Common Risk Factors in the US Treasury and Corporate Bond Markets: An Arbitrage-free Dynamic Nelson-Siegel Modeling Approach†

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Abstract

The vast majority of the term structure literature has focused on modeling the risk-free term structure as implied by Treasury bond yields. As fixed-income markets should be interconnected, we combine the modeling of Treasury yields with a modeling of the common factors present in representative risky credit spread term structures derived from Bloomberg corporate bond data. The question we address is whether we can improve our understanding of, and our ability to forecast, Treasury yields by incorporating information from corporate bond market. We use the arbitrage-free dynamic version of the Nelson-Siegel yield-curve model derived Christensen, Diebold and Rudebusch (2007) to model Treasury yields and corporate bond spreads across rating and industry categories. In addition to the three-factor Nelson-Siegel factors for Treasury yields, we find two common factors—a level and a slope factor—are required to capture the time series dynamics of aggregated credit spreads. We find that the preferred specifications of the joint dynamics of all five factors have feedback effects from the Treasury factors to the credit risk factors, but we also find feedback effects from the credit risk factors to the Treasury factors. To determine the significance of these feedback effects, we perform an out-of-sample forecast exercise. The results so far suggest that the preferred Treasury yield model can easily beat the random walk and that adding the information from the credit markets allows us to improve forecast performance even further for forecast horizons up to 26-weeks.

†The views expressed are those of the authors and should not be interpreted as reflecting the views of the Federal Reserve Bank of San Francisco or the Board of Governors of the Federal Reserve System. We thank Robert Goldstein for many valuable comments.

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1 Introduction

Treasury bond yields are the main mechanism through which monetary policy affects the real economy. However, as shown by the recent turmoil in the credit markets since August 2007, the credit markets and their interactions with the factors driving changes in Treasury yields can be equally important. In this paper, we address the question of whether information contained in the credit spreads of corporate bonds can be useful in forecasting Treasury yields. To the best of our knowledge, this paper is the first to conduct a joint estimation of both risk-free and risky factors with feedback effects from the Treasury market to the credit markets and from the credit markets to the Treasury market.

How might these feedback effects come about? The Treasury bond markets are impacted by the variation of the business cycle, mainly through bond investors' expectations about the anticipated response of the Federal Reserve to such variation. The credit markets are impacted by business cycle variation in two ways. First, monetary policy changes create market risk in the form of interest rate uncertainty in much the same way as observed in the Treasury bond market. Second, the business cycle variation itself has a direct influence on credit risk in that it impacts the ability of firms to pay their debts (i.e., their probability of default). The key question is, with forward looking investors in the credit markets, can information from the credit markets improve our understanding of the dynamics of the Treasury bond market.

Going back at least to Litterman and Scheinkman (1991), it is an established fact that close to 99.9% of the variation in Treasury yields can be explained by three latent variables often referred to as a level, slope, and curvature factor, respectively. Thus, the starting point for our analysis is a three-factor affine arbitrage-free term structure model of Treasury bond yields. In the choice of model, we focus on the Nelson-Siegel model, which is a robust, yet flexible model with a very compelling interpretation of the latent state variables as level, slope, and curvature factor. Christensen, Diebold, and Rudebusch (2007) derive the affine arbitrage-free class of Nelson-Siegel term structure models (i.e., AFDNS model). Thus, building on their results, we are able to overcome the theoretical deficiency of the Nelson-Siegel model and present a model with a full continuous-time dynamic description of the state variables that is arbitrage-free while maintaining the Nelson-Siegel factor loading structure for the state variables in the yield function.

Focusing directly on credit risk, the variation of the business cycle naturally causes the default probability of all firms to exhibit some amount of co-variation that, by implication, will show up as co-variation in their credit spreads. In addition, researchers have found that credit spreads are above and beyond what can be justified by the actual expected default loss risk. This has been termed the 'credit spread puzzle', which holds across rating and industry categories.[1] Even correcting for the difference in tax treatment and liquidity relative to the Treasury bond market, a significant residual component remains in credit spreads, which is likely to be a risk premium that reflects overall bond investor sentiment. In this paper, we take this as evidence that systematic factors are common to the credit spreads of firms across business sectors and rating categories. Since the nature of any such systematic risk factors is so far unknown, we use latent factor models to extract them and examine their interaction with the systematic components of Treasury bond yields.[2]

[1] See Elton et al. (2001), Collin-Dufresne et al. (2001), Berndt et al. (2005) for discussions.

[2] Two recent studies have motivated this choice. First, Krishnan, Ritchken, and Thomson (2007) use the standard

The specifics of our model are as follows. Treasury bond yields are assumed to follow a three-factor model with a level, slope, and curvature factor like in the AFDNS model. We proceed in two estimation steps. We first estimate the three Treasury factors using Treasury yields only. This gives us a preferred model that can be used as a benchmark in terms of forecasting Treasury bond yields. In a second step, we estimate jointly the three Treasury factors and the two credit risk factors by combining Treasury yields with credit spreads from each of the four business sectors in our data set. By imposing the Nelson-Siegel factor structure on the latent factors, the estimation of even the five factor models is greatly facilitated and allows us to perform weekly re-estimations of our models for forecastin gpurposes over a two-year period. We then can study the out-of-sample forecast performance of each model and let this be part of our model selection criteria. As shown by Christensen, Diebold, and Rudebusch (2007), in-sample measures of fit can be misleading and, by implication, do not constitute a sufficient model selection criterion. We supplement our in-sample analysis with the out-of-sample forecast evaluation in order to better judge the appropriateness of the models analyzed.

Our major empirical finding is that there are two common credit risk factors in all the four industry sectors we analyze: one is a level factor, the other is a slope factor. Furthermore, the results indicate that the common credit risk factors across sectors are very similar both in their estimated paths as well as in their estimated parameters for the factor dynamics. One possible interpretation of this finding is that these are two economy-wide systematic risk factors impacting the credit spreads of all firms independent of rating, maturity, and business sector. As far as forecasting Treasury bond yields goes, our preferred Treasury bond yield model easily beats the random walk assumption at the 26- and 52-week forecast horizon. More importantly, adding information from the credit markets (i.e., these two economy-wide systematic factors influencing the corporate sector) appears to improve on the forecast performance of our model for forecast horizons up to 26-weeks.

The remainder of the paper is structured as follows. Section 2 describes the related literature. Section 3 describes the data and the data-driven motivation for the modeling approach. Section 4 contains the model description. Section 5 details the estimation method. Section 6 has the in-sample estimation results, while section 7 contains the evaluation of the forecast performance. Section 8 describes some additional misspecification tests. Finally, Section 9 concludes the paper. The appendix contains additional technical details.

2 Related literature

Our study explicitly assumes that there exist some, as of yet, unidentified common factors that drive the movement of credit spreads across different industries, rating categories, and maturities. The existing credit risk literature provides evidence of the existence of such common factors, and

three-factor Nelson-Siegel yield curve model to fit both the risk-free Treasury yields as well as the credit spread curves of individual firms, and they find that firm-level credit spread curves combined with the risk-free yield curve contain all of the information necessary for predicting future credit spreads. Thus, the Nelson-Siegel based approach appears to be just as relevant in a credit risk setting as the NS model has proven to be for the modeling of government bond yield term structures at central banks around the world, see BIS (2005) for evidence on the latter. Second, Diebold, Li, and Yue (2007) introduce a global factor model for government bond yields based on the Nelson-Siegel approach with a global level, slope, and curvature factor. In that model, each currency area can load more or less heavily on the latent global level, slope, and curvature factor. Here, we apply this setup to each sector of the economy assuming that there are latent sector-specific factors that each rating category within the sector is more or less sensitive to.

it even provides some indirect evidence of the nature of these factors. The major findings are briefly summarized in the following.

