The Shape of Things in a Currency Trio

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Abstract: This article provides an example of how geometric concepts can help visualize and interpret the sometimes complex relations between financial variables. We illustrate the power and the elegance of the geometric approach to statistical concepts in finance by analyzing the volatilities and correlations in a currency trio, i.e., a set of three currencies. We expand previous work in this area by providing further insight into the relationship between volatilities and correlations in a currency trio and by analyzing differences in the correlation structure across currency trios and over time. We also present a graphical method for comparing the predictive ability of correlation forecasts from several competing models. The geometric approach towards analyzing correlation structures and correlation forecasts may be particularly helpful for financial institutions. As these institutions survive on their ability to react to the massive amount of data generated by financial markets and management information systems, they can take advantage of the human capability to instantaneously understand pictures by transforming such data into graphics.

Key Words: Foreign exchange rates, Volatility, Correlation, Options

Acknowlegements: The views expressed here are those of the authors and do not necessarily reflect those of the Credit Suisse Group, the Federal Reserve Bank of San Francisco, or the Federal Reserve System.

1. Introduction

The power and elegance of a geometric approach to statistical concepts has proven helpful in various areas of modern financial management, such as illustrating the trade-off between a portfolio's risk and return in the mean-variance framework pioneered by Markowitz (1952) or visualizing the effects of individual trades on a portfolio's risk, as per Litterman (1997). In this article, we expand the work of Zerolis (1996) and Singer, Terhaar, and Zerolis (1998) on the geometric interpretation of volatilities and correlations in a currency trio, which is simply a set of three currencies. First, we briefly review why the volatility and correlation structure of a currency trio may be represented by a triangle. Second, we present, both analytically and geometrically, the interrelationships between these volatilities and correlations. Third, we provide further insight into the geometric interpretation of correlations by mapping them onto the set of possible valid combinations in a 3-by-3 correlation matrix. Finally, we use these geometric tools to examine the behavior of several volatility and correlation forecasts; specifically, we examine forecasts as implied from over-the-counter option prices and from a standard GARCH model.

2. Representing volatilities and correlations in a currency trio as a triangle

In the absence of arbitrage opportunities in the foreign exchange market, the volatility of a cross-rate – defined as the standard deviation of the relative change in the exchange rate between two non-US dollar currencies – is related to the volatilities of the two US dollar exchange rates and their correlation by the following equation:

$$\sigma(A_B)^2 = \sigma(A_{USD})^2 + \sigma(B_{USD})^2 - 2\,\rho(A_{USD}, B_{USD})\,\sigma(A_{USD})\,\sigma(B_{USD}),$$

where A_{USD} and B_{USD} denote the US dollar exchange rates of currencies A and B, respectively, A_B is the cross-rate, $\sigma(X_Y)$ denotes the standard deviation of the relative change in the exchange rate between currencies X and Y, and $\rho(X_Y, Z_Y)$ is the correlation between the relative changes of the exchange rates X_Y and Z_Y. Since any of the three currencies may serve as the "base-currency", we can use the law of cosine in trigonometry, which has an identical structure to this equation, to visualize the relationships between these volatilities and correlations as a triangle.[1]

To illustrate these relationships, consider the currency trio consisting of the US dollar (USD), the German mark (DEM), and the Japanese yen (JPY). Exhibit 1 gives a graphical representation of the volatilities and correlations that were realized over the 3 months prior to September 8, 1998.[2] The length of a given side is equivalent to the volatility of the exchange rate between the two currencies indicated at the side's endpoints. The cosine of a given angle is equivalent to the correlation between the two other currencies indicated on the opposite side of the triangle and expressed in terms of the base currency in that corner.[3]

Exhibit 1:
Volatility and correlation triangle for the USD/DEM/JPY trio (8-SEP-1998)

