Solving for Optimal Simple Rules in Rational Expectations Models*

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Abstract

This paper presents techniques to solve for optimal simple monetary policy rules in rational expectations models. Both the pre-commitment and the discretionary solutions are considered. The techniques described are notable for the flexibility they provide over the structure of the policy rule being solved for. Specifically, not all state variables need enter the rule. This allows rules optimal, conditional on a specified information set, or structure, to be easily constructed. The algorithms are illustrated through application to the models in Clarida, Gali, and Gertler (1999) and Rudebusch (2000).

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1) Introduction

It is widely known from Kydland and Prescott (1977), Calvo (1978), and Barro and Gordon (1983) that the issue of time-inconsistency must be confronted when considering policy in rational expectations models. In particular, Kydland and Prescott (1977) show that in models with forward-looking rational expectations, dynamic programming does not generate an optimal policy program and that optimal policy programs are unlikely to be implemented because they are not time-consistent.

In the absence of some pre-commitment technology the optimal policy program is not time-consistent. However, it is often still relevant and useful to assume that some pre-commitment technology exists and to solve for optimal pre-commitment rules (Kydland and Prescott, 1980). The optimal pre-commitment solution provides a natural benchmark against which other rules can be compared, thereby quantifying the welfare loss arising from not having a pre-commitment technology. Kydland and Prescott (1980) were the first to describe the optimal policy solution in rational expectations models, using an example of optimal taxation. However, they did not implement their solution method because they found it to be computationally infeasible at that time. Cohen and Michel (1984) analytically solved a one state variable problem before Oudiz and Sachs (1985) presented and implemented a general numerical solution method for discrete-time[1] models with a linear-quadratic objective function[2] (see also Backus and Driffill, 1986).

Around the same time, numerical methods were developed to solve for optimal time-consistent rules. The appendix to Kydland and Prescott (1977) outlines one algorithm. Further algorithms are developed in Oudiz and Sachs (1985), Backus and Driffill (1986), McKibbin and Sachs (1991), and Krusell, *et al*, (1997). A short list of applications would include McKibbin and Sachs (1988), McKibbin (1993), Svensson (1994, 2000), Krusell, *et al*, (1997), and Cooley and Quadrini (1999), who collectively examine issues as diverse as economic growth, monetary policy rules, and exchange rate regimes.

[1] See Currie and Levine (1985) for the solution in continuous-time models.

In many cases it is of interest to solve for optimal pre-commitment and optimal time-consistent (discretionary) rules. However, in the monetary policy literature there is a long and well established tradition of examining simple rules and *optimal* simple rules (Bryant, *et al*, 1993), in part driven by the fact that empirically based policy models often contain large numbers of state variables. The curse of dimensionality can then make solving for optimal pre-commitment rules or optimal discretionary rules infeasible. And where optimal rules can be solved they are often large and unwieldy, doing little to aid transparency and the communication of policy. Optimal simple rules place precise restrictions on the information set and structure of the policy rule and optimize over the remaining free parameters. Examples include optimal Henderson-McKibbin rules (Henderson and McKibbin, 1993) and optimal Taylor type rules (Taylor, 1993). One of the aims of the optimal simple rules literature is to construct rules that contain just a few variables and whose performance is comparable to the optimal policy rule. A natural way to construct these optimal simple rules is to begin with the optimal policy rule and to then eliminate state variables whose contribution to optimal policy is 'small.'

One technique for solving for optimal simple pre-commitment rules is sketched in Oudiz and Sachs (1985). Soderlind (1999) describes the solution method in detail. Applications include Fuhrer (1997), Batini and Haldane (1999), Rudebusch (2000), Dennis (2000), Leitemo and Soderstrom (2001), and Batini, *et al*, (2001), all of whom explore optimized rules for setting short-term nominal interest rates. It is important, however, to also be able to solve for optimal simple rules under discretion. In this paper we address this issue and present a numerical method that solves for optimal simple discretionary rules. Being able to solve for optimal simple discretionary rules allows us to quantify the benefits of pre-commitment in the context of specific policy rules. Further, it facilitates construction of high performance simple (discretionary) rules by sequentially omitting state variable from the optimal discretionary rule. Moreover, unless one is prepared to impose some pre-commitment mechanism the discretionary solution is the natural case to focus on.

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[2] Applications include Lansing (1998), who examines optimal taxation, and King and Wolman (1999) and Khan, *et al*, (2000) who investigate optimal monetary policy.

We begin in Section 2 by introducing the class of economic models to which the algorithms we develop can be applied. This class is relatively broad, encompassing models with lots of lagged variables as well as models with expectations of future variables based on different information sets. Section 2 also presents the form of the policy objective function. In Section 3 we describe how to solve for optimal simple pre-commitment rules. The pre-commitment algorithm we present parallels Soderlind (1999), but uses a different representation of the stochastic dynamic policy constraints. Presenting the pre-commitment solution allows us to build up the apparatus needed to solve for optimal simple discretionary rules, and makes clear how the pre-commitment and discretionary solutions differ.

With the pre-commitment solution behind us, Section 4 turns to the discretionary case. The basic solution approach is to distinguish the policy rule to be applied today from that to be applied in the future, thereby building in the condition that today's policy rule does not constrain future policymakers. The value function is expressed as a function of both the future rule and today's rule and then optimized with respect to the coefficients in today's rule. We then iterate 'backward through time' to solve for the perfect Nash equilibrium.

In Section 5 the techniques developed in Sections 3 and 4 are applied to two benchmark models drawn from the literature. We numerically solve for optimal discretionary rules in the Clarida, *et al*, (1999) model and replicate their results, demonstrating that the proposed algorithm can solve for optimal discretionary rules as well as optimal *simple* discretionary rules. We then go on to solve for optimal simple discretionary rules in Rudebusch's (2000) model. Section 6 concludes. An Appendix shows how objective functions based on linear combinations of unconditional variances (that are very common in the monetary policy rules literature) relate to the discounted quadratic objective function that is more standard elsewhere in the control literature (see also Svensson, 1998, Appendix E).

