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Frequency Shifting

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Abstract

What determines the frequency domain properties of a stochastic process? How much risk comes from high frequencies, business cycle frequencies or low frequency swings? If these properties are under the influence of an agent, who is compensated by a principal according to the distribution of risk across frequencies, then the nature of this contracting problem will affect the spectral properties of the endogenous outcome. We imagine two thought experiments: in the first, the principal (or 'regulator') is myopic with regard to certain frequencies - he is characterized by a filter - and the agent ('bank') chooses to hide risk by shifting power from frequencies to which the regulator is attuned to those to which he is not. Thus, the regulator is fooled into thinking there has been an overall reduction in risk when, in fact, there has simply been a frequency shift. In the second thought experiment, the regulator is not myopic, but simply cares more about risk from certain frequencies, perhaps due to the preferences of the constituents he represents or because certain types of market incompleteness make certain frequencies of risk more damaging. We model this intuition by positing a filter design problem for the agent and also by a particular type of portfolio selection problem, in which the agent chooses among investment projects with different spectral properties. We discuss implications of these models for macroprudential policy and regulatory arbitrage.

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1 Introduction

This paper develops a framework for mechanism design in the frequency domain. There are two main strands of motivation. The first motivation is the idea that a principal (or 'regulator') might be more attuned to certain frequencies of risk than others, perhaps through myopia, mandate or unwitting choice/treatment of data. The principal might wish unconditional variance to be reduced (and compensates the agent accordingly) but the principal's perception of the properties of the stochastic process controlled by the agent is inaccurate. Assuming that the agent finds it 'difficult' to reduce overall variance but 'easy' to shift power between frequencies, we show that the agent chooses to manipulate the spectral properties of the process so as to reduce (increase) power in the ranges of frequencies to which the regulator is (is not) attuned. The regulator might then fail to distinguish an overall reduction in variance from simply a shift in power across frequencies, which leaves variance unchanged.[1]

We characterize the regulator's sensitivity to different frequencies with the use of particular linear filters. For example, if one wishes to model the idea that there is a certain band of frequencies to which the regulator is attuned, then one would use a bandpass filter. We then derive the agent's best response, in terms of a filter to be applied to an uncontrolled process, subject to the restrictions that the filter is causal, of finite span and leaves the variance of the uncontrolled (before application of the agent's filter) and the controlled (after application of the agent's filter) equal. The final restriction is to concentrate the model on issues of 'frequency shifting' rather than changes in overall unconditional variance. In order to capture the idea that the agent (in some abstract sense) faces a 'technology' for frequency shifting we set up the agent's filter design problem such that he is penalized for selecting a filter that deviates from a 'do-nothing' filter. This reflects that idea that it is perhaps it is more difficult to amplify or attenuate certain frequencies than others, due to market structure or other business practices.[2]

Given these findings, we argue that the margin for manipulation or 'moral hazard' available to the agent should raise concerns over model misspecification by macroprudential regulators at particular frequencies, naive applications of filters and excessive attention paid to certain frequencies. In particular, recent proposals to require banks to hold buffers dependent on an estimate of cyclical credit ratios, seem to require the application of statistical filters

[1]See Crowley and Hughes Hallett (2011) for evidence of the Great Moderation being a type of frequency shift, from high to low frequencies, perhaps leading to misguided beliefs in an enduring reduction of risk.

[2]Or some deeper design limits that are implied by some underlying structural model (left unmodeled in the current version of this paper) - see Brock, Durlauf, and Rondina (2008).

that may induce the type of frequency shifting illustrated in the model.

The second strand of motivation does not depend on a story of frequency-dependent myopia. Instead, rather than asserting that regulators are more attuned to certain frequencies, we argue that they may wish to overweight certain frequencies of risk due to their being differentially important for the welfare of the constituents that the regulator represents. These 'frequency-sensitive' preferences may reflect 'direct utility' of the constituents if they have preferences that embed aversion to particular frequencies of risk in their consumption streams, such as an aversion to higher frequencies of risk because of the presence of habits in preferences. Alternatively, they may reflect 'indirect utility' in the sense that constituents may implicitly have preferences over income processes with different spectral properties, even if their preferences over consumption streams exhibit indifference. The reason being, that certain frequencies of risk in income may be particularly damaging or difficult to share in an incomplete markets setting. In this case, the filter design problem described above can be thought of as a way of designing a mechanism to induce the agent (say, a bank in a financial system) to shift power to frequencies that are less damaging for constituents.

Recognizing that agents do not, in practice, choose filters, we propose an alternative frequency shifting problem to complement the filter design problem discussed above. Specifically, we propose a setup in which an agent can invest in different risky assets (or 'projects') which have the same unconditional variance but have power concentrated at different frequencies. In this environment, we also see the tendency of the agent to invest in assets in such a way to tune the frequency domain properties of the overall portfolio in such a way as to reduce power through the range of frequencies that the regulator is more attuned to or cares more about.

2 Literature

Frequency domain methods are (largely) known to macroeconomists through the application of statistical filters to prepare data for analysis. One of the most common applications is to apply a band pass (Baxter and King (1999) and Christiano and Fitzgerald (2003)) or Hodrick-Prescott (Hodrick and Prescott (1997)) filter to obtain a data series that can be more easily interpreted as a theoretical model object, either because of the supposed presence of noise (seasonal or measurement error) or because the theoretical model in hand is not specified to make predictions about the properties of data at certain frequencies, such as the nature of the trend.[3] Powerful and useful though these methods are, the inappropriate application of filters or failure to account for the underlying motivation for their specification can lead to various problems (see Harvey and Jaeger (1993), Harvey and Trimbur (2008), Harvey (1993), Cogley and Nason (1995), Canova (1998) and Canova (2010) for discussions). Important pitfalls can arise if filters are applied in such a way as to introduce spurious cyclicality in data and if concentration on a particular band of frequencies hides (or perhaps even induces - see below) misspecification at other frequencies (see Christiano and Vigfusson (2003) for example). Unthinking applications of filters often also presume a general applicability of standard filter calibrations across countries, variables and time periods, that may not actually be justified by the data generating process.

Beyond these longstanding concerns about the effects of filtering on time series and inference, there are recent examples of regulatory reform and research on macro-prudential policy that place significant emphasis on extracting measures of cyclicality in financial data as a way of identifying whether fluctuations represent unsustainable booms in asset prices or credit. For example, the work of Mendoza and Terrones (2008), Mendoza and Terrones (2012), Borio and Drehmann (2009) and Chen and Christensen (2010) examines the nature of credit booms and their implications for the macroeconomy, using filtered data to characterize what constitutes deviations from trend growth in credit availability and other measures of financial conditions. This literature has particular relevance due to the proposed counter-cyclical capital buffers discussed in Basel III documentation (Supervision (2010)). Thus, a parameter of the regulatory environment that financial institutions may face would be a particular filter or set of filters. Although the framework below will be very abstract, it is intended to draw attention to the possibility that, like many other aspects of regulation, there may be unintended consequences from applying regulations dependent on naive filtering when the regulated are aware of this behavior and have an incentive to game it.[4]

One interesting and related debate concerns the treatment of seasonality and whether or not models should be estimated using seasonally adjusted data discussed in Sims (1976) and Hansen and Sargent (1993). One (tentative) conclusion from this debate is captured in the following quote

> . . . some of our examples illustrate that in situations where the model is correctly specified, using seasonally adjusted data induces no asymptotic biases even when

[3]For a clear and accessible discussion of some of these matters, see Cogley (2008).

[4]This leaves aside other important statistical concerns that one might have with applying filters in this context, such as those discussed in Edge and Meisenzahl (2011).

> some of the key parameters are those governing behavior primarily at the seasonal frequencies. Strong cross-equation, cross-frequency overidentifying restrictions permit identification of all parameters, even when seasonal frequencies are deemphasized (or even deleted) via seasonal adjustment. - Hansen and Sargent (1993)

In what follows, we will not be within the realm of rational expectations equilibrium modeling and estimation of deep structural parameters under seasonal adjustment. However, the 'frequency dependent myopia' story we tell for a principal (regulator) and the 'frequency shifting' it induces by the agent ('bank') are partly attempts to imagine what might happen if 'seasonality' (here interpreted more generally as the spectral structure of a controlled process under the influence of an agent) were to 'fight back' in response to the principal filtering out certain frequencies. Thus, the frequency shifting *induces* a particular type of misspecification that the principal ('regulator') labors under.

Setting aside these stories of frequency sensitive myopia, one of the more fruitful literatures to apply frequency domain techniques is that of modeling frequency sensitive preferences, arising from time-non-separable utility. Early work by Bowden (1977), Whiteman (1985) and Taub (1989) using spectral utility functions and more recent work by Otrok (2001a) and Otrok (2001b) have emphasized that the utility assigned to a particular stochastic consumption stream may be dramatically different depending on the frequencies at which power is concentrated, if preferences, say, exhibit habits or durability in the felicity derived from consumption. Consequently, calculations of the welfare costs of business cycles, *à la* Lucas Jr. (1987), that depend only on unconditional variance, may mis-characterize the gains from eliminating variation and, in particular, obscure gains that may be possible from redistributing power from one set of frequencies to others. Indeed, as noted in Otrok (2001b), an agent with frequency sensitive preferences may be prepared to accept a higher overall variance if that increase is accompanied by a shift in power away from frequencies that are particularly painful to him. Similar attempts to allow for differential costs of fluctuations, according to frequency, are found in Alvarez and Jermann (2004), Otrok, Ravikumar, and Whiteman (2007) and Dew-Becker and Giglio (2013), with particular emphasis on asset pricing implications.