Duffie, Saita, and Wang (2007) find that the probability of default for individual firms is well determined by two firm-specific variables (the distance-to-default in a Moody's KMV EDF-style and the trailing 12-month return on its stock) and two variables common to all 2770 US industrial firms in their sample. The two common factors are the 3-month US T-Bill rate and the 12-month trailing return on the S&P 500-index. Thus, the slope-factor from the Treasury data and the latent factor driving the S&P 500-index are key candidates as common factors driving the actual default probability across different rating categories in the US industrial sector. Equally important for our study, they find that neither the 10-year Treasury yield, US personal income growth, US GDP growth, nor the AAA-BBB bond yield spread add any significant explanatory power beyond that of the four variables mentioned above.

Collin-Dufresne, Goldstein, and Martin (2001) study a sample of 688 corporate bonds and they have two major findings. First, including all firm-specific default-related variables only explains about 25% of the observed variation in credit spreads across industries, ratings, and maturities. Second, in the residuals they find a significant cross-correlation across all 688 bonds 80% of which can be explained by the first principal component. Furthermore, the factor loadings on the eigenvector for the first principal component indicate that credit spreads of all ratings and maturities have a positive loading on this factor of about the same size. The only notable variation in the loadings on this factor across firms is that firms of lower credit quality tend to load slightly more heavily on this factor than higher rated firms. This points to the existence of a common level factor in credit spreads that is not related to the actual default probability of firms. Their interpretation is that it reflects local supply and demand shocks (local here means confined to the corporate bond market). An alternative interpretation is that it reflects a risk premium factor specific to the corporate bond market. Finally, in all their regressions, changes in the 10-year Treasury yield come across as highly significant, while changes in the 10-yr vs. 2-yr slope of the Treasury yield curve rarely show up as significant. The last observation is at odds with the results in Duffie et al. (2007) and with the results in this paper. One explanation for this difference may be that they are analyzing credit spread changes and yield changes whereas this paper and Duffie et al. (2007) are explaining the levels of credit spreads and default probabilities, respectively. The differencing may eliminate some of the level effects that we observe.

Driessen (2005) includes common credit risk factors in his analysis of credit spread dynamics across 104 firms. He finds that two common credit risk factors are needed in addition to his two Treasury factors in order to explain the joint movement in credit spreads across the firms in his sample. Unfortunately, he provides few details about the characteristics of his common credit risk factors.

In summary, there is evidence that Treasury factors have a role in determining movements in credit spreads. The exact role, however, is not clear. The 3-month yield, which is a proxy for the slope factor, is observed to affect the actual default probability which is most likely tied to the fact that the slope factor reflects the current state of the business cycle which naturally impacts the default probability of firms independent of industry and rating. Changes in the 10-year Treasury yield, which is a proxy for changes in the level factor of the Treasury bond yield curve, also seem to impact credit spreads. Whether this effect is related to the default probability or the risk

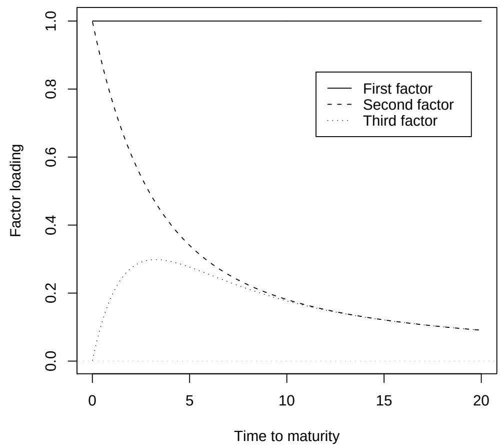


Figure 1: **Factor loadings in the Nelson-Siegel yield function.**
Illustration of the factor loadings on the three state variables in the Nelson-Siegel model. The value for λ is 0.55 and maturity is measured in years.

premium on corporate bonds is not clear from the study by Collin-Dufresne et al. (2001). Finally, credit spread dynamics appear to be driven by two additional common credit risk factors. One factor impacts the part of the credit spread that accounts for the default probability, and this factor should be correlated with the returns observed in the general stock market as observed by Duffie et al. (2007). The intuition behind this is that the development of the general stock market reflects the current general business conditions that would impact any firm independent of industry and rating. The other factor is some kind of risk premium factor. In theory, all bonds should be subject to the same marginal investor. In that case it may be reasonable that a risk premium factor would be close to a level factor impacting the credit spread of all corporate bonds in much the same way independent of their industry, rating, and maturity. This would be in line with the observations in Collin-Dufresne et al. (2001).

The purpose of this paper is to propose a simple and parsimonious model that can tie all these pieces together.

3 Data description and model motivation

The Nelson-Siegel model fits the yield curve at any point in time with the simple functional form[3]

$$y(\tau) = \beta_0 + \beta_1 \Big(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \Big) + \beta_2 \Big(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \Big), \tag{1}$$

where $y(\tau)$ is the zero-coupon yield with τ denoting the time to maturity, and β_0, β_1, β_2, and λ are model parameters. The three β parameters can be interpreted as factors and their corresponding factor loadings in the Nelson-Siegel yield curve function are illustrated in Figure 1.

Due to its flexibility this model is able to provide a good fit of the cross section of yields at a given point in time which is the primary reason for its popularity amongst financial market

[3]This is Equation (2) in Nelson and Siegel (1987).

Maturity	Mean	St.dev.	Skewness	Kurtosis
3	0.0401	0.0175	-0.5557	1.7322
6	0.0408	0.0178	-0.5516	1.7545
12	0.0422	0.0175	-0.5356	1.8332
24	0.0446	0.0160	-0.4778	1.9655
36	0.0464	0.0144	-0.3891	2.0197
60	0.0494	0.0121	-0.1941	2.0182
84	0.0518	0.0105	-0.0350	2.0060
120	0.0548	0.0090	0.1199	2.0627

Table 1: **The summary statistics for the Treasury bond yields.**
Summary statistics for the sample of weekly observed Treasury zero-coupon bond yields covering the period from January 6, 1995 to August 4, 2006.

practitioners. Although for some purposes such a static representation appears useful, a dynamic version is required to understand the evolution of the bond market over time. Diebold and Li (2006) achieve this by introducing time-varying parameters

$$y_t(\tau) = L_t + S_t\Big(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\Big) + C_t\Big(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\Big), \tag{2}$$

where L_t, S_t, and C_t can be interpreted as level, slope, and curvature factors (given their associated Nelson-Siegel factor loadings). Furthermore, once the model is viewed as a factor model, a dynamic structure can be postulated for the three factors, which yields a fully dynamic version of the Nelson-Siegel model. In the following we will argue that the features of this model are relevant for modeling both the Treasury bond yields as well as the corporate bond credit spreads considered in this paper.

We use Treasury yields as a proxy for the risk-free rate. This choice deserves some comments given the fact that we will be using Treasury bond yields to extract the credit spreads that goes into the estimation of the credit risk models. Blanco, Brennan, and Marsh (2005) find that for accurate pricing of credit risk the choice of proxy for the risk-free rate can be critical. In their study they match corporate bond credit spreads with the corresponding CDS rates and find that swap rates are the most appropriate choice as risk-free benchmark. However, for our purposes of estimating common risk factors we think that this will be a second-order effect, and even if using Treasury yields should introduce a bias, the credit spreads for all ratings in each industry will be biased in the same direction. Thus, the relative behavior, and in particular, the relative factor loadings on the common credit risk factors across the different rating categories in each industry should not be impacted in any significant way by this choice of risk-free benchmark. Finally, given that one purpose of this paper is to improve our understanding of the Treasury bond market and our ability to forecast those yields, we feel that it is natural to center the analysis around Treasury yields and measure the credit spreads of corporate bonds relative to the corresponding Treasury bond yields.

