Houses as ATMs? Mortgage Refinancing and Macroeconomic Uncertainty

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The Question

What is the role of housing collateral in financing consumption?

- Mortgages in U.S. household liabilities, 2012: over 70%
- Home equity extraction over 1993 2010: \$1.7 trillion

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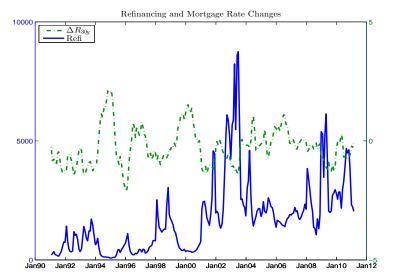
- Mortgages in U.S. household liabilities, 2012: over 70%
- Home equity extraction over 1993 2010: \$1.7 trillion
- "Great Moderation" (Campbell and Hercowitz 2004)
- Home-equity based borrowing the main force behind run-up in household leverage from 2002 to 2006 (Mian and Sufi 2010)
- Subsequent decline in consumption stronger in high leverage areas (Mian, Rao, Sufi 2013)

Cash-out Refinancing and Consumption

The very low level of interest rates ... encouraged household spending through a variety of channels. ... The lowest home mortgage rates in decades were a major contributor ... engendering a large extraction of cash from home equity. A significant part of that cash supported personal consumption expenditures and home improvement. In addition, many households took out cash in the process of refinancing, often using the proceeds to substitute for higher-cost consumer debt.

- Alan Greenspan, Congressional Testimony, February 11, 2004

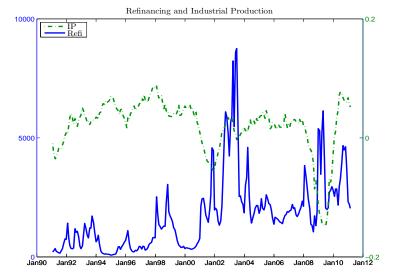
Refinancing Comoves with Interest Rates







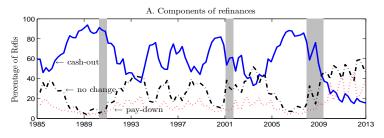
Refinancing Comoves with the Business Cycle

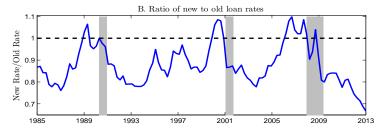






Cash-out and Rate Ratios over the Business Cycle









- Develop and estimate a rational model of home-equity based borrowing by liquidity-constrained households:
 - counter-cyclical idiosyncratic labor income uncertainty
 - long-term mortgages + borrowing constraints
 - targeting assets, debt, and refinancing behavior

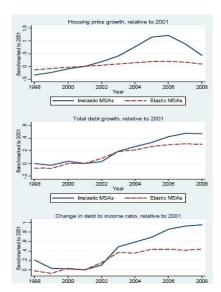
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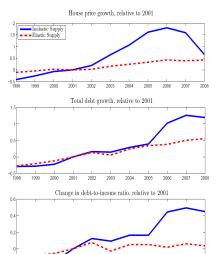
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 - heterogeneous consumption paths for households with different boom-time leverage
- Policy implications: refi sensitivity to monetary policy



Preview: Leverage Run-up





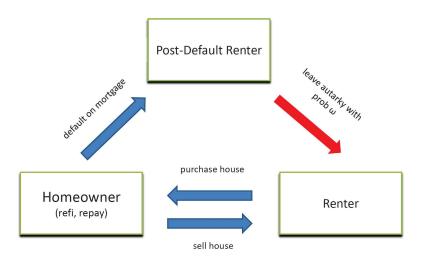
Year

2001 2002 2003 2004 2005 2006 2007 2008

Related Literature

- Mortgage refinancing: Boudoukh, Richardson, Stanton and Whitelaw (1997), Stanton (1995), Downing, Stanton and Wallace (2005), Deng et. al. (2000), Gabaix, Krishnamurthy, and Vigneron (2007), Duarte, Longstaff and Yu (2007)
- Housing wealth and consumption: Campbell and Cocco (2007), Caplin, Freeman, and Tracy (1997), Lustig and Van Nieuwerburgh (2010), Attanasio, Leicester, and Wakefield (2011), Carroll, Otsuka, and Slacalek (2011), Case, Quigley, and Shiller (2011), Calomiris, Longhofer, and Miles (2012)
- Consumption smoothing and cash-out: Hurst and Stafford (2004)