One may also ask whether or not, even under 'standard' preferences, *income* processes with different spectral properties may be more or less desirable. If one imagines a world in which there is no aggregate risk and, in the presence of complete markets, all idiosyncratic income risk is shared away, then the persistence of shocks to individuals' incomes, like all other properties of the risky income stream of any given individual, are irrelevant (post contract). However, suppose that due to a particular form of market incompleteness, it is more difficult to insure oneself against shocks with persistent (low frequency) impacts, than those with transient (high frequency) impacts, then one could imagine an induced preference ordering that would imply a preference for an income process that more heavily weights a component operating at high frequency, than one operating at low frequency, holding overall unconditional variance fixed. Indeed, Deaton (1991) and, more recently, Kaplan and Violante (2010) are suggestive in this respect in that, in the first case, varying the degree of autocorrelation in income affects the ability of a buffer stock consumer to smooth consumption and, in the second, the ability of agents to insulate their consumption from transitory idiosyncratic income shocks is estimated to be far larger than in the case of innovations to permanent shocks. Thus, if one were to imagine a policymaker who (by some means) can influence the relative weight of different frequencies in the 'income' process, we might think that structuring a regulatory environment that incentivizes agents to shift risk between frequencies might be desirable - perhaps even at the cost of increasing overall variance.

3 The Frequency Domain

In this section we outline some of the methods of frequency domain analysis, since much of what follows will be expressed in these terms. Most papers using frequency domain techniques, use them as just that: 'techniques'. In our case, however, it is not just the methodology that exists in this space, but also the economic intuition. Readers comfortable in the frequency domain may only wish to glance at this section to fix notation.

3.1 Cycles and Frequencies

Let us first consider a process x_t, which we refer to as a deterministic cycle

$$x_t = \rho \cos(\omega t - \theta)$$

where ω is measured in radians and is referred to as the (angular) frequency. x_t is a cyclical function of time that repeats with period, $\frac{2\pi}{\omega}$. We refer to ρ as the amplitude and θ as the phase shift of the associated cycle.[5]

In figure 1 we illustrate x_t as a function of time, for different parameterizations. In the first case we posit the base cycle with unity amplitude and frequency, so that it repeats every

[5] Much of this section and the manner in which I approach frequency domain modeling draws heavily on the treatment in Harvey (1993) chapter 6, which is an excellent and compact introduction to these methods.

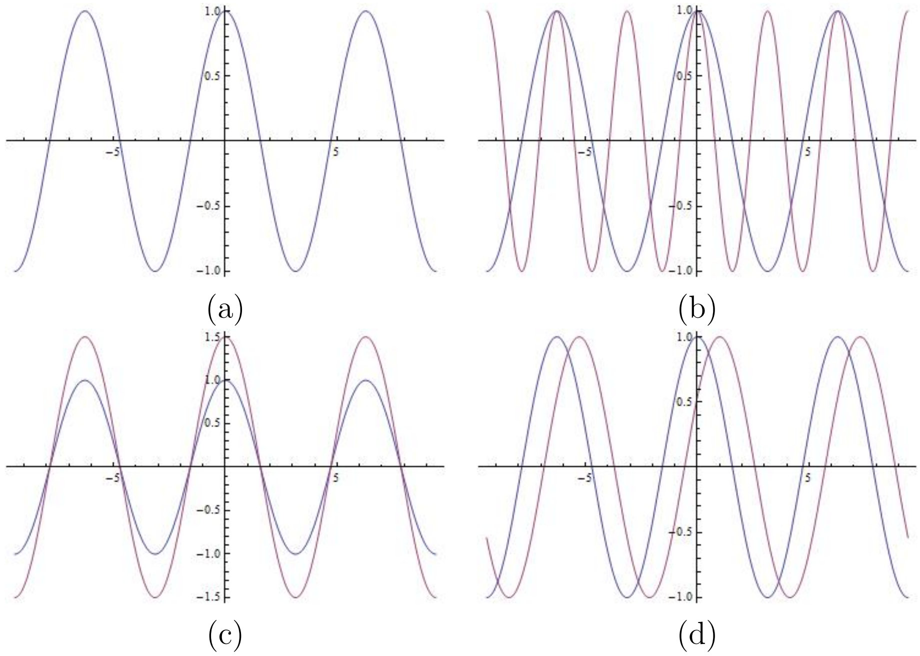


Figure 1: Cycles plotted for different (ω, ρ, θ): (a) $(1, 1, 0)$, (b) $(2, 1, 0)$, (c) $(1, 1.5, 0)$, (d) $(1, 1, 1)$

2π units of time. In the second panel, we illustrate a related cycle repeating operating at twice the base frequency, with the same amplitude and, in the third, we illustrate a cycle with unchanged frequency but increased amplitude. In the fourth panel we plot a cycle that is the same as the base cycle in terms of frequency and amplitude but is 'phase shifted' and lags the base cycle by 1 radian or, alternatively expressed, is shifted back by $\frac{\theta}{\omega} = 1$ units of time.

3.2 Time and Frequency Domain Representations of a Stochastic Process

Allowing a purely indeterministic, covariance stationary process, $x(t)$ to be complex valued we obtain the corresponding spectral density function, f_x, as the Fourier transform of the autocovariances, $\gamma(\tau)$,

$$f_x(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} \gamma(\tau) e^{-i\omega\tau} \tag{1}$$

where $i^2 = -1$ and $\gamma(\tau) = E\left[x_t \overline{x_{t-\tau}}\right]$.

3.2.1 The Spectral Representation

In our applications, we employ only real processes, so that f_x will be symmetric around 0. Nevertheless, it is notationally convenient to work with the more general framework that accommodates complex valued processes. The spectral representation of the process is given by

$$x(t) = \int_{-\pi}^{\pi} e^{i\omega t} z(\omega)\, d\omega$$

where for any small interval, $d\omega$, in the range $-\pi \leq \omega \leq \pi$

$$\begin{aligned} E[z(\omega)\, d\omega] &= 0 \\ E\left[z(\omega)\, d\omega \cdot \overline{z(\omega)\, d\omega}\right] &= f_x(\omega) \end{aligned}$$

and for any two non-overlapping intervals, $d\omega_1$ and $d\omega_2$, centered on ω_1 and ω_2 respectively,

$$E\left[z(\omega_1)\, d\omega_1 \cdot \overline{z(\omega_2)\, d\omega_2}\right] = 0$$

Thus, recalling the previous section, we can envisage x_t as being a sum of cycles of different frequencies with stochastic amplitudes and phases. We have, $E[x(t)] = 0$ and

$$\gamma(\tau) = \int_{-\pi}^{\pi} e^{i\omega\tau} f_x(\omega)\, d\omega$$

In particular, the variance of x_t can be decomposed into the variation or 'power' derived from different frequencies.

$$\gamma(0) = \int_{-\pi}^{\pi} f_x(\omega)\, d\omega$$

3.2.2 Examples of Spectral Densities

In this paper we will make much use of Gaussian $ARMA$ processes as primitives in our analysis. An $ARMA(p, q)$ process x_t is characterized by

$$\Phi(L)\, x_t = \Theta(L)\, \varepsilon_t$$

where $\Phi(L)$ and $\Theta(L)$ are polynomials in the lag operator of order p and q respectively, and ε_t is a homoscedastic and mean zero Gaussian White Noise process with standard deviation σ. We can express the spectral density of such a process as

$$f_x(\omega) = \frac{\sigma^2}{2\pi} \frac{|\Theta(e^{-i\omega})|^2}{|\Phi(e^{-i\omega})|^2}$$

As particular cases, then, for an $AR(1)$ where $x_t = \phi_1 x_{t-1} + \varepsilon_t$ we would have

$$f_x(\omega) = \frac{\sigma^2}{2\pi} \frac{1}{1 + \phi_1^2 - 2\phi_1 \cos\omega}$$

or in the case of an $MA(1)$ where $x_t = \varepsilon_t + \theta_1 \varepsilon_{t-1}$ we would have

$$f_x(\omega) = \frac{\sigma^2}{2\pi} \left(1 + \theta_1^2 + 2\theta_1 \cos\omega\right)$$

In figure 2 we plot spectral densities corresponding to several $ARMA$ processes. In the top left quadrant we have the spectral density of a White Noise process, a special case of the $ARMA$ where only the current innovation features. The term 'White Noise' can be seen to correspond to the idea that power is equal at all frequencies, much as white light comprises all colors. In the top right quadrant we observe the case of an $MA(1)$ process with $\theta_1 = 0.5$. We see that the power at low frequencies is greater than at high, capturing the feature that the process exhibits an 'averaging' property that smoothes the underlying sequence of innovations. In contrast, the bottom left quadrant features the opposite pattern, with the higher frequencies relatively accentuated. This is due to the fact that the assumed process is an $AR(1)$ with $\phi_1 = -0.5$. Thus, the negative autoregressive parameter induces greater churning in the process that, in the frequency domain, manifests itself as power at high frequencies. Finally, in the bottom right quadrant we show an example of a spectral density of an $AR(2)$ process that has a 'peak' at an interior point in $[0, \pi]$.[6] This process has $\phi_1 = 0.7$ and $\phi_2 = -0.5$ which leads to pseudo-cyclical behavior in the process, which type of phenomenon is often observed in economic time series.