2) The General Setup

Consider the following stochastic dynamic macroeconomic structure

$$\mathbf{A_0 y_t} = \mathbf{A_1 y_{t-1}} + \mathbf{A}_2 \mathrm{E}_t \mathbf{y_{t+1}} + \mathbf{A}_3 \mathbf{x_t} + \mathbf{v_t}, \qquad \mathbf{v_t} \sim \mathrm{iid}[\mathbf{0},\Sigma] \qquad (1)$$

where $\mathbf{y_t}$ is an n×1 vector of jump and predetermined variables, $\mathbf{x_t}$ is an p×1 vector of policy instruments, and $\mathbf{v_t}$ is an n×1 vector of stochastic innovations. $\mathbf{A_0}$, $\mathbf{A_1}$, $\mathbf{A_2}$, and $\mathbf{A_3}$ are matrices of policy- and time-invariant coefficients. The variance-covariance matrix Σ will in general be singular. Finally, E_t is the mathematical expectation operator conditional upon period t information, $I_t = \{\mathbf{y_t}, I_{t-1}\}$, and the model's structure and parameters are assumed known.

Equation (1) is more general than may first appear. Systems with more general lead and lag structures in $\mathbf{y_t}$ and $\mathbf{x_t}$ can be manipulated into the form (1). Moreover, by redefining variables, expanding the state vector, and exploiting the law of iterated expectations, expectations of current and/or future variables conditional on period t-s (s > 0) information are also possible (Binder and Pesaran, 1995). We provide an example of this in Section 5.2 when we solve for optimal simple policy rules in Rudebusch's (2000) model.

Let $\mathbf{z_t} = [\mathbf{y_t}^T \ \mathbf{x_t}^T]^T$, then the central bank's policy objective function is taken to be

$$L_t = E_t \sum_{i=0}^{\infty} \beta^i [\mathbf{z}_{\mathbf{t+i}}^T \mathbf{W} \mathbf{z}_{\mathbf{t+i}}] \qquad 0 < \beta < 1. \qquad (2)$$

where $\mathbf{W}$ is a symmetric, positive semi-definite, time-invariant, matrix of known policy weights. Equation (2) can be written recursively as

$$L_t = \mathbf{z}_\mathbf{t}^T \mathbf{W} \mathbf{z}_\mathbf{t} + \beta E_t L_{t+1} \qquad 0 < \beta < 1. \qquad (3)$$

It is well known (Sargent, 1987, chapter one) that the value function for infinite horizon linear-quadratic problems takes the form $L_t = \mathbf{z}_t^T \mathbf{P} \mathbf{z}_t + d$, which allows (3) to be re-expressed as

$$\mathbf{z}_t^T \mathbf{P} \mathbf{z}_t + d = \mathbf{z}_t^T \mathbf{W} \mathbf{z}_t + \beta E_t[\mathbf{z}_{t+1}^T \mathbf{P} \mathbf{z}_{t+1} + d] \qquad 0 < \beta < 1. \tag{4}$$

To continue further requires knowing the recursive equilibrium law of motion describing how the economy evolves over time. But the economy's evolution through time depends on whether the policymaker can pre-commit to a policy strategy or not. We begin by discussing optimal simple pre-commitment rules in Section 3 before turning to optimal simple discretionary rules in Section 4.

3) Optimal Simple Pre-commitment Rules

The information available to the policymaker when it sets policy is contained in the vector of predetermined variables, $\mathbf{y}_{t-1}$, and the vector of innovations, $\mathbf{v}_t$. Accordingly it is these variables that form the basis of the policy rule. Excluding $\mathbf{v}_t$ from the rule imposes the restriction that the rule is formed using period t-1 information. With simple rules some state variables and/or innovation terms are omitted from the rule, or otherwise constrained. Alternatively, at the expense of elements in $\mathbf{y}_{t-1}$ and $\mathbf{v}_t$ expectations of future variables can be built into the reaction function.[3]

Let the policy rule take the following general linear form

$$\mathbf{x}_t = \varphi_1 \mathbf{y}_{t-1} + \varphi_2 E_t \mathbf{y}_{t+1} + \varphi_3 \mathbf{v}_t, \tag{5}$$

[3] Note that additional information can only be obtained from expected future variables if some elements in $\mathbf{y}_{t-1}$ or $\mathbf{v}_t$ are directly excluded from the rule. In which case placing expected future variables in the policy rule amounts to an indirect way of accessing this information.

where φ_1, φ_2, and φ_3, are feedback coefficient matrices whose values have yet to be determined. Elements in these coefficient matrices can be restricted to zero or constrained in other ways. We are looking for a stationary policy rule in which the feedback matrices are time invariant.

Augmenting equation (1) with equation (5) gives

$$\begin{bmatrix} \mathbf{A_0} & -\mathbf{A}_3 \\ \mathbf{0}_{\mathbf{p\times n}} & \mathbf{I}_\mathbf{p} \end{bmatrix}\begin{bmatrix} \mathbf{y_t} \\ \mathbf{x_t} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0}_{\mathbf{n\times p}} \\ \varphi_1 & \mathbf{0}_{\mathbf{p\times p}} \end{bmatrix}\begin{bmatrix} \mathbf{y_{t-1}} \\ \mathbf{x_{t-1}} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_2 & \mathbf{0}_{\mathbf{n\times p}} \\ \varphi_2 & \mathbf{0}_{\mathbf{p\times p}} \end{bmatrix} \mathrm{E}_t \begin{bmatrix} \mathbf{y_{t+1}} \\ \mathbf{x_{t+1}} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & \mathbf{0}_{\mathbf{n\times p}} \\ \varphi_3 & \mathbf{0}_{\mathbf{p\times p}} \end{bmatrix}\begin{bmatrix} \mathbf{v_t} \\ \mathbf{0} \end{bmatrix},$$

which in terms of $\mathbf{z_t}$ can be written as

$$\mathbf{B_0 z_t} = \mathbf{B_1 z_{t-1}} + \mathbf{B_2 E_t z_{t+1}} + \mathbf{B_4 u_t}, \qquad (6)$$

where $\mathbf{u_t} = [\mathbf{v_t}^T\ \mathbf{0}^T]^T$ has variance-covariance matrix Φ, and $\mathbf{B_0}$, $\mathbf{B_1}$, $\mathbf{B_2}$, and $\mathbf{B_4}$ are defined as $\begin{bmatrix} \mathbf{A_0} & -\mathbf{A}_3 \\ \mathbf{0}_{\mathbf{p\times n}} & \mathbf{I}_\mathbf{p} \end{bmatrix}$, $\begin{bmatrix} \mathbf{A}_1 & \mathbf{0}_{\mathbf{n\times p}} \\ \varphi_1 & \mathbf{0}_{\mathbf{p\times p}} \end{bmatrix}$, $\begin{bmatrix} \mathbf{A}_2 & \mathbf{0}_{\mathbf{n\times p}} \\ \varphi_2 & \mathbf{0}_{\mathbf{p\times p}} \end{bmatrix}$, and $\begin{bmatrix} \mathbf{I} & \mathbf{0}_{\mathbf{n\times p}} \\ \varphi_3 & \mathbf{0}_{\mathbf{p\times p}} \end{bmatrix}$ respectively. Equation (6) can be rearranged into first order form

