(p_t^o)}]$, and $Var\,[\Delta \log{(p_t^*)}]$. When $\alpha = 1$, representing logarithmic utility, all three variance statistics are equal. When $\alpha < 1$, the variance of log price-changes declines with more information, analogous to the risk-neutral results obtained by Engel (2005) and LeRoy (1984). When the $\alpha > 1$, the variance of log price-changes can either increase or decrease with more information. Specifically, the variance ordering is changed such that $Var\,[\Delta \log{(p_t)}] < Var\,[\Delta \log{(p_t^*)}] < Var\,[\Delta \log{(p_t^o)}]$. Furthermore, this ordering is sensitive to the parameters of the ARMA $(1,1)$ process. For an alternative calibration that implies more persistence in dividend growth, the variance ordering undergoes a further change for $\alpha \gtrsim 3.9$ such that $Var\,[\Delta \log{(p_t^*)}] < Var\,[\Delta \log{(p_t)}] < Var\,[\Delta \log{(p_t^o)}]$.

When the investor's utility function is logarithmic, the income and substitution effects of future dividend growth innovations exactly cancel such that the price-dividend ratio is constant regardless of the information set. In this case, any variation in the log-price change must be driven solely by variation in dividend growth, which is the same across information sets. The sensitivity of the variance ordering to changes in the risk aversion coefficient is again linked to the discounting mechanism. When $\alpha < 1$, the stochastic discount factors applied to successive future dividend growth innovations decay gradually. By taking the log-difference of the perfect foresight price series, the terms involving future innovations tend to cancel out, thus shrinking the magnitude of $Var\left[\Delta \log\left(p_t^*\right)\right]$. When $\alpha > 1$, the stochastic discount factors applied to successive future innovations decay rapidly, so these terms do not tend to difference out, resulting in a higher value for $Var\left[\Delta \log\left(p_t^*\right)\right]$. The decay rate of the terms involving future innovations is similarly influenced by the parameters of the ARMA$(1,1)$ process for dividend growth.

The results presented here complement those of Lansing and LeRoy (2014) who, among other things, show that the volatility of log equity returns is not generally a monotonic decreasing function of investors' information about future dividends. The behavior of log price-changes is similar to the behavior of log returns, as can be demonstrated using the Campbell and Shiller (1988) approximation of the equity return identity. This paper goes beyond Lansing and LeRoy (2014) by considering the influence of cum-dividend versus ex-dividend stock prices, allowing for a more-general dividend growth process, i.e., ARMA$(1,1)$ versus AR(1), and reconciling the results about price-change variance with those of Engel (2005) and LeRoy (1984).

2 Asset Pricing Model

Equity shares are priced using the frictionless pure exchange model of Lucas (1978). There is a representative investor who can purchase shares to transfer wealth from one period to another. Each share pays an exogenous stream of stochastic dividends in perpetuity. The investor's problem is to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\alpha}-1}{1-\alpha} | I_0\right], \tag{1}$$

subject to the budget constraint

$$c_t + p_t s_t = \left(p_t + d_t\right) s_{t-1}, \qquad c_t,\ s_t > 0 \tag{2}$$

where c_t is the investor's consumption in period t , α is the coefficient of relative risk aversion (the inverse of the intertemporal elasticity of substitution), s_t is the number of shares held in period t, and d_t is the dividend per share. I use the notation $E_t\left(\cdot \mid I_t\right)$ to represent the mathematical expectation operator, conditional on the investor's information set I_t, to be described more completely below. The symbol p_t denotes the ex-dividend stock price conditional on the investor's information.

The first-order condition that governs the investor's share holdings is given by

$$p_t = E_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\alpha} (p_{t+1} + d_{t+1}) \,|I_t \right]. \tag{3}$$

The first-order condition can be iterated forward to substitute out p_{t+1+j} for $j = 0, 1, 2, ...$ Applying the law of iterated expectations and imposing a transversality condition that excludes bubble solutions yields the following expression for the ex-dividend stock price

$$p_t = E_t \left\{ \sum_{j=1}^{\infty} M_{t,\,t+j}\, d_{t+j} | I_t \right\}, \qquad (p_t \text{ is ex-dividend}), \tag{4}$$

where $M_{t,\,t+j} \equiv \beta^j \left(c_{t+j}/c_t\right)^{-\alpha}$ is the stochastic discount factor. Equity shares are assumed to exist in unit net supply. Market clearing therefore implies $s_t = 1$ for all t. Substituting this equilibrium condition into the budget constraint (2) yields, $c_t = d_t$ for all t.

To consider the case of cum-dividend stock prices, the budget constraint (2) is rewritten as follows

$$c_t + (p_t - d_t)\, s_t = p_t s_{t-1}, \tag{5}$$

where p_t now represents the cum-dividend price and $p_t - d_t$ is the ex-dividend price. Proceeding as before, the expression for the cum-dividend stock price is

$$p_t = E_t \left\{ \sum_{j=0}^{\infty} M_{t,\,t+j}\, d_{t+j} | I_t \right\}, \qquad (p_t \text{ is cum-dividend}), \tag{6}$$

where the only difference is that the infinite sum now starts at $j = 0$ rather than $j = 1$.

2.1 Investor Information

I define the following information sets, where each set contains progressively increasing amounts of information about the dividend process:

$$I_t \equiv \{d_t,\ d_{t-1},\ d_{t-2},\ ...\}, \tag{7}$$

$$I_t^o \equiv \{d_{t+1},\ d_t,\ d_{t-1},\ d_{t-2},\ ...\}, \tag{8}$$

$$I_t^* \equiv \{...d_{t+2},\ d_{t+1},\ d_t,\ d_{t-1},\ d_{t-2},\ ...\}, \tag{9}$$

such that $I_t \subseteq I_t^o \subseteq I_t^*$. Set I_t allows investors to observe current and past dividends and thereby discover the underlying stochastic process that governs the evolution of dividends. Set I_t^* provides the maximum amount of investor information, corresponding to perfect knowledge about the entire stream of past and future dividends. In between these two, set $I_t^o = I_t \cup d_{t+1}$ provides more information than set I_t by allowing investors to have one-period foresight regarding dividends at time $t+1$. Along the lines of LeRoy and Parke (1992), set I_t^o entertains the possibility that investors receive some news that allows them to forecast dividends over the near-term without error. Given the above definitions, we can write $p_t = E_t(p_t^o|I_t) = E_t(p_t^*|I_t)$.

3 Risk-Neutral Investor

Following Engel (2005) and LeRoy (1984), I initially consider an economy where the representative investor is risk neutral $(\alpha = 0)$. To keep things as simple as possible, I assume that dividends follow an arithmetic AR(1) process that allows for a unit root as a special case. I begin with the more-standard case of ex-dividend stock prices and then show how the results are changed with cum-dividend stock prices.

3.1 Ex-Dividend Stock Prices

When $\alpha = 0$, the pricing equations can be written as follows

$$p_t = E_t\left\{\beta d_{t+1} + \beta^2 d_{t+2} + \beta^3 d_{t+3} + ...|I_t\right\}, \tag{10}$$

$$\begin{aligned} p_t^o &= \beta d_{t+1} + E_t\left\{\beta^2 d_{t+2} + \beta^3 d_{t+3} + ...|I_t^o\right\}, \\ &= \beta\left(d_{t+1} + p_{t+1}\right), \end{aligned} \tag{11}$$

$$p_t^* = \beta d_{t+1} + \beta^2 d_{t+2} + \beta^3 d_{t+3} + ..., \tag{12}$$

where the assumption of one-period foresight for information set I_t^o implies that d_{t+1} is known at time t. However, going forward from time $t+1$, the investor will be faced with information set I_{t+1} where d_{t+2} is not known. Hence, p_{t+1} is the equilibrium price that will prevail at time $t+1$. In the case of information set I_t^*, all future dividends are known so the expectation operator may be dropped.

To facilitate analytical solutions for p_t, and p_t^o, and p_t^*, I assume that dividends are governed by the following AR(1) process

$$d_{t+1} = \rho\, d_t + (1-\rho)\,\overline{d} + \varepsilon_{t+1}, \qquad \begin{array}{l} \varepsilon_{t+j} \sim N\left(0, \sigma_\varepsilon^2\right), \\ |\rho| \leq 1 \end{array}, \tag{13}$$

which allows for a unit root when $\rho = 1$.