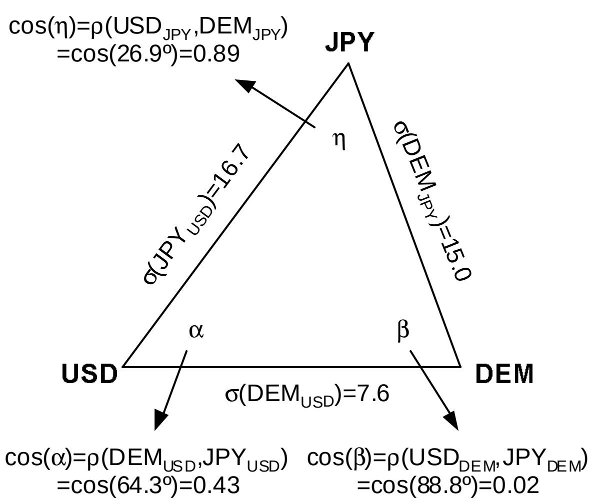
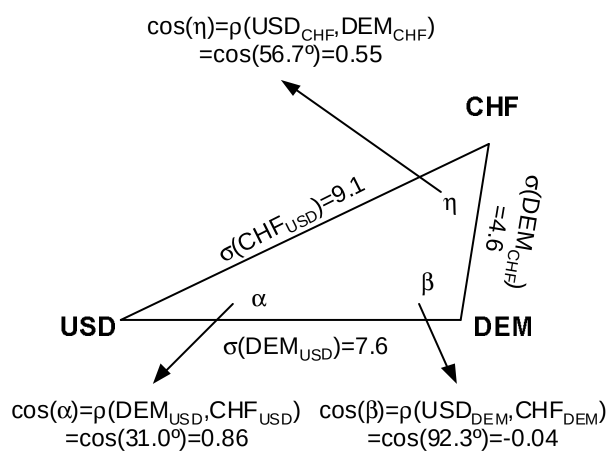


Exhibit 2:
Volatility and correlation triangle for the USD/DEM/CHF trio (8-SEP-1998)



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[1] The law of cosines states that if the angles of a triangle are lettered α, β and γ, respectively, and the lengths of the sides opposite the angles are labeled as a, b and c, respectively, then $c^2 = a^2 + b^2 - 2\cos(\gamma)$. (Analogous expressions for cos(α) and cos(β) are obtained by the appropriate permutations of the letters.)

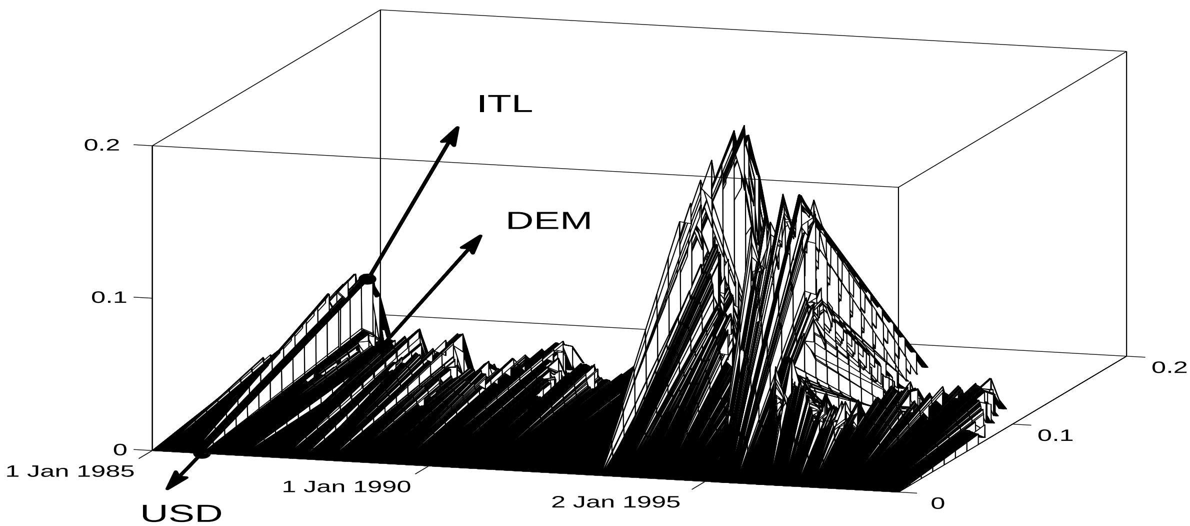
[2] Following market practice, volatilities are expressed as annualized standard deviations of logarithmic returns. Note that $\sigma(X_Y)=\sigma(Y_X)$.