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{bmatrix}\begin{bmatrix} \mathbf{z_t} \\ \mathrm{E}_t \mathbf{z_{t+1}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{B}_1 & \mathbf{B_0} \end{bmatrix}\begin{bmatrix} \mathbf{z_{t-1}} \\ \mathbf{z_t} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_4 & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{u_t} \\ \mathbf{0} \end{bmatrix}. \qquad (7)$$

Equation (7) cannot be solved directly using the methods in Blanchard and Kahn (1980) because $\mathbf{B_2}$ is singular. However, it can be solved using many other methods, such as those described Anderson and Moore (1985), Binder and Pesaran (1995), Sims (1996), King and Watson (1998), McCallum, (1983, 1999), Uhlig, (1999), and Klein (2000). Provided coefficient feedback matrices φ_1, φ_2, and φ_3, exist such that the number of eigenvalues in the system with modulus greater than one equals the number of jump variables in $\mathbf{z_t}$, then the system has a unique stable solution. The solution to the rational expectations model (7) gives equations describing how the predetermined variables evolve over time and how the jump variables respond to the predetermined variables and the shocks. These equations can be used to show (see Uhlig, 1999, equation 3.15 for example) that $\mathbf{z_t}$ evolves according to[4]

$$\mathbf{z_t} = \boldsymbol{\theta}_1 \mathbf{z}_{t-1} + \boldsymbol{\theta}_2 \mathbf{u_t} \tag{8}$$

From equation (8) the evolution of the system is determined as an implicit function of the policy feedback coefficients $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$, and $\boldsymbol{\varphi}_3$. Now advancing (8) one time period and substituting into (4) gives

$$\mathbf{z_t}^{\mathrm{T}} \mathbf{P} \mathbf{z_t} + \mathrm{d} = \mathbf{z_t}^{\mathrm{T}} \mathbf{W} \mathbf{z_t} + \beta \mathrm{E}_t [(\boldsymbol{\theta}_1 \mathbf{z_t} + \boldsymbol{\theta}_2 \mathbf{u_t})^{\mathrm{T}} \mathbf{P} (\boldsymbol{\theta}_1 \mathbf{z_t} + \boldsymbol{\theta}_2 \mathbf{u_t}) + \mathrm{d}], \tag{9}$$

where $\mathbf{z_t}^{\mathrm{T}} \mathbf{P} \mathbf{z_t} + \mathrm{d}$ gives the value of the loss function for given coefficient matrices $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$, and $\boldsymbol{\varphi}_3$. Equation (9) holds for all realizations of $\mathbf{z_t}$ and $\mathbf{u_t}$. Thus the value of the loss function for a given policy rule is

$$\mathrm{L}_t = \mathbf{z_t}^{\mathrm{T}} \mathbf{P} \mathbf{z_t} + \beta(1-\beta)^{-1} \mathrm{tr}[\boldsymbol{\theta}_2^{\mathrm{T}} \mathbf{P} \boldsymbol{\theta}_2 \boldsymbol{\Phi}], \tag{10}$$

where $\mathbf{P}$ is the fixed point of

$$\mathbf{P} = \mathbf{W} + \beta \boldsymbol{\theta}_1^{\mathbf{T}} \mathbf{P} \boldsymbol{\theta}_1. \tag{11}$$

By construction the spectral radius of θ_1 is less than one and the recursive equilibrium law of motion, equation (8), is stable. With $\mathbf{W}$ known, θ_1 and θ_2 known conditional on $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$, and $\boldsymbol{\varphi}_3$, $0 < \beta < 1$, and the spectral radius of θ_1 less than one, standard fixed point solution methods can be applied to (11) to solve for $\mathbf{P}$. Equation (10) can then be evaluated and optimized with respect to the policy feedback coefficients $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$, and $\boldsymbol{\varphi}_3$.

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[4] Notice that the variables in $\mathbf{z_{t-1}}$ are predetermined. However, it is not necessarily the case that all variables in $\mathbf{z_t}$ are jump variables. Those variables in $\mathbf{z_t}$ that are free to jump will be related to

Computationally the solution procedure is as follows. First values for φ_1, φ_2, and φ_3, are postulated such that the rational expectations solution to equation (7) is unique and stable. The rational expectations model, equation (7), is solved using methods such as Klein (2000) with the solution to the system given by (8). We then solve[5] for the fixed point of equation (11) and substitute this value of **P** into (10), which is now an implicit function of φ_1, φ_2, and φ_3. Equation (10) is then optimized with respect to φ_1, φ_2, and φ_3. In general, the feedback coefficients in the optimal simple pre-commitment rule depend on the initial state of the economy, through $\mathbf{z}_t^T \mathbf{P} \mathbf{z}_t$ in (10), and on the variance-covariance matrix of the shocks, through Φ in (10).

Finally, the unconditional variance-covariance matrix for $\mathbf{z}_t$, Ω, can be obtained by applying standard fixed-point solution techniques to

$$\boldsymbol{\Omega} = \boldsymbol{\theta}_1 \boldsymbol{\Omega} \boldsymbol{\theta}_1^T + \boldsymbol{\theta}_2 \boldsymbol{\Phi} \boldsymbol{\theta}_2^T .$$

4) Optimal Simple Discretionary Rules

Under discretion the central bank cannot commit itself to apply the same policy rule in the future that it applies today. As time passes opportunities for the central bank to re-optimize arise and the policy rule implemented today, or in the past, do not constrain future policy decisions. To capture these ideas we draw a distinction between the rule the central bank proposes to follow today and that it proposes to follow in the future. Distinguishing between today's rule and the future rule captures the notion that today's

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$\mathbf{z}_{t-1}$ and $\mathbf{u}_t$ through the solution to the RE model. By construction, predetermined variables in $\mathbf{z}_t$ will be related to $\mathbf{z}_{t-1}$ through identities.