Repeated substitution of equation (13) into equation (10) and then imposing $E_t \varepsilon_{t+j} = 0$ for $j = 1, 2, ...$ yields the following expression for p_t:

$$\begin{aligned} p_t &= d_t \left\{\beta\rho + (\beta\rho)^2 + (\beta\rho)^3 + ...\right\} + \overline{d} \left\{\beta(1-\rho) + \beta^2\left(1-\rho^2\right) + \beta^3\left(1-\rho^3\right) + ...\right\}, \\ &= d_t \left[\frac{\beta\rho}{1-\beta\rho}\right] + \overline{d} \left[\frac{\beta(1-\rho)}{(1-\beta)(1-\beta\rho)}\right], \end{aligned} \tag{14}$$

which shows that the equilibrium price-dividend ratio p_t/d_t is constant when $\rho = 1$ or $\overline{d} = 0$.

Iterating the solution for p_t ahead one period and then substituting into equation (11) yields

$$p_t^o = d_{t+1} \left[\frac{\beta}{1-\beta\rho}\right] + \overline{d} \left[\frac{\beta^2(1-\rho)}{(1-\beta)(1-\beta\rho)}\right]. \tag{15}$$

Repeated substitution of equation (13) into equation (12) yields the following expression for the perfect foresight price

$$p_t^* = \underbrace{d_t \left[\frac{\beta\rho}{1-\beta\rho}\right] + \overline{d} \left[\frac{\beta(1-\rho)}{(1-\beta)(1-\beta\rho)}\right]}_{p_t} + \frac{\beta}{1-\beta\rho} \sum_{j=1}^{\infty} \beta^{j-1} \varepsilon_{t+j}, \tag{16}$$

Since p_t is the rational forecast of p_t^*, Shiller (1981) argued that market efficiency requires $Var\left(p_t\right) \leq Var\left(p_t^*\right)$. Marsh and Merton (1986) and Kleidon (1986) later pointed out that neither variance will exist if dividends (and hence prices) are nonstationary. Shiller's derivation assumed that prices and dividends were rendered stationary by removing a common deterministic time trend. However when $\rho = 1$, the trend in prices and dividends is stochastic, so Shiller's detrending procedure would not eliminate the unit root. To allow for the $\rho = 1$ case, we can take the first difference of the respective price series. Taking the first difference of equations (14) through (16) yields the following relationships

$$\rho\, \Delta p_t^o = \Delta p_{t+1} \tag{17}$$

$$\Delta p_t^* = \Delta p_t - \left[\frac{\beta}{1-\beta\rho}\right] \varepsilon_t + \left[\frac{\beta(1-\beta)}{1-\beta\rho}\right] \left[\varepsilon_{t+1} + \beta\varepsilon_{t+2} + \beta^2\varepsilon_{t+3} + ...\right]. \tag{18}$$

Proposition 1. *When the representative investor is risk neutral and dividends are governed by the AR(1) process (13), then the variance of stock price changes will increase with investor information when the following parameter conditions hold:*

$$Var\left(\Delta p_t\right) \quad < \quad Var\left(\Delta p_t^o\right) \quad \text{if} \quad |\rho| < 1,$$

$$Var\left(\Delta p_t\right) \quad < \quad Var\left(\Delta p_t^*\right) \quad \text{if} \quad \rho\left(1+\beta\right) < 1.$$

Proof: Taking the variance of both sides of equation (17) yields $Var\left(\Delta p_t\right) = \rho^2 Var\left(\Delta p_t^o\right).$ Taking the variance of both sides of equation (18) yields

$$\begin{aligned} Var\left(\Delta p_t^*\right) \quad = \quad & Var\left(\Delta p_t\right) + \frac{\beta^2\sigma_\varepsilon^2}{\left(1-\beta\rho\right)^2} - \frac{2\beta\, Cov\left(\Delta p_t,\, \varepsilon_t\right)}{\left(1-\beta\rho\right)} \\ & + \frac{\beta^2\left(1-\beta\right)^2\sigma_\varepsilon^2}{\left(1-\beta\rho\right)^2}\left[1+\beta^2+\beta^4+\beta^6+\ldots\right]. \end{aligned}$$

From equations (13) and (14), we have $Cov\left(\Delta p_t, \varepsilon_t\right) = \beta\rho\,\sigma_\varepsilon^2/\left(1-\beta\rho\right).$ The infinite sum inside the square brackets of the above expression is equal to $1/\left(1-\beta^2\right).$ Inserting these results into the variance expression and then simplifying yields the following result

$$Var\left(\Delta p_t^*\right) = Var\left(\Delta p_t\right) + \frac{2\beta^2\sigma_\varepsilon^2}{\left(1-\beta\rho\right)^2\left(1+\beta\right)}\left[1-\rho\left(1+\beta\right)\right],$$

which shows that the direction of the variance inequality is governed by the sign of the term $\left[1-\rho\left(1+\beta\right)\right].$ ■

Proposition 1 shows that when $\rho\left(1+\beta\right) > 1$, we have $Var\left(\Delta p_t\right) > Var\left(\Delta p_t^*\right)$ which is consistent with numerical results obtained by LeRoy (1984). LeRoy employed the values $\rho \in (0.8,\ 0.99)$ and $\beta = 0.9$, which satisfy the condition $\rho\left(1+\beta\right) > 1$. However, when $\rho\left(1+\beta\right) < 1$, the variance inequality is reversed such that $Var\left(\Delta p_t\right) < Var\left(\Delta p_t^*\right).$

The intuition for the variance inequality reversal can be understood from equations (16) and (18). When the parameters ρ and β are both close to unity, the discounting weights applied to future dividend innovations in the solution for p_t^* decay gradually, as shown by equation (16). By taking the first-difference of the p_t^* series to obtain Δp_t^*, the terms involving future innovations tend to cancel each other out, as can be seen from equation (18), where these terms are multiplied by the coefficient $\beta(1-\beta)/(1-\beta\rho)$. However, equation (18) shows that the *current* dividend innovation ε_t continues to have a strong impact on Δp_t^*. The negative covariance between Δp_t and the term involving ε_t serves to shrink the variance of Δp_t^* relative to the variance of Δp_t. In contrast, the variance of *future* innovations ε_{t+j}, $j = 1, 2, ...$serves to magnify the variance of Δp_t^* relative to the variance of Δp_t. The negative influence of ε_t on the variance of Δp_t^* dominates the positive influence of ε_{t+j}, $j = 1, 2, ...$when the discounting weights in the solution for p_t^* decay sufficiently gradually, as measured by the condition $\rho(1+\beta) > 1$.

3.2 Cum-Dividend Stock Prices

Unlike the more-standard setup where observed stock prices are viewed as ex-dividend, Engel (2005) and West (1988) employ cum-dividend pricing equations where the stock price at time t includes a guaranteed dividend. The cum-dividend versions of equations (10) through (12) are:

$$p_t = d_t + E_t\left\{\beta d_{t+1} + \beta^2 d_{t+2} + \beta^3 d_{t+3} + ...|I_t\right\}, \tag{19}$$

$$\begin{aligned} p_t^o &= d_t + \beta d_{t+1} + E_t\left\{\beta^2 d_{t+2} + \beta^3 d_{t+3} + ...|I_t^o\right\}, \\ &= d_t + \beta p_{t+1}, \end{aligned} \tag{20}$$

$$p_t^* = d_t + \beta d_{t+1} + \beta^2 d_{t+2} + \beta^3 d_{t+3} + ..., \tag{21}$$

which differ in small but important ways from their ex-dividend counterparts. Starting from the ex-dividend pricing equations (10) through (12), we can obtain the cum-dividend versions by substituting in for d_{t+1} from (13) and then imposing $\rho\beta \simeq 1$. By effectively imposing $\rho\beta \simeq 1$, the cum-dividend pricing equations ensure that the condition $\rho(1+\beta) > 1$ from Proposition 1 is satisfied.