Household States: Homeownership, Default, and Renting



• Real aggregate income growth: $Z_{t+1} = Y_{t+1}/Y_t$

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- Individual labor income: $y_{i,t} = P_t Y_t \ \tilde{y}_{i,t}$
 - $\tilde{y_i}$ idiosyncratic labor income

$$\log \tilde{y}_{i,t+1} = \log \mu_y(Z_t) + \rho_y \log \tilde{y}_{i,t} + \sigma(Z_t)\epsilon_{i,t+1}^y$$

Preferences

Epstein-Zin Preferences:

$$egin{aligned} U_t &= \left[\left(1 - \delta
ight) X_t^{rac{1 - \gamma}{ heta}} + \delta \mathbb{E}_t \left[U_{t+1}^{1 - \gamma}
ight]^{rac{1}{ heta}}
ight]^{rac{
u}{1 - \gamma}} \ & heta &= rac{1 - \gamma}{1 - rac{1}{\imath b}} \end{aligned}$$

 X_t: Cobb-Douglas aggregator of nonhousing consumption and housing services

$$X_t = (h_t Y_t)^{\nu} (c_t / P_t)^{1-\nu}$$

Households

- Taxes: labor income and interest income taxed at rate au
- Liquid assets: $a_{i,t}$, earning interest at rate r_t
- House:
 - House size: h_{i,t}
 - Transaction cost: proportional cost ϕ_h
- Homeowners:
 - Short-term secured borrowing (HELOC): at rate $r_t^{HL} = r_t + \vartheta$
 - Long-term (and illiquid) mortgage: $b_{i,t}$, with mortgage rate $k_{i,t}$
- Renters: aggregate rent-to-income ratio ϖ

Long-term mortgages

- Interest-only mortgage:
 - Fixed-rate
 - Interest payments $k_{i,t}b_{i,t}$ are tax deductible
- Refinancing:
 - Option to refinance: reset $k_{i,t}$ to market rate $R_t = R(S_t)$
 - Transaction cost: $\phi(b) = P_t Y_t \phi_0 + \phi_1 b$
- (P)repayment:
 - Option to reduce mortgage balance costlessly
- Option to default (on mortgage and HELOC jointly):
 - Lose house and portion $1-\zeta$ of liquid assets
 - ullet Temporarily excluded from housing market; rate of re-entry ω



Borrowing constraints

• Collateral (LTV) constraints:

$$b_{i,t+1} + \mathsf{HELOC}_{i,t+1} \le \xi_{LTV} P_t^H h_{i,t}$$

Debt service (LTI) constraints:

$$b_{i,t+1} + \mathsf{HELOC}_{i,t+1} \leq \xi_{LTI} \ y_{i,t}$$

HELOC limit:

$$\mathsf{HELOC}_{i,t+1} \leq \underline{a}P_tY_t$$

 Long vs. short-term debt: LTV and LTI imposed on HELOC every period; only at refinancing and origination for mortgage

Preemptive refinancing

- Households with no immediate liquidity needs might preemptively refinance before the constraints become binding
- LTI: cash-out when aggregate labor income growth drops, and when idiosyncratic labor income uncertainty rises
- LTV: cash-out when house prices are sufficiently high

Simulated moments estimation

Three-step estimation procedure:

- 1. Estimate/calibrate exogenous state variable dynamics
- 2. Calibrate pre-set institutional parameters
- 3. Estimate structural parameters of interest by targeting auxiliary statistics of simulated data
 - moments of assets, debt, and consumption
 - · dynamics of refinancing and cash-out

Calibration and Estimation

Household state vector:

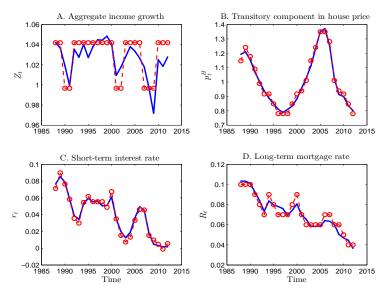
$$\underbrace{a_i, h_i, b_i, k_i}_{endogenous}, \underbrace{y_i, Z, r, p^H}_{exogenous}$$

