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Abstract: From a macroeconomic perspective, the short-term interest rate is a policy instrument under the direct control of the central bank. From a finance perspective, long rates are risk-adjusted averages of expected future short rates. Thus, as illustrated by much recent research, a joint macro-finance modeling strategy will provide the most comprehensive understanding of the term structure of interest rates. We discuss various questions that arise in this research, and we also present a new examination of the relationship between two prominent dynamic, latent factor models in this literature: the Nelson-Siegel and affine no-arbitrage term structure models.

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From a macroeconomic perspective, the short-term interest rate is a policy instrument under the direct control of the central bank, which adjusts the rate to achieve its economic stabilization goals. From a finance perspective, the short rate is a fundamental building block for yields of other maturities, which are just risk-adjusted averages of expected future short rates. Thus, as illustrated by much recent research, a joint macro-finance modeling strategy will provide the most comprehensive understanding of the term structure of interest rates. In this paper, we discuss some salient questions that arise in this research, and we also present a new examination of the relationship between two prominent dynamic, latent factor models in this literature: the Nelson-Siegel and affine no-arbitrage term structure models.

I. Questions about Modeling Yields

(1) Why use factor models for bond yields? The first problem faced in term structure modeling is how to summarize the price information at any point in time for the large number of nominal bonds that are traded. In fact, since only a small number of sources of systematic risk appear to underlie the pricing of the myriad of tradable financial assets, nearly all bond price information can be summarized with just a few constructed variables or factors. Therefore, yield curve models almost invariably employ a structure that consists of a small set of factors and the associated factor loadings that relate yields of different maturities to those factors. Besides providing a useful compression of information, a factor structure is also consistent with the celebrated "parsimony principle," the broad insight that imposing restrictions—even those that are false and may degrade in-sample fit—often helps both to avoid data mining and to produce good forecasting models. For example, an unrestricted

Vector Autoregression (VAR) provides a very general linear model of yields, but the large number of estimated coefficients renders it of dubious value for prediction (Diebold and Calin Li, 2005). Parsimony is also a consideration for determining the number of factors needed, along with the demands of the precise application. For example, to capture the time series variation in yields, one or two factors may suffice since the first two principal components account for almost all (99%) of the variation in yields. Also, for forecasting yields, using just a few factors may often provide the greatest accuracy. However, more than two factors will invariably be needed in order to obtain a close fit to the entire yield curve at any point in time, say, for pricing derivatives.

(2) How should bond yield factors and factor loadings be constructed? There are a variety of methods employed in the literature. One general approach places structure only on the estimated factors. For example, the factors could be the first few principal components, which are restricted to be mutually orthogonal, while the loadings are relatively unrestricted. Indeed, the first three principal components typically closely match simple empirical proxies for level (e.g., the long rate), slope (e.g., a long minus short rate), and curvature (e.g., a mid-maturity rate minus a short and long rate average). A second approach, which is popular among market and central bank practitioners, is a fitted Nelson-Siegel curve (introduced in Charles Nelson and Andrew Siegel, 1987). As described by Diebold and Li (2005), this representation is effectively a dynamic three-factor model of level, slope, and curvature. However, the Nelson-Siegel factors are unobserved, or latent, which allows for measurement error, and the associated loadings have plausible economic restrictions (forward rates are always positive, and the discount factor approaches zero as maturity increases). A third approach is the no-arbitrage dynamic latent factor model, which is the model of choice

in finance. The most common subclass of these models postulates flexible linear or affine forms for the latent factors and their loadings along with restrictions that rule out arbitrage strategies involving various bonds.

(3) How should macroeconomic variables be combined with yield factors? Both the Nelson-Siegel and affine no-arbitrage dynamic latent factor models provide useful statistical descriptions of the yield curve, but they offer little insight into the nature of the underlying economic forces that drive its movements. To shed some light on the fundamental determinants of interest rates, researchers have begun to incorporate macroeconomic variables into these yield curve models.