The specific Treasury yields we use are zero-coupon yields constructed by the method described in Gürkaynak, Sack, and Wright (2006)[4] and briefly detailed here. For each business day from

[4]The Board of Governors in Washington DC frequently updates the factors and parameters of this method, see the related website http://www.federalreserve.gov/pubs/feds/2006/index.html

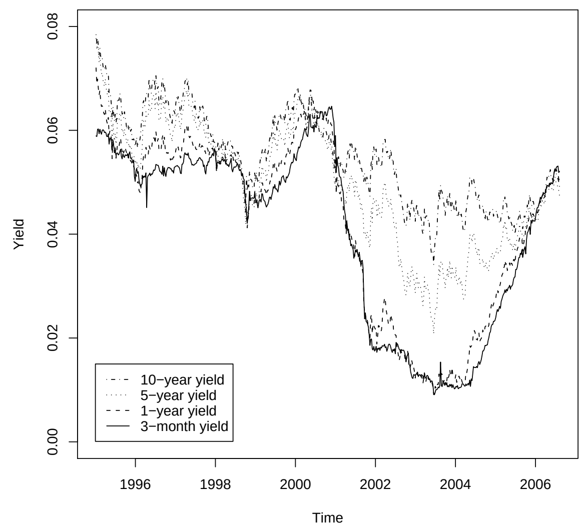


Figure 2: **Time series of Treasury bond yields.**
Illustration of the weekly observed treasury zero-coupon bond yields covering the period from January 6, 1995 to August 4, 2006. The yields shown have maturities: 3-month, 1-year, 5-year and 10-year.

June 14, 1961 to the present,[5] a zero-coupon yield curve of the Svensson (1994)-type

$$y_t(\tau) = \beta_0 + \frac{1-e^{-\lambda_1\tau}}{\lambda_1\tau}\beta_1 + \left[\frac{1-e^{-\lambda_1\tau}}{\lambda_1\tau} - e^{-\lambda_1\tau}\right]\beta_2 + \left[\frac{1-e^{-\lambda_2\tau}}{\lambda_2\tau} - e^{-\lambda_2\tau}\right]\beta_3$$

is fitted to price a large pool of underlying off-the-run Treasury bonds. Thus, for each business day since June 1961 we have the fitted values of the four factors $(\beta_0(t), \beta_1(t), \beta_2(t), \beta_3(t))$ and the two parameters $(\lambda_1(t), \lambda_2(t))$. From this data set zero-coupon yields for any relevant maturity can be calculated as long as the maturity is within the range of maturities used in the fitting process. As demonstrated by Gürkaynak, Sack, and Wright (2006), this model fits the underlying pool of bonds extremely well. By implication, the zero-coupon yields derived from this approach constitute a very good approximation to the true underlying Treasury zero-coupon yield curve.

In order to match the maturity spectrum for the corporate bond yield data we construct Treasury zero-coupon bond yields with the following maturities: 3-month, 6-month, 1-year, 2-year, 3-year, 5-year, 7-year, and 10-year, and we limit our sample to weekly observations (Fridays) over the period from January 6, 1995 to August 4, 2006. The summary statistics are provided in Table 1, while Figure 2 illustrates the constructed time series of the 3-month, 1-year, 5-year, and 10-year Treasury zero-coupon yields.

In the literature on US Treasury bond yields researchers have found that three factors are sufficient to model the time-variation in cross-sections of such yields, for an early example see Litterman and Scheinkman (1991). This holds for our Treasury bond yield data as well. However, to get some insights about the characteristics of these three factors we focus on the eigenvectors that correspond to the first three principal components. They are reported in Table 2.

[5]The latest update has data until December 28, 2007 at the time of this writing.

The first principal component explains 95.6% of the variation in the Treasury bond yields and its loading across maturities is uniformly negative. Thus, when there is a shock to the first principal component it changes all yields in the same direction independent of maturity. We refer to this as a level factor.

Likewise, it follows from the table that the second principal component explains about 4.1% of the variation in this data set. This factor has large negative loadings for the shorter maturities while it has large positive loadings for the long maturities. Thus, a positive shock to this factor causes short-term yields to move lower while long-term yields go up, effectively creating a steepening of the yield curve. In case of a negative shock to the second principal component we get the reverse movements, short-term yields go up while long-term yields move down leading to a flattening of the yield curve. We refer to this as a slope factor as it determines the slope of the yield curve.

Finally, the third principal component explains an additional 0.3% of the variation in the data. Its factor loading is a V-shaped function of maturity with large positive loadings for the short and long maturities, while its loading is large negative for the medium-term maturities. Thus, when there is a negative shock to the third principal component, the short and long end of the yield curve moves down while the medium-term yields move up, effectively creating a hump shaped yield curve. Similarly, a positive shock to this factor will lead to an inverted hump shaped yield curve. For these reasons this factor is naturally interpreted as a curvature factor.

In summary, for Treasury bond yields we can easily explain 99.9% of the total variation with just three factors. Focusing on the eigenvectors corresponding to the first three principal components we see a clear pattern that is well approximated by the factor loadings of the level, slope, and curvature factor in the Nelson-Siegel model as illustrated in Figure 1.

The corporate bond data consists of representative zero-coupon yields for a number of rating categories across four different US industries. The data is downloaded from Bloomberg. We focus on the following maturities: 3-month, 6-month, 1-year, 2-year, 3-year, 5-year, 7-year and 10-year. We neglect the 20-year and 30-year maturities primarily because the Nelson-Siegel models used in this paper are not able to match yields beyond the 10-year maturity. Therefore, little is lost estimation-wise by discarding the long-term yields.[6]

The four sectors we have data for are the following: US Industrial firms, US Financial firms, US Banks, and US Utility firms. For each sector a number of different rating categories are available. For the US Industrial firms we have a full sample of corporate bond yields for the BBB, A, AA, and AAA rating categories. For the US Financial firms we have a full sample for the A, AA, and AAA rating categories while the BBB+ category have a 20-month subperiod with missing data from April 22, 2000 to January 17, 2002. For the US Banks the BBB and A rating categories are represented by a full sample while the AA category is only available as of Sept. 21, 2001. Finally, for the US Utility firms only the A and BBB rating categories are available. However, both are represented with a full sample. For all four sectors 'full sample' means weekly observations (Fridays) covering the period from January 6, 1995 to August 4, 2006.

Since the Bloomberg data are annual discrete interest rates, we convert the corporate bond yields into continuously compounded yields. Denote the n-year yield at time t by $r_t(n)$, then the

[6] Controlling estimations performed with and without the long-term yields in the data sample confirm this.

Maturity	First	Second	Third
3	-0.421	-0.403	0.544
6	-0.432	-0.347	0.167
12	-0.429	-0.202	-0.259
24	-0.392	0.043	-0.469
36	-0.352	0.201	-0.370
60	-0.286	0.376	-0.043
84	-0.239	0.463	0.211
120	-0.190	0.527	0.452
Explain	0.956	0.041	0.003

Table 2: **The eigenvectors of the first three principal components in the Treasury bond yields.**
The loadings of each maturity on the three eigenvectors that correspond to the first three principal components in the weekly Treasury zero-coupon bond yield data covering the period from January 6, 1995 to August 4, 2006.

corresponding zero-coupon bond price is given by

$$P_t(n) = \frac{1}{(1 + r_t(n))^n}.$$

The annual yields are converted into continuously compounded yields through the equation

$$P_t(n) = \frac{1}{(1 + r_t(n))^n} = e^{-y_t(n)n} \iff y_t(n) = -\frac{1}{n} \ln \frac{1}{(1 + r_t(n))^n} = \ln(1 + r_t(n)).$$

For maturities shorter than one year we assume the standard convention of linear interest rates. Thus, the zero-coupon bond price corresponding to the 6-month yield is calculated by

$$P_t(6m) = \frac{1}{1 + 0.5r_t(6m)} = e^{-0.5y_t(6m)}$$

and the corresponding continuously compounded yield is given by

$$y_t(6m) = -2 \ln \frac{1}{1 + 0.5r_t(6m)} = 2 \ln(1 + 0.5r_t(6m)).$$

A similar method is applied to the 3-month yields.