- Aggregate state $S = (Z, r, p^H)$: restricted VAR(1) in logs; estimated using GDP, 1-year T-bill rate, and S&P Case-Shiller HPI
- Idiosyncratic labor income process: AR(1) process in logs, with heteroscedasticity (Storesletten, Telmer, Yaron, 2004)
- Mortgage rate R: function of aggregate states

$$\log R(S) = \kappa_0 + \kappa_1' \log S + \kappa_2 (\log p_t^H)^2$$

Estimated using 30-year FRM rate

Discretized aggregate state variables





Exogenous Institutional Parameters

Parameter	Value	Description
au	0.25	Income tax rate
Ħ	4	Average house price to income ratio
ξ_{LTV}	8.0	Collateral constraint
ξιτι	3.5	Debt service constraint
− <u>a</u>	30%	Max HELOC balance as fraction of avg. income
ω	0.15	Probability of return to credit market after default
ζ	1	Retention of liquid assets upon default
ϑ	0.04	Interest rate premium on HELOC

Estimation Approach: 8 parameters/14 moments

Simulated Method of Moments:

Estimate the vector of model parameters $\Theta \equiv (\delta, \gamma, \psi, \eta, \phi_0, \phi_1, \phi_h)$ such that

$$\hat{\Theta} = arg \min_{\Theta} (M - m(\Theta))' \mathbf{W} (M - m(\Theta))$$

	Parameter	Description
Preferences	δ	Subjective discount rate
	γ	Risk aversion
	ψ	Intertemporal elasticity of substitution
	u	Utility share of housing
	$ar{\eta}$	Indirect (dis)utility of renting (vs home-ownership)
Transaction	ϕ_0	Fixed cost of issuing new mortgage
Costs	ϕ_1	Proportional cost of issuing new mortgage
	$\phi_{\it h}$	Proportional cost of buying/selling a house

Use pre-specified weighting matrix \mathbf{W} ; simulation-based inference

Estimation Approach: 8 parameters/14 moments

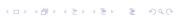
	Model	Data
Consumption/Income, average	c/pY	0.66
Consumption growth volatility, average	$\sigma(\Delta \log c_{t+1}^i)$	9%
Homeownership rate	$E[I^h]$	60%
Liquid Asset Holdings/Income (homeowners)	a/pY	0.28
Mortgage Balance/Income	b/pY	0.98
Refinancing rate	REFI	8%
HELOC Balance/Income	$-a^{-}/pY$	0.07
Refinancing Loan/Income	b'/pY	1.41
Dollar Cash-out (as a share of Refi)	$(b'-b)^+/b'$	0.12
Liquid Asset Holdings/Income (renters)	a/pY	0.18
Refinancing Regression: Details		
Coefficient on Z	eta_{Z}^{REFI}	-0.25
Coefficient on $\Delta \log H$	β_H^{REFI}	0.15
Cashout Regression: Details		
Coefficient on Z	eta_{Z}	-0.13
Coefficient on $\Delta \log H$	eta_{H}	0.06

Estimated parameters

	Parameter	Value	Description
Preferences	δ	0.920	Subjective discount rate
		(0.007)	
	γ	3.036	Risk aversion
		(0.347)	
	ψ	0.301	Intertemporal elasticity of substitution
		(0.020)	
	ν	0.134	Housing utility share
		(0.004)	
	$ar{\eta}$	0.750	Disutility of renting versus home-ownership
		(0.006)	
Institutional	ϕ_0	0.154	Fixed cost of issuing new mortgage
		(0.020)	
	ϕ_1	0.014	Proportional cost of issuing new mortgage
		(800.0)	
	ϕ_{h}	0.135	Proportional cost of buying/selling a house
		(0.017)	