[3] Accordingly, positive correlations correspond to angles between 0 and 90 degrees (in degree measure) or 0 and π/2 (in radian measure), and negative correlations map into angles from 90 to 180 degrees or π/2 to π, respectively.

Translating the volatilities and correlations in a currency trio into a triangle clearly allows for a better intuitive understanding of their economic interrelationships. The geometric approach is particularly helpful in visualizing the differences in volatility and correlation structures across currency trios. Exhibit 2, for instance, shows that the currency trio consisting of the USD, DEM, and the Swiss franc (CHF) has a markedly different structure than the USD/DEM/JPY trio. It is characterized by the low variability of the DEM/CHF rate, indicated by the short, right-hand side of the triangle, and the high correlation between DEM/USD and CHF/USD exchange rates, indicated by the angle α. Note that the negative correlation between USD/DEM and CHF/DEM implies that angle β is obtuse. Hence, for a USD-based investor holding part of a portfolio in DEM, the diversification benefits offered by CHF are limited. In contrast, for a DEM-based investor holding USD assets, CHF offers substantial diversification benefits due to the low and often negative correlation between USD/DEM and CHF/DEM.

This visual framework can also be used to analyze the correlation structure within a currency- trio over time. Exhibit 3 shows the time-series of 3521 triangles for the currency trio consisting of the USD, DEM, and the Italian lira (ITL) from January 1, 1985, to June 30, 1998. The picture clearly visualizes the dramatic impact of the lira's exit from the exchange rate mechanism of the European Monetary System (EMS) on the volatilities and correlations in the USD/DEM/ITL trio. Specifically, up to October 1992, the volatilities and correlations in this trio were reflective of the highly credible exchange rate band between the ITL and the DEM. That month, the confidence in the sustainability of the EMS parities eroded, which led to a massive speculative attack and the lira's eventual exit from the exchange rate mechanism on October 16. Until the lira's re-entry into the exchange rate mechanism on November 25, 1996, the volatilities and correlations in the trio exhibited markedly different characteristics from the ones in the pre-October 1992 period, reflecting the fact that – geometrically speaking – the distance between the ITL and the DEM had increased substantially. That is, $\sigma(ITL_{DEM})$ increased sharply and $\rho(DEM_{USD}, ITL_{USD})$ dropped.

Exhibit 3:
Series of volatility and correlation triangle for the currency trio USD/DEM/ITL
(90 day historical volatilities and correlations)



3. Geometric interrelationships between volatilities and correlations in a currency trio

Not only does the geometric approach provide an intuitive graphical representation of the volatility and correlation structure of a currency trio, it also highlights three important analytical relationships between them.

First, for a triangle, knowledge of the lengths of its three sides implies knowledge of its three angles as well. However, the converse is not true. Correspondingly, knowledge of the volatilities in a currency trio implies knowledge of the respective correlations, but not vice-versa. This fact is obviously important for implied asset correlations derived from the implied volatilities of traded options. Second, since a triangle can have only one obtuse angle (i.e., greater than 90 degrees), only one of the three correlations in a currency trio can be negative.

Finally, the three correlations in a currency trio must satisfy the condition that the sum of their arcus cosines equals 180 degrees (in degree measure) or pi (in radian measure); i.e.,