4 Linear Filters

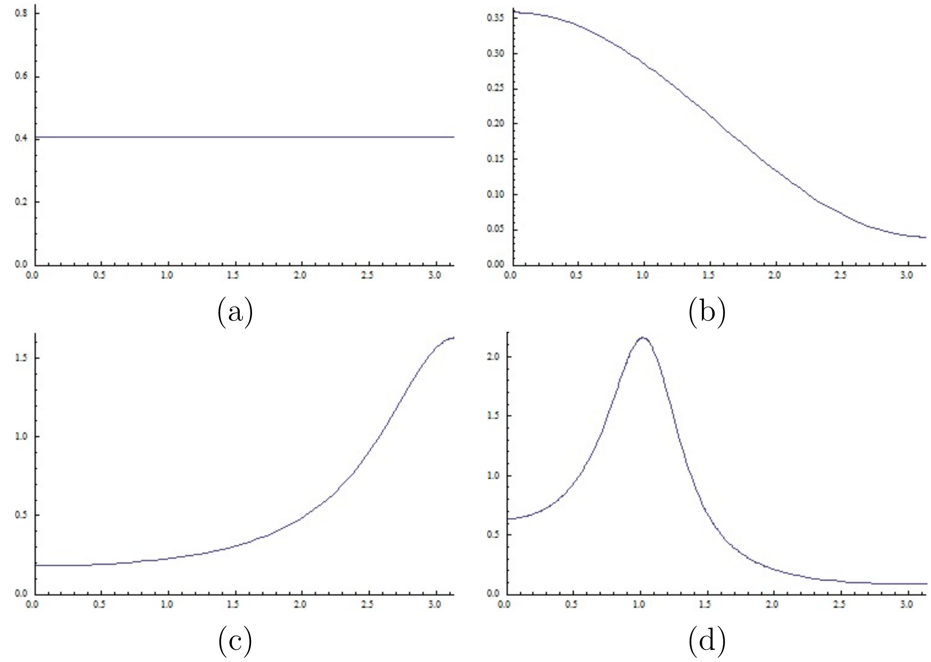
We will make heavy use of the concept of linear filters in understanding the framework in which the agent and principal operates and in terms of modeling the agent's control and the principal's information or preferences. In this section we outline the basic theory of linear filtering and some examples of particular filters.

4.1 Some Preliminaries

We say that z_t is the linearly filtered version of x_t where

$$z_t = \sum_{u=r}^{s} g_u x_{t-u}$$

[6] Here we exploit the symmetry of the spectral densities in only plotting for $\omega \in [0, \pi]$.



(a) (b) (c) (d)

Figure 2: Spectral densities for different processes:(a) White Noise, (b) MA(1) $\theta = 0.5$, (c) AR(1) $\phi = -0.5$, (d) AR(2) $\phi_1 = 0.7$, $\phi_2 = -0.5$

Thus a linear filter is (under one representation) characterized by the weights $\{g_r, g_{r+1}, ...g_s\}$, which are also referred to as the 'Impulse Response Function' (IRF). The effect of the filter at frequency ω is given by the transfer function, $G(\omega)$ where

$$\begin{aligned} G(\omega) &= \sum_{u=r}^{s} g_u e^{-i\omega u} \\ &= |G(\omega)| e^{i\theta(\omega)} \end{aligned}$$

which implicitly defines 'Gain' and 'Power Transfer' functions of the filter to be $|G(\omega)|$ and $|G(\omega)|^2$ respectively. The implicitly defined function $\theta(\omega)$ is referred to as the 'Phase'.

It is instructive to consider the application of a linear filter to a process, x_t, that can be represented as a linear combination of complex exponentials, $x_t = \sum\limits_{\omega} \alpha_\omega e^{i\omega t}$. The effect of a linear filter on a complex exponential at a given frequency ω is

$$L\left(e^{-i\omega t}\right) = G(\omega) e^{i\omega t}$$

Thus by the linearity of the filter, we obtain

$$L(x_t) = \sum_{\omega} \alpha_\omega |G(\omega)| e^{i(\omega t + \theta(\omega))}$$

so that the components of x_t corresponding to different frequencies have their amplitudes and phases amended by the filter and the resulting series is the linear combination of these filtered components. Consequently, the spectral density of a (linearly) filtered process at a particular frequency, ω, is given by the product of the power transfer function (PTF) of the filter and the spectral density of the unfiltered process, both evaluated at ω. That is

$$f_z(\omega) = |G(\omega)|^2 f_x(\omega)$$

It can be shown that the IRF of a filter obtained by sequentially applying two filters is equal to the convolution of the weights of the two composite filters. Thus if z_t^1 is obtained by applying the first filter to x_t and z_t^2 is obtained by applying the second filter to z_t^1 then we have

$$\begin{aligned} z_t^1 &= \sum_{u=r^1}^{s^1} g_u^1 x_{t-u} \\ z_t^2 &= \sum_{u=r^2}^{s^2} g_u^2 z_{t-u}^1 = \sum_{u=r^1+r^2}^{s^1+s^2} \tilde{g}_u x_{t-u} \\ \tilde{g}_k &= \sum_{u=r^2}^{s^2} g_u^2 g_{k-u}^1 \end{aligned}$$

We note that, the spectral density of z_t^1 is given by $f_{z^1}(\omega) = |G^1(\omega)|^2 f_x(\omega)$ and that of z_t^2 is given by $f_{z^2}(\omega) = |G^2(\omega)|^2 f_{z^1}(\omega) = |G^2(\omega)|^2 |G^1(\omega)|^2 f_x(\omega)$. Thus the power transfer function of the composite filter implied by sequential application of two filters is equal to the product of the power transfer functions.

4.2 Types of Filters

We now provide some examples of particular types of filter and demonstrate their effects.

4.2.1 Moving Average and Difference Filters

A (centered) moving average filter is obtained when $s = -r$ in the definition of a linear filter above. We will typically be working with the case in which the moving average weights are equal, so $g_u = \frac{1}{2s+1}$ for all u. A τ-difference filter yields $z_t = y_t - y_{t-\tau}$, that is, g_u is unity lag 0 and negative unity at lag τ. The first row of figure3 shows the power transfer functions for a 5-period moving average and a 5-difference filter. One notes that both filters eliminate power at frequencies corresponding to periods of 5 time units (and an additional harmonic

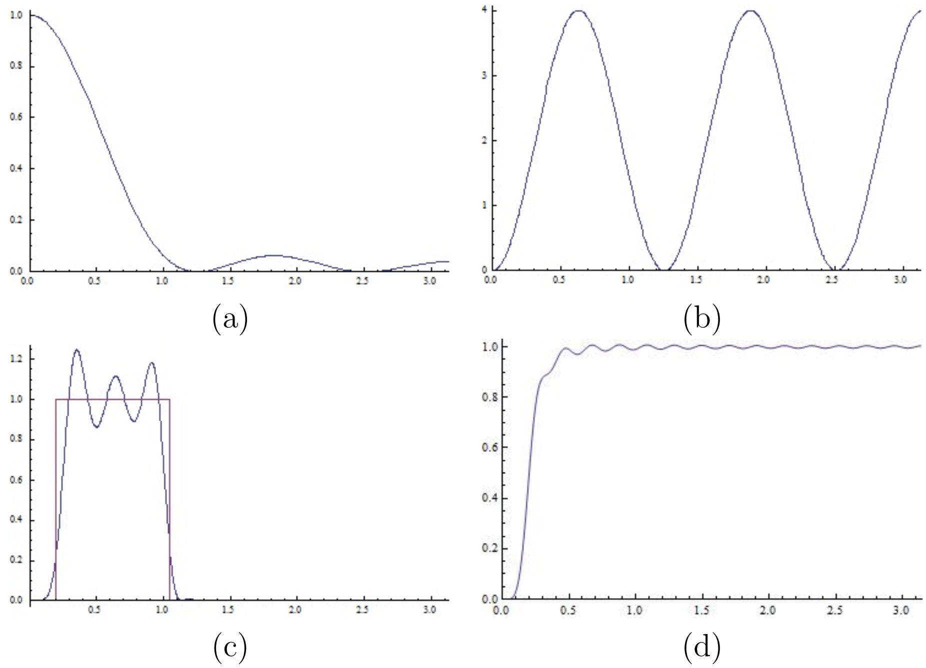


Figure 3: Power Transfer Functions (PTF) for different filters: (a) 5-period moving average, (b) 5-period difference, (c) Band pass (ideal and approx), (d) Hodrick-Prescott (approx)

at higher frequency). An important contrast between the two types of filter, however, is that the moving average filter passes the low frequency component (and returns the frequency zero component unchanged) while the difference filter, intuitively, reduces power at low frequencies (and eliminates the frequency zero component).