Estimation results: targeted moments

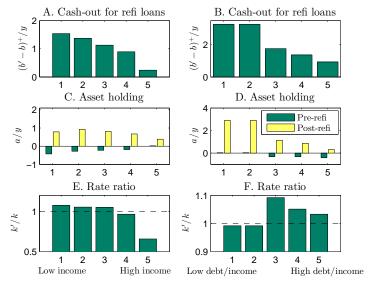
Moment	Variable	Data	Model	s.e.
All Households:				
1. Consumption/Income	c_i/y_i	0.66	0.71	0.01
2. Consumption growth volatility, %	$\sigma(\Delta \log c_{i,t+1})$	9(18)	16.4	0.01
3. Homeownership rate, %	$E[I^h]$	66.0	67.5	0.08
Homeowners:				
4. Liquid assets/Income	a_i^+/y_i	0.28	0.24	0.04
5. Mortgage/Income	b_i/y_i	0.98	0.96	0.08
6. HELOC/Income	$-a_i^-/y_i$	0.07	0.08	0.01
7. Refinancing rate, % of homeowners	REFI	8.0	11.3	0.02
8. Refi loan/Income	b_i'/y_i	1.41	2.74	0.14
Dollar cash-out/Refi loan	$(b_i^\prime - b_i)^+/b_i^\prime$	0.12	0.51	0.03
Renters:				
10. Liquid assets/Income	a_i^+/y_i	0.18	0.15	0.06
Refinancing Regression:				
11. Coefficient on Z	β_Z^{REFI}	-0.25	-0.24	0.41
12. Coefficient on $\Delta \log H$	$eta_{H}^{ ilde{R}EFI}$	0.15	0.08	0.14
Cashout Regression:				
13. Coefficient on Z	β_{Z}	-0.12	-0.23	0.43
14. Coefficient on $\Delta \log H$	β_H	0.06	0.11	0.15



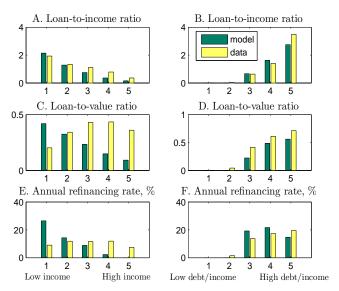
Estimation results: additional moments

Moment	Variable	Data	Model	s.e.
Volatility of agg. consumption growth, %	$\sigma(\Delta \log C_{t+1})$	2.7	3.9	0.01
Sensitivity of consumption to Z shocks Sensitivity of consumption to H shocks	$eta_{\mathcal{L}}^{\mathcal{C}} \ eta_{\mathcal{H}}^{\mathcal{C}}$	0.46 0.06	1.30 0.09	0.20 0.05
Sensitivity of consumption to lagged r Sensitivity of consumption to lagged R	$eta_{r}^{C} \ eta_{R}^{C}$	0.07 0.09	0.09 0.10	0.43 0.65
Refinancing regression coefficient on R Cashout regression coefficient on R	$eta_{\it R}^{\it REFI}$ $eta_{\it R}$	-1.91 -0.43	-1.09 -0.83	0.67 0.73

Quintiles sorted on income and debt/income

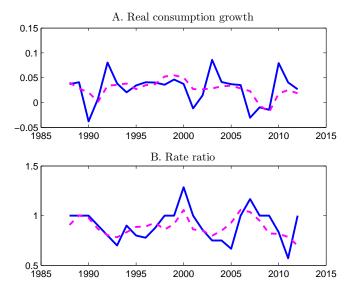


Model vs. Data (sorted on income/house value and debt/income, SCF)





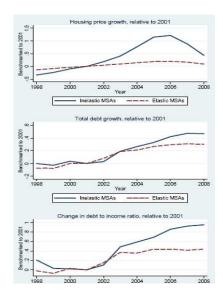
Aggregate time series

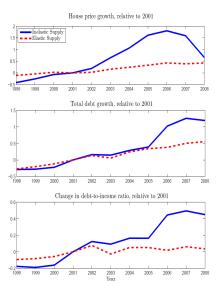


Experiment: replicating Mian-Sufi evidence

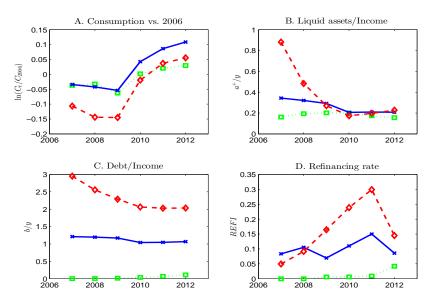
- Feed in alternative time series
- "Inelastic MSAs": model with 2x volatility of p^H shocks
- "Elastic MSAs": $p^H = 1$ (house prices comove with income)

Mian-Sufi experiment: Leverage Run-up





Leverage-sorted groups during crisis (model)