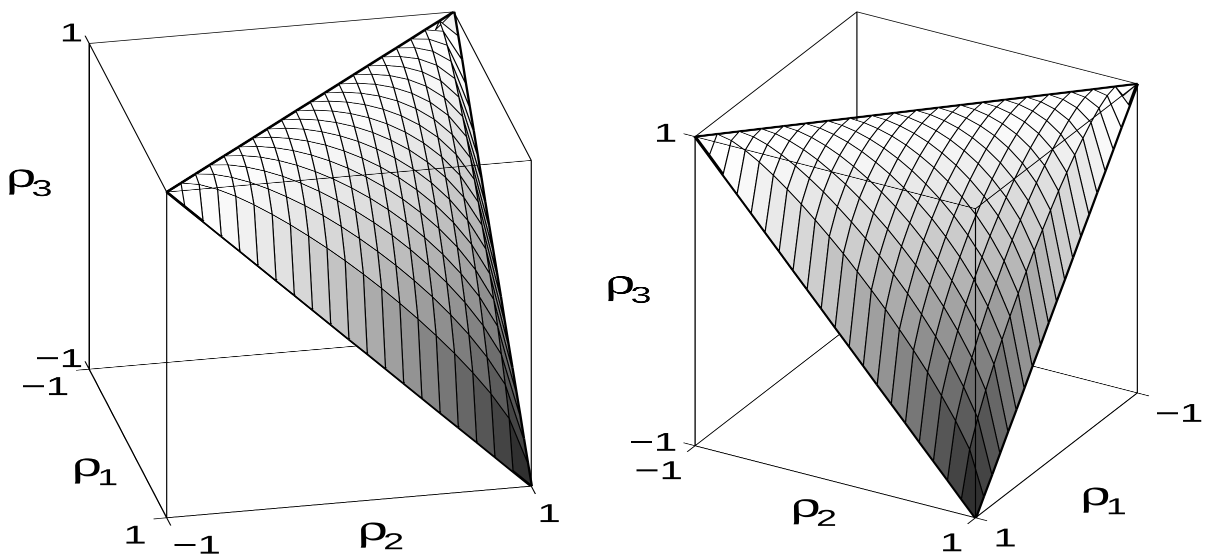
$$\arccos(\rho_1) + \arccos(\rho_2) + \arccos(\rho_3) = \pi.$$

Thus, for a currency trio, we can express a correlation as a function of the two other correlations using the formula

$$\rho_3 = \cos(\pi - \arccos[\rho_1] - \arccos[\rho_2]).$$

Given this relationship, the set of all possible correlations in a currency trio can be visually represented in the $[-1,1]^3$ cube in three-dimensional space. Exhibit 4 shows the set of possible correlation combinations in a currency trio from two perspectives.

Exhibit 4:
Possible correlation combinations in a currency trio



In fact, the combinations in exhibit 4 are a subset of all possible correlation combinations in a valid 3-by-3 correlation matrix. Note that the 3-by-3 correlation matrix, denoted

$$Q = \begin{pmatrix} 1 & \rho_{XY} & \rho_{XZ} \\ \rho_{XY} & 1 & \rho_{YZ} \\ \rho_{XZ} & \rho_{YZ} & 1 \end{pmatrix},$$

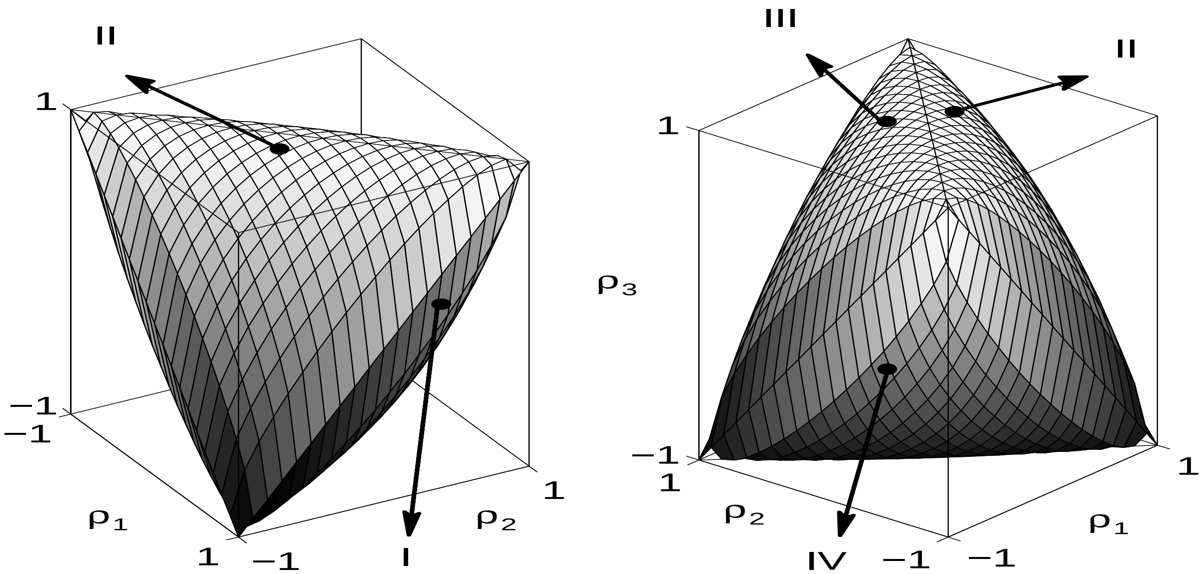
where X, Y, and Z are the variables of interest, is a valid correlation matrix if and only if it is positive semidefinite; i.e., if and only if