For example, Diebold, Rudebusch, and S. Boragan Aruoba (2005) provide a macroeconomic interpretation of the Nelson-Siegel representation by combining it with VAR dynamics for the macroeconomy. Their maximum likelihood estimation approach extracts three latent factors (essentially level, slope, and curvature) from a set of 17 yields on U.S. Treasury securities and simultaneously relates these factors to three observable macroeconomic variables (specifically, real activity, inflation, and a monetary policy instrument).

The role of macroeconomic variables in a no-arbitrage affine model is explored by several papers. In Piazzesi (2005), the key observable factor is the Federal Reserve's interest rate target. The target follows a step function or pure jump process, with jump probabilities that depend on the schedule of policy meetings and three latent factors, which also affect risk premiums. The short rate is modeled as the sum of the target and short-lived deviations from target. The model is estimated with high-frequency data and provides a new identification scheme for monetary policy. The empirical results show that relative to standard latent factor models using macroeconomic information can substantially lower pricing errors. In particular, including the Fed's target as one of four factors allows the model to match both the short and the long end of the yield curve.

In Andrew Ang and Piazzesi (2003) and Ang, Sen Dong, and Piazzesi (2004), the macroeconomic factors are measures of inflation and real activity. The joint dynamics of these macro factors and additional latent factors are captured by VARs. In Ang and Piazzesi (2003), the measures of real activity and inflation are each constructed as the first principal component of a large set of candidate macroeconomic series, to avoid relying on specific macro series. Both papers explore various methods to identify structural shocks. They differ in the dynamic linkages between macro factors and yields, discussed further below.

Finally, Rudebusch and Tao Wu (2004a) provide an example of a macro-finance specification that employs more macroeconomic structure and includes both rational expectations and inertial elements. They obtain a good fit to the data with a model that combines an affine no-arbitrage dynamic specification for yields and a small fairly standard macro model, which consists of a monetary policy reaction function, an output Euler equation, and an inflation equation.

(4) What are the links between macro variables and yield curve factors? Diebold, Rudebusch, and Aruoba (2005) examine the correlations between Nelson-Siegel yield factors and macroeconomic variables. They find that the level factor is highly correlated with inflation, and the slope factor is highly correlated with real activity. The curvature factor appears unrelated to any of the main macroeconomic variables. Similar results with a more structural interpretation are obtained in Rudebusch and Wu (2004a); in their model, the level factor reflects market participants' views about the underlying or medium-term inflation target of the central bank, and the slope factor captures the cyclical response of the central bank,

which manipulates the short rate to fulfill its dual mandate to stabilize the real economy and keep inflation close to target. In addition, shocks to the level factor feed back to the real economy through an ex ante real interest rate.

Piazzesi (2005), Ang and Piazzesi (2003) and Ang, Dong, and Piazzesi (2004) examine the structural impulse responses of the macro and latent factors that jointly drive yields in their models. For example, Piazzesi (2005) documents that monetary policy shocks change the slope of the yield curve, because they affect short rates more than long ones. Ang and Piazzesi (2003) find that output shocks have a significant impact on intermediate yields and curvature, while inflation surprises have large effects on the level of the entire yield curve. They also find that better interest rate forecasts are obtained in an affine model in which macro factors are added to the usual latent factors.

For estimation tractability, Ang and Piazzesi (2003) only allow for unidirectional dynamics in their arbitrage-free model, specifically, macro variables help determine yields but not the reverse. Diebold, Rudebusch, and Aruoba (2005) consider a general bidirectional characterization of the dynamic interactions and find that the causality from the macroeconomy to yields is indeed significantly stronger than in the reverse direction but that interactions in both directions can be important. Ang, Dong, and Piazzesi (2004) also allow for bidirectional macro-finance links but impose the no-arbitrage restriction as well, which poses a severe estimation challenge that is solved via Markov Chain Monte Carlo methods. The authors find that the amount of yield variation that can be attributed to macro factors depends on whether or not the system allows for bidirectional linkages. When the interactions are constrained to be unidirectional (from macro to yield factors), macro factors can only explain a small portion of the variance of long yields. In contrast, the bidirectional system attributes over half of the variance of long yields to macro factors.