4.2.2 High, Low and Band Pass Filters

An important class of filters are intended to pass a range of frequencies without effect, while eliminating power at all remaining frequencies. These ideal filters are referred to as 'high', 'low' or 'band' pass filters according to whether the filter passes all frequencies above, below or between certain cutoffs. Of course, one can also envisage filters that pass power in a set of frequency bands. Now, these filters are not implementable since they require a doubly infinite set of observations. However, they can be approximated with finite span filters which have the same filter coefficients as the ideal versions, up to the points at which they are truncated to have finite span.[7] In the bottom left quadrant of figure 3 we plot an ideal and

[7] This is subject to the caveat that there may be necessary adjustments if, say, one want to ensure zero pass at frequency zero - see Baxter and King (1999).

approximated (using a span of 20 periods) band pass filter, with passed band $\omega \in [0.20, 1.05]$. We observe that the approximations are imperfect, with the fluctuating form of the errors typically referred to as the 'Gibbs effect'. Nevertheless, for our purposes it will be convenient to work with approximations to the ideal filters.

4.2.3 Hodrick-Prescott Filter

A now commonly used filter was proposed by Hodrick and Prescott (1997) as a way of extracting a cyclical component from a series, x_t, identified by solving a particular optimization problem that seeks to identify a 'smooth' trend around which the series fluctuates.[8] The concept of trend, τ_t, proposed is that which solves

$$\min_{\{\tau_t\}} \sum_{t=-\infty}^{\infty} \left[(x_t - \tau_t)^2 + \lambda \left(\tau_{t+1} - 2\tau_t + \tau_{t-1} \right)^2 \right]$$

which yields a filter defined by a particular infinite span, two sided moving average

$$z_t = \frac{\lambda (1-L)^2 (1-L^{-1})^2}{1 + \lambda (1-L)^2 (1-L^{-1})^2} x_t$$

where L is the back-shift operator. The filter can be interpreted as a type of 'high pass' filter where the parameter λ controls the approximate cutoff of the passed frequency range. In applications, the filter can only be approximated using finite span filters that truncate the implicit filter weights beyond some finite lag. In the bottom right quadrant of figure 3 we plot the PTF where we have used $\lambda = 1600$ and a finite span of 30 lags.

4.3 A Filtering Example

To better understand the nature of linear filters and their effect on the processes to which they are applied we now present a simple example. We simulate data for 1000 periods under an $AR(1)$ process $x_t = \rho x_{t-1} + \sigma \epsilon_t$ with $\rho = 0.2$ and $\sigma = 1$ and $\epsilon_t \sim N(0, 1)$. At the end of the 1000 periods we engineer a structural break by setting $\rho = 0.8$ and adjusting σ to maintain the same unconditional variance. We then simulate for a further 1000 periods. Thus the data generating process undergoes a structural break half way through the simulation, from a 'noisy' AR(1) to a (relatively) 'smooth' AR(1). In the first row of figure 4 we show the spectral density functions for the two AR(1) specifications.

Now we consider the effects of applying two different linear filters to the underlying process: a first-difference filter and a centered moving average filter with a window of 21 periods.

[8] This section draws on Cogley (2008).

The difference filter tends to pass power at high frequencies, while the moving average filter tends to pass power at low frequencies, which is reflected in the shape of their PTFs in the second row of figure 4. Recalling that the spectral density of a filtered process is given by the product of the PTF with the spectral density of the unfiltered process, we plot the filtered and unfiltered spectral densities in the 3^{rd} and 4^{th} rows of the figure.

In figure 5 we plot the simulated data pre-filtering, and after applying the two filters. The first sub-figure illustrates (up to some sampling variability) the nature of the structural break since we observe that, unconditionally, the variance in the two regimes is rather similar, but we see much more churning in the first 1000 periods (the 'noisy' regime), whereas we see more persistent swings later in the simulation (the 'smooth' regime). Since the MA filter passes only relatively low frequencies the reduction in variance it implies is greater in the noisy AR(1) regime since there is much less power at those frequencies in the underlying unfiltered process. In contrast, because the difference filter passes only relatively high frequencies, we see a particularly dramatic reduction in variability in the smooth AR(1) regime since there is relatively little power at high frequencies in the unfiltered process.

5 Frequency Shifting

In this section we will construct a particular filter design problem. As aforementioned, one interpretation of the problem will be that a principal is myopic with regard to risks coming from certain frequencies and this, via a moral hazard type of argument, induces the agent shift power to frequencies unobserved by the principal, thereby 'hiding' risk. In this context we will often adopt the language of 'bank' and 'regulator' in place of 'agent' and 'principal'. A second interpretation, not based on any form of myopia, is that the problem captures the design of incentives to induce an agent to shift power from frequencies that are particularly undesirable for the regulator or the constituents the regulator represents.

5.1 Statement of Problem

The regulator is characterized by a (possibly two sided) linear filter, denoted Φ^R.[9] The bank is assumed to face a situation in which, if it adopts a 'do-nothing' policy, the spectral density of the 'uncontrolled' process (which we loosely interpret as asset value or some other

[9] We could make the filter one sided without changing much of the analysis that follows. Whether or not we make the filter one sided (or 'causal') will depend on whether we regard this problem as one of a regulator anticipating the application of real time filtering to data or whether we simply regarded it mechanically as a way of capturing certain preferences over spectra.

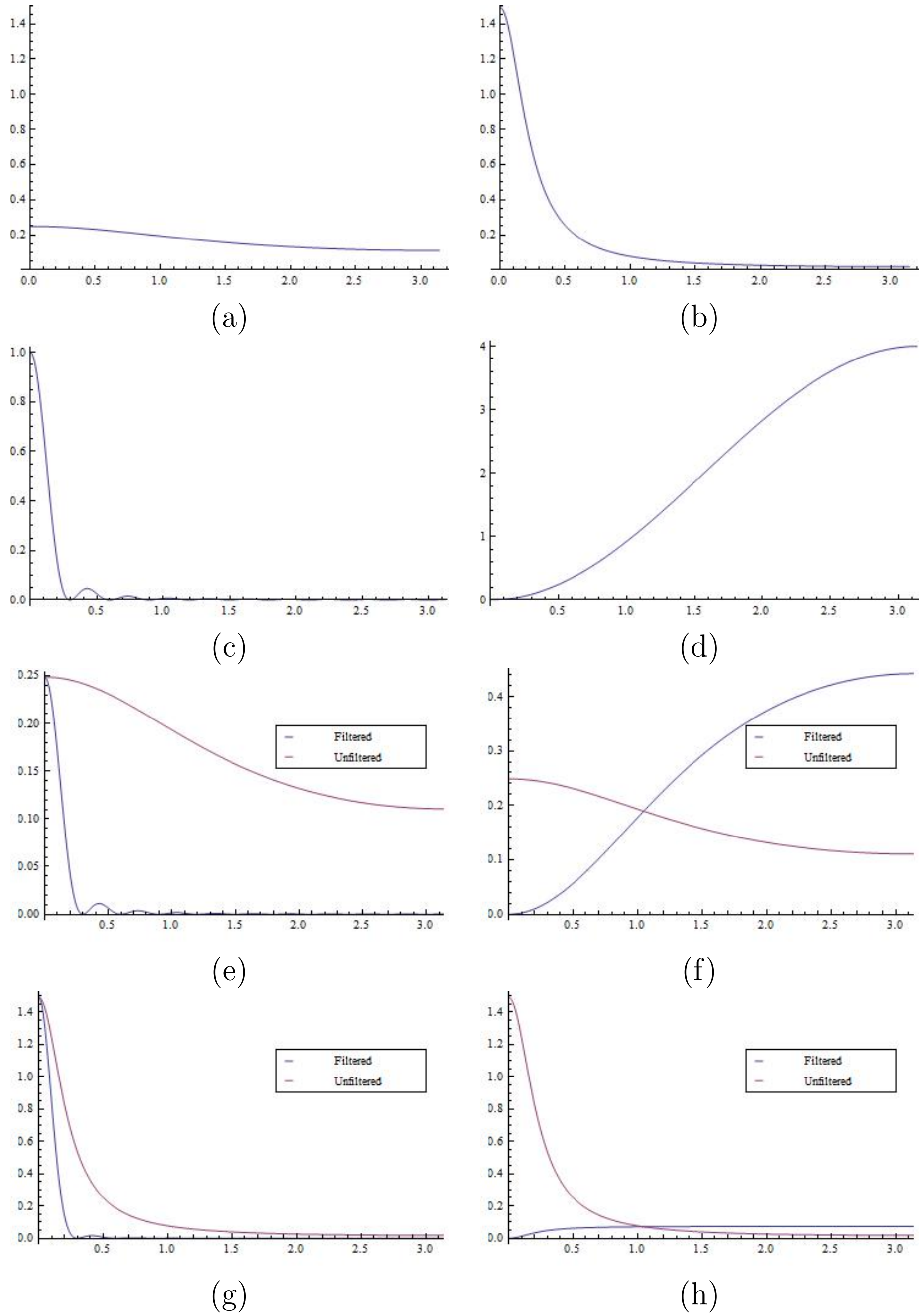
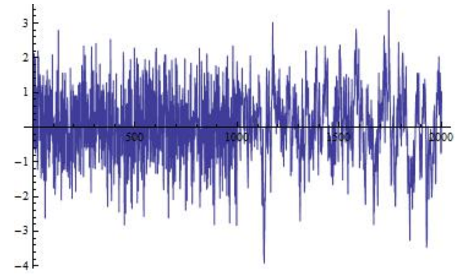
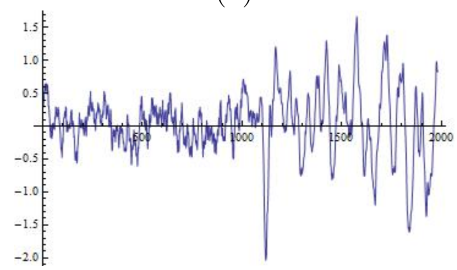