Finally, in order to convert the continuously compounded corporate zero-coupon bond yields into continuously compounded credit spreads, we deduct the corresponding observed Treasury zero-coupon yield. Thus, the credit spreads for each rating category c and industry i are given by

$$s_t^{i,c}(\tau) = y_t^{i,c}(\tau) - y_t^T(\tau), \qquad \begin{array}{l} i \in \{\text{Industrials, Financials, Banks, Utilities}\}, \\ c \in \{BBB, A, AA, AAA\}. \end{array}$$

Summary statistics for all the credit spreads are provided in Table 3. Across industries the credit spreads appear to share some general characteristics, both the average spread and the credit spread volatility for a given maturity increase as credit quality deteriorates.[7]

[7] There is one exception to this rule: the volatility of the credit spreads of AA US Industrials is marginally lower than those of AAA US Industrials for the maturities from 3 to 12 months. This may be tied to the fact that there is a very limited number of industrial firms with a AAA rating in the period analyzed here. Thus, this rating category may be subject to a higher degree of idiosyncratic variation in the underlying pool of bonds.

Maturity	US Industrials							
	BBB		A		AA		AAA	
	Mean	St. dev.	Mean	St. dev.	Mean	St. dev.	Mean	St. dev.
3	77.27	31.25	46.99	20.52	28.61	16.70	21.62	16.90
6	85.85	30.67	55.09	19.55	37.08	15.41	30.05	15.50
12	84.30	35.75	52.79	20.23	34.86	12.61	28.23	12.88
24	90.26	38.70	56.11	22.11	37.54	12.97	30.60	12.64
36	98.53	41.71	63.17	27.78	42.28	16.21	35.20	15.78
60	108.16	41.20	71.23	27.11	50.08	18.29	43.34	16.39
84	117.69	42.83	78.03	29.14	54.10	22.05	46.03	19.78
120	112.44	38.86	71.54	28.56	48.35	24.06	41.25	21.62
No. obs	605		605		605		605	

Maturity	US Financials							
	BBB+		A		AA		AAA	
	Mean	St. dev.	Mean	St. dev.	Mean	St. dev.	Mean	St. dev.
3	64.03	28.48	44.94	21.97	30.69	19.84	23.10	19.00
6	73.62	26.36	53.29	20.12	38.61	16.49	31.51	15.65
12	73.72	32.31	54.68	21.44	40.84	14.95	33.28	12.66
24	86.60	45.90	62.71	24.65	48.16	16.71	41.39	14.43
36	97.59	52.27	73.94	31.54	57.55	22.85	50.18	19.54
60	112.66	56.80	86.92	35.28	70.78	26.45	59.76	21.37
84	121.63	56.34	96.99	35.66	79.00	29.27	67.13	26.11
120	116.54	48.06	93.54	35.57	74.71	29.41	60.69	22.55
No. obs	515		605		605		605	

Maturity	US Banks							
	BBB		A		AA		AAA	
	Mean	St. dev.	Mean	St. dev.	Mean	St. dev.	Mean	St. dev.
3	54.65	23.27	40.79	20.26	20.80	14.34	n.a.	n.a.
6	68.14	23.63	52.04	20.55	29.75	13.62	n.a.	n.a.
12	69.91	22.11	52.84	17.62	43.35	17.01	n.a.	n.a.
24	80.37	31.53	61.47	23.04	52.65	16.88	n.a.	n.a.
36	90.15	38.86	73.09	32.41	65.77	29.10	n.a.	n.a.
60	102.14	39.61	81.23	31.59	67.32	21.92	n.a.	n.a.
84	117.23	47.92	96.40	38.34	93.24	30.93	n.a.	n.a.
120	104.28	42.11	88.10	33.27	71.24	19.60	n.a.	n.a.
No. obs	605		605		255		n.a.	

Maturity	US Utilities							
	BBB		A		AA		AAA	
	Mean	St. dev.	Mean	St. dev.	Mean	St. dev.	Mean	St. dev.
3	71.84	40.15	48.03	24.82	n.a.	n.a.	n.a.	n.a.
6	81.36	40.92	56.59	24.87	n.a.	n.a.	n.a.	n.a.
12	83.49	44.41	57.99	26.18	n.a.	n.a.	n.a.	n.a.
24	91.58	48.48	65.38	30.34	n.a.	n.a.	n.a.	n.a.
36	97.88	49.21	73.20	34.73	n.a.	n.a.	n.a.	n.a.
60	110.07	52.74	84.12	39.55	n.a.	n.a.	n.a.	n.a.
84	115.44	50.99	87.66	40.56	n.a.	n.a.	n.a.	n.a.
120	115.84	48.92	83.60	44.17	n.a.	n.a.	n.a.	n.a.
No. obs	605		605		n.a.		n.a.	

Table 3: **The summary statistics for the credit spreads.**
Summary statistics for the weekly zero-coupon credit spreads for US Industrial firms, US Financial firms, US Banks, and US Utility firms, respectively, for 8 fixed maturities covering the period from January 6, 1995 to August 4, 2006. All numbers are measured in basis points.

To provide some preliminary, non-parametric evidence of the existence and characteristics of common factors in the credit spread data, we pool together the 11 (industry, rating)-combinations for which we have a full sample, a total of $11 \times 8 = 88$ time series. In a second step, we derive the covariance matrix for these 88 time series and make a principal component analysis. The results are provided in Table 4.

The principal component analysis reveals that the first three principal components each explain 78.9%, 6.8%, and 6.0% of the total variation in the observed credit spreads. By implication, a three-factor model may explain as much as 91% of the observed variation in these 88 time series. Beyond this point each additional factor contributes very little. For example, it takes 8 factors to explain 95% of the variation, and to explain 99% requires 29 factors.

The approximately 79% of the variation explained by the first principal component has a somewhat remarkable resemblance to the results in Collin-Dufresne, Goldstein, and Martin (2001) who find that their residuals across firms are highly correlated and that some 80% of that variation can be explained by a single systematic factor that has a positive loading for all (rating, maturity)-combinations, i.e. this factor is most easily interpreted as a systematic credit spread level factor. Whether the first principal component in this data is in fact identical to the systematic residual factor observed by Collin-Dufresne et al. (2001) is an open question, but we can confirm that the first principal component is a level factor based on the observation that its loading is uniformly negative for all 88 time series. Furthermore, for each (sector, rating)-combination the loading structure exhibits a hump shaped pattern across maturities with a peak between the 5- and 7-year maturity and within each sector the loading increases monotonically as credit quality deteriorates. Overall, this pattern supports a modeling strategy with a common level factor for each business sector where the loadings on the common factor for the rating categories within each sector are scaled up as credit quality deteriorates.

For the second principal component we also see a systematic pattern. For all (sector, rating)-combinations the loading on this component is monotonically increasing with maturity,[8] and it always starts with a large negative factor loading on the 3-month maturity and ends with a large positive factor loading on the long-term maturities. Thus, this factor describes the slope of the credit spread curves. Across rating categories within each of the four sectors we see a tendency for a larger loading at the short and long maturities when credit quality deteriorates, but this pattern is not as systematic as the pattern observed for the first principal component. However, for our modeling purposes the important thing is that each (sector, rating)-combination exhibits the pattern of a slope factor. The variation across ratings within each sector we adjust for by allowing the rating categories to load differently on the common slope factor.

Finally, for the third principal component there is no clearly discernible pattern. For some (sector, rating)-combinations it appears to be similar to a level factor. For other combinations it has changes in sign and non-systematic variation in the size of the factor loading. Most importantly for our analysis, however, it does not exhibit the pattern of a curvature factor similar to the one observed in Table 2 for the Treasury curvature factor. Based on this evidence we do not include a third factor in the form of a curvature factor in the modeling of credit spreads.[9]

In summary, we have found that there are two common risk factors across the 11 different

[8]There are three minor exceptions to this rule in Table 4: the 10-year maturity for the BBB category of US Banks and the 10-year maturity for the BBB and A categories of US Utility firms.

[9]Controlling estimations performed with a third curvature factor included did not bring any meaningful results and therefore supports this conclusion.