Conclusion

- "Standard" model able to account for the dynamics of household leverage and consumption over the "boom" and the "bust" periods
- Financing frictions have quantitatively large effects on household finance and consumption
 - Long vs. short-term debt: deleveraging effect substantial even with long-maturity debt
 - Precautionary savings in liquid assets vs. illiquid home equity
- Substantial heterogeneity in refi and consumption behavior in response to monetary shocks and government programs (e.g., HARP and FHA loans)

Aggregate-level regression: REFI

Regression:

$$REFI_{t} = b_{0} + b_{IP}\Delta IP_{t} + b_{r30}R_{t}^{30Y} + b_{HPI}\Delta HPI_{t}$$
$$+b_{r}R_{t}^{3M} + b_{\Delta r30}\Delta R_{t}^{30Y} + \epsilon_{t}$$

- MBA REFI Index (# of loans, refinancing only)
- Monthly data, January 1990 March 2011.



Aggregate Refinancing Activity

ΔIP_t	-0.422	-0.253	-0.196	-0.268
	(0.161)	(0.087)	(0.097)	(0.091)
ΔHPI_t		0.148	0.156	0.155
		(0.098)	(0.095)	(0.095)
R_t^{M30}		-1.914	-1.982	-2.700
		(0.667)	(0.675)	(0.601)
$R_t^{M30} - R_{t-12}^{M30}$			-1.464	
			(0.845)	
$R_t^{M30} - R_{avg,t}^{M30}$				-2.609
				(1.247)
r_t^{1Y}		-1.156	-0.986	-0.278
		(0.611)	(0.566)	(0.496)
Adj. R ²	0.060	0.654	0.673	0.687

Aggregate-level regression: \$ Home Equity Withdrawal

Regression:

$$\begin{split} HEW_t^j &= \beta_0^j + \beta_Z^j \Delta P I_t + \beta_H^j \Delta H P I_t + \beta_R^j R_t^{M30} + \beta_{RI}^j \Delta R_t^{M30} + \beta_r^j r_t^{1Y} + \epsilon_t. \\ \text{where } j \in \{\text{Cash-out}, \text{HELOC}\} \end{split}$$

- Freddie Mac, \$ Cash-out (over year-ago personal income)
- Fed Flow of Funds Accounts, \$ Home equity loans and lines of credit (over year-ago personal income)
- Quarterly data, Q1 1993 Q1 2011

Back to Intro Structural Estimation

Home Equity Withdrawal

	Prime, first-lien mortgage			HEL(OC)s			
ΔPI_t	-0.003	-0.116	-0.132	-	0.056	-0.013	-0.027
	(0.051)	(0.041)	(0.042)		(0.041)	(0.032)	(0.031)
ΔHPI_t		0.061	0.063			0.062	0.064
		(0.023)	(0.021)			(0.018)	(0.016)
R_t^{M30}		-0.430	-0.431			-0.038	-0.039
		(0.146)	(0.133)			(0.112)	(0.099)
$R_t^{M30} - R_{t-1}^{M30}$			0.207				0.185
			(0.084)				(0.063)
r_t^{1Y}		0.279	0.262			0.045	0.030
-		(0.099)	(0.087)			(0.076)	(0.065)
Adj. R ²	-0.055	0.487	0.545		0.111	0.611	0.679

Exploiting State-Level Variation

- At the state level, macroeconomic conditions are less likely to comove with interest rates
- Variation in the ability to use housing collateral (prices vary)
 - Caplin, Freeman and Tracy (1997)
 - Lustig and Van Nieuwerburgh (2010)
 - Mian and Sufi (2010)
 - Case, Quigley and Shiller (2011)
 - Midrigan and Philippon (2011)
- Use state-level aggregations of HMDA data
 - all originated loans

Business Cycle and Refinancing: State Level Variation

• Quarterly refi loans (scaled by population):

$$\begin{split} REFI_{t}^{State} &= b_{Cycle}Cycle_{t}^{State} + b_{HPI}\Delta HPI_{t}^{State} \\ &+ b_{CH}Cycle_{t}^{State} \times HPI_{t}^{State} + b_{\bar{R}}\bar{R}_{t}^{State} + b_{w}WAC_{t}^{State} \\ &+ b_{r}R_{t}^{3M} + b_{r30}R_{t}^{30Y} + b_{r30}\Delta R_{t}^{30Y} + \mathbf{b}_{t} + \mathbf{b}_{State} + \epsilon_{t}, \end{split}$$