$$|Q| = 1 + 2\,\rho_{XY}\,\rho_{XZ}\,\rho_{YZ} - \rho_{XY}^2 - \rho_{XZ}^2 - \rho_{YZ}^2 \geq 0.$$

Thus, we cannot pick any three numbers in the [-1, 1] interval and expect that they will form a valid correlation matrix; the above conditions must be met.

The set of correlation combinations that form a valid 3-by-3 correlation matrix can be pictured in three-dimensional space within the $[-1, 1]^3$ cube. Exhibit 5 shows the set of possible correlations arising in this way (seen from two perspectives). The surface of this convex body is made up of the correlation matrices with a zero determinant, and strictly positive definite correlation matrices are represented by points inside the body; see Rousseeuw and Molenberghs (1994) for further discussion.

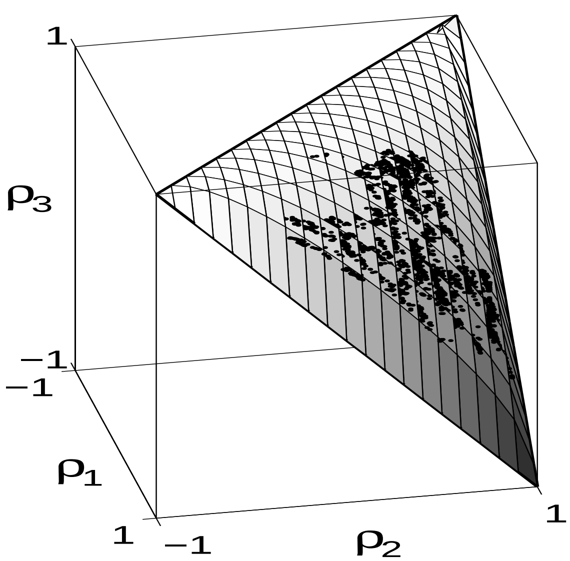
Exhibit 5:
Set of all possible correlation combinations in a valid 3-by-3 correlation matrix