Starting from the cum-dividend pricing equations (19) through (21) and then following the same methodology as before, it is straightforward to derive the following relationships between the variance of stock price changes under the different information sets:

$$Var\left(\Delta p_t^o\right) = Var\left(\Delta p_t\right) - \frac{2\beta\left(1-\beta\right)\sigma_\varepsilon^2}{\left(1-\beta\rho\right)^2} \tag{22}$$

$$Var\left(\Delta p_t^*\right) = Var\left(\Delta p_t^o\right) - \frac{2\beta^3\sigma_\varepsilon^2}{\left(1-\beta\rho\right)^2\left(1+\beta\right)} \tag{23}$$

which implies $Var\left(\Delta p_t\right) \geq Var\left(\Delta p_t^o\right) \geq Var\left(\Delta p_t^*\right)$, in agreement with Propositions 1 and 2 in Engel (2005). Hence, the assumption of cum-dividend stock prices is crucial for Engel's results.

4 Risk Averse Investor and Growing Dividends

I now expand the modeling framework to consider risk aversion and exponentially growing dividends. The growth rate of dividends $x_t \equiv \log\left(d_t/d_{t-1}\right)$ is governed by the following ARMA (1, 1) process

$$x_{t+1} = \overline{x} + \rho\left(x_t - \overline{x}\right) + \varepsilon_{t+1} - \phi\varepsilon_t, \qquad \begin{array}{l}\varepsilon_{t+j} \sim N\left(0, \sigma_\varepsilon^2\right),\\ |\rho| < 1,\end{array} \tag{24}$$

where the special case of $\phi = 1$ implies a deterministic trend in log dividends.[4] With $E\left(x_t\right) = \overline{x}$, equation (24) implies the following unconditional moments:

$$Var\left(x_t\right) = \frac{\left(1+\phi^2-2\rho\phi\right)\,\sigma_\varepsilon^2}{1-\rho^2}, \tag{25}$$

$$Corr\left(x_t,\, x_{t-1}\right) = \frac{\left(\rho-\phi\right)\left(1-\rho\phi\right)}{1+\phi^2-2\rho\phi}. \tag{26}$$

$$Corr\left(x_t,\, x_{t-2}\right) = \rho\, Corr\left(x_t,\, x_{t-1}\right) \tag{27}$$

The price-dividend ratios under the three information sets are denoted by $y_t \equiv p_t/d_t$, $y_t^o \equiv p_t^o/d_t$, and $y_t^* \equiv p_t^*/d$. Starting from equation (3) and imposing $c_t = d_t$ for all t, the first-order conditions under the three information sets be written as

$$y_t = E_t\left[\beta\exp\left(\theta x_{t+1}\right)\left(y_{t+1}+1\right)|I_t\right], \tag{28}$$

$$y_t^o = E_t\left[\beta\exp\left(\theta x_{t+1}\right)\left(y_{t+1}^o+1\right)|I_t^o\right] \tag{29}$$

$$y_t^* = \beta\exp\left(\theta x_{t+1}\right)\left(y_{t+1}^*+1\right), \tag{30}$$

[4] The $\phi = 1$ case corresponds to the following dividend specification: $\log(d_t) = \rho\log(d_{t-1}) + \mu t + \varepsilon_t$, where μt is the the deterministic time trend. Lagging this equation by one period and then subtracting one equation from the other yields equation (24) with $\phi = 1$, where $(1-\rho)\,\overline{x} = \mu$.

where $\theta \equiv 1 - \alpha$.

The corresponding expressions for the log price-change can be written as follows:

$$\Delta \log (p_t) = \Delta \log (y_t) + x_t, \tag{31}$$

$$\Delta \log (p_t^o) = \Delta \log (y_t^o) + x_t, \tag{32}$$

$$\Delta \log (p_t^*) = \Delta \log (y_t^*) + x_t. \tag{33}$$

The above expressions show that differences in the variance of the log price-change across information sets can only be due to differences in the variance of the change in the log price-dividend ratio. Lansing and LeRoy (2014) show that more information about future dividends increases the variance of the *level* of the log price-dividend ratio. But, as we shall see, there is no monotonic relationship between information and the variance of the *change* in the log price-dividend ratio.

4.1 Information Set I_t

I now derive an approximate analytical solution for the price-dividend ratio y_t under information set I_t. Given this solution, it is straightforward to derive an analytical expression for $Var\left[\Delta \log (p_t)\right]$ using the relationship (31). The solution for y_t is obtained by solving the first-order condition (28), subject to the dividend growth process (24). It is convenient to transform the first-order condition using a nonlinear change of variables to obtain

$$z_t = \beta \exp(\theta x_t) \left[E_t(z_{t+1}|I_t) + 1 \right], \tag{34}$$

where $z_t \equiv \beta \exp(\theta x_t)(y_t + 1)$. Under this formulation, z_t represents a composite variable that depends on both the growth rate of dividends and the price-dividend ratio. By making use of the definition of z_t, the first-order condition (28) can be rewritten as $y_t = E_t(z_{t+1}|I_t)$. Hence, the price-dividend ratio under information set I_t is simply the conditional forecast of the composite variable z_{t+1}. The following proposition presents an approximate analytical solution for the composite variable z_t.[5]

[5] Lansing (2010) demonstrates the accuracy of this solution method for the level of the price-dividend ratio by comparing the approximate solution to the exact theoretical solution derived by Burnside (1998) for the case of $\phi = 0$. Here I focus on the variance of the log price-change which is not affected by the constant term in the approximate solution. The constant term can be an important source of approximation error when the point of approximation is the deterministic steady state (Collard and Juillard 2001). As in Lansing (2010), the point of approximation for the solution here is the ergodic mean, not the deterministic steady state, which helps to improve accuracy.

Proposition 2. *An approximate analytical solution for the composite variable* $z_t \equiv \beta \exp(\theta x_t)(y_t + 1)$ *under information set* I_t *is given by*

$$z_t \;=\; \exp\left[a_0 + a_1\left(x_t - \overline{x}\right) + a_2\,\varepsilon_t\right],$$

where a_1 *and* a_2 *solve the following system of nonlinear equations*

$$a_1 \;=\; \frac{\theta}{1-\rho\beta\exp\left[\theta\,\overline{x} + \frac{1}{2}\left(a_1 + a_2\right)^2\sigma_\varepsilon^2\right]},$$

$$a_2 \;=\; -\,a_1\phi\beta\exp\left[\theta\,\overline{x} + \frac{1}{2}\left(a_1 + a_2\right)^2\sigma_\varepsilon^2\right]$$

and $a_0 \equiv E\left[\log\left(z_t\right)\right]$ *is given by*

$$a_0 \;=\; \log\left\{\frac{\beta\exp\left(\theta\,\overline{x}\right)}{1-\beta\exp\left[\theta\,\overline{x} + \frac{1}{2}\left(a_1 + a_2\right)^2\sigma_\varepsilon^2\right]}\right\},$$

provided that $\beta\exp\left[\theta\,\overline{x} + \frac{1}{2}\left(a_1 + a_2\right)^2\sigma_\varepsilon^2\right] < 1$.

Proof: See appendix.

Two values of a_1 satisfy the nonlinear system in Proposition 2. The inequality restriction selects the value of a_1 with lower magnitude to ensure that $\exp(a_0)$ is positive. Given the solution for the composite variable z_t, we can recover the solution for the price-dividend ratio as follows

$$y_t = E_t(z_{t+1}|I_t) = \exp\left[a_0 + a_1\rho\left(x_t - \overline{x}\right) - a_1\phi\,\varepsilon_t + \frac{1}{2}\left(a_1 + a_2\right)^2\sigma_\varepsilon^2\right]. \tag{35}$$

As shown in the appendix, the approximate solution can be used to derive the following unconditional variances:

$$Var\left[\log\left(y_t\right)\right] \;=\; \frac{\left[a_1\left(\rho - \phi\right)\right]^2\sigma_\varepsilon^2}{1-\rho^2}, \tag{36}$$

$$\begin{aligned} Var\left[\Delta\log\left(p_t\right)\right] \;=\; & \left[\left(a_1\rho\right)^2 + \left(1 + a_1\rho\right)^2 - 2a_1\rho\left(1 + a_1\rho\right)Corr\left(x_t,\, x_{t-1}\right)\right]Var\left(x_t\right) \\ & -2a_1\phi\left[a_1\left(\rho - \phi\right) + \left(1 - \rho + \phi\right)\left(1 + a_1\rho\right)\right]\sigma_\varepsilon^2 \end{aligned} \tag{37}$$

where $Var\left(x_t\right)$ and $Corr\left(x_t,\, x_{t-1}\right)$ are given by equations (25) and (26).