[5] There are several ways to go about solving for the fixed point of equation (11). One approach is to make an initial guess of **P**, $\mathbf{P}_0$, (using $\mathbf{P}_0 = \mathbf{W}$ is a useful choice) and then iterate over (11) until convergence, exploiting the properties of the Contraction Mapping Theorem. A further approach is to solve for the fixed point directly using methods like Guass-Seidel. Alternatively, we can recognize that equation (11) is an example of a Sylvester Equation and exploit the literature for solving such equations (See Golub and Van Loan, 1996, pp367-369).

policymaker cannot tie the hands of future policymakers. Accordingly we set up today's rule as

$$\mathbf{x_t} = \varphi_1 \mathbf{y_{t-1}} + \varphi_2 \mathrm{E}_t \mathbf{y_{t+1}} + \varphi_3 \mathbf{v_t}, \tag{12}$$

and the future rule as[6]

$$\mathbf{x_{t+j}} = \boldsymbol{\psi}_1 \mathbf{y_{t+j-1}} + \boldsymbol{\psi}_2 \mathrm{E}_{t+j} \mathbf{y_{t+j+1}} + \boldsymbol{\psi}_3 \mathbf{v_{t+j}} \qquad \forall\ j > 0. \tag{13}$$

Several aspects of (12) and (13) require clarification. First, the structure of the rule cannot change over time, so constraints on φ_1, φ_2, and φ_3 must be matched with identical constraints on ψ_1, ψ_2, and ψ_3. This prevents the central bank from proposing to follow a nominal GDP growth targeting rule in the future while implementing a Taylor type rule today (for example). Second, an important special case of (12) and (13) is that where policy responds to every predetermined variable and every shock. In this case expected future variables contain no additional information and φ_2 and ψ_2 would naturally be restricted to equal null matrices. The special case identified relates to the optimal discretionary rule, illustrating that the methods developed below can be used to solve for optimal discretionary rules as well as optimal *simple* discretionary rules. We demonstrate this is Section 5.1 were we solve for the optimal discretionary state-contingent rule in the Clarida, Gali, and Gertler (1999) model. Third, because the two rules are state-contingent they describe how policy will be set in any given state, regardless of how the economy arrived at that state. In particular, equation (13) describes how future policy will be set, given the state, regardless of whether the economy arrived at that state along an equilibrium path or not.

[6] Assuming the same policy rule for all future periods is without loss of generality. The optimization problem is time-invariant and consequently in equilibrium all policymakers end up applying the same policy rule. In solving for this equilibrium policy rule the essential property to capture is that policy decisions made today do not constrain future policymakers. Distinguishing between today's rule and the future rule captures this property.

Before moving on to the solution algorithm itself, we note that although today's policymaker sets policy for today with the understanding that it cannot constrain future policymakers' behavior, it still cares and accounts for the consequences of its decisions in all future periods. Thus *discretionary policy is not to be confused with myopic policy*, in which a policymaker only accounts for the losses that occur over the period for which that policymaker's rule is applied.

4.1) Solving the System

The optimization problem is to find the rule that minimizes the objective function, (2), subject to equation (1), and the requirement that no policymaker has an incentive to depart from this rule as time passes and opportunities to re-optimize arrive. Thus the rule solved for is the outcome of a perfect Nash equilibrium in the game played between current and future policymakers; it is therefore time-consistent. We begin by considering period t+1, solve for the period t optimal policy rule conditional on the policy rule followed in periods t+1 onward, and then iterate 'backward through time' to reach a fixed point. Following Section 2, advance the time subscript on (1) and augment (1) with (13), with j = 1, giving

$$\mathbf{B}_0^*\mathbf{z}_{t+1} = \mathbf{B}_1^*\mathbf{z}_t + \mathbf{B}_2^*\mathrm{E}_{t+1}\mathbf{z}_{t+2} + \mathbf{B}_4^*\mathbf{u}_{t+1}, \qquad \mathbf{u}_{t+1} \sim \mathrm{iid}[\mathbf{0},\Phi] \tag{14}$$

where $\mathbf{B}_0^* = \begin{bmatrix} \mathbf{A}_0 & -\mathbf{A}_3 \\ \mathbf{0}_{p\times n} & \mathbf{I}_p \end{bmatrix}$, $\mathbf{B}_1^* = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0}_{n\times p} \\ \psi_3 & \mathbf{0}_{p\times p} \end{bmatrix}$, $\mathbf{B}_2^* = \begin{bmatrix} \mathbf{A}_2 & \mathbf{0}_{n\times p} \\ \psi_2 & \mathbf{0}_{p\times p} \end{bmatrix}$, and $\mathbf{B}_4^* = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times p} \\ \psi_3 & \mathbf{0}_{p\times p} \end{bmatrix}$.

The variance-covariance matrix of $\mathbf{u}_{t+t} = [\mathbf{v}_{t+1}^{\mathrm{T}}\ \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}$, Φ, is directly and clearly related to the variance-covariance matrix of $\mathbf{v}_{t+1}$, Σ. Continuing to follow Section 2, equation (14), can be expressed in first order form

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2^* \end{bmatrix}\begin{bmatrix} \mathbf{z}_{t+1} \\ \mathrm{E}_{t+1}\mathbf{z}_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{B}_1^* & \mathbf{B}_0^* \end{bmatrix}\begin{bmatrix} \mathbf{z}_t \\ \mathbf{z}_{t+1} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_4^* & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{u}_{t+1} \\ \mathbf{0} \end{bmatrix}.$$

Provided matrices ψ_1, ψ_2, and ψ_3 exist such that the system has the same number of eigenvalues with modulus greater than one as there are jump variables in $\mathbf{z_{t+1}}$, the system has a unique stable solution that, as earlier, can be represented as

$$\mathbf{z_{t+1}} = \boldsymbol{\theta}_1^* \mathbf{z_t} + \boldsymbol{\theta}_2^* \mathbf{u_{t+1}}, \tag{15}$$

where θ_1^* and θ_2^* are functions of ψ_1, ψ_2, and ψ_3. By construction, the spectral radius of θ_1^* is less than one. Thus far the solution procedure has followed step-by-step that used to solve for optimal simple pre-commitment rules, albeit applied to period t+1 onward rather than period t onward. It is at this point that the discretionary solution algorithm departs from the pre-commitment solution algorithm. Take the mathematical expectation of equation (15) conditional on period t information, and substitute the resulting expression into equation (1). Provided $[\mathbf{B_0} - \mathbf{B}_3\boldsymbol{\theta}_1^*]$ has full rank this results in

$$\mathbf{z_t} = [\mathbf{B_0} - \mathbf{B}_3\boldsymbol{\theta}_1^*]^{-1}[\mathbf{B}_2\mathbf{z_{t-1}} + \mathbf{B}_4\mathbf{u_t}] \stackrel{\text{def}}{=} \boldsymbol{\theta}_1\mathbf{z_{t-1}} + \boldsymbol{\theta}_2\mathbf{u_{t+1}}. \tag{16}$$