- BC = Payroll, Coincident Economic Activity Index or Personal Income Growth
- Quarterly data, March 1993 December 2007

Refi Loan Originations

ВС	ΔHPI_t	$BC \times HPI_t$	WACt	\bar{R}_t	R_t^{30Y}	R_t^{3M}	ΔR_t^{30Y}	R^2	
BC = Payroll									
-0.29	0.17	-1.85	0.62	1.50	-1.70	-0.75	-0.20	0.61	
(0.05)	(0.01)	(0.39)	(0.05)	(0.22)	(0.12)	(0.06)	(0.12)		
-0.24	0.10	-0.64	-2.74	0.32				0.89	
(0.05)	(0.01)	(0.20)	(0.67)	(0.37)					
BC = CEAI									
-0.10	0.16	-1.29	0.64	1.56	-1.79	-0.80	-0.23	0.60	
(0.03)	(0.01)	(0.34)	(0.05)	(0.23)	(0.12)	(0.07)	(0.12)		
-0.14	0.10	-0.47	-2.62	0.36				0.89	
(0.03)	(0.01)	(0.13)	(0.69)	(0.37)					
BC = Personal Income									
0.01	0.15	-1.89	0.61	1.84	-1.89	-1.00	-0.32	0.60	
(0.03)	(0.01)	(0.37)	(0.05)	(0.26)	(0.13)	(0.07)	(0.13)		
-0.10	0.09	-0.36	-2.63	0.18				0.89	
(0.03)	(0.01)	(0.22)	(0.70)	(0.39)					

Household Problem: Home-owner

$$U_i^h(a_i,b_i,k_i,s_i) = \max_{a_i',b_i',l_i^{RF}} \left[(1-\delta)(c_i/p)^{\frac{1-\gamma}{\theta}} + \delta \mathbb{E}\left[\max\left(U_i^{h'},U_i^{hr'},U_i^{hd'}\right)^{1-\gamma}\right]^{\frac{1}{\theta}} \right]^{\frac{\nu}{1-\gamma}}$$

subject to

$$c_{i} + rac{a_{i}^{+\prime}}{1 + (1 - au)r} + rac{a_{i}^{-\prime}}{1 + r^{HL}} + b_{i} = (1 - au)(y_{i} - k_{i}b_{i}) + a_{i} + b_{i}^{\prime} - \phi(b_{i}^{\prime}) \ I_{i}^{RF}, \ (b_{i}^{\prime} - b_{i}) \ (1 - I_{i}^{RF}) \leq 0, \ c_{i}, b_{i}^{\prime} \geq 0,$$

and the borrowing constraints

Household Problem: Renter

Renters:

• Incur a rental expense: share $\frac{\eta}{1+\eta}$ of per period income

$$U_i^r(a_i,s_i) = \max_{a_i^r} \quad \left[(1-\delta)(c_i/\rho)^{\frac{1-\gamma}{\theta}} + \delta \mathbb{E}\left[\max\left(U_i^{rh'},U_i^{r'}\right)^{1-\gamma}\right]^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}}$$

subject to,

$$c_i = \underbrace{(1- au)rac{y_i}{1+\eta}}_{After-Tax\ Income} + \underbrace{a_i - rac{a_i'}{1+(1- au)r}}_{Change\ in\ Savings}$$

$$a_i', c_i \geq 0$$

Household Problem: Default State

Households in Default State:

- Become renters and stay in that state (w.p. $1-\omega$)
- Are not allowed to buy a house

$$U_{i}^{d} = \max_{a_{i}^{r}} \left[(1 - \delta)(c_{i}/p)^{\frac{1 - \gamma}{\theta}} + \delta \mathbb{E} \left[(1 - \omega) \left(U_{i}^{d'} \right)^{1 - \gamma} + \omega \max \left(U_{i}^{rh'}, U_{i}^{r'} \right)^{1 - \gamma} \right]^{\frac{1}{\theta}} \right]^{\frac{\theta}{1 - \gamma}}$$
 subject to,

$$c_i = \underbrace{(1- au)rac{y_i}{1+\eta}}_{After-Tax\ Income} + \underbrace{a_i - rac{a_i'}{1+(1- au)r}}_{Change\ in\ Savings}$$
 $a_i', c_i > 0$