(5) How useful are no-arbitrage modeling restrictions? The assumption of no arbitrage ensures that, after accounting for risk, the dynamic evolution of yields over time is consistent with the cross-sectional shape of the yield curve at any point in time. This consistency condition is likely to hold, given the existence of deep and well-organized bond markets. However, if the underlying factor model is misspecified, such restrictions may actually degrade empirical performance. (Of course, the ultimate goal is a model that is both internally consistent and correctly specified.) Ang and Piazzesi (2003) present some empirical evidence favorable to imposing no-arbitrage restrictions because of improved forecasting performance. As discussed below, this issue is worthy of further investigation.

(6) What is the appropriate specification of term premiums? With risk-neutral investors, yields are equal to the average value of expected future short rates (up to Jensen's inequality terms), and there are no expected excess returns on bonds. In this setting, the expectations hypothesis, which is still a mainstay of much casual and formal macroeconomic analysis, is valid. However, it seems likely that bonds, which provide an uncertain return, are owned by the same investors who also demand a large equity premium as compensation for holding risky stocks. Furthermore, as suggested by many statistical tests in the literature, these risk premiums on nominal bonds appear to vary over time, contradicting the assumption of risk-neutrality. To model these premiums, Ang and Piazzesi (2003) and Rudebusch and Wu (2004a, b) specify time-varying "prices of risk," which translate a unit of factor volatility into a term premium. This time variation is modeled using business cycle indicators such as the slope of the yield curve or measures of real activity. However, Diebold, Rudebusch, and Aruoba (2005) suggest that the importance of the statistical deviations from the expectations

hypothesis may depend on the application.

II. Example: An Affine Interpretation of Nelson-Siegel

In this section, we develop a new example to illustrate several of the above issues, particularly the construction of yield curve factors and the imposition of the no-arbitrage restrictions. By showing how to impose no-arbitrage restrictions in a Nelson-Siegel representation of the yield curve, we outline a methodology to judge these restrictions. The Nelson-Siegel model is a popular model that performs well in forecasting applications, so it would be interesting to compare its accuracy with and without these restrictions (a subject of our ongoing research).

The 2-factor Nelson-Siegel model specifies the yield on a τ-period bond as

$$y_t^{(\tau)} = a_\tau^{NS} + b_\tau^{NS} \cdot x_t, \tag{1}$$

where x_t is a 2-dimensional vector of latent factors (or state variables) and the yield coefficients depend only on the time to maturity τ :

$$a_\tau^{NS} = 0 \tag{2}$$

$$b_\tau^{NS} = \begin{bmatrix} 1 & \frac{1-\exp(-k\tau)}{k\tau} \end{bmatrix}^\top . \tag{3}$$

The two coefficients in b_τ^{NS} give the loadings of yields on the two factors. The first loading is unity, so the first factor operates as a level shifter and affects yields of all maturities one-for-one. The second loading goes to one as $\tau \to 0$ and goes to zero as $\tau \to \infty$ (assuming $k > 0$), so the second factor mainly affects short maturities and, hence, the slope. Furthermore, as maturity τ goes to zero, the yield in equation (1) approaches the instantaneous short rate of interest, denoted r_t, and, since the second component of b_τ^{NS} goes to 1, the short rate is just the sum of the two factors,

$$r_t = x_t^1 + x_t^2, \tag{4}$$

and is latent as well. Finally, as in Diebold and Li (2005), we augment the cross-sectional equation (1) with factor dynamics; specifically, both components of x_t are independent AR(1)'s:

$$x_t^i = \mu_i + \rho_i x_{t-1}^i + \upsilon_i \varepsilon_t^i, \tag{5}$$

with Gaussian errors ε_t^i, $i = 1, 2$. Therefore, the complete Nelson-Siegel dynamic representation, (1), (2), (3), (5), has 7 free parameters: k, $\mu_1, \rho_1, \upsilon_1, \mu_2, \rho_2$, and υ_2.