The recursive equilibrium law of motion, equation (16), is a function of future policy through θ_1^* and today's policy through $\mathbf{B_2}$, $\mathbf{B_1}$, and $\mathbf{B_4}$. Advancing equation (16) one time period and substituting into (4) gives the policy loss conditional on the proposed policy rules

$$L_t = \mathbf{z_t^T P z_t} + \beta(1-\beta)^{-1}\mathrm{tr}[\boldsymbol{\theta}_2^T \mathbf{P} \boldsymbol{\theta}_2 \boldsymbol{\Phi}], \tag{17}$$

where $\mathbf{P}$ is again the fixed point of

$$\mathbf{P} = \mathbf{W} + \beta\boldsymbol{\theta}_1^T \mathbf{P} \boldsymbol{\theta}_1.$$

Equation (17) is a function of the feedback parameters in the future policy rule as well as those in today's policy rule. The procedure is to now differentiate equation (17) with respect to φ_1, φ_2, and φ_3 while holding ψ_1, ψ_2, and ψ_3 constant, then set $\varphi_1 = \psi_1$, $\varphi_2 = \psi_2$, $\varphi_3 = \psi_3$ and solve for the fixed-point of the first order equation. Notice, however, that, in general, the feedback coefficients in the optimal simple discretionary rule depend on the initial state of the economy and on the variance-covariance matrix of the stochastic innovations.

Computationally, this fixed point can be solved for as follows. Given values of φ_1, φ_2, φ_3, ψ_1, ψ_2, and ψ_3 that satisfy the necessary uniqueness and stability conditions required to construct θ_1, θ_2, and **P**, equation (17) can easily be evaluated and then numerically optimized with respect to φ_1, φ_2, and φ_3 holding ψ_1, ψ_2, and ψ_3 constant. Recognizing that the future policymaker faces the same optimization problem as the current policymaker we can use the newly optimized values of φ_1, φ_2, and φ_3 as revised guesses at the feedback coefficients in the future policy rule. That is we set $\psi_1 = \varphi_1$, $\psi_2 = \varphi_2$, and $\psi_3 = \varphi_3$, and with this new guess at the future policy reaction function re-solve the future period rational expectations model to obtain updated values of θ_1^* and θ_2^*. A new recursive equilibrium law of motion, equation (16), is obtained and (17) is again optimized with respect to φ_1, φ_2, and φ_3 while again holding ψ_1, ψ_2, and ψ_3, constant. This iterative procedure continues until a fixed point is obtained in which the newly optimized feedback parameters in today's policy rule equal those proposed in the future policy rule: $\varphi_1 = \psi_1$, $\varphi_2 = \psi_2$, and $\varphi_3 = \psi_3$. The resulting fixed point is a perfect Nash equilibrium and the policy rule is the optimal (simple) time-consistent (discretionary) rule. This solution procedure is illustrated through application in Sections 5.1 and 5.2.

As with the pre-commitment solution, the unconditional variance-covariance matrix of $\mathbf{z_t}$, Ω, is easily obtained by solving for the fixed point of

$$\mathbf{\Omega} = \boldsymbol{\theta}_1 \mathbf{\Omega} \boldsymbol{\theta}_1^{\mathrm{T}} + \boldsymbol{\theta}_2 \mathbf{\Phi} \boldsymbol{\theta}_2^{\mathrm{T}}. \tag{18}$$

Note that in the perfect Nash equilibrium $\theta_1 = \theta_1^*$, and by construction the spectral radius of θ_1^* is less than one. Consequently Ω can be solved from equation (18) using standard fixed-point solution methods.

5) Two Applications

In this Section we apply the algorithms developed in Sections 3 and 4 to two models drawn from the literature. The first model that we apply the algorithms to is that in Clarida, Gali, and Gertler (1999) (CGG). The CGG model is very useful as a testing ground because the model is simple, forward-looking, and contains just two state variables. Moreover, the model is simple enough for analytic solutions to be obtained making it particularly easy to test the algorithms. We numerically solve for the optimal discretionary rule and the optimal state-contingent pre-commitment rule. This latter rule assumes pre-commitment, and contains information from both state variables, but excludes the Lagrange multiplier terms that would enter the rule if we were to solve for the optimal pre-commitment rule.

The second model that we consider is that in Rudebusch (2000). Rudebusch's model contains eleven independent state variables making it a slightly more challenging numerical problem. We consider optimal simple Taylor type rules and optimal simple nominal GDP growth rules under both pre-commitment and discretion. The solution for the optimal simple Taylor type rule under pre-commitment for an objective function without discounting is presented in Rudebusch (2000).

5.1) Example One – Clarida, Gali, and Gertler (1999)

Our example is taken from Clarida, Gali and Gertler's (1999) *Journal of Economic Literature* paper. CGG's analysis is theoretical, but we take their New Keynesian model and parameterize it for simulation purposes. They provide analytic solutions that we use to verify our results with theirs. Their model has two key equations: those for the output gap and inflation. Both demand and supply shocks are persistent, modeled as simple

AR(1) processes. The model also has a policy reaction function determined optimally, so the complete system has five equations. The system is

$$y_t = E_t y_{t+1} - \gamma[i_t - E_t \pi_{t+1}] + g_t \qquad \gamma > 0$$

$$\pi_t = \beta E_t \pi_{t+1} + \lambda y_t + u_t \qquad 0 < \beta < 1, \lambda > 0$$

where y_t is the output gap, π_t is inflation, g_t is a demand shock, and u_t is a supply shock. The demand and supply shocks are modeled respectively as

$$g_t = \mu g_{t-1} + \varepsilon_t \qquad 0 \le \mu < 1$$

$$u_t = \rho u_{t-1} + v_t . \qquad 0 \le \rho < 1$$

In matrix notation the CGG model can be written

$$\begin{bmatrix} 1 & 0 & \gamma \\ -\lambda & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & \beta & 0 \\ 0 & \varphi_1 & 0 \end{bmatrix} E_t \begin{bmatrix} y_{t+1} \\ \pi_{t+1} \\ i_{t+1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varphi_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_t \\ u_t \\ 0 \end{bmatrix}, \qquad (19)$$

with the error vector driven by

$$\begin{bmatrix} g_t \\ u_t \\ 0 \end{bmatrix} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_{t-1} \\ u_{t-1} \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ v_t \\ 0 \end{bmatrix}.$$