From Proposition 2, the magnitude of the solution coefficient a_1 increases as the risk aversion coefficient α rises above unity. An increase in the magnitude of a_1 serves to magnify the volatility of the price-dividend ratio, as shown by equation (36) which depends on $(a_1)^2$.[6] For the case of log utility $(\alpha = 1)$, we have $\theta = 1 - \alpha = 0$, such that $a_1 = a_2 = 0$. In this case, the price-dividend ratio is constant at the value $y_t = \beta / (1 - \beta)$. This result obtains because the income and substitution effects of an innovation to dividend growth are exactly offsetting.

4.2 Information Set $I_t^o = I_t \cup d_{t+1}$

Information set $I_t^o = I_t \cup d_{t+1}$ entertains the possibility that investors may have some auxiliary information that helps to predict future dividends. An example of such auxiliary information might be company-provided guidance about future financial performance that is typically disseminated to investors via quarterly conference calls. To capture this idea, I consider an environment where investors can see dividends one period ahead without error, as in LeRoy and Parke (1992).

Information set I_t^o implies the following relationships:

$$p_t^o = M_{t,t+1} \left(d_{t+1} + p_{t+1} \right), \tag{38}$$

$$\begin{aligned} y_t^o &= \beta \exp \left(\theta \, x_{t+1} \right) \left(y_{t+1} + 1 \right), \\ &= z_{t+1} = \exp \left[a_0 + a_1 \left(x_{t+1} - \mu \right) + a_2 \, \varepsilon_{t+1} \right], \end{aligned} \tag{39}$$

where p_t^o and y_t^o are the price and price-dividend ratio under information set I_t^o, while p_t and y_t are the counterparts under set I_t. Under set I_t^o, the discount factor $M_{t,\,t+1}$ and the dividend growth rate x_{t+1} are both known to investors at time t. In equation (39), I have employed the definition of z_{t+1} and the corresponding solution implied by Proposition 2 at time $t+1$. Since $y_t = E_t(z_{t+1}|I_t)$ and $y_t^o = z_{t+1}$, it follows directly that $y_t = E_t(y_t^o|I_t)$, which in turn implies $Var\left(y_t\right) \leq Var\left(y_t^o\right)$.

As shown in the appendix, equations (39) and (32) can be used to derive the following

[6] LeRoy and LaCivita (1981) demonstrate that risk aversion magnifies the volatility of the price-dividend ratio in a Lucas-type model where the level of dividends is governed by a two-state Markov process.

relationships

$$\log\left(y_t^o\right) - E\left[\log\left(y_t^o\right)\right] = \log\left(y_t\right) - E\left[\log\left(y_t\right)\right] \;+\; \left(a_1 + a_2\right)\,\varepsilon_{t+1} \tag{40}$$

$$\Delta\log\left(p_t^o\right) = \Delta\log\left(p_t\right) \;+\; \left(a_1 + a_2\right)\left(\varepsilon_{t+1} - \varepsilon_t\right) \tag{41}$$

$$Var\left[\log\left(y_t^o\right)\right] = Var\left[\log\left(y_t\right)\right] \;+\; \left(a_1 + a_2\right)^2\,\sigma_\varepsilon^2, \tag{42}$$

$$Var\left[\Delta\log\left(p_t^o\right)\right] = Var\left[\Delta\log\left(p_t\right)\right] + 2\left(a_1 + a_2\right)\left[a_1\left(1 - \rho + \phi\right) + a_2 - 1\right]\,\sigma_\varepsilon^2, \tag{43}$$

where equation (42) confirms $Var\left[\log\left(y_t\right)\right] \leq Var\left[\log\left(y_t^o\right)\right]$, i.e., news about d_{t+1} serves to increase the variance of the log price-dividend ratio. In contrast, equation (43) reveals the possibility of a complex ordering between $Var\left[\Delta\log\left(p_t^o\right)\right]$ and $Var\left[\Delta\log\left(p_t\right)\right]$. Depending on parameter values, the second term on the right side of equation (43) can be either positive or negative because it includes the covariance between $\Delta\log\left(p_t\right)$ and $-\varepsilon_t$.

4.3 Information Set I_t^*

The perfect foresight price-dividend ratio is governed by equation (30), which is a nonlinear law of motion. To derive analytical expressions for the perfect foresight variances, I approximate equation (30) using the following log-linear law of motion (details are contained in the appendix):

$$\log\left(y_t^*\right) - E\left[\log\left(y_t^*\right)\right] \;\simeq\; \theta\left(x_{t+1} - \overline{x}\right) + \beta\exp\left(\theta\,\overline{x}\right)\left\{\log\left(y_{t+1}^*\right) - E\left[\log\left(y_t^*\right)\right]\right\}. \tag{44}$$

As shown in the appendix, the approximate law of motion (44) and the dividend growth process (24) can be used to derive the following unconditional moments

$$Var\left[\log\left(y_t^*\right)\right] \;=\; \frac{\theta^2\;\left\{1 + \beta\exp\left(\theta\,\overline{x}\right)\left[2Corr\left(x_t,\,x_{t-1}\right) - \rho\right]\right\}}{\left[1 - \rho\beta\exp\left(\theta\,\overline{x}\right)\right]\left[1 - \beta^2\exp\left(2\theta\,\overline{x}\right)\right]}\,Var\left(x_t\right), \tag{45}$$

$$\begin{aligned} Var\left[\Delta\log\left(p_t^*\right)\right] \;&=\; \left[1 - \beta\exp\left(\theta\,\overline{x}\right)\right]^2\;Var\left[\log\left(y_t^*\right)\right] \\ &\quad + \left\{\alpha^2 + \frac{2\alpha\theta\left[1 - \beta\exp\left(\theta\,\overline{x}\right)\right]Corr\left(x_t,\,x_{t-1}\right)}{\left[1 - \rho\beta\exp\left(\theta\,\overline{x}\right)\right]}\right\}Var\left(x_t\right), \end{aligned} \tag{46}$$

where $Var\left(x_t\right)$ and $Corr\left(x_t,\,x_{t-1}\right)$ are again given by equations (25) and (26).

4.4 Volatility Comparison

Given the complexity of the variance expressions for the log price-change, the variance ordering across information sets is not obvious. To gain some insight, it is helpful to consider some special cases.

Proposition 3. *For the special case of logarithmic utility* $(\alpha = 1)$*, we have:*

$$Var\left[\Delta \log\left(p_t\right)\right] \;=\; Var\left[\Delta \log\left(p_t^o\right)\right] \;=\; Var\left[\Delta \log\left(p_t^*\right)\right] \;=\; Var\left(x_t\right).$$

Proof: With log utility, we have $\theta = 1 - \alpha = 0$. Proposition 1 then implies $a_1 = a_2 = 0$ for any values of ρ and ϕ. Plugging these values into the appropriate expressions yields the above result. ■

When the utility function is logarithmic, the income and substitution effects of dividend growth innovations exactly cancel. As a result, the price-dividend ratios y_t, y_t^o, and y_t^* are all constant, as can be seen from the variance expressions (36), (42), and (45) when $a_1 = a_2 = 0$ and $\theta = 0$, respectively. Given that the price-dividend ratios are constant, any variation in the stock price must be driven solely by variation in the stream of dividends which is common across information sets.