Consider now the 2-factor affine no-arbitrage term structure model. This model starts from the linear short rate equation (4); however, rather than postulating a particular functional form for the factor loadings as above, the loadings are derived from the short rate equation (4) and the factor dynamics (5) under the assumption of an absence of arbitrage opportunities. In particular, if there are risk-neutral investors, they are indifferent between buying a long bond that pays off \$1 after τ periods and an investment that rolls over cash at the short rate during those τ periods and has an expected payoff of \$1. Thus, risk-neutral investors would engage in arbitrage until the τ-period bond price equals the expected roll-over amount, so the yield on a τ-period bond will equal the expected average future short rate over the τ periods (plus a Jensen's inequality term.) Risk-averse investors will need additional compensation for holding risky positions, but the same reasoning applies after correcting for risk premiums. Therefore, to make the Nelson-Siegel model internally consistent under the assumption of no-arbitrage, yields computed from expected average future short rates using (4) with the factor dynamics (5) must be consistent with the cross-sectional specification in equations (1) through (3).

To enforce this no-arbitrage internal consistency, we switch to continuous time and fix the sampling frequency so that the interval $[t-1,t]$ covers, say, one month. The continuous-time AR(1) process corresponding to (5) is

$$dx_t^i = \kappa_i \left(\theta_i - x_t^i\right) dt + \sigma_i dB_t^i, \tag{6}$$

where κ_i, θ_i and σ_i are constants and B^i is a Brownian motion (which means that dB^i is normally distributed with mean zero and variance dt). (In continuous time, the Nelson-Siegel has 7 parameters: k, $\kappa_1, \theta_1, \sigma_1, \kappa_2, \theta_2$, and σ_2.)

We first consider the model with risk-neutral investors, which consists of the linear short rate equation (4) and the factor dynamics (6) and has 6 parameters: $\kappa_1, \theta_1, \sigma_1, \kappa_2, \theta_2$, and σ_2. Investors engage in arbitrage until the time-t price $P_t^{(\tau)}$ of the τ-bond is given by

$$P_t^{(\tau)} = E_t \left(\exp \left(- \int_t^{t+\tau} r_s ds \right) \right). \tag{7}$$

This expectation can be computed by hand, since the short rate is the sum of two Gaussian AR(1)'s and is thus normally distributed. (The appendix details these calculations.) The resulting τ-period yield is

$$\begin{aligned} y_t^{(\tau)} &= -\frac{\log P_t^{(\tau)}}{\tau} \\ &= a_\tau^{NA} + b_\tau^{NA} \cdot x_t, \end{aligned} \tag{8}$$

with the no-arbitrage factor loadings given by

$$b_\tau^{NA} = \left[\begin{array}{cc} \frac{1-\exp(-\kappa_1\tau)}{\kappa_1\tau} & \frac{1-\exp(-\kappa_2\tau)}{\kappa_2\tau} \end{array} \right]^\top. \tag{9}$$

The equations (4), (6), (8), and (9) constitute a canonical affine term-structure specification with two Gaussian factors. Intuitively, in the risk-neutral world, where yields are based only on expected future short rates, the cross-sectional factor-loading coefficients b_τ^{NA} are restricted to be functions of the time series parameters κ_1 and κ_2. The constant a_τ^{NA} absorbs any Jensen's inequality terms. In general, the Nelson-Siegel representation does not impose this dynamic consistency restriction because the loadings b_τ^{NS} are not related to the time series parameters from the AR(1). However, the no-arbitrage restriction can be applied to the Nelson-Siegel model under two conditions. First, let κ_1 go to zero and set $\kappa_2 = k$, since for these parameter values, $b_\tau^{NA} = b_\tau^{NS}$. Second, it will have to be case that the constant a_τ^{NA}, which embeds the Jensen's inequality terms, is close to zero for reasonable parameter values, i.e., $a_\tau^{NA} \approx a_\tau^{NS} = 0$. (As a rule, macroeconomists often ignore Jensen's terms; however, with near-random walk components in the short rate process as κ_1 goes to zero, the Jensen's terms may be large, especially for long maturities τ.)