As indicated in (19) the monetary authority sets the level of the nominal interest rate as a linear function of the demand disturbance g_t and expected future inflation $E_t\pi_{t+1}$. This rule makes use of all information in the state vector and is therefore the optimal discretionary rule.[7] The policy objective function is

[7] In general we do not recommend solving for optimal discretionary rules using the techniques developed in this paper. The solution methods developed in this paper rely on numerical

$$L_t = (1-\beta)E_t \sum_{i=0}^{\infty} \beta^i [(1-\alpha)(\pi_{t+i})^2 + \alpha(y_{t+i})^2] \qquad 0 \le \alpha < 1.$$

The scalar (1-β) outside the conditional expectation simply re-normalizes the objective function without changing the optimal policy. Provided the system is stabilizable, in the limit as $\beta \to 1$ this objective function tends to a linear combination of the unconditional variances of inflation and the output gap (see the Appendix). CGG's model assumes that period t expectations are formed, and period t policy decisions made, with all agents knowing the structure of the economy and aware of all variables dated period t or earlier. Thus policymakers know the demand and supply shocks before they set policy.[8]

Because demand shocks move output and inflation pro-cyclically, and they are observed, it is always optimal for policymakers to eliminate the influence of these demand shocks from the system – regardless of policy preferences. Consequently the coefficient applied to the demand disturbance in the optimal policy reaction function is invariant to the weight placed on output in the policy objective function, α.

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derivatives and are better suited for solving for optimal *simple* discretionary rules, which typically contain just a few feedback coefficients. To solve for optimal discretionary rules better numerical accuracy can be obtained using methods such as those in Oudiz and Sachs (1985), Backus and Driffill (1986), or Soderlind (1999), which make full use of analytic derivatives.

[8] To simulate the model we set: β = 0.99; γ = 0.8; λ = 0.4; μ = ρ = 0.5; σ_y = σ_v = 1; and $\sigma_{y\pi}$ = 0.

Table One: Clarida-Gali-Gertler (1999) Model						
Solution under Discretion						
	Feedback Coefficients		**Standard Deviations %**			
1-α	**g_t**	**$E_t\pi_{t+1}$**	**y_t**	**π_t**	**i_t**	**Loss**
0.00	1.250	1.000	0.000	2.287	1.841	0.000
0.20	1.250	1.125	0.212	2.119	1.872	0.921
0.40	1.250	1.333	0.503	1.888	1.915	1.557
0.60	1.250	1.750	0.930	1.550	1.981	1.764
0.80	1.250	3.000	1.614	1.008	2.091	1.317
1.00	1.250	∞	2.887	0.000	2.311	0.000
Solution under Pre-commitment						
	Feedback coefficients		**Standard Deviations %**			
1-α	**g_t**	**$E_t\pi_{t+1}$**	**y_t**	**π_t**	**i_t**	**Loss**
0.00	1.250	1.000	0.000	2.287	1.841	0.000
0.20	1.250	1.248	0.391	1.977	1.898	0.892
0.40	1.250	1.660	0.851	1.612	1.968	1.455
0.60	1.250	2.485	1.400	1.178	2.056	1.595
0.80	1.250	4.960	2.064	0.652	2.167	1.176
1.00	1.250	∞	2.887	0.000	2.311	0.000

Table One presents the optimal policy reaction functions for a range of values for α. In solving for these policy rules we assumed that in the initial period the economy was at steady state. The upper panel corresponds to discretion, the lower panel to pre-commitment. As expected the coefficient applied to the demand disturbance, g_t, is invariant to α for both pre-commitment and discretion, and its value serves to eliminate the effects of the demand shock from the system. Also observe that in the extreme cases where $\alpha = 0, 1$ all volatility in inflation and output (respectively) is completely eliminated. This is a consequence of agents and policymakers knowing the disturbances prior to making their decisions.

Unsurprisingly, in the cases where $\alpha = 0, 1$ the optimal pre-commitment and the optimal discretionary solutions coincide. In these two special cases the problem collapses to that where there is one instrument matched against a single policy goal. With one instrument and one goal the system is controllable, ruling out time-inconsistent behavior. Outside these two extreme cases, in which the system is controllable, the loss under discretion is as much as 12% ($\alpha = 0.2$) more than the loss under pre-commitment.[9]

CGG show analytically that with discretion the solution to the system is[10] (using our parameterization of the objective function)

$$y_t = \frac{-\lambda(1-\alpha)}{(1-\alpha)\lambda^2 + \alpha(1-\beta\rho)} u_t \tag{20}$$

$$\pi_t = \frac{\alpha}{(1-\alpha)\lambda^2 + \alpha(1-\beta\rho)} u_t \tag{21}$$

$$i_t = \left[1 + \frac{(1-\alpha)(1-\rho)\lambda}{\alpha\rho\gamma}\right] E_t \pi_{t+1} + \frac{1}{\gamma} g_t \tag{22}$$

With our assigned parameter values in equations (20) - (22), varying α generates results identical to those obtained numerically in the upper panel of Table One. Note also that in the limiting case where $\alpha \to 0$ the interest rate's response to expected future inflation becomes infinite. However, even in this limiting case the policy rule can be written in terms of the state variables u_t and g_t (the α in the numerator of (21) cancels with the α in the denominator of (22)), and with the rule in this form unconditional variances for y_t, π_t, and i_t can be obtained. An identical cancellation occurs in the pre-commitment solution.

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[9] Here we are comparing the loss under the optimal discretionary rule to that under the optimal state-contingent pre-commitment rule. If we instead compared the optimal discretionary rule to the optimal pre-commitment rule, then the relative performance of the discretionary rule would be even worse.