Proposition 4. *For the special case of iid dividend growth* $(\rho = \phi = 0)$ *we have* $Var\left[\Delta \log\left(p_t\right)\right] > Var\left[\Delta \log\left(p_t^o\right)\right]$ *when* $0 < \alpha < 1$*, but* $Var\left[\Delta \log\left(p_t\right)\right] \leq Var\left[\Delta \log\left(p_t^o\right)\right]$ *when* $\alpha \geq 1$*. Similarly, we have* $Var\left[\Delta \log\left(p_t\right)\right] > Var\left[\Delta \log\left(p_t^*\right)\right]$ *when* $0 < \alpha < 1$*, but* $Var\left[\Delta \log\left(p_t\right)\right] \leq Var\left[\Delta \log\left(p_t^*\right)\right]$ *when* $\alpha \geq 1$*.*

Proof: When $\rho = \phi = 0$, Proposition 1 implies $a_1 = \theta$ and $a_2 = 0$. From equations (37), (43), and (46), we then have

$$\begin{aligned}
Var\left[\Delta \log\left(p_t\right)\right] &= Var\left(x_t\right), \\
Var\left[\Delta \log\left(p_t^o\right)\right] &= \left(1 - 2\theta\alpha\right) Var\left(x_t\right), \\
Var\left[\Delta \log\left(p_t^*\right)\right] &= \left\{\alpha^2 + \theta^2 \left[\frac{1 - \beta \exp\left(\theta\, \overline{x}\right)}{1 + \beta \exp\left(\theta\, \overline{x}\right)}\right]\right\} Var\left(x_t\right),
\end{aligned}$$

where $\theta \equiv 1-\alpha$. By inspection, the term multiplying multiplying $Var\left(x_t\right)$ in the expression for $Var\left[\Delta \log\left(p_t^o\right)\right]$ is less than unity when $0<\alpha<1$ but greater than unity when $\alpha>1$. Similarly, the term multiplying multiplying $Var\left(x_t\right)$ in the expression for $Var\left[\Delta \log\left(p_t^*\right)\right]$ is less than unity when $0<\alpha<1$ but greater than unity when $\alpha>1$. ■

When $0<\alpha<1$, Proposition 4 shows $Var\left[\Delta \log\left(p_t\right)\right] > Var\left[\Delta \log\left(p_t^*\right)\right]$, analogous to the risk-neutral result obtained by LeRoy (1984) and Engel (2005). When $\alpha>1$, the price-change variance inequality is reversed such that $Var\left[\Delta \log\left(p_t\right)\right] < Var\left[\Delta \log\left(p_t^*\right)\right]$. A similar reversal occurs in the ordering between $Var\left[\Delta \log\left(p_t\right)\right]$ and $Var\left[\Delta \log\left(p_t^o\right)\right]$. Hence, news about future dividends can either increase or decrease the variance of stock price changes. The variance of stock price changes can be interpreted as a measure of the "smoothness" of the underlying price series. According to Proposition 4, a risk aversion coefficient above unity is needed to cause p_t (the forecast) to appear smoother than either p_t^o or p_t^* (the objects being forecasted). Proposition 4 further shows that there is no obvious ordering between $Var\left[\Delta \log\left(p_t^o\right)\right]$ and $Var\left[\Delta \log\left(p_t^*\right)\right]$ even in the special case with $\rho=\phi=0$.

Some intuition for the variance inequality reversal in Proposition 4 can be obtained by writing out equations (28) and (30) for the case of *iid* dividend growth, which implies $x_{t+j} = \overline{x}+\varepsilon_{t+j}$ for $j=1,2,\ldots$ We have

$$p_t = d_t E_t\left\{\beta \exp\left[\theta\overline{x}+\theta\varepsilon_{t+1}\right]+\beta^2 \exp\left(2\theta\overline{x}+\theta\varepsilon_{t+1}+\theta\varepsilon_{t+2}\right)+\ldots | I_t\right\} \tag{47}$$

$$p_t^* = d_t\left\{\beta \exp\left[\theta\overline{x}+\theta\varepsilon_{t+1}\right]+\beta^2 \exp\left(2\theta\overline{x}+\theta\varepsilon_{t+1}+\theta\varepsilon_{t+2}\right)+\ldots\right\}. \tag{48}$$

Since $E_t\left\{\exp\left(\theta\varepsilon_{t+j}\right)|I_t\right\} = \exp\left(\theta^2\sigma_\varepsilon^2/2\right)$ for $j=1,2,\ldots$, the price-dividend ratio p_t/d_t under information set I_t is constant in this case. If we neglect the higher-order terms in the above expression for p_t^*, then the corresponding log price changes can be compared directly as follows

$$\begin{aligned}\Delta \log\left(p_t\right) &= \log\left(d_t\right)-\log\left(d_{t-1}\right)\\ &= \overline{x}+\varepsilon_t \end{aligned}\tag{49}$$

$$\begin{aligned}\Delta \log\left(p_t^*\right) &\simeq \log\left(d_t\right)-\log\left(d_{t-1}\right)+\theta\varepsilon_{t+1}-\theta\varepsilon_t\\ &\simeq \Delta \log\left(p_t\right)+\left(1-\alpha\right)\left(\varepsilon_{t+1}-\varepsilon_t\right). \end{aligned}\tag{50}$$

where I have made use of the definition $\theta \equiv 1-\alpha$. The above expressions imply

$$\begin{aligned} Var\left[\Delta \log\left(p_t^*\right)\right] &\simeq Var\left[\Delta \log\left(p_t\right)\right] + 2\left(1-\alpha\right)^2 \sigma_\varepsilon^2 - 2\left(1-\alpha\right) \underbrace{Cov\left[\Delta \log\left(p_t\right),\, \varepsilon_t\right]}_{\sigma_\varepsilon^2}, \\ &\simeq Var\left[\Delta \log\left(p_t\right)\right] - 2\left(1-\alpha\right)\alpha\, \sigma_\varepsilon^2, \qquad \text{when } \rho = \phi = 0. \end{aligned} \tag{51}$$

The two variance statistics in equation (51)are equal when $\alpha = 1$. When $0 < \alpha < 1$, the covariance term involving the current innovation ε_t serves to shrink the magnitude of $Var\left[\Delta \log\left(p_t^*\right)\right]$ relative to $Var\left[\Delta \log\left(p_t\right)\right]$, whereas the variance of the future innovation ε_{t+1} always serves to magnify $Var\left[\Delta \log\left(p_t^*\right)\right]$ relative to $Var\left[\Delta \log\left(p_t\right)\right]$. The negative influence of ε_t dominates the positive influence of ε_{t+1} when $0 < \alpha < 1$. The differential impact of current versus future innovations is similar to the effect noted earlier in describing the intuition for Proposition 1. Recall that the above approximation neglects the variance impact of the higher order terms which involve the future innovations ε_{t+2}, $\varepsilon_{t+3}, \ldots$ etc. But when $0 < \alpha < 1$, the stochastic discount factors applied to these future innovations decay gradually in equation (48). By taking the log-difference of p_t^* in equation (48) to obtain $\Delta \log\left(p_t^*\right)$, the terms involving the future innovations tend to cancel out, rendering the above approximation valid, provided that α is not too far from unity.

4.5 Model Calibration

I now turn to a quantitative analysis of the model's predictions for the volatility of the log price-change. There are six parameter values to be chosen: four pertain to the dividend process ($\overline{x}$, ρ, ϕ, and σ_ε) and two pertain to the investor's preferences (α and β).

Given that an equity share in the model represents a consumption claim, I calibrate the process for x_t in equation (24) using U.S. data on real per capita aggregate consumption expenditures (services and nondurable goods) from 1930 to 2012.[7] For the baseline calibration, I choose values for $\overline{x}$, σ_ε, ρ, and ϕ to match the mean, standard deviation, and the first two autocorrelation statistics for U.S. real per capita consumption growth. I also consider an alternative calibration that simply imposes $\rho = 0.8$ and then calibrates the values for $\overline{x}$, σ_ε, and ϕ to match the mean, standard deviation, and the first autocorrelation statistic for U.S. real per capita consumption growth. The calibrated parameters are shown in Table 1.

[7] Data on nominal consumption expenditures for services and nondurable goods are from the Bureau of Economic Analysis, NIPA Table 2.3.5, lines 8 and 13. The corresponding price indices are from Table 2.3.4, lines 8 and 13. Population data are from Table 2.1, line 40.

Table 1: Calibrated Parameter Values

Parameter	Baseline	Higher ρ
$\overline{x}$	0.0186	0.0186
σ_ε	0.01863	0.01822
ρ	0.385	0.800
ϕ	-0.154	0.420

Table 2 compares the moments of U.S. real per capita consumption growth versus those in the model. By construction, the baseline calibration matches both $Corr\ (x_t,\, x_{t-1})$ and $Corr\ (x_t,\, x_{t-2})$ in the data while the alterative calibration with $\rho = 0.8$ matches $Corr\ (x_t,\, x_{t-1})$ but not $Corr\ (x_t,\, x_{t-2})$.