Now consider the more general case of no-arbitrage with risk-averse investors. To accommodate departures from risk-neutrality, we parametrize the risk premiums used to adjust expectations. For example, suppose the pricing kernel solves

$$\frac{dm_t}{m_t} = -r_t dt - \lambda_t^1 dB_t^1 - \lambda_t^2 dB_t^2,$$

where

$$\lambda_t^i = \lambda_0^i + \lambda_1^i x_t^i$$

and λ_0^i, λ_1^i are constants. The variables λ_t^i are the prices of risk for each Brownian motion and are affine functions of the factors and so vary over time. The no-arbitrage factor loadings are given by

$$b_\tau^{NA} = \left[\begin{array}{cc} \frac{1-\exp\left(-\kappa_1^* \tau\right)}{\kappa_1^* \tau} & \frac{1-\exp\left(-\kappa_2^* \tau\right)}{\kappa_2^* \tau} \end{array} \right]^\top, \tag{10}$$

where

$$\kappa_i^* = \kappa_i + \sigma_i \lambda_1^i.$$

This 2-factor Gaussian model has 10 parameters $\lambda_0^1, \lambda_1^1, \lambda_0^2, \lambda_1^2, \kappa_1, \theta_1, \sigma_1, \kappa_2, \theta_2$, and σ_2. Now it is possible to pick the slope parameters, λ_1^i, so that the loadings, b_τ^{NA}, equal the Nelson-Siegel loadings, b_τ^{NS}. The values for λ_1^i that meet this condition are obtained by setting $\kappa_1^* = 0$ and $\kappa_2^* = k$, and these imply that

$$\lambda_1^1 = -\frac{\kappa_1}{\sigma_1} \text{ and } \lambda_1^2 = \frac{k - \kappa_2}{\sigma_2}.$$

The constant terms in the market prices of risk are unrestricted, so we can set $\lambda_0^1 = \lambda_0^2 = 0$. Again, it will have to be case that the Jensen's inequality terms should be close to zero, so $a_\tau^{NA} \approx a_\tau^{NS} = 0$.

III. The Future

The macro-finance term structure literature is in its infancy with many unresolved but promising issues to explore. For example, as suggested above, the appropriate specification for the time-series forecasting of bond yields is an exciting area for additional research, especially in a global context (Diebold, Li, and Vivian Yue 2005). In addition, the goal of an estimated no-arbitrage macro-finance model specified in terms of underlying preference and technology parameters (so the asset-pricing kernel is consistent with the macrodynamics) remains a major challenge.

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Appendix

To derive the affine bond pricing formulas and yield curve equations, consider the case with prices of risk $\lambda_t = \left[\lambda_t^1 \ \ \lambda_t^2\right]^\top$. (Note that equation (9) can be obtained from (10) by setting the prices of risk to zero.) There are two ways to derive thes formulas. First, we can construct a risk-neutral probability measure under which the risk-neutral pricing formula (7) holds. Second, we can start from the Euler equation $E\left[d\left(m_t F_t\right)\right] = 0$.

Risk-neutral probability

Under the risk-neutral probability measure, the process B^* which solves $dB_t^* = dB_t + \lambda_t dt$ is a Brownian motion. By solving for dB_t and inserting this expression into the AR(1) dynamics of the factors (6), we get

$$dx_t^i = \kappa_i\left(\theta_i - x_t^i\right)dt + \sigma_i(dB_t^{*i} - \lambda_t^i dt) \tag{11}$$

$$= \left(\kappa_i\theta_i - \kappa_i x_t^i - \sigma_i\lambda_0^i - \sigma_i\lambda_1^i x_t^i\right)dt + \sigma_i dB_t^{*i} \tag{12}$$

$$= \left(\kappa_i\theta_i - \sigma_i\lambda_0^i - (\kappa_i + \sigma_i\lambda_1^i)x_t^i\right)dt + \sigma_i dB_t^{*i} \tag{13}$$