Maturity in months	US Industrials											
	BBB			A			AA			AAA		
	First	Second	Third	First	Second	Third	First	Second	Third	First	Second	Third
3	-0.107	-0.154	0.090	-0.048	-0.192	-0.012	-0.019	-0.172	-0.042	-0.015	-0.174	-0.050
6	-0.112	-0.122	0.067	-0.053	-0.169	-0.028	-0.027	-0.152	-0.039	-0.024	-0.152	-0.046
12	-0.133	-0.053	0.157	-0.068	-0.100	0.036	-0.035	-0.075	0.013	-0.030	-0.082	-0.009
24	-0.150	0.027	0.080	-0.084	-0.024	-0.040	-0.043	-0.020	-0.062	-0.032	-0.018	-0.093
36	-0.164	0.040	0.052	-0.107	-0.006	-0.047	-0.058	-0.004	-0.080	-0.042	-0.003	-0.114
60	-0.161	0.072	-0.020	-0.101	0.015	-0.121	-0.065	0.015	-0.096	-0.055	0.005	-0.095
84	-0.168	0.080	-0.014	-0.110	0.030	-0.110	-0.080	0.044	-0.091	-0.069	0.028	-0.095
120	-0.140	0.121	-0.152	-0.093	0.052	-0.207	-0.072	0.069	-0.165	-0.061	0.069	-0.139

Maturity in months	US Financials											
	BBB+			A			AA			AAA		
	First	Second	Third	First	Second	Third	First	Second	Third	First	Second	Third
3	n.a.	n.a.	n.a.	-0.032	-0.233	-0.113	-0.012	-0.210	-0.138	-0.011	-0.199	-0.113
6	n.a.	n.a.	n.a.	-0.038	-0.193	-0.105	-0.013	-0.164	-0.124	-0.011	-0.156	-0.104
12	n.a.	n.a.	n.a.	-0.073	-0.115	-0.004	-0.041	-0.088	-0.054	-0.032	-0.082	-0.035
24	n.a.	n.a.	n.a.	-0.096	-0.033	-0.019	-0.061	-0.018	-0.062	-0.050	-0.010	-0.045
36	n.a.	n.a.	n.a.	-0.125	0.006	-0.019	-0.088	0.025	-0.006	-0.071	0.044	0.005
60	n.a.	n.a.	n.a.	-0.138	0.063	-0.060	-0.101	0.067	-0.033	-0.075	0.073	0.013
84	n.a.	n.a.	n.a.	-0.139	0.068	-0.055	-0.110	0.100	-0.032	-0.093	0.114	0.012
120	n.a.	n.a.	n.a.	-0.127	0.088	-0.184	-0.099	0.118	-0.156	-0.071	0.119	-0.060

Maturity in months	US Banks											
	BBB			A			AA			AAA		
	First	Second	Third	First	Second	Third	First	Second	Third	First	Second	Third
3	-0.051	-0.215	-0.098	-0.028	-0.201	-0.125	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
6	-0.057	-0.198	-0.130	-0.035	-0.184	-0.159	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
12	-0.078	-0.084	-0.016	-0.057	-0.094	-0.044	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
24	-0.119	-0.034	0.057	-0.086	-0.020	-0.067	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
36	-0.149	0.004	-0.018	-0.126	0.014	-0.024	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
60	-0.151	0.035	-0.061	-0.119	0.041	-0.101	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
84	-0.183	0.122	-0.046	-0.150	0.058	-0.023	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
120	-0.149	0.061	-0.215	-0.112	0.063	-0.207	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.

Maturity in months	US Utilities											
	BBB			A			AA			AAA		
	First	Second	Third	First	Second	Third	First	Second	Third	First	Second	Third
3	-0.131	-0.189	0.216	-0.076	-0.177	0.018	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
6	-0.136	-0.190	0.221	-0.079	-0.176	0.019	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
12	-0.156	-0.106	0.279	-0.094	-0.088	0.066	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
24	-0.179	-0.025	0.271	-0.118	-0.026	0.061	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
36	-0.186	0.011	0.228	-0.136	0.000	0.062	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
60	-0.203	0.060	0.194	-0.155	0.048	0.033	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
84	-0.197	0.098	0.154	-0.159	0.063	0.013	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
120	-0.186	0.030	0.068	-0.168	0.041	-0.081	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.

Table 4: **The eigenvectors of the first three principal components in the credit spreads.** The loading of each maturity on the three eigenvectors that correspond to the first three principal components in the weekly zero-coupon credit spreads for US Industrial firms, US Financial firms, US Banks, and US Utility firms covering the period from January 6, 1995 to August 4, 2006. The analysis is based on 88 time series, each with 605 weekly observations.

(sector, rating)-combinations analyzed here. As for the characteristics of the two common credit risk factors we find that the first factor has the properties of a level factor within each (sector, rating)-combination, and the other has properties normally associated with a slope factor. Given this pattern in the first two principal components we feel that the credit spread data analyzed here is well suited for the generalized common factor decomposition based on the Nelson-Siegel model introduced in Diebold, Li, and Yue (2007). The idea is to replace the three global yield factors in that paper with our two sector-wide common credit risk factors and give them an interpretation of level and slope. Subsequently, each rating category within the industry can load more or less intensely on these sector-wide common credit risk factors. As a further elaboration, we choose to modify that model to make it arbitrage-free as detailed in Christensen et al. (2007). This motivates the modeling framework to be presented in the following section.

4 Model description

In this section the arbitrage-free model of Treasury and corporate bond yields that we are going to use in the empirical analysis is described. The first step is to choose a model framework. To that end we choose to work within the reduced-form credit risk modeling framework[10] using the assumption of Recovery of Market Value (See Duffie and Singleton (1999) for details). Denote the risk-free short rate by r_t, the default intensity by λ_t^Q, and the recovery rate by π_t^Q.[11] Under these assumptions the price of a representative zero-coupon bond is given by

$$V(t,T) = E^Q[e^{-\int_t^T (r_t + (1-\pi_s^Q)\lambda_s^Q) ds}].$$

Since the loss rate $1-\pi_t^Q$ and the default intensity λ_t^Q only appear as a product under the RMV modeling assumption, we replace the product $(1-\pi_s^Q)\lambda_s^Q$ by the instantaneous credit spread s_t without any loss of generality. It is this credit spread process that we are interested in analyzing.

One important thing to note here is that for each rating category in a given sector we will only be making a marginal or isolated pricing of the corporate bonds in that category in the sense that we do not take the rating transitions into consideration. We recognize that this is a theoretical inconsistency of our approach, and we caution that it limits the applicability of the model for accurate pricing purposes. However, for the purpose of extracting any common risk factors across rating categories and industries, which is the goal here, the model will clearly be able to catch any such effects if present in the data. Taking the rating transitions into consideration, we believe, is a second-order effect and a fine-tuning that will not change our results in any significant way. Therefore, we leave that issue for future research.[12]

With the general modeling framework settled, the next step is to decide on the assumed dynamics of the risk-free rate r_t and the credit spread s_t. The details are provided in the following subsections.

[10] See Lando (1998) for the technical details on the reduced-form modeling approach.

[11] If a jump risk premium exists the default intensity under the P-measure may deviate by a factor from the default intensity under the Q-measure. Since we only observe bond yields, the model-implied default intensities and recovery rates are only meaningful when interpreted under the Q-measure as indicated by the notation.

[12] However, the model framework presented here allows for such extensions. For example the method used by Feldhütter and Lando (2006) can be applied immediately in this setting under the restriction that each rating category has the same factor loading for all the common credit risk factors. For now we want to avoid such restrictions and therefore leave this for future research.