Table 2. Moments of Consumption Growth: Data versus Model

Statistic	U.S. Data 1930 to 2012	Baseline	Higher ρ
$Mean\ (x_t)$	0.0186	0.0186	0.0186
$Std\ Dev\ (x_t)$	0.0216	0.0216	0.0216
$Corr\ (x_t,\, x_{t-1})$	0.50	0.50	0.50
$Corr\ (x_t,\, x_{t-2})$	0.19	0.19	0.40

Note: Data source is Bureau of Economic Analysis, NIPA tables 2.1, 2.3.4, and 2.3.5.

Given the parameter values from Table 1 and the expression for the price-dividend ratio (35) under set I_t, I choose the value of the subjective time discount factor β to achieve $E\left[\log\left(y_t\right)\right] = 3.36$, consistent with the sample mean of the log price-dividend ratio for the S&P 500 stock index from 1930 to 2012, i.e., the same sample period used to measure U.S. consumption growth. When the coefficient of relative risk aversion is $\alpha = 2$, this procedure yields $\beta = 0.9840$ for the baseline calibration and $\beta = 0.9835$ for the alternative calibration that imposes $\rho = 0.8$.[8] The same value of β is used for all information sets. Whenever α or the parameters of the dividend process are changed, the value of β is recalibrated to maintain $E\left[\log\left(y_t\right)\right] = 3.36$. When α exceeds a value slightly above 3, achieving the target value $E\left[\log\left(y_t\right)\right] = 3.36$ requires a β value greater than unity. Nevertheless, for all values of α examined, the mean value of the stochastic discount factor $E\left[\beta\left(c_{t+1}/c_t\right)^{-\alpha}\right]$ remains below unity.[9]

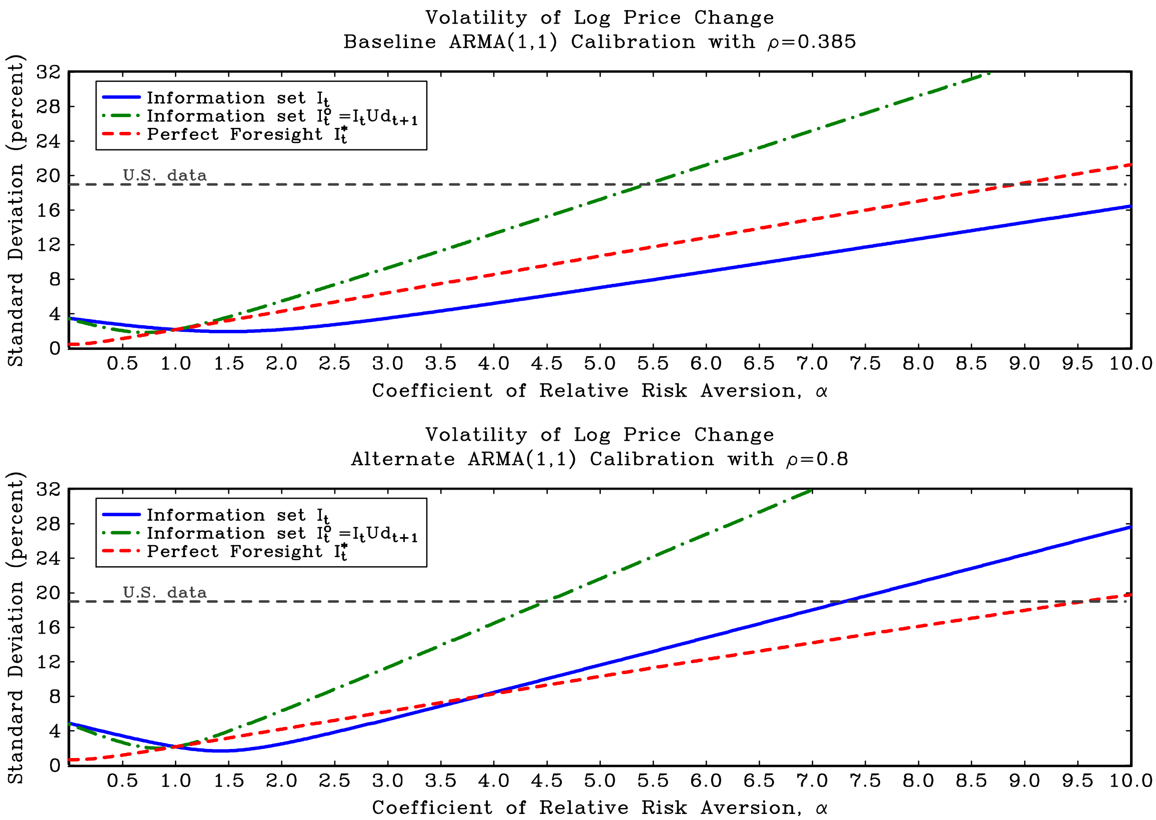

Figure 1: Investor information and the volatility of the change in the log stock price. Providing investors with more information about the dividend process can either increase or decrease the volatility of the log price-change. Moreover, under the alternative calibration with $\rho = 0.8$ (bottom panel), the volatility lines can cross at two different values of α, implying reversals in the variance ordering at the crossing point. Under the baseline calibration (top panel), model-predicted volatility can match the data volatility with $\alpha \leq 10$ under information sets I_t^o and I_t^*.

5 Quantitative Analysis

Figure 1 plots the standard deviation in percent of $\Delta \log(p_t)$, $\Delta \log(p_t^o)$, and $\Delta \log(p_t^*)$ as a function of the risk aversion coefficient over the range $0 \leq \alpha \leq 10$.[10] The top panel shows the results for the baseline calibration while the bottom panel shows the results for the alternative calibration that imposes $\rho = 0.8$. The horizontal dashed lines at 18.98% show the standard

[8] Cochrane (1992) employs a similar calibration procedure. For a given discount factor β, he chooses the risk aversion coefficient α to match the mean price-dividend ratio in the data.

[9] Kocherlakota (1990) shows that a well-defined competitive equilibrium with positive interest rates can still exist in growth economies when $\beta > 1$.

[10] Mehra and Prescott (1985) argue that risk aversion coefficients that fall within this range are plausible.

deviation of the log real stock price change in U.S. data for the period 1930 to 2012.[11]

Figure 1 shows that the volatility of the log price-change is U-shaped with respect to α for information sets I_t and I_t^o. In contrast, volatility is approximately linear in α under set I_t^*. Regardless of the calibration, all three lines intersect at $\alpha = 1$, consistent with Proposition 3. When $0 < \alpha < 1$, the volatility of the log price-change is lowest under set I_t^* and highest under set I_t, with volatility under set I_t^o in between the other two. Hence, for $0 < \alpha < 1$, the variance ordering is similar in spirit to the results obtained by Engel (2005) and LeRoy (1984) in risk neutral settings. However, when $\alpha > 1$, the ordering of the three variances is different, with set I_t^o now exhibiting the highest volatility under both model calibrations.

Both panels illustrate the reversal in the price-change variance inequality as α crosses unity. Recall that Proposition 4 considered the special case of *iid* dividend growth $(\rho = \phi = 0)$. But even in that special case, there was no obvious ordering between $Var\left[\Delta \log\left(p_t^o\right)\right]$ and $Var\left[\Delta \log\left(p_t^*\right)\right]$. Both calibrations in Figure 1 employ non-zero values for ρ and ϕ. Under both calibrations, we have $Var\left[\Delta \log\left(p_t^o\right)\right] > Var\left[\Delta \log\left(p_t^*\right)\right]$ whenever $\alpha \neq 1$. So this is an example where providing investors with more information about dividends, i.e., moving from information set I_t^o to set I_t^*, will decrease the variance of the log price-change—similar in spirit to the results obtained by Engel (2005) and LeRoy (1984). But as we have seen, this is not a general result.