Figure 4: Two processes, two filters - an example: (a) Spectrum of noisy AR(1), (b) Spectrum of smooth AR1, (c) PTF of MA filter, (d) PTF of difference filter, (e) MA filtered and unfiltered noisy AR1, (f) Difference filtered and unfiltered noisy AR1, (g) MA filtered and unfiltered smooth AR1, (h) Difference filtered and unfiltered smooth AR1



(a)



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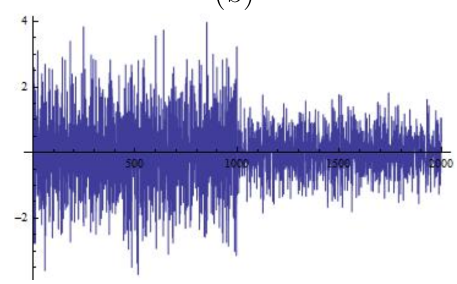


(c)

Figure 5: Base and filtered simulations with a structural break: (a) Base simulation, (b) MA filtered, (c) Difference filtered

measure of bank performance) is given by, f_U. However, the bank is assumed to be able to take actions to affect the properties of the process an this is captured by allowing the bank to choose a filter, Φ^B. This implies a 'controlled' process with spectral density f_C, where $f_C(\omega) = |\Phi^B(\omega)|^2 f_U(\omega)$.

The bank is assumed to seek to minimize the variance of the process obtained by applying the regulator's filter to the controlled process, perhaps because it is required by the regulator to hold capital or undertake some other privately costly activity that is increasing in the regulator's perception of variance. Intuitively, the filter the bank chooses in response to the regulator will reduce power at the frequencies where the regulator's filter passes the most power. One might imagine that if, say, the regulator's filter is a business cycle band pass, the bank would ideally wish to cut power to zero in the passed region, which seems pathological. We wish to avoid such 'bang bang' cases. In the real world, there are operational/technological constraints or costs facing a bank that might mitigate its manipulation of the underlying process. Thus we will model the bank as facing a penalty based on weighted deviations of Φ^B from a 'do nothing' filter, using a weighting function $\varphi(\omega)$. We also impose the constraint that Φ^B leaves unconditional variance unchanged under application of Φ^B. The last requirement is to focus the model on issues of 'frequency shifting' rather than changes in variance.

The idea (under the first motivation) is to show that a regulator can be 'tricked' into thinking that variance has declined, when in fact all it observes is a reduction in power at the frequencies the bank knows it is attuned to, with power at other frequencies actually rising to offset those gains. The formal expression of the intuition above is to have the bank choose Φ^B to minimize

$$\int_{-\pi}^{\pi} |\Phi^R(\omega)|^2 \overbrace{|\Phi^B(\omega)|^2 f_U(\omega)}^{f_C(\omega)} + \frac{\varphi(\omega)}{2} |\Phi^B(\omega) - 1|^2 d\omega$$

subject to

$$\underbrace{\int_{-\pi}^{\pi} |\Phi^B(\omega)|^2 f_U(\omega)\, d\omega}_{\gamma_C(0)} = \underbrace{\int_{-\pi}^{\pi} f_U(\omega)\, d\omega}_{\gamma_U(0)}$$

5.2 Reformulation of Problem

Let us impose that Φ^R and Φ^B are finite span linear filters. In the case of Φ^B, the filter is assumed to be one sided (or 'causal'), capturing the idea that it relies only on current and past data. In the case of the regulator's filter, we allow for two sided filtering. Thus we have

$$\begin{aligned}\Phi^{B}(\omega) &= \sum_{u=0}^{M^{B}-1} g_u^B e^{-i\omega u} \\ \Phi^{R}(\omega) &= \sum_{u=-M^{R}}^{M^{R}} g_u^R e^{-i\omega u}\end{aligned}$$

Therefore, we can model the bank's choice of filter in terms of choosing $g^B \equiv \left\{g_u^B\right\}$.[10] Similarly, we can take the Fourier transforms of the weighting function, $\varphi(\omega)$ and the spectral density of the uncontrolled process:

$$\begin{aligned}\varphi(\omega) &= \frac{1}{2\pi}\sum_{s=-\infty}^{\infty} k_s e^{is\omega} \\ f_U(\omega) &= \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty} \gamma_\tau e^{i\tau\omega}\end{aligned}$$

Consequently, we can re-express the problem above as

$$\min_{\{g_u^B\}} \frac{1}{2\pi}\left[\begin{array}{c} \int\limits_{-\pi}^{\pi}\left(\sum\limits_{u^B,v^B} g_{u^B}^B g_{v^B}^B e^{-i\omega\left(u^B-v^B\right)}\right)\left(\sum\limits_{u^R,v^R} g_{u^R}^R g_{v^R}^R e^{-i\omega\left(u^R-v^R\right)}\right)\left(\sum\limits_{\tau=-\infty}^{\infty}\gamma_\tau e^{i\tau\omega}\right) \\ + \\ \frac{1}{2}\left(\sum\limits_{s=-\infty}^{\infty} k_s e^{is\omega}\right)\left(\sum\limits_{u^B,v^B} g_{u^B}^B g_{v^B}^B e^{-i\omega\left(u^B-v^B\right)} - \sum\limits_{u^B} g_{u^B}^B\left(e^{-i\omega u^B}+e^{i\omega u^B}\right)+1\right)\end{array}\right]$$

subject to

$$\left(\sum_{u^B,v^B} g_{u^B}^B g_{v^B}^B e^{-i\omega\left(u^B-v^B\right)} - 1\right)\left(\sum_{\tau=-\infty}^{\infty}\gamma_r e^{ir\omega}\right) = 0$$

Exploiting the orthogonality properties of complex exponentials under integration over $\omega \in [-\pi,\pi]$, we can obtain an alternative expression, more amenable to numerical analysis

$$\begin{aligned}\min_{g^B} f\left(g^B; g^R, \gamma, k\right) &= t_1\left(g^B;\gamma,g^R\right)+t_2\left(g^B;k\right) \\ & s.t. \\ h\left(g^B;\gamma\right) &= 0\end{aligned}$$

[10] I thank Cliff Hurvich for extremely helpful advice on how to reformulate this problem - any remaining errors are my own.

where

$$
\begin{aligned}
t_1\left(g^B;\gamma,g^R\right) &= \sum_{u^B-v^B+u^R-v^R-r=0} \gamma_r g^B_{u^B} g^B_{v^B} g^R_{u^R} g^R_{v^R} \\
t_2\left(g^B;k\right) &= \frac{1}{2}\left(\sum_{u^B-v^B-s=0} k_s g^B_{u^B} g^B_{v^B} - \sum_{u^B-s=0} k_s g^B_{u^B} - \sum_{u^B+s=0} k_s g^B_{u^B} + k_0\right) \\
h\left(g^B;\gamma\right) &= \sum_{u^B-v^B-r=0} \gamma_r g^B_{u^B} g^B_{v^B} - \gamma_0
\end{aligned}
$$

Thus, taking the regulator's filter, the properties of the uncontrolled process and the penalty weighting function (captured by g^R, γ and k respectively), the bank chooses a vector of weights, g^B, to minimize $f\left(g^B; g^R, \gamma, k\right)$, subject to equating the variance of the controlled and uncontrolled processes. The function $f\left(g^B; g^R, \gamma, k\right)$ is given by the sum of two components: $t_1\left(g^B;\gamma,g^R\right)$ and $t_2\left(g^B;k\right)$. The first component captures the variance of the controlled process under the regulator's filter and the second captures the assumed penalization deriving from the deviation of the bank's filter from a 'do-nothing' filter. The latter penalty is intended, through judicious choice of weighting function, $\varphi(\omega)$, to capture technological constraints or other incentives the bank faces when deciding how to influence the spectral density of the controlled process.

5.3 Penalty Functions

The penalty term weights squared absolute deviations of the filter's transfer function from that of a 'do-nothing' filter. If the penalty function is constant then the penalty is simply the L^2 distance. However, it is useful to consider non-constant weighting functions as a way of imagining how it might be more difficult or costly to reallocate power from certain frequencies to others.[11]

In figure 6 we plot several possible weighting functions and their (often indistinguishable) finite span approximations. All of the functions have been scaled to integrate to the same value as the constant unity penalty function. The top left quadrant is obtained by weighting deviations equally at all frequencies. The next two penalty functions are designed to be fairly uniform across frequencies but to penalize deviations at lower (higher) frequencies relatively heavily. Finally, in the lower right quadrant, we observe a penalty function that is

[11] Note that deviations in the phase dimension will also be penalized although, as yet, I have not examined the phase shifts implicit in the bank's chosen filter. The analysis undertaken so far is largely in terms of the effects on amplitudes.