4.1 The model for the risk-free rate

For the risk-free rate we choose to build on the affine three-factor arbitrage-free approximation of the well-known Nelson-Siegel term structure model presented in Christensen, Diebold, and Rudebusch (2007). This is a three-factor model where the latent state variables $X_t^T = (L_t^T, S_t^T, C_t^T)$ can be given the interpretation of level, slope, and curvature by imposing a fixed set of restrictions on the Q-dynamics of a general three-factor affine Gaussian term structure model[13]

$$\begin{aligned} r_t &= \delta_0 + \delta_1' X_t^T, \\ dX_t^T &= K^{Q,T}(\theta^{Q,T} - X_t^T)dt + \Sigma^T dW_t^{Q,T}. \end{aligned}$$

The first critical assumption is to define the instantaneous risk-free rate as the sum of the level and the slope factor

$$r_t = L_t^T + S_t^T.$$

The second critical assumption is that the mean-reversion matrix under the Q-matrix must have the following simple form

$$K^{Q,T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda^T & -\lambda^T \\ 0 & 0 & \lambda^T \end{pmatrix},$$

where λ^T will be identical to λ in the standard Nelson-Siegel model in Equation (1). These two assumptions also highlight the role of the curvature factor C_t^T. This factor is absent from the instantaneous risk-free rate and therefore does not impact short-term yields. Instead, its sole role (under the Q-measure) is to act as a stochastic mean for the slope factor S_t^T.

Finally, we follow Christensen et al. (2007) and fix the mean vector under the Q-measure at zero, $\theta^{Q,T} = 0$. They show that this way of identification is without loss of generality.

Imposing the above structure on the general affine model default risk-free zero-coupon yields will be given by

$$y_t^T(\tau) = L_t^T + \frac{1 - e^{-\lambda^T \tau}}{\lambda^T \tau} S_t^T + \Big[\frac{1 - e^{-\lambda^T \tau}}{\lambda^T \tau} - e^{-\lambda^T \tau}\Big] C_t^T + \frac{A^T(\tau)}{\tau},$$

where we recognize the Nelson-Siegel factor loadings for the level, slope, and curvature factors. In addition, the yield function contains the maturity dependent term $\frac{A^T(\tau)}{\tau}$ which arises from imposing absence of arbitrage on the dynamic Nelson-Siegel model.[14] Given our way of identifying the model, i.e. fixing $\theta^{Q,T} = 0$, the yield-adjustment term is entirely determined by the parameterization of the volatility matrix Σ^T. Christensen et al. (2007) find that allowing for a maximally flexible parameterization of the volatility matrix Σ improves in-sample fit but deteriorates the out-of-sample forecast performance as the added flexibility is used to fit idiosyncratic variations in-sample that do not appear to repeat themselves when we move out of sample. For that reason we only consider diagonal volatility matrices in this paper.

[13] Since the affine property refers to bond prices, affine models only impose restrictions on the factor dynamics under the pricing measure.

[14] The analytical formula for $\frac{A^T(\tau)}{\tau}$ is provided in Christensen et al. (2007).

4.2 The five-factor credit spread model

Krishnan, Ritchken, and Thomson (2007) include the risk-free level, slope, and curvature factor from their Treasury bond yield analysis in their model for firm-specific credit spreads. They find that this combination performs well for forecasting purposes. They interpret their results as evidence that once they have decomposed the observed Treasury and corporate bond yields into the risk-free level, slope, and curvature factors and into the credit risk-related level, slope, and curvature factors, they have extracted all the essential information contained in the current bond prices, and provided financial markets are close to being efficient, these prices will reflect all currently available information. Furthermore, Krishnan, Ritchken, and Thomson (2007) demonstrate that it does not improve on the ability of their model to forecast to include additional factors like for example macroeconomic time series in the estimation. However, once they exclude the Treasury factors from their model, the forecasting performance *is* deteriorated.

Inspired by the results in Krishnan, Ritchken, and Thomson (2007), we include the factors of the risk-free rate directly in the function for the instantaneous credit spread. As the Treasury curvature factor is absent in the instantaneous short-rate process, it will also be absent from the instantaneous credit spread process. Thus, the instantaneous credit spread is assumed to be a function of the level and the slope factor from the risk-free yield term structure in addition to two sector-specific common credit risk factors

$$s_t^{i,c} = \alpha_0^{i,c} + \alpha_{L^T}^{i,c} L_t^T + \alpha_{S^T}^{i,c} S_t^T + \alpha_L^{i,c} L_t^S(i) + \alpha_S^{i,c} S_t^S(i).$$

This structure implies that each rating category c within industry i can load separately on each of these four factors and do it independently of the remaining rating categories within the sector.

We impose the Nelson-Siegel factor loading structure on the two common credit risk factors $(L_t^S(i), S_t^S(i))$. For that reason the dynamics of the common credit risk factors under the Q-measure must be assumed to be given by

$$\begin{pmatrix} dL_t^S(i) \\ dS_t^S(i) \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 0 & \lambda^S(i) \end{pmatrix} \begin{pmatrix} L_t^S(i) \\ S_t^S(i) \end{pmatrix} dt + \begin{pmatrix} \sigma_{L^S}^i & 0 \\ 0 & \sigma_{S^S}^i \end{pmatrix} \begin{pmatrix} dW_t^{Q,L^S}(i) \\ dW_t^{Q,S^S}(i) \end{pmatrix}.$$

Thus, the dynamic description of the five factors under the Q-measure is given by the following system of stochastic differential equations

$$\begin{pmatrix} dL_t^S(i) \\ dS_t^S(i) \\ dL_t^T \\ dS_t^T \\ dC_t^T \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^S(i) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^T & -\lambda^T \\ 0 & 0 & 0 & 0 & \lambda^T \end{pmatrix} \begin{pmatrix} L_t^S(i) \\ S_t^S(i) \\ L_t^T \\ S_t^T \\ C_t^T \end{pmatrix} dt + \begin{pmatrix} \sigma_{L^S}^i & 0 & 0 & 0 & 0 \\ 0 & \sigma_{S^S}^i & 0 & 0 & 0 \\ 0 & 0 & \sigma_{L^T} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{S^T} & 0 \\ 0 & 0 & 0 & 0 & \sigma_{C^T} \end{pmatrix} \begin{pmatrix} dW_t^{Q,L^S}(i) \\ dW_t^{Q,S^S}(i) \\ dW_t^{Q,L^T} \\ dW_t^{Q,S^T} \\ dW_t^{Q,C^T} \end{pmatrix}.$$

Given this dynamic structure under the Q-measure the common risk factors will preserve the Nelson-Siegel two-factor structure. Thus, the yield of a representative zero-coupon bond from industry i with rating c and maturity in τ years is given by

$$\begin{aligned}
y_t^{i,c}(\tau) &= L_t^T + \frac{1-e^{-\lambda^T\tau}}{\lambda^T\tau}S_t^T + \Big[\frac{1-e^{-\lambda^T\tau}}{\lambda^T\tau} - e^{-\lambda^T\tau}\Big]C_t^T + \frac{A^T(\tau)}{\tau} \\
&+\alpha_{L^T}^{i,c}L_t^T + \alpha_{S^T}^{i,c}\frac{1-e^{-\lambda^T\tau}}{\lambda^T\tau}S_t^T + \alpha_{S^T}^{i,c}\Big[\frac{1-e^{-\lambda^T\tau}}{\lambda^T\tau} - e^{-\lambda^T\tau}\Big]C_t^T + \frac{A^{T,i,c}(\tau;\alpha_{L^T}^{i,c},\alpha_{S^T}^{i,c})}{\tau} \\
&+\alpha_0^{i,c} + \alpha_L^{i,c}L_t^S(i) + \alpha_S^{i,c}\frac{1-e^{-\lambda^S(i)\tau}}{\lambda^S(i)\tau}S_t^S(i) + \frac{A^{i,c}(\tau)}{\tau}.
\end{aligned}$$