The bottom panel of Figure 1 shows that the volatility lines for information sets I_t and I_t^* can cross at two different values of α, implying reversals in the variance ordering at the crossing point. The first crossing point occurs at $\alpha = 1$ while the second occurs at $\alpha = 3.86$. This result shows that providing investors with more information about future dividends can either increase or decrease the variance of the log price-change, depending on the level of risk aversion and the parameters that govern the dividend growth process. The value of α at the second crossing point depends on the value of ρ. Given the intuition from the risk neutral case in Section 3, it is not surprising that a dividend growth persistence parameter can influence the direction of the variance inequality in the model with risk aversion.[12]

Another interesting result shown in Figure 1 is that, under the baseline calibration (top panel), model-predicted volatility can match the data volatility with $\alpha \leq 10$ under information

[11] The real stock price in the data is measured as the real value of the S&P 500 index from Robert Shiller's website.

[12] A similar reversal pattern occurs when plotting the variance of log equity returns, as shown by Lansing and LeRoy (2014).

sets I_t^o and I_t^*. Under information set I_t^o the model can match the volatility in the data when $\alpha \simeq 5.4$. Put another way, volatility in the data does not appear excessive if $\alpha > 5.4$ and one is willing to accept the idea that investors can predict dividends accurately one year in advance.

6 Conclusion

This paper showed that providing investors with more information about future dividends can either increase or decrease the variance of stock price changes, depending on some key parameters, namely, those governing the properties of dividends and the stochastic discount factor. I reconcile this result with Engel (2005) and LeRoy (1984) who found that more information decreases the variance of stock price changes in specialized model settings. The results derived here are important because it means that news about future dividends can help account for the high variance of stock price changes in the data.

A Appendix: Information Set I_t

A.1 Proof of Proposition 2

Iterating ahead the conjectured law of motion for z_t and taking the conditional expectation yields

$$E_t\left(z_{t+1}|I_t\right)=\exp\left[a_0+\rho a_1\left(x_t-\overline{x}\right)-a_1\phi\,\varepsilon_t+\tfrac{1}{2}\left(a_1+a_2\right)^2\sigma_\varepsilon^2\right]. \tag{A.1}$$

Substituting the above expression into the first order condition (34) and then taking logarithms yields

$$\begin{aligned}\log\left(z_t\right)=F\left(x_t,\,\varepsilon_t\right) &= \log\left(\beta\right)+\theta x_t \\ &\quad +\ \log\left\{\exp\left[a_0+\rho a_1\left(x_t-\overline{x}\right)-a_1\phi\,\varepsilon_t+\tfrac{1}{2}\left(a_1+a_2\right)^2\sigma_\varepsilon^2\right]+1\right\}, \\ &\simeq a_0+a_1\left(x_t-\overline{x}\right)+a_2\,\varepsilon_t,\end{aligned} \tag{A.2}$$

where the Taylor-series coefficients $a_0\equiv E\left[\log\left(z_t\right)\right]$, a_1, and a_2 are given by

$$a_0 = F\left(\overline{x},\,0\right) = \log\left(\beta\right)+\theta\,\overline{x}+\log\left\{\exp\left[a_0+\tfrac{1}{2}\left(a_1+a_2\right)^2\sigma_\varepsilon^2\right]+1\right\} \tag{A.3}$$

$$a_1 = \left.\frac{\partial F}{\partial x_t}\right|_{\overline{x},\,0} = \theta\ +\ \frac{\rho a_1\exp\left[a_0+\frac{1}{2}\left(a_1+a_2\right)^2\sigma_\varepsilon^2\right]}{\exp\left[a_0+\frac{1}{2}\left(a_1+a_2\right)^2\sigma_\varepsilon^2\right]+1}. \tag{A.4}$$

$$a_2 = \left.\frac{\partial F}{\partial \varepsilon_t}\right|_{\overline{x},\,0} = \frac{-a_1\phi\exp\left[a_0+\frac{1}{2}\left(a_1+a_2\right)^2\sigma_\varepsilon^2\right]}{\exp\left[a_0+\frac{1}{2}\left(a_1+a_2\right)^2\sigma_\varepsilon^2\right]+1} \tag{A.5}$$

Solving equation (A.3) for a_0 yields

$$a_0\ =\ \log\left\{\frac{\beta\exp\left(\theta\,\overline{x}\right)}{1-\beta\exp\left[\theta\,\overline{x}+\frac{1}{2}\left(a_1+a_2\right)^2\sigma_\varepsilon^2\right]}\right\}, \tag{A.6}$$

which can be substituted into equations (A.4) and (A.5) to yield the following system of nonlinear equations that determines a_1 and a_2:

$$a_1 = \theta\ +\ \rho a_1\beta\exp\left[\theta\,\overline{x}+\tfrac{1}{2}\left(a_1+a_2\right)^2\sigma_\varepsilon^2\right] \tag{A.7}$$

$$a_2 = -a_1\phi\beta\exp\left[\theta\,\overline{x}+\tfrac{1}{2}\left(a_1+a_2\right)^2\sigma_\varepsilon^2\right] \tag{A.8}$$

Solving equation (A.7) for a_1 yields the expression shown in Proposition 1. When $\rho \neq 0$, the above equations can be combined to obtain the following explicit expression for a_2

$$a_2 = \phi\left(\theta - a_1\right)/\rho, \qquad \left(\rho \neq 0\right), \tag{A.9}$$

which can be substituted back into equation (A.7). There are two solutions, but only one solution satisfies the condition $\beta \exp\left[\theta\,\overline{x} + \frac{1}{2}\left(a_1 + a_2\right)^2 \sigma_\varepsilon^2\right] < 1.$ ■

A.2 Asset Pricing Moments

This section briefly outlines the derivation of equations (36) and (37). Starting from equation (35) and taking the unconditional expectation of $\log\left(y_t\right)$ yields

$$\log\left(y_t\right) - E\left[\log\left(y_t\right)\right] \;=\; a_1\rho\left(x_t - \overline{x}\right) \;-\; a_1\phi\,\varepsilon_t, \tag{A.10}$$

which in turn implies

$$Var\left[\log\left(y_t\right)\right] \;=\; \left(a_1\rho\right)^2 Var\left(x_t\right) \;+\; \left(a_1\phi\right)^2 \sigma_\varepsilon^2 \;-\; 2\left(a_1\right)^2 \rho\,\phi\, \underbrace{Cov\left(x_t,\, \varepsilon_t\right)}_{=\sigma_\varepsilon^2}. \tag{A.11}$$

The above expression can be simplified to obtain equation (36).

An expression for $\Delta \log\left(p_t\right)$ can be obtained using equation (31). Substituting for $\Delta \log\left(y_t\right)$ and then subtracting the unconditional expectation of $\Delta \log\left(p_t\right)$ yields

$$\Delta \log\left(p_t\right) - E\left[\Delta \log\left(p_t\right)\right] \;=\; \left(1 + a_1\rho\right)\left(x_t - \overline{x}\right) \;-\; a_1\rho\left(x_{t-1} - \overline{x}\right) - a_1\phi\,\varepsilon_t \;+\; a_1\phi\,\varepsilon_{t-1}. \tag{A.12}$$

Taking the square of the above expression and then taking the unconditional expectation yields equation (37).

B Appendix: Information Set $I_t^o = I_t \cup d_{t+1}$

Substituting the law of motion for dividend growth (24) into equation (39) yields

$$y_t^o = z_{t+1} = \exp[a_0 + \underbrace{a_1\rho\left(x_t - \mu\right) - a_1\phi\,\varepsilon_t}_{\log(y_t) - E[\log(y_t)]} + \left(a_1 + a_2\right)\,\varepsilon_{t+1}]. \tag{B.1}$$

where $a_0 = E\left[\log\left(z_t\right)\right] = E\left[\log\left(y_t^o\right)\right]$. Taking logs of equation (B.1) and then comparing to the approximate solution for $\log\left(y_t\right)$ from equation (35) yields the relationship shown in equation

(40). Taking the first difference of $\log(y_t^o)$ and $\log(y_t)$ and making use of equations (31) and (32) yields the relationship shown in equation (41).

Taking the square of equations (40) and (41) and then taking the unconditional expectation yields equations (42) and (43).