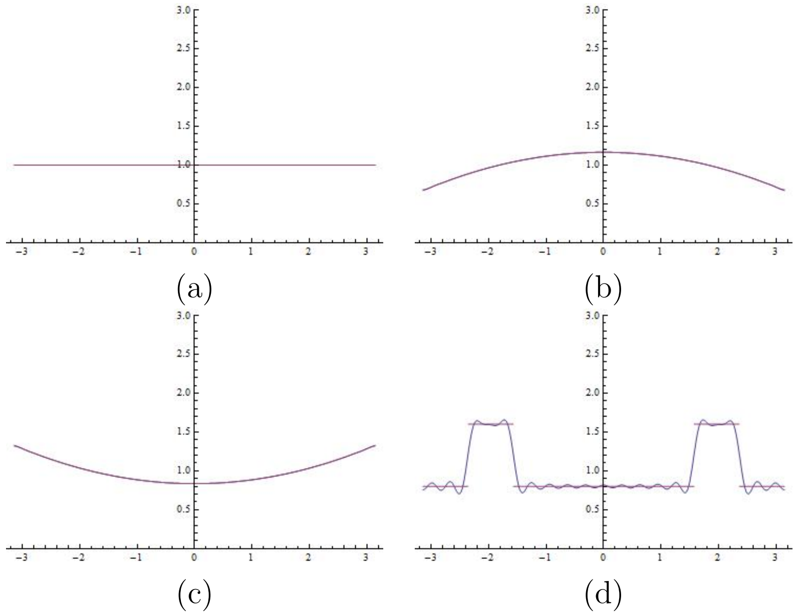


Figure 6: A collection of illustrative penalty functions (and finite approximations): (a) Equal across ω, (b) Greater at low ω, (c) Greater at high ω, (d) Heavy in a particular band

more discrete in its concentration on a certain band of frequencies where deviations are to be particularly penalized. Ultimately one would hope to relate the form of penalty function used to observed real world constraints facing banks, but at the moment, most of the analysis that follows will assume an equal weight across frequencies.

6 Portfolio Selection

In this section we describe a framework for characterizing investment projects, or assets, by their spectral properties. We show how one can imagine a bank inducing a portfolio with particular spectral properties by assembling a weighted collection of assets with power at different frequencies. This is an attempt to take the intuition from the previous filter design problem and transfer it to a slightly more realistic setting. Clearly banks are not in the business of explicitly choosing filters but one can imagine a portfolio selection problem that leads to the tuning of spectral properties of the overall portfolio.

6.1 Stochastic Cycles

Primitives within our analysis will be 'stochastic cycles', as discussed in Harvey (1993). To construct such processes, we first envisage a cyclical component, φ_t, given by

$$\varphi_t = \alpha \cos \lambda t + \beta \sin \lambda t$$

We can render this deterministic cycle stochastic by allowing α and β to evolve over time. To allow for this we posit a recursion for φ_t as follows

$$\begin{pmatrix} \varphi_t \\ \varphi_t^* \end{pmatrix} = \begin{pmatrix} \cos \lambda & \sin \lambda \\ -\sin \lambda & \cos \lambda \end{pmatrix} \begin{pmatrix} \varphi_t \\ \varphi_t^* \end{pmatrix}$$

where $\varphi_0 = \alpha$ and $\varphi_0^* = \beta$. The auxiliary variable φ_t^* is of no intrinsic importance. To introduce randomness, we add two white noise disturbances to the model, κ_t and κ_t^*, which we assume to be uncorrelated and of equal variance. Finally, we also introduce a damping parameter, ρ, to allow for additional flexibility in the φ_t process. Thus we have an augmented process given by

$$\begin{pmatrix} \varphi_t \\ \varphi_t^* \end{pmatrix} = \rho \begin{pmatrix} \cos \lambda & \sin \lambda \\ -\sin \lambda & \cos \lambda \end{pmatrix} \begin{pmatrix} \varphi_t \\ \varphi_t^* \end{pmatrix} + \begin{pmatrix} \kappa_t \\ \kappa_t^* \end{pmatrix}$$

Consequently, we have that

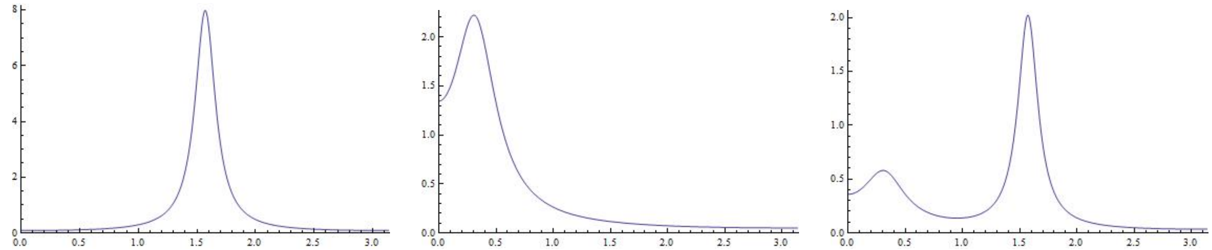
$$\begin{pmatrix} \varphi_t \\ \varphi_t^* \end{pmatrix} = \begin{pmatrix} 1 - \rho \cos \lambda L & -\rho \sin \lambda L \\ \rho \sin \lambda L & 1 - \rho \cos \lambda L \end{pmatrix}^{-1} \begin{pmatrix} \kappa_t \\ \kappa_t^* \end{pmatrix}$$

Thus, we can express φ_t as

$$\varphi_t = \frac{1 - \rho \cos \lambda L}{1 - 2\rho \cos \lambda L + \rho^2 L^2} \kappa_t + \frac{\rho \sin \lambda L}{1 - 2\rho \cos \lambda L + \rho^2 L^2} \kappa_t^*$$

which is the sum of two uncorrelated ARMA(2,1) processes, where the AR and MA coefficients are particular functions of the primitive parameters, ρ and λ. This allows us very easily to calculate the spectral density of φ_t and, in fact, any linear combination of uncorrelated stochastic cycles.

By varying ρ and λ one can tune the properties of the stochastic cycle in question. The parameter, λ, controls the location of a 'peak' in the associated spectral density, whereas ρ controls how sharply concentrated is the power around the peak. As $\rho \to 1$ the peak becomes sharper and, at $\rho = 1$, assuming non-degenerate innovations, the cyclical process becomes non-stationary and the spectral density is no longer well defined. In figure 7 we depict the spectral densities of two stochastic cycles as well as those of two linear combinations of the



(a) (b) (c)

Figure 7: Two stochastic cycles and their combination: (a) $\lambda = \frac{\pi}{2}$, $\rho = 0.9$, (b) $\lambda = \frac{\pi}{10}$, $\rho = 0.8$, (c) $(0.5, 0.5)$ weights

cycles. In the top row we see the spectral densities of stochastic cycles with $\{\rho, \lambda\} = \left\{0.9, \frac{\pi}{2}\right\}$ and $\left\{0.8, \frac{\pi}{10}\right\}$ respectively. On the second row we observe the spectral densities obtained by linear combinations of the two stochastic cycles, with weights of 0.6 (0.4) and 0.4 (0.6) on the first (second) cycle.

6.2 Portfolios

We will think of 'projects' or 'assets' that banks can invest in, in terms of stochastic cycles. Clearly this is an abstraction but it is intended to capture the idea that different risky ventures may exhibit different spectral properties and, by combining them, the bank can tune the spectral properties of an overall portfolio. Thus, in the same way that models of shareholder myopia and executive short-termism often posit stylized projects with different time profiles of returns, here we propose a different set of projects, characterized instead by the frequencies at which they have power.

We model the banks as having access to a set of investment projects, each of which is a stochastic cycle. All of the projects are normalized such that they share the same unconditional variance and the projects are independent of each other. Let this set be denoted (with some abuse of notation)

$$\Xi = \left\{\varphi_t^{i,j} : \rho_i \in \varrho, \lambda_j \in \Lambda\right\}$$

where I posit sets, ϱ and Λ of possible values of the damping and spectral peak parameters, respectively, and adjust the (common) standard deviation of κ_t and κ_t^*, to maintain a particular unconditional variance of the cycle, given (ρ_i, λ_j). I refer to Ξ as a 'generating set' for the bank's portfolio problem.

As discussed above, we wish to model the response of a bank to regulatory incentives, given technological constraints, in terms of the choice of a filter. Applying the resulting filter to the uncontrolled process yields a filtered or controlled process with a particular spectral density. Banks do not explicitly choose filters but one may envisage an approximation of the filtered spectral density through appropriate combination of cycles drawn from a given generating set, Ξ. We give a parsimonious example of such an approximation problem below.

Let us define the distance between two spectral densities in an L^2 sense

$$\|f_1, f_2\| \equiv \int_{-\pi}^{\pi} |f_1(\omega) - f_2(\omega)|^2 d\omega$$

then, if we have a 'target' spectral density, f_T, and a generating set Ξ, then we seek a vector of weights, $w \geq 0$ that minimizes $\|f_T, f_P(w)\|$ where $f_P(\omega) = \sum_{i,j} w_{i,j}^2 f_{i,j}(\omega)$ and $f_{i,j}$ is the spectral density associated with stochastic cycle $\varphi_t^{i,j} \in \Xi$.