$$= (\kappa_i + \sigma_i\lambda_1^i)\left(\frac{\kappa_i\theta_i - \sigma_i\lambda_0^i}{(\kappa_i + \sigma_i\lambda_1^i)} - x_t^i\right)dt + \sigma_i dB_t^{*i} \tag{14}$$

$$= \kappa_i^*\left(\theta_i^* - x_t^i\right)dt + \sigma_i dB_t^{*i}, \tag{15}$$

where

$$\kappa_i^* = \kappa_i + \sigma_i\lambda_1^i$$

$$\theta_i^* = \frac{\kappa_i\theta_i - \sigma_i\lambda_0^i}{\kappa_i + \sigma_i\lambda_1^i}$$

The price of the τ-period bond is equal to

$$P_t^{(\tau)} = E_t^*\left(\exp\left(-\int_t^{t+\tau} r_s ds\right)\right),$$

where the expectation operator E^* uses the risk-neutral probability measure. Since the vector $x = (x_1, x_2)^\intercal$ is Markov, this expectation is a function of the state today x_t. Thus, the bond price is a function

$$P_t^{(\tau)} = F\left(x_t, \tau\right)$$

of the state vector x_t and time-to-maturity τ. The Feynman-Kac formula says that F solves the partial differential equation

$$F_t r_t = -\frac{\partial F}{\partial \tau} + \sum_{i=1}^{2}\left[\frac{\partial F}{\partial x^i}\kappa_i^*\left(\theta_i^* - x_t^i\right) + \frac{1}{2}\frac{\partial^2 F}{\partial x^{i2}}\sigma_i^2\right]$$

with terminal condition $F\left(x, 0\right) = 1$.

We guess the solution

$$F\left(x_t, \tau\right) = \exp\left(A\left(\tau\right) + B\left(\tau\right)\cdot x_t\right) \tag{16}$$

which means that

$$\begin{aligned}
\frac{\partial F}{\partial x^i} &= B_i\left(\tau\right)F \\
\frac{\partial^2 F}{\partial x^{i2}} &= B_i\left(\tau\right)^2 F \\
\frac{\partial F}{\partial \tau} &= \left(A'\left(\tau\right) + B'\left(\tau\right)\cdot x_t\right)F.
\end{aligned}$$

Insert these expressions into the partial differential equation and get

$$\begin{aligned}
x_t^1 + x_t^2 &= -A'\left(\tau\right) - B_1'\left(\tau\right)x_t^1 - B_2'\left(\tau\right)x_t^2 \\
&\quad + \sum_{i=1}^{2}\left[B_i\left(\tau\right)\kappa_i^*\left(\theta_i^* - x_t^i\right) + \frac{1}{2}B_i\left(\tau\right)^2\sigma_i^2\right].
\end{aligned}$$

Matching coefficients results in

$$\begin{aligned} A'(\tau) &= \sum_{i=1}^{2} B_i(\tau)\kappa_i^*\theta_i^* + \frac{1}{2}B_i(\tau)^2\sigma_i^2 \\ 1 &= -B_1'(\tau) - B_1(\tau)\kappa_1^* \\ 1 &= -B_2'(\tau) - B_2(\tau)\kappa_2^*. \end{aligned}$$

The boundary conditions are

$$\begin{aligned} A(0) &= 0 \\ B(0) &= 0_{2\times 1}. \end{aligned}$$

The solution to these ODE's are

$$\begin{aligned} B_1(\tau) &= \frac{(\exp(-\kappa_1^*\tau) - 1)}{\kappa_1^*} \qquad (17a) \\ B_2(\tau) &= \frac{(\exp(-\kappa_2^*\tau) - 1)}{\kappa_2^*}. \end{aligned}$$

We can plug these solutions into the yield equation

$$\begin{aligned} y_t^{(\tau)} &= -\frac{A(\tau)}{\tau} - \frac{B_1(\tau)}{\tau}x_t^1 - \frac{B_2(\tau)}{\tau}x_t^2 \qquad (18) \\ &= a^{NA}(\tau) + b_1^{NA}(\tau)x_t^1 + b_2^{NA}(\tau)x_t^2 \end{aligned}$$

and get equations (9).