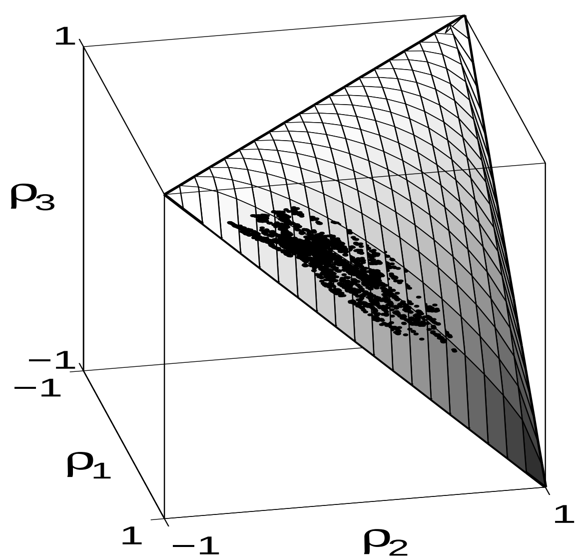


To see how the correlations in a currency trio relate to this surface, define the variables X, Y and Z as the relative changes of the USD exchange rate of currency A (A_{USD}), the USD exchange rate of currency B (B_{USD}), and the exchange rate A measured in units of currency B (A_B), respectively. Since the relative change of a cross-rate is a linear combination of the returns of the two underlying exchange rates (i.e., Z = X - Y), the correlation matrix for the three variables has a zero determinant. Hence, the set of possible correlation combinations forms a subset of all the valid 3-by-3 correlation matrices, consisting of the correlation combinations that are located on the part of the surface labeled “I” in exhibit 5. Note that area I implies the same restrictions on the correlations in a currency trio as the surface in exhibit 4, although the areas are located in different parts of the $[-1, 1]^3$ cube. The reason for this lies in the different definitions of the correlations analyzed. The three correlations pictured in exhibit 4 are defined as the correlations between the relative changes of the two foreign currencies expressed in units of the relevant home-currency, such that $\rho_1 = \rho(A_{USD}, B_{USD})$, $\rho_2 = \rho(USD_A, B_A)$, and $\rho_3 = \rho(USD_B, A_B)$. In contrast, the correlations depicted in exhibit 5 are defined in terms of the series $\{x(A_{USD}); y(B_{USD}); z(A_B)\}$, such that $\rho_1' = \rho(A_{USD}, B_{USD})$, $\rho_2' = \rho(A_{USD}, A_B)$, and $\rho_3' = \rho(B_{USD}, A_B)$. Whereas $\rho_1 = \rho_1'$ and $\rho_2 = \rho_2'$ (since $\rho(X_Z, Y_Z) = \rho(Z_X, Z_Y)$), we have $\rho_3 = -\rho_3'$ (since $\rho(X_Z, Y_Z) = -\rho(Z_X, Y_Z)$). Thus, every correlation combination in exhibit 4 corresponds to a correlation combination in exhibit 5 with identical first two correlations, but a reverse sign on the third correlation. Geometrically, this implies a reflection on the plane given by zero values for the third correlation.

Plotting specific correlation combinations on this surface allows for another intuitively appealing visualization of the characteristics of the correlation structure in a currency trio. Exhibit 6 gives an example, depicting the three-month correlation combinations in the USD/DEM/JPY and USD/DEM/CHF currency trios for the 3521 trading days between January 1, 1985, to June 30, 1998. The historical correlations are based on a 90-day window of past observations. For the JPY trio, $\rho_1=\rho(DEM_{USD}, JPY_{USD})$, $\rho_2=\rho(USD_{DEM}, JPY_{DEM})$, $\rho_3=\rho(USD_{JPY}, DEM_{JPY})$; and for the CHF trio, $\rho_1=\rho(DEM_{USD}, CHF_{USD})$, $\rho_2=\rho(USD_{DEM}, CHF_{DEM})$, $\rho_3=\rho(USD_{CHF}, DEM_{CHF})$. Among other things, the graphics clearly show that the correlation structure in the CHF trio tends to be more stable (more clustered) than the correlation structure in the JPY trio.

Exhibit 6:
Three-month historical correlations in the currency trios USD/DEM/JPY and USD/DEM/CHF (January 1, 1985, through June 30, 1998)





4. Geometric comparisons of correlation forecasts

Forecasting correlations between financial variables has emerged over the past few years as an important topic of financial research and practice. This is not surprising, given the practical benefits offered by accurate correlation forecasts to investors optimizing portfolios or risk managers calculating value-at-risk estimates.

Although forecasts of the volatilities and correlations in a currency trio can be generated in a variety of ways, there are two general categories of forecasts. The first category is time-series forecasts; that is, the past behavior of the exchange rates in question are used to forecast their future behavior. With respect to the variance-covariance matrix, a very popular time-series model is the bivariate GARCH(1,1) model with standard normal innovations. The second category of forecasts are those derived from derivative prices, such as over-the-counter foreign exchange options. Since today's option prices incorporate the market's forecast of volatility and correlation over the maturity of the option, the volatilities and correlations implied in these prices can be extracted and used as forecasts.