Consider first the case where Ξ comprises stochastic cycles with $\{\rho, \lambda\} = \left\{0.9, \frac{\pi}{2}\right\}$ and $\left\{0.8, \frac{\pi}{10}\right\}$ and where the target spectral density is that given by combining these two cycles with $w = (0.2, 0.8)$. In this pathological case, we can replicate the target spectral density perfectly by searching for the weights that minimize $\|f_T, f_P(w)\|$, since we recover $w = (0.2, 0.8)$. Now, however, suppose that Ξ is unchanged but we now generate the target spectral density by combing cycles given by $\{\rho, \lambda\} = \left\{0.9, \frac{\pi}{2}\right\}$, $\left\{0.8, \frac{\pi}{10}\right\}$ and $\{0.3, 0.3\}$, using weights $w = (0.31, 0.31, 0.38)$. We can no longer perfectly replicate the spectral density by combining cycles from Ξ (since we have created the target spectral by weighting 'independent' cycles from $\Xi \cup \{\{0.3, 0.3\}\}$). Thus we will face approximation error at the minimizing w. In this case, solving the minimization problem yields a minimizing portfolio weights vector of $w = (0.3798, 0.3114)$. Figure 8 shows the approximation to the target spectral density obtained by applying these weights. With a rich enough set of generating cycles one can approximate a target spectral density extremely closely.[12]

7 Results

In this section we report our results and findings. We begin with the results obtained from solving the frequency shifting problem laid out in section 5 and then discuss those obtained from a particular case of the portfolio problem laid out in section 6

[12]This is loose - there is probably a way to formalize this in the context of vector spaces and using projections.

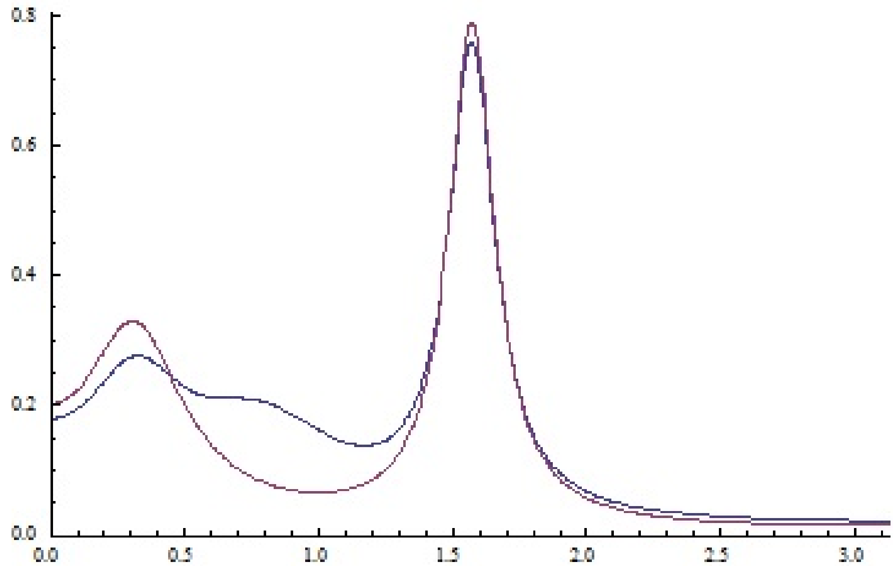


Figure 8: Approximation to a particular spectral density obtained by a combination of stochastic cycles.

7.1 Frequency Shifting

We first treat the case where, as discussed above, the bank's filter choice reduces to choosing the coefficients of a one sided, finite span filter. We will examine a particular case of this problem in which $\varphi(\omega)$ is constant at unity, so deviations from a do nothing filter are weighted equally by frequency. We consider two Φ^R: both are band pass filters pass with pass bands $[0.20, 1.05]$ and $[1.57, 2.34]$, respectively.

All else equal, the bank will wish to reduce power within the passed band, and, to maintain overall variance, raise it at others. However, it will not choose to reduce power within the passed band to a pathological degree due to the presence of the penalization. This is reflected in the PTFs implied by the bank's filter weights (obtained through numerical optimization), displayed in figure 9. We observe that the bank chooses to reduce power through the passed range but that the deviation from the uncontrolled spectral density is fairly modest, reflecting the restriction of using a finite span filter but also, importantly, the incentives implied by the penalty function. Due to the iso-variance constraint, attenuation though the passed frequencies must be offset by amplification at others. Due to the penalty for manipulation, the deviations from the 'do-nothing' filter are smoothed.

To clarify the role of the penalty component, we illustrate the PTFs obtained under different (constant) penalty weights. Figure 10 shows that the attenuation of power through

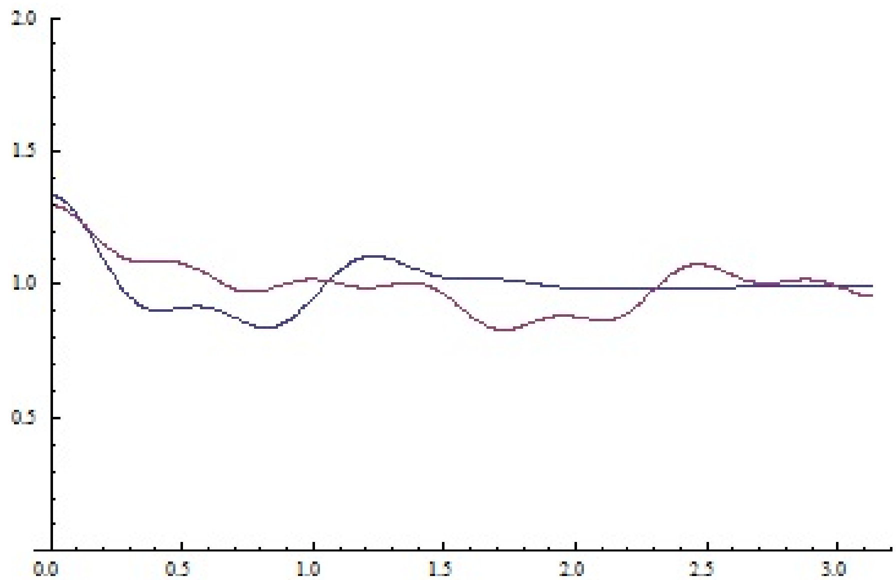
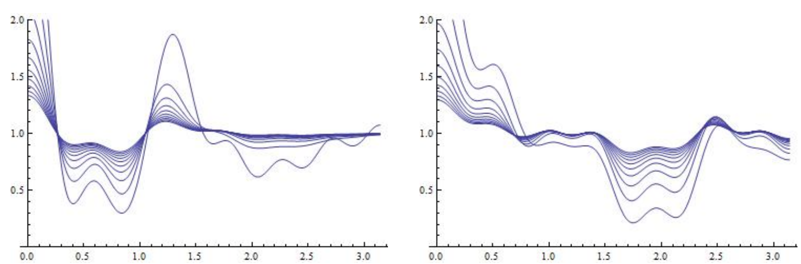


Figure 9: Power transfer functions implied by the bank's filter choice

the regulator's passed frequency range is much more extreme as we reduce the penalty weight.

Motivated by the prevalence of Hodrick-Prescott (H-P) filtering (and the various controversies over its inappropriate application) we also consider a bank's response to different types of H-P filter, for different values of λ. In figure 11 we see the PTFs of three filters in the first three quadrants and, in the bottom right quadrant, the PTFs implied by the bank's respective best responses. As one might expect, the bank shifts power to the low frequencies that are attenuated under the H-P filters. Given the proposed use of H-P filters in distinguishing cyclical and unsustainable fluctuations in credit and other financial measures, there may be



(a) (b)

Figure 10: Power transfer functions implied by the bank's filter choice - under different penalization weights: (a) Regulator band $[0.20, 1.05]$ (b) Regulator band $[1.57, 2.34]$

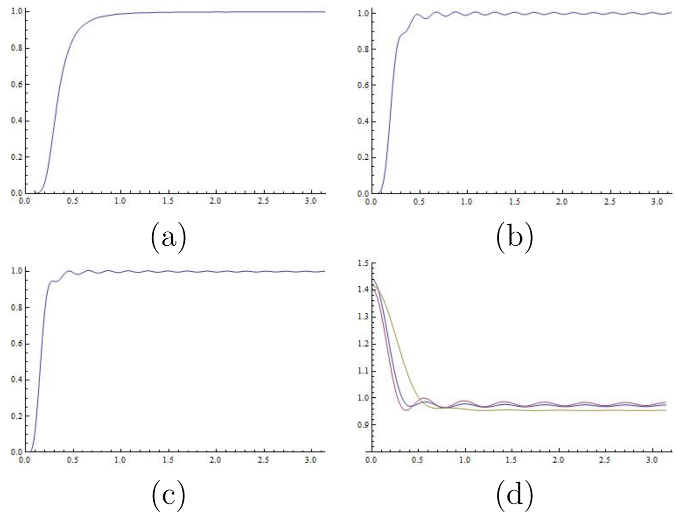


Figure 11: Hodrick-Prescott (approx.) filters and the associated power transfer function responses of the bank: (a) $\lambda = 200$, (b) $\lambda = 1600$, (c) $\lambda = 4600$ (d) PTF

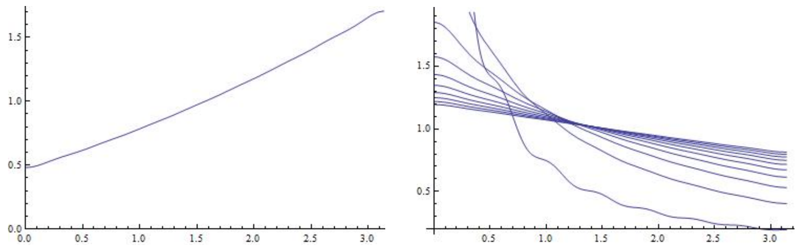
the concern that this sort of frequency shifting may be undertaken as a form of regulatory arbitrage.