Euler equation approach

The Euler equation is

$$P_t^{(\tau)} = E_t\left[\frac{m_{t+\tau}}{m_t}\right]$$

and the instantaneous equation is

$$E\left[d\left(m_t F_t\right)\right] = 0. \qquad (19)$$

The bond price is a function $F(x,\tau)$ and we can apply Ito's Lemma

$$dF = \mu_F dt + \sigma_F dB_t,$$

where the drift and volatility of F are given by

$$\begin{aligned}
\mu_F &= -\frac{\partial F}{\partial \tau} + \sum_{i=1}^{2}\left[\frac{\partial F}{\partial x_i}\kappa_i\left(\theta_i - x^i\right) + \frac{1}{2}\frac{\partial^2 F}{\partial x^{i2}}\sigma_i^2\right] \\
\sigma_F &= \sum_{i=1}^{2}\frac{\partial F}{\partial x^i}\sigma_i
\end{aligned}$$

Both m_t and F_t are Ito processes, so their product solves

$$\begin{aligned}
d(m_t F_t) &= -r_t m_t F_t dt + m_t \mu_t^F dt - m_t \lambda_t \sigma_t^F dt \\
&\quad -F_t m_t \lambda_t dB_t + m_t \sigma_t^F dB_t
\end{aligned}$$

We use the Euler equation (19) and get

$$\begin{aligned}
0 &= -r_t m_t F_t + m_t \mu_t^F - m_t \lambda_t \sigma_t^F \qquad (20) \\
F_t r_t &= \left(-\frac{\partial F}{\partial \tau} + \sum_{i=1}^{2}\left[\frac{\partial F}{\partial x^i}\kappa_i\left(\theta_i - x_t^i\right) + \frac{1}{2}\frac{\partial^2 F}{\partial x^{i2}}\sigma_i^2\right]\right) - \sum_{i=1}^{2}\frac{\partial F}{\partial x^i}\sigma_i \lambda_t^i
\end{aligned}$$

Again, guess the exponential-affine solution (16) and insert the expressions into (20), we get

$$\begin{aligned}
x_t^1 + x_t^2 &= -A'(\tau) - B_1'(\tau)x_t^1 - B_2'(\tau)x_t^2 \\
&\quad + \sum_{i=1}^{2}\left[B_i(\tau)\kappa_i\left(\theta_i - x_t^i\right) + \frac{1}{2}B_i(\tau)^2\sigma_i^2\right] \\
&\quad - \sum_{i=1}^{2}B_i(\tau)\sigma_i\left(\lambda_0^i + \lambda_1^i x_t^i\right).
\end{aligned}$$

Matching coefficients, we get the ordinary differential equations:

$$\begin{aligned}
A'(\tau) &= \sum_{i=1}^{2}B_i(\tau)\left(\kappa_i\theta_i - \sigma_i\lambda_0^i\right) + \frac{1}{2}B_i(\tau)^2\sigma_i^2 \\
1 &= -B_1'(\tau) - B_1(\tau)\left(\kappa_1 + \sigma_1\lambda_1^1\right) \\
1 &= -B_2'(\tau) - B_2(\tau)\left(\kappa_2 + \sigma_2\lambda_1^2\right).
\end{aligned}$$

From this expression, we can see that we get the coefficients (17a) with risk neutral parameters

$$\begin{aligned}
\kappa_i^* &= \kappa_i + \sigma_i \lambda_1^i \\
\kappa_i^* \theta_i^* &= \kappa_i \theta_i - \sigma_i \lambda_0^i \Longrightarrow \theta_i^* = \frac{\kappa_i \theta_i - \sigma_i \lambda_0^i}{\kappa_i + \sigma_i \lambda_1^i}.
\end{aligned}$$