Complementary to the statistical methods for comparing the accuracy of correlation forecasts (as discussed in Walter and Lopez, 1997), we may again use graphics to improve our intuitive grasp of their performance. For example, the performance of a specific forecasting method may be visualized by plotting the realized values of the three correlations against their predicted values. However, a more succinct way to graphically compare the performance of different forecasting methods is to depict the series of 3-by-1 forecast error vectors (one for each correlation in the trio) in three-dimension space. This method gives rise to a cloud of points in the $[-1, 1]^3$ cube, where each point represents the three forecast errors. Similar to before, this graphical approach allows for a better intuitive understanding of the forecasts' accuracy and is particularly helpful in visualizing the differences between the accuracy of different sets of forecasts.

Exhibit 7 illustrates the power of the visual approach by comparing the forecasting performance of one-month implied correlation forecasts across two currency trios. The graph on the left depicts the forecast errors for the USD/DEM/JPY trio (using the 1679 daily observations data from October 2, 1990 through April 2, 1997), and the graph on the right presents the forecast errors for the USD/DEM/CHF trio (using the 910 daily observations from September 13, 1993 through April 2, 1997). Clearly, the forecasts errors in the second trio exhibit markedly different properties from the first trio. Specifically, the option-based forecast errors for $\rho(DEM_{USD}, CHF_{USD})$ (denoted FE1 on the graph on the right) are small in comparison to both the forecast errors in the USD/DEM/JPY trio and the forecast errors with respect to the two other correlations in the USD/DEM/CHF trio.

Exhibit 7:
Three-dimensional plot of the forecast errors of one-month implied correlation with respect to the USD/DEM/JPY trio (left side) and the USD/DEM/CHF trio (right side)