By implication, the corresponding zero-coupon credit spread is given by

$$\begin{aligned}
s_t^{i,c}(\tau) &= y_t^{i,c}(\tau) - y_t^T(\tau) \\
&= \alpha_{L^T}^{i,c}L_t^T + \alpha_{S^T}^{i,c}\frac{1-e^{-\lambda^T\tau}}{\lambda^T\tau}S_t^T + \alpha_{S^T}^{i,c}\Big[\frac{1-e^{-\lambda^T\tau}}{\lambda^T\tau} - e^{-\lambda^T\tau}\Big]C_t^T + \frac{A^{T,i,c}(\tau;\alpha_{L^T}^{i,c},\alpha_{S^T}^{i,c})}{\tau} \\
&\quad+\alpha_0^{i,c} + \alpha_L^{i,c}L_t^S(i) + \alpha_S^{i,c}\frac{1-e^{-\lambda^S(i)\tau}}{\lambda^S(i)\tau}S_t^S(i) + \frac{A^{i,c}(\tau)}{\tau},
\end{aligned}$$

where[15]

$$\begin{aligned}
\frac{A^{T,i,c}(\tau;\alpha_{L^T}^{i,c},\alpha_{S^T}^{i,c})}{\tau} &= -\frac{\sigma_{L^T}^2(\alpha_{L^T}^{i,c})^2}{6}\tau^2 - \sigma_{S^T}^2(\alpha_{S^T}^{i,c})^2\Big[\frac{1}{2(\lambda^T)^2} - \frac{1}{(\lambda^T)^3}\frac{1-e^{-\lambda^T\tau}}{\tau} + \frac{1}{4(\lambda^T)^3}\frac{1-e^{-2\lambda^T\tau}}{\tau}\Big] \\
&\quad -\sigma_{C^T}^2(\alpha_{S^T}^{i,c})^2\Big[\frac{1}{2(\lambda^T)^2} + \frac{1}{(\lambda^T)^2}e^{-\lambda^T\tau} - \frac{1}{4\lambda^T}\tau e^{-2\lambda^T\tau} - \frac{3}{4(\lambda^T)^2}e^{-2\lambda^T\tau}\Big] \\
&\quad -\sigma_{C^T}^2(\alpha_{S^T}^{i,c})^2\Big[\frac{5}{8(\lambda^T)^3}\frac{1-e^{-2\lambda^T\tau}}{\tau} - \frac{2}{(\lambda^T)^3}\frac{1-e^{-\lambda^T\tau}}{\tau}\Big], \\
\frac{A^{i,c}(\tau)}{\tau} &= -\frac{(\sigma_{L^S}^i)^2(\alpha_L^{i,c})^2}{6}\tau^2 - (\sigma_{S^S}^i)^2(\alpha_S^{i,c})^2\Big[\frac{1}{2(\lambda^S(i))^2} - \frac{1}{(\lambda^S(i))^3}\frac{1-e^{-\lambda^S(i)\tau}}{\tau} + \frac{1}{4(\lambda^S(i))^3}\frac{1-e^{-2\lambda^S(i)\tau}}{\tau}\Big].
\end{aligned}$$

A final, equally important point to take away from the result above is that there are no restrictions on the dynamic drift components under the empirical P-measure. Therefore, beyond the requirement of constant volatility, we are free to choose the dynamics under the P-measure. However, to facilitate the empirical implementation we limit our focus to the essentially affine risk premium specification introduced in Duffee (2002). In the Gaussian framework that we are working within this specification implies that the risk premiums are given by

$$\Gamma_t = \gamma_0 + \gamma_1 X_t$$

where $\gamma_0 \in \mathbf{R}^5$ and $\gamma_1 \in \mathbf{R}^{5\times 5}$ contain unrestricted parameters. Thus, in general, we can write the P-dynamics of the state variables as

$$dX_t = K^P(\theta^P - X_t)dt + \Sigma dW_t^P,$$

where both K^P and θ^P are allowed to vary freely relative to their counterparts under the Q-measure.

[15]See Appendix A for details.

5 Model estimation

We estimate the models introduced in the previous subsections using the Kalman filter. Since all models considered in this paper are affine Gaussian models the Kalman filter is an efficient and consistent estimator. In addition, the Kalman filter requires a minimum of assumptions about the observed data and it easily handles missing data. This motivates our use of the standard Kalman filter in this setting.

The measurement equation for the Treasury bond yields is given by

$$y_t^T = A^T + B^T \begin{pmatrix} L_t^T \\ S_t^T \\ C_t^T \end{pmatrix} + \varepsilon_t^T,$$

while the measurement equation for the credit spreads can be written as

$$s_t(i) = A^S(i) + B^S(i) \begin{pmatrix} L_t^S(i) \\ S_t^S(i) \\ L_t^T \\ S_t^T \\ C_t^T \end{pmatrix} + \varepsilon_t^S(i).$$

Expanding on this equation for a given business sector we obtain[16]

$$\begin{pmatrix} s_t^{BBB}(\tau_1) \\ \vdots \\ s_t^{BBB}(\tau_N) \\ s_t^{A}(\tau_1) \\ \vdots \\ s_t^{A}(\tau_N) \\ s_t^{AA}(\tau_1) \\ \vdots \\ s_t^{AA}(\tau_N) \\ s_t^{AAA}(\tau_1) \\ \vdots \\ s_t^{AAA}(\tau_N) \end{pmatrix} = \begin{pmatrix} \alpha_0^{BBB} \\ \vdots \\ \alpha_0^{BBB} \\ \alpha_0^{A} \\ \vdots \\ \alpha_0^{A} \\ \alpha_0^{AA} \\ \vdots \\ \alpha_0^{AA} \\ \alpha_0^{AAA} \\ \vdots \\ \alpha_0^{AAA} \end{pmatrix} + \begin{pmatrix} \alpha_L^{BBB} \\ \vdots \\ \alpha_L^{BBB} \\ \alpha_L^{A} \\ \vdots \\ \alpha_L^{A} \\ \alpha_L^{AA} \\ \vdots \\ \alpha_L^{AA} \\ \alpha_L^{AAA} \\ \vdots \\ \alpha_L^{AAA} \end{pmatrix} L_t^S + \begin{pmatrix} \alpha_S^{BBB} \frac{1-e^{-\lambda^S \tau_1}}{\lambda^S \tau_1} \\ \vdots \\ \alpha_S^{BBB} \frac{1-e^{-\lambda^S \tau_N}}{\lambda^S \tau_N} \\ \alpha_S^{A} \frac{1-e^{-\lambda^S \tau_1}}{\lambda^S \tau_1} \\ \vdots \\ \alpha_S^{A} \frac{1-e^{-\lambda^S \tau_N}}{\lambda^S \tau_N} \\ \alpha_S^{AA} \frac{1-e^{-\lambda^S \tau_1}}{\lambda^S \tau_1} \\ \vdots \\ \alpha_S^{AA} \frac{1-e^{-\lambda^S \tau_N}}{\lambda^S \tau_N} \\ \alpha_S^{AAA} \frac{1-e^{-\lambda^S \tau_1}}{\lambda^S \tau_1} \\ \vdots \\ \alpha_S^{AAA} \frac{1-e^{-\lambda^S \tau_N}}{\lambda^S \tau_N} \end{pmatrix} S_t^S + \begin{pmatrix} \frac{A^{BBB}(\tau_1)}{\tau_1} \\ \vdots \\ \frac{A^{BBB}(\tau_N)}{\tau_N} \\ \frac{A^{A}(\tau_1)}{\tau_1} \\ \vdots \\ \frac{A^{A}(\tau_N)}{\tau_N} \\ \frac{A^{AA}(\tau_1)}{\tau_1} \\ \vdots \\ \frac{A^{AA}(\tau_N)}{\tau_N} \\ \frac{A^{AAA}(\tau_1)}{\tau_1} \\ \vdots \\ \frac{A^{AAA}(\tau_N)}{\tau_N} \end{pmatrix} + \begin{pmatrix} \varepsilon_t^{BBB}(\tau_1) \\ \vdots \\ \varepsilon_t^{BBB}(\tau_N) \\ \varepsilon_t^{A}(\tau_1) \\ \vdots \\ \varepsilon_t^{A}(\tau_N) \\ \varepsilon_t^{AA}(\tau_1) \\ \vdots \\ \varepsilon_t^{AA}(\tau_N) \\ \varepsilon_t^{AAA}(\tau_1) \\ \vdots \\ \varepsilon_t^{AAA}(\tau_N) \end{pmatrix}.$$

It follows from this structure that not all the parameters are identified. For this reason we choose the A-rating category to be the benchmark category in the sense that we fix the constant term at $\alpha_0^A = 0$ and let the factor loadings on the two credit risk factors be $\alpha_L^A = 1$ and $\alpha_S^A = 1$. This choice is motivated by the fact that the A-category is represented by a full sample of data in all four sectors, but beyond that this choice is without consequences. The implication is that the sensitivities to changes in the two credit risk factors are measured relative to those of the A-rating category in that same industry, while the estimated values of the two credit risk factors represent the absolute sensitivity of the benchmark A-rating to those factors.