Now, as discussed above, even if one does not find arguments for frequency-sensitive myopia as a justification for characterizing a regulator by a filter, one can also motivate this framework by envisaging a regulator who cares about risk at certain frequencies, more than at others. The spectral welfare cost functions of Otrok (2001b) for agents with habits, for example, are shown to imply monotonically increasing costs of power as frequency increases. This could be taken as a justification for a regulatory filter that weights higher frequencies more heavily, and thus has a PTF qualitatively like that in the first panel of figure 12. The second panel indicates the PTFs of the agent's filter for different weights on the penalty term.[13] This shows that the agent has been induced to reallocate power to the less costly frequencies, which might be thought desirable.[14]

Going beyond appealing to deep preferences over the spectral properties of consumption, one might also think that particular forms of market incompleteness may induce a preference ordering over the spectral properties of income. Inspired partly by Deaton (1991) I propose a very simple income fluctuation problem in which there is a no-borrowing constraint, an agent only has access to a riskless bond and faces a stochastic income stream given by an

[13]Given our change in interpretation it is perhaps more suitable to use 'agent' rather than 'bank' here

[14]There are loads of shadow prices implicitly floating around here, by frequency, but due to my using numerical optimization, I'm not sure how to characterize them cleanly.



(a) (b)

Figure 12: Regulator using a filter increases in frequency - inspired by Otrok habits example: (a) PTF of regulator, (b) Bank's PTF by penalty weights

ρ	$SD\left(w\exp\left\{z_t\right\}\right)$	$SD\left(c\right)$
0	0.26	0.07
0.3	0.26	0.09
0.6	0.26	0.12
0.9	0.27	0.24

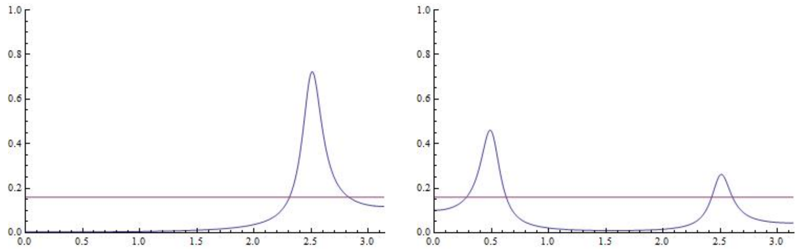
Table 1: The effects of changing the persistence of income fluctuations on consumption.

AR(1) process. The period payoff function is assumed to be $u\left(c\right) = \frac{c^{1-\varphi}-1}{1-\varphi}$, assets evolve as $\frac{a_t}{1+r} + c_t = w\exp\left\{z_t\right\} + a_{t-1}$ and $z_t = \rho z_{t-1} + \sigma\epsilon_t$. We will vary ρ and adjust σ to maintain the same unconditional variance of z_t and examine the effect on the variability of consumption.[15] The results of this experiment are displayed in table 1 where we observe that buffer stock saving is less successful at smoothing consumption when for (approximately) the same variance, the income process becomes more persistent. This suggests a regulator might wish to apply a filter that emphasizes lower frequencies.

7.2 Portfolio Problem

Now we turn to our results for the portfolio problem discussed in section 6. In this case we envisage the bank as forming a portfolio from $\Xi = (0.5, 0.9) \times (0.5, 2.5)$, so from a set of four stochastic cycles. The regulator's filter, again, is a band pass, but with passed frequency range of $(0.20, 1.05)$, which are frequencies corresponding to periodicities of 32 and 6 time units. In quarterly units, this is the range typically denoted 'business cycle' frequencies. We again use the constant unity penalty function. In the first panel of figure 13 we display the controlled spectral density were the bank not penalized for deviating from a do-nothing filter

[15]The process for z_t will actually be approximated using the method proposed by Tauchen (1986)



(a) (b)

Figure 13: Spectral densities of portfolios compared to uncontrolled spectral density
(a) No penalty, (b) Constant penalty

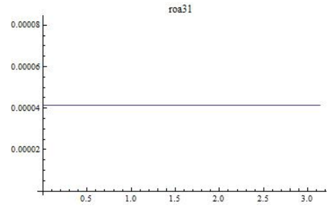
(though still subject to maintaining the same variance as the uncontrolled process), as well as the uncontrolled (white noise) spectral density. We observe that, in this case, the bank would load on a stochastic cycle that leaves almost no power in the passed frequency range, while having substantial power at high frequencies. Under the penalized case, however, in the second panel of the figure, the bank still is seen to reduce power through the passed region substantially, but does so by combining stochastic cycles in a way that leaves power allocated more evenly between low and high frequencies.

In ongoing work we will attempt to identify practical investment strategies or business practices that can be real world applications of the above problem. At this preliminary stage, we have only contrasted the spectral properties of the return on assets associated with particular (SIC-defined) industries. In figure 14 we plot the spectral densities implied by estimated ARMA processes for four SIC categories. We observe differing allocations of power across frequencies and, thus, one might think that investing in these industries to differing degrees might allow an investor to tune the frequency domain properties of a portfolio.

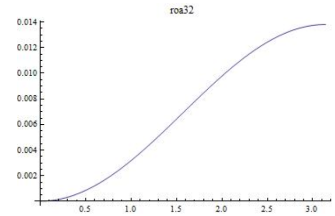
7.3 Extensions

Trying to identify types of investor who might wish to avoid power at certain frequencies (e.g. because they value smooth returns and are prepared to trade off high frequency noise for, say, assets that are exposed to long run risk) is also an important avenue that we are currently exploring. A possible financial product that one might envisage would be a twist on a standard variance swap, which we would denote a 'frequency swap'. A variance swap pays off

$$N_v \left(\sigma_r^2 - \sigma_s^2\right)$$



(a)



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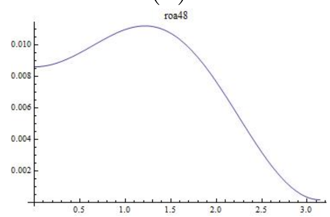
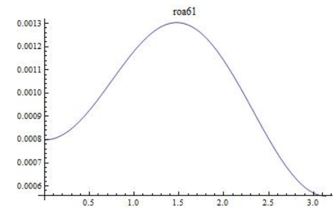


(c)



(d)

Figure 14: ARMA-implied spectral densities: (a) Leather etc., (b) Stone etc., (c) Communications (d) Non-depository Credit Institutions

where N_v is the notional variance, σ_r^2 is realized variance over the duration of the swap and σ_s^2 is the variance strike. It seems fairly simple to conceive of adding an additional dimension to this product, which is a band of frequencies over which the realize variance applies. Thus, if we imagine contracting based on a particular band, $\omega \in [\underline{\omega}, \overline{\omega}]$, we could define a frequency swap's payoff as

$$N_{v,\underline{\omega},\overline{\omega}} \left(\sigma^2_{r,\underline{\omega},\overline{\omega}} - \sigma^2_{s,\underline{\omega},\overline{\omega}} \right)$$

with obvious notation for the notional and strike and where $\sigma^2_{r,\underline{\omega},\overline{\omega}}$ is the realized variance of a band pass filtered version of the underlying process over the duration of the swap. One might imagine that certain market participants might naturally be more exposed to certain frequencies of risk and, thus, would write this sort of asset as a hedge. Alternatively, if a speculator happens not to have a view on overall variance but does have a view on what frequencies the variance will come from, then this product could conceivably allow a position to be taken based on this view. It is possible that such products already exist or can be synthesized.

8 Conclusions

In this paper we propose frameworks for endogenizing the spectral properties of stochastic processes that are under the influence of some agent. We motivate this in two ways. Firstly, we imagine that the agent is seeking to hide risk from a principal who wishes variance to be reduced but is myopic with regard to particular frequency ranges. Under this interpretation we think of the agent as being a 'bank' and the principal being a macroprudential 'regulator'. Secondly, we imagine that the principal may be equally well aware of all frequencies but is more concerned about risk at certain frequencies, perhaps because of the deep preferences of its constituents or because of differential ability of the constituents to insure themselves against different frequencies of risk.

The first framework entails characterizing the regulator as a 'filter' and then deriving a best response of the agent, also in terms of a filter that is regarded as the agent's control. In the second framework, we envisage the agent choosing to invest in projects with different spectral properties, thereby tuning the characteristics of his portfolio. In both cases we observe that the agent attempts to shift power from frequencies emphasized by the regulator's filter to those that are not. Depending on the 'frequency shifting' technology the agent faces, the manipulation of the spectral properties appears to be smoothed across frequencies.

We argue that this framework helps us evaluate possible problems, from a strategic perspective, that might arise from unthinkingly basing regulation on filtered series or from a particular concentration on certain frequencies, such as an emphasis on smoothing a business cycle. In addition, the framework may eventually help provide a way of designing incentives that, rather than inducing regulated agents to reduce risk at all frequencies, can induce more targeted reduction in power at particularly salient frequencies.

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