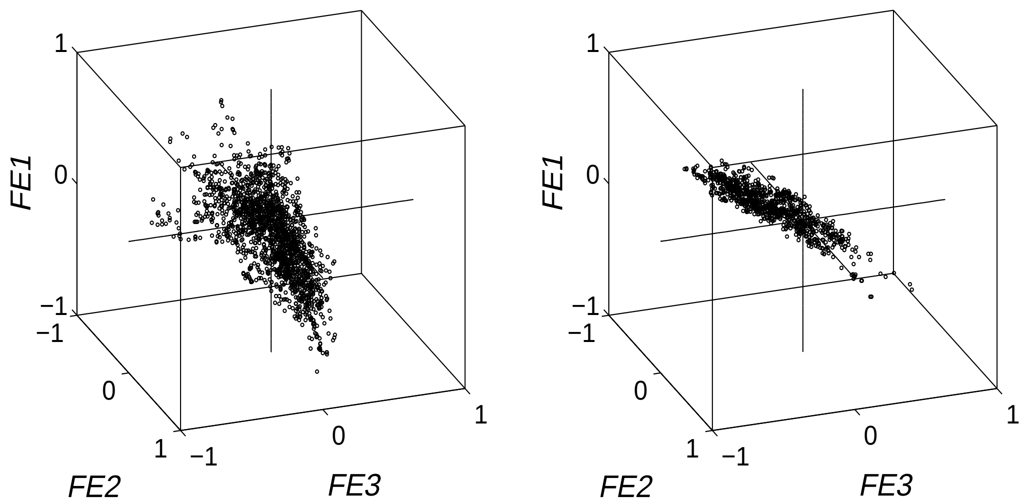
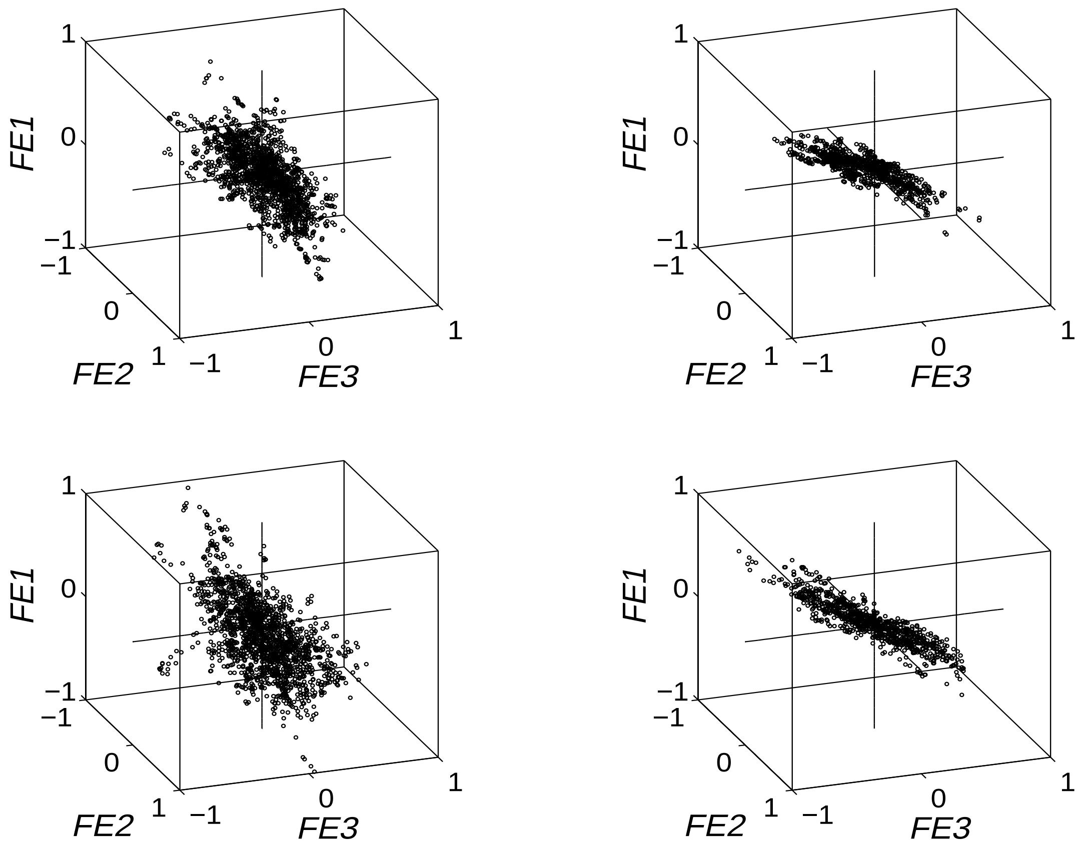


Exhibit 8 depicts the forecast error vectors for two time-series forecasting methods; the upper graphs show the error vectors for correlation forecasts generated by bivariate GARCH(1,1) models, and the lower part gives the error vectors for a particularly simple forecasting method - the historical correlations over the last 20 trading days just prior to the formulation of the forecast. The same general patterns observed in Exhibit 7 are present here, which indicate that these three sets of forecasted correlations share some basic characteristics. However, certain differences are clear. The 20-day historical forecast errors have a wider dispersion and many more outlying values than those of the GARCH and implied forecasts. This result suggests larger errors and thus less reliable forecasts. The GARCH forecasts appear quite similar to the implied correlations in terms of forecast accuracy, a result that is generally borne out in the statistical results presented in Walter and Lopez (1997; Tables 4A and 4C).

Exhibit 8:
Three-dimensional plots of the forecast errors of two alternative forecasting methods



5. Conclusion

Financial institutions survive on their ability to react to the massive amount of data generated by financial markets, risk management systems, and other databases. By transforming such data into graphics, financial institutions can take advantage of the human capability to instantaneously understand pictures. As discussed by Singer, Terhaar and Zerolis (1998), the geometric approach to the correlation structure between financial assets in general and exchange rates in particular allows for an intuitive visual approach. In this article, we expand previous work in this area to analyze correlations across currency trios and over time. We also present a graphical method for analyzing correlation forecast errors from several competing models. All of these tools can add valuable graphical intuition to the variety of numerical tools available for analyzing covariance matrices and correlation forecasts.

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