5.2) Example Two – Rudebusch (2000)

In this subsection we examine the forward-looking macro-econometric model developed in Rudebusch (2000). The model consists of the output gap and inflation equations[11]

$$y_t = 1.15y_{t-1} - 0.27y_{t-2} - 0.09(i_{t-1} - E_{t-1}\pi^a_{t+3}) + \eta_t$$
$$\pi_t = 0.29E_{t-1}\pi^a_{t+3} + 0.71(0.67\pi_{t-1} - 0.14\pi_{t-2} + 0.40\pi_{t-3} + 0.13\pi_{t-4}) + 0.13y_{t-1} + \varepsilon_t$$

where η_t ~ iid[0, 0.694], ε_t ~ iid[0, 1.024], and $\pi^a_t = \sum_{j=0}^{3}\pi_{t-j}$. Rudebusch (2000) uses the policy objective function

$$\text{Loss}[0,\infty] = (1-\alpha)\text{Var}[y_t] + \alpha\text{Var}[\pi^a_t] + \nu\text{Var}[\Delta i_t], \quad (23)$$

focusing on the benchmark case in which $\alpha = 0.5$ and $\nu = 0.25$. The Appendix shows how this policy objective function, which is in terms of unconditional variances, relates to the discounted quadratic objective function, equation (2). We solve for optimal Taylor type rules (Taylor, 1993) and optimal nominal GDP growth rules under both pre-commitment and discretion. The form of the policy objective function, equation (23), means that the feedback coefficients in these optimal simple rules do not depend on the initial state of the economy, see the Appendix. We also solve for policy rules using the discounted objective function, equation (2), in which the output gap, annual inflation, and the change in the Federal funds rate are the target variables entering the function. In this case we assume that the economy is initially in steady state and use the same values for α and ν as above. The two rules examined take the form

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[10] The following three equations are equivalent to equations 3.3, 3.5, and 3.6 in Clarida, *et al*, (1999).

[11] Rudebusch's model can be expressed in terms of equation (1) by defining:
$\mathbf{y_t} = [\pi_t^a\ E_t\pi_{t+4}\ E_t\pi_{t+3}\ E_t\pi_{t+2}\ E_t\pi_{t+1}\ \pi_t\ \pi_{t-1}\ \pi_{t-2}\ \pi_{t-3}\ y_t\ y_{t-1}\ \Delta i_t]^T$ and $\mathbf{x_t} = i_t$.

$$i_t = \varphi_1 \pi_t^a + \varphi_2 y_t \qquad \text{(Taylor type rule)}$$

$$i_t = \varphi_3 [y_t - y_{t-1} + \pi_t] \qquad \text{(Nominal GDP growth rule)}$$

Results are presented in Table Two.

Table Two – Rudebusch (2000) Model								
Taylor type rule: α **= 0.5,** κ **= 0.25**								
Discretion		β	φ_1	φ_2	**sd[y_t]**	**sd[π_t^a]**	**sd[i_t]**	**Loss**
		1.000	2.720	1.746	1.761	1.688	4.391	3.821
		0.990	2.657	1.707	1.763	1.714	4.364	3.638
		0.975	2.565	1.649	1.767	1.753	4.326	3.382
Pre-commitment								
		1.000	2.870	1.740	1.790	1.634	4.448	3.811
		0.990	2.808	1.701	1.793	1.656	4.421	3.629
		0.975	2.717	1.644	1.797	1.691	4.382	3.374
Nominal GDP growth rule: α **= 0.5,** κ **= 0.25**								
Discretion		β	φ_3		**sd[y_t]**	**sd[π_t]**	**sd[i_t]**	**Loss**
		1.000	1.884		2.221	2.536	4.855	7.951
		0.990	1.839		2.218	2.595	4.845	7.373
		0.975	1.771		2.214	2.694	4.839	6.618
Pre-commitment								
		1.000	1.937		2.225	2.472	4.870	7.940
		0.990	1.889		2.222	2.530	4.856	7.363
		0.975	1.817		2.217	2.626	4.842	6.610

$$\text{Loss} = \begin{cases} \alpha \text{Var}(\pi_t^a) + (1-\alpha)\text{Var}(y_t) + \nu \text{Var}(\Delta i_t) & \beta = 1 \\ (1-\beta) E_t \sum_{i=0}^{\infty} \beta^i [\alpha(\pi_{t+i}^a)^2 + (1-\alpha)(y_{t+i})^2 + \nu(\Delta i_{t+i})^2] & 0 < \beta < 1 \end{cases}$$

As expected, policy under pre-commitment outperforms policy under discretion. Strikingly, however, the gains to pre-commitment are small, much smaller than those obtained for the CGG model earlier. This probably stems from the fact that Rudebusch's model is not as forward-looking as the CGG model. Finally note that the performance of the nominal GDP targeting rules is much worse than that of the Taylor type rules. They generate a good deal more volatility in annual inflation and the output gap, as well as greater interest rate volatility.

6) Conclusions

A large portion of the monetary policy rules literature examines the properties of optimal simple rules. Where these rules have been constructed and evaluated in forward-looking models it has always been assumed that the policymaker could pre-commit to the rule. This paper extends the literature on solving for optimal simple rules to the case where the policymaker cannot pre-commit. It is desirable to be able to solve for optimal simple discretionary rules for three principle reasons. First, to assess a variable's marginal contribution to optimal policy it is necessary to solve for policy rules that both include and exclude the variable in question. Second, it is informative to be able to assess the advantages to pre-commitment in the context of specific policy rules. Third, unless one is prepared to impose some pre-commitment mechanism the discretionary solution arguably would appear to be the most relevant.

Section 2 described the class of economic model to which the proposed algorithm can be applied. Section 3 showed how to solve for optimal simple pre-commitment rules by numerically choosing the feedback coefficients in the policy rule to minimize the policy objective function. Using the apparatus established for the pre-commitment case, Section 4 turned to the case of discretion. We showed that distinguishing between today's rule and the future rule, and not allowing today's policymakers to tie the hands of future policymakers, allows us to solve for optimal simple discretionary rules. We indicated how the initial state of the economy could affect the coefficients in these optimal simple rules. Furthermore, we discussed the relationship between the standard discounted quadratic objective function and the linear combination of unconditional variances most commonly used in the policy rules literature.

Section 5 applied the techniques developed in Sections 3 and 4 to two models in the literature. For the first of these models, the Clarida, *et al*, (1999) model, we used the algorithms to solve for optimal state-contingent rules and replicated their results. We further showed that the dynamic gains to pre-commitment were potentially large. For the second model, developed in Rudebusch (2000), we solved for optimal simple Taylor type rules and optimal nominal GDP growth rules and showed that Taylor types rules were vastly superior to nominal GDP growth rules and that, for these simple rules, the advantages to pre-commitment were minor. This latter funding is intriguing because it stands in sharp relief to the results obtained from the CGG model. While their convergence properties have yet to be explored in detail, the solution methods presented are simple to apply and appear to converge without difficulty.