C Appendix: Information Set I_t^*

C.1 Log-linearized Law of Motion

Taking logarithms of the nonlinear law of motion (30) yields

$$\begin{aligned} \log(y_t^*) &= G\left[x_{t+1}, \log\left(y_{t+1}^*\right)\right] = \log(\beta) + \theta x_{t+1} + \log\left\{\exp\left[\log\left(y_{t+1}^*\right)\right] + 1\right\} \\ &\simeq b_0 + b_1\left(x_{t+1} - \overline{x}\right) + b_2\left[\log\left(y_{t+1}^*\right) - b_0\right], \end{aligned} \tag{C.1}$$

where the Taylor-series coefficients $b_0 \equiv E\left[\log(y_t^*)\right]$, b_1, and b_2 are given by

$$b_0 = G(\overline{x}, b_0) = \log(\beta) + \theta\,\overline{x} + \log\left[\exp(b_0) + 1\right], \tag{C.2}$$

$$b_1 = \left.\frac{\partial G}{\partial x_t}\right|_{\overline{x},\, b_0} = \theta, \tag{C.3}$$

$$b_2 = \left.\frac{\partial G}{\partial \log\left(y_{t+1}^*\right)}\right|_{\overline{x},\, b_0} = \frac{\exp(b_0)}{\exp(b_0) + 1}. \tag{C.4}$$

Solving equation (B.2) for b_0 yields

$$b_0 = \log\left\{\frac{\beta\exp(\theta\,\overline{x})}{1 - \beta\exp(\theta\,\overline{x})}\right\}, \tag{C.5}$$

which can be substituted into equation (B.4) to obtain $b_2 = \beta\exp(\theta\,\overline{x})$.

C.2 Asset Pricing Moments

This section briefly outlines the derivation of equations (45) and (46). Since $b_0 \equiv E\left[\log(y_t^*)\right]$, equation (C.1) implies

$$\log(y_t^*) - E\left[\log(y_t^*)\right] = \theta\left(x_{t+1} - \overline{x}\right) + \beta\exp(\theta\,\overline{x})\left\{\log\left(y_{t+1}^*\right) - E\left[\log(y_t^*)\right]\right\}, \tag{C.6}$$

which in turn implies

$$Var\left[\log\left(y_t^*\right)\right] = \frac{\theta^2 Var\left(x_t\right) + 2\theta\beta\exp\left(\theta\,\overline{x}\right)\,Cov\left[\log\left(y_t^*\right),\,x_t\right]}{1-\beta^2\exp\left(2\theta\,\overline{x}\right)}. \tag{C.7}$$

The next step is to compute $Cov\left[\log\left(y_t^*\right),\,x_t\right]$ which appears in equation (C.7). Starting from equation (C.6), we have

$$Cov\left[\log\left(y_t^*\right),\,x_t\right] = \theta Cov\left(x_t,\,x_{t-1}\right) + \beta\exp\left(\theta\,\overline{x}\right)\underbrace{Cov\left[\log\left(y_{t+1}^*\right),\,x_t\right]}_{=Cov\left[\log\left(y_t^*\right),\,x_{t-1}\right]}, \tag{C.8}$$

$$Cov\left[\log\left(y_t^*\right),\,x_{t-1}\right] = \theta\underbrace{Cov\left(x_t,\,x_{t-2}\right)}_{=\rho Cov\left(x_t,\,x_{t-1}\right)} + \beta\exp\left(\theta\,\overline{x}\right)\underbrace{Cov\left[\log\left(y_{t+1}^*\right),\,x_{t-1}\right]}_{=Cov\left[\log\left(y_t^*\right),\,x_{t-2}\right]}, \tag{C.9}$$

where repeated substitution is used to eliminate $Cov\left[\log\left(y_t^*\right),\,x_{t-j}\right]$ for $j = 1, 2, ...$ Applying a transversality condition yields

$$\begin{aligned} Cov\left[\log\left(y_t^*\right),\,x_t\right] &= \theta Cov\left(x_t,\,x_{t-1}\right)\sum_{j=0}^{\infty}\left[\rho\beta\exp\left(\theta\,\overline{x}\right)\right]^j \\ &= \frac{\theta Cov\left(x_t,\,x_{t-1}\right)}{1-\rho\beta\exp\left(\theta\,\overline{x}\right)}. \end{aligned} \tag{C.10}$$

where the infinite sum converges provided that $\rho\beta\exp\left(\theta\,\overline{x}\right) < 1$. Substituting equation (C.10) into equation (C.7), together with $Cov\left(x_t,\,x_{t-1}\right) = Corr\left(x_t,\,x_{t-1}\right)\times Var\left(x_t\right)$ and then simplifying yields equation (45).

Starting from equation (33) and subtracting the unconditional expectation of the perfect foresight log price change and then substituting for $\Delta\log\left(y_t^*\right)$ computed using the approximate law of motion (C.6) yields

$$\Delta\log\left(p_t^*\right) - E\left[\Delta\log\left(p_t^*\right)\right] = \alpha\left(x_t-\overline{x}\right) + \left[1-\beta\exp\left(\theta\,\overline{x}\right)\right]\left\{\log\left(y_t^*\right) - E\left[\log\left(y_t^*\right)\right]\right\}. \tag{C.11}$$

Taking the square of the above expression, followed by the unconditional expectation and then once again making use of equation (C.10) yields equation (46).

References

[1] Burnside, C. 1998 Solving asset pricing models with Gaussian shocks, *Journal of Economic Dynamics and Control* 22, 329-340.

[2] Barsky, R.B., and E.R. Sims, 2011 News shocks and business cycles, *Journal of Monetary Economics* 58, 273-289.

[3] Campbell, J.Y. and R.J. Shiller 1988 The dividend-price ratio and expectations of future dividends and discount factors, *Review of Financial Studies* 1, 195-228.

[4] Cochrane, J.H. 1992 Explaining the variance of price-dividend ratios, *Review of Financial Studies* 5, 243-280.

[5] Collard, F. and M. Juillard 2001 Accuracy of stochastic perturbation methods: The case of asset pricing models. *Journal of Economic Dynamics and Control* 25, 979-999.

[6] Engel, C. 2005 Some new variance bounds for asset prices, *Journal of Money Credit and Banking* 37, 949-955.

[7] Engel, C. 2014 Exchange rates and interest parity, in G. Gopinath, E. Helpman, and K. Rogoff, (eds.), *Handbook of International Economics Volume 4.* Amsterdam: Elsevier, pp. 453-522.

[8] Gilles, C. and S.F. LeRoy 1991 Econometric aspects of variance-bounds tests: A survey, *Review of Financial Studies* 4, 753-791.

[9] Kleidon, A.W. 1986 Variance bounds tests and stock price valuation models, *Journal of Political Economy* 94, 953-1001.

[10] Kocherlakota, N.R. 1990 On the discount factor in growth economies. *Journal of Monetary Economics* 25, 43-47.

[11] Lansing, K.J. 2010 Rational and near-rational bubbles without drift, *Economic Journal* 120, 1149-1174.

[12] Lansing, K.J. and S.F. LeRoy 2014 Risk aversion, investor information, and stock market volatility, *European Economic Review* 70, 88-107.

[13] LeRoy, S.F. and R.D. Porter 1981 The present-value relation: Tests based on implied variance bounds, *Econometrica* 49, 555-577.

[14] LeRoy, S.F. and C.J. LaCivita 1981 Risk aversion and the dispersion of asset prices, *Journal of Business* 54, 535-547.

[15] LeRoy, S.F. 1984 Efficiency and the variability of asset prices, *American Economic Review, Papers and Proceedings* 74, 183-187.

[16] LeRoy, S.F. and W.R. Parke 1992 Stock price volatility: Tests based on the geometric random walk, *American Economic Review* 82, 981-992.

[17] Lucas, R.E. 1978 Asset prices in an exchange economy, *Econometrica* 46, 1429-1445.

[18] Marsh, T.A. and R.C. Merton 1986 Dividend variability and variance bounds tests for the rationality of stock prices, *American Economic Review* 76, 483-498.

[19] Mehra, R., and E. Prescott 1985 The equity premium: a puzzle, *Journal of Monetary Economics* 14, 145-161.

[20] Shiller, R.J. 1981 Do stock prices move too much to be justified by subsequent changes in dividends? *American Economic Review* 71, 421-436.

[21] Shiller, R.J. 2003 From efficient markets theory to behavioral finance, *Journal of Economic Perspectives* 17 (Winter), 83-104.

[22] West, K.D. 1988b Dividend innovations and stock price volatility, *Econometrica* 56, 37-61.