Appendix One – On the discounted quadratic loss function as $\beta \rightarrow 1$.

In this appendix we elaborate on a result in Svensson (1998) to show how the objective function Rudebusch (2000), among many others, uses relates to the standard discounted quadratic objective function, equation (2).

Recall that under both pre-commitment and discretion the value function, with discounting, was

$$\mathrm{L}_t = \mathbf{z}_t^{\mathrm{T}}\mathbf{P}\mathbf{z}_t + \beta(1-\beta)^{-1}\mathrm{tr}[\boldsymbol{\theta}_2^{\mathrm{T}}\mathbf{P}\boldsymbol{\theta}_2\boldsymbol{\Phi}], \quad \text{(B1)}$$

with **P** the solution to

$$\mathbf{P} = \mathbf{W} + \beta\boldsymbol{\theta}_1^{\mathbf{T}}\mathbf{P}\boldsymbol{\theta}_1. \quad \text{(B2)}$$

Differences between pre-commitment and discretion arose in the specific θ_1, θ_2, and $\mathbf{P}$ matrices entering (B_1) and (B_2), but the two algorithms share (B_1) and (B_2) in their solution structure. Recall also that the unconditional variance-covariance matrix of $\mathbf{z_t}$ is the fixed point of

$$\boldsymbol{\Omega} = \boldsymbol{\theta}_1\boldsymbol{\Omega}\boldsymbol{\theta}_1^{\mathbf{T}} + \boldsymbol{\theta}_2\boldsymbol{\Phi}\boldsymbol{\theta}_2^{\mathbf{T}} \tag{B3}$$

Multiplying equation (B1) through by (1-β) gives

$$(1-\beta)L_t = (1-\beta)\mathbf{z}_\mathbf{t}^{\mathrm{T}}\mathbf{P}\mathbf{z}_\mathbf{t} + \beta\mathrm{tr}[\boldsymbol{\theta}_2^{\mathrm{T}}\mathbf{P}\boldsymbol{\theta}_2\boldsymbol{\Phi}]. \tag{B4}$$

Equation (B4) is equivalent to

$$(1-\beta)L_t = (1-\beta)\mathbf{z}_\mathbf{t}^{\mathrm{T}}\mathbf{P}\mathbf{z}_\mathbf{t} + \beta\mathrm{tr}[\mathbf{P}\boldsymbol{\theta}_2\boldsymbol{\Phi}\boldsymbol{\theta}_2^{\mathrm{T}}], \tag{B5}$$

which, employing (B3), can be re-expressed as

$$(1-\beta)L_t = (1-\beta)\mathbf{z}_\mathbf{t}^{\mathrm{T}}\mathbf{P}\mathbf{z}_\mathbf{t} + \beta\mathrm{tr}[\mathbf{P}(\boldsymbol{\Omega} - \boldsymbol{\theta}_1\boldsymbol{\Omega}\boldsymbol{\theta}_1^{\mathrm{T}})]. \tag{B6}$$

Expanding terms in equation (B6) gives

$$(1-\beta)L_t = (1-\beta)\mathbf{z}_\mathbf{t}^{\mathrm{T}}\mathbf{P}\mathbf{z}_\mathbf{t} + \beta\mathrm{tr}[\mathbf{P}\boldsymbol{\Omega}] - \beta\mathrm{tr}[\mathbf{P}\boldsymbol{\theta}_1\boldsymbol{\Omega}\boldsymbol{\theta}_1^{\mathrm{T}}]. \tag{B7}$$

Now equation (B7) is equivalent to

$$(1-\beta)L_t = (1-\beta)\mathbf{z}_\mathbf{t}^{\mathrm{T}}\mathbf{P}\mathbf{z}_\mathbf{t} + \beta\mathrm{tr}[\mathbf{P}\boldsymbol{\Omega}] - \beta\mathrm{tr}[\boldsymbol{\theta}_1^{\mathrm{T}}\mathbf{P}\boldsymbol{\theta}_1\boldsymbol{\Omega}]$$

or

$$(1-\beta)L_t = (1-\beta)\mathbf{z}_t^T\mathbf{P}\mathbf{z}_t + \beta \mathrm{tr}[(\mathbf{P} - \boldsymbol{\theta}_1^T\mathbf{P}\boldsymbol{\theta}_1)\boldsymbol{\Omega}]. \quad \text{(B8)}$$

Finally, substituting equation (B2) into equation (B8) produces

$$(1-\beta)L_t = (1-\beta)\mathbf{z}_t^T\mathbf{P}\mathbf{z}_t + \beta \mathrm{tr}[(\mathbf{W} - (1-\beta)\boldsymbol{\theta}_1^T\mathbf{P}\boldsymbol{\theta}_1)\boldsymbol{\Omega}].$$

In the limit as $\beta \to 1$, $(1-\beta)L_t \to \mathrm{tr}[\mathbf{W}\boldsymbol{\Omega}]$, provided $(1-\beta)\mathbf{z}_t^T\mathbf{P}\mathbf{z}_t \to 0$ and $(1-\beta)\boldsymbol{\theta}_1^T\mathbf{P}\boldsymbol{\theta}_1 \to 0$. These two terms converge to zero provided the solution to equation (B2) is bounded as $\beta \to 1$. It is clear that the fixed-point solution to equation (B2) will be bounded when the spectral radius of θ_1 is less than one, which from Hamiltion (1994, pp186) amounts to the requirement that when subject to control $\mathbf{z}_t$ be weakly stationary and ergodic. In control theory terms, $\mathbf{z}_t$ must be stabilizable. But the condition that the spectral radius of θ_1 be less than one arises naturally from the solution algorithms in Sections 3 and 4. Consequently, the optimization problem is well defined in the limiting case where $\beta \to 1$ and the objective function can be expressed in terms of a linear combination of unconditional variances, as per Rudebusch (2000).

Note that the term $\mathbf{z}_t^T\mathbf{P}\mathbf{z}_t$ in equation (B1) describes how the feedback coefficients in the optimal simple rule depend on the initial state vector. In the limiting case where $\beta \to 1$ the dependence of the policy rule on the initial state vector vanishes.

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