[16] Here, the Treasury factors have been left out, and the industry identifier is suppressed in the notation.

For continuous-time Gaussian models the conditional mean vector and the conditional covariance matrix are given by

$$\begin{aligned} E^P[X_T|\mathcal{F}_t] &= (I-\exp(-K^P\Delta t))\mu^P+\exp(-K^P\Delta t)X_t, \\ V^P[X_T|\mathcal{F}_t] &= \int_0^{\Delta t} e^{-K^P s}\Sigma\Sigma' e^{-(K^P)'s}ds, \end{aligned}$$

where $\Delta t = T-t$ and $\exp(-K^P\Delta t_i)$ is a matrix exponential.

Stationarity of the system under the P-measure is ensured provided the real component of all the eigenvalues of K^P is positive. This is imposed in all estimations. For this reason we can start the Kalman filter at the unconditional mean and covariance matrix[17]

$$\widehat{X}_0 = \mu^P \quad \text{and} \quad \widehat{\Sigma}_0 = \int_0^{\infty} e^{-K^P s}\Sigma\Sigma' e^{-(K^P)'s}ds.$$

The state equation in the Kalman filter is given by

$$X_{t_i} = \Phi^0_{\Delta t_i} + \Phi^1_{\Delta t_i} X_{t_{i-1}} + \eta_{t_i}$$

where

$$\Phi^0_{\Delta t_i} = (I-\exp(-K^P\Delta t_i))\mu^P, \quad \Phi^1_{\Delta t_i} = \exp(-K^P\Delta t_i), \quad \text{and} \quad \eta_{t_i} \sim N\Big(0, \int_0^{\Delta t_i} e^{-K^P s}\Sigma\Sigma' e^{-(K^P)'s}ds\Big)$$

with $\Delta t_i = t_i - t_{i-1}$.

In the Kalman filter estimations all measurement errors are assumed to be i.i.d. white noise. Thus, the error structure is in general given by

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix}\right].$$

In the estimation each maturity of the Treasury bond yields has its own measurement error standard deviation, $\sigma^2_{\varepsilon^T}(\tau_i)$, while the measurement errors for the credit spreads are assumed to have a uniform standard deviation $\sigma^2_{\varepsilon^S}$ across all ratings and maturities in each industry. Thus, given the range of maturities in the Treasury bond yield data considered in this paper the H-matrix in general takes the following form

$$H = diag(\sigma^2_{\varepsilon^T}(3m), \sigma^2_{\varepsilon^T}(6m), \sigma^2_{\varepsilon^T}(1y), \sigma^2_{\varepsilon^T}(2y), \sigma^2_{\varepsilon^T}(3y), \sigma^2_{\varepsilon^T}(5y), \sigma^2_{\varepsilon^T}(7y), \sigma^2_{\varepsilon^T}(10y), \sigma^2_{\varepsilon^S}, \ldots, \sigma^2_{\varepsilon^S}).$$

The linear least-squares optimality of the Kalman filter requires that the white noise transition and measurement errors be orthogonal to the initial state, i.e.

$$E[f_0\eta_t'] = 0, \quad E[f_0\varepsilon_t'] = 0.$$

[17]In the estimation $\int_0^\infty e^{-K^P s}\Sigma\Sigma' e^{-(K^P)'s}ds$ is approximated by $\int_0^{10} e^{-K^P s}\Sigma\Sigma' e^{-(K^P)'s}ds$.

Finally, the standard deviations of the estimated parameters are calculated as

$$\Sigma(\widehat{\psi}) = \frac{1}{T}\Big[\frac{1}{T}\sum_{t=1}^{T}\frac{\partial \log l_t(\widehat{\psi})}{\partial \psi}\frac{\partial \log l_t(\widehat{\psi})}{\partial \psi}'\Big]^{-1},$$

where $\widehat{\psi}$ denotes the optimal parameter set.

6 In-sample estimation results

The purpose of this section is to find appropriate specifications of the P-dynamics of the state variables in our term structure models based on in-sample measures of fit. First, we investigate the dynamics of the three Treasury factors using Treasury yields only. This leads us to a preferred specification that can be used as benchmark for the performance of the five-factor credit spread models studied subsequently. Second, we analyze the appropriate specification of the joint P-dynamics of the three Treasury risk factors and the two credit risk factors from each of the four business sectors in our data.

6.1 The P-dynamics of the Treasury factors

We begin the search for the most appropriate specification of the dynamic interaction of the three Treasury factors by estimating the model that allows for the maximum flexibility in terms of dynamic interaction between the state variables as they revert back to steady state. This model is represented by the following stochastic differential equation for the P-dynamics

$$\begin{pmatrix} dL_t^T \\ dS_t^T \\ dC_t^T \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{P,T} & \kappa_{12}^{P,T} & \kappa_{13}^{P,T} \\ \kappa_{21}^{P,T} & \kappa_{22}^{P,T} & \kappa_{23}^{P,T} \\ \kappa_{31}^{P,T} & \kappa_{23}^{P,T} & \kappa_{33}^{P,T} \end{pmatrix} \left[\begin{pmatrix} \theta_1^{P,T} \\ \theta_2^{P,T} \\ \theta_3^{P,T} \end{pmatrix} - \begin{pmatrix} L_t^T \\ S_t^T \\ C_t^T \end{pmatrix} \right] + \begin{pmatrix} \sigma_{L^T} & 0 & 0 \\ 0 & \sigma_{S^T} & 0 \\ 0 & 0 & \sigma_{C^T} \end{pmatrix} \begin{pmatrix} dW_t^{P,L^T} \\ dW_t^{P,S^T} \\ dW_t^{P,C^T} \end{pmatrix}.$$

We denote this "Case I: full $K^{P,T}$".

Christensen, Diebold, and Rudebusch (2007) find that the parsimonious model with independent factors under the P-measure performed well in terms of forecasting out of sample. We will refer to this as "Case II: diagonal $K^{P,T}$" which is characterized by a mean-reversion matrix given by a diagonal matrix

$$K^{P,T} = \begin{pmatrix} \kappa_{11}^{P,T} & 0 & 0 \\ 0 & \kappa_{22}^{P,T} & 0 \\ 0 & 0 & \kappa_{33}^{P,T} \end{pmatrix}.$$

However, from the estimation results for these two specifications it is evident that the Treasury factors are correlated and have some interesting dynamic interactions between, in particular, the slope and the curvature factor.

To take this evidence into consideration we also analyze the cases where the mean-reversion matrix under the P-measure is specified as upper- and lower-triangular matrices

$$K^{P,T} = \begin{pmatrix} \kappa_{11}^{P,T} & \kappa_{12}^{P,T} & \kappa_{13}^{P,T} \\ 0 & \kappa_{22}^{P,T} & \kappa_{23}^{P,T} \\ 0 & 0 & \kappa_{33}^{P,T} \end{pmatrix} \quad \text{and} \quad K^{P,T} = \begin{pmatrix} \kappa_{11}^{P,T} & 0 & 0 \\ \kappa_{21}^{P,T} & \kappa_{22}^{P,T} & 0 \\ \kappa_{31}^{P,T} & \kappa_{32}^{P,T} & \kappa_{33}^{P,T} \end{pmatrix}.$$

Model description	Treasury bond yield models			
	Max log likelihood	D.f.	LR	p-value
Case I: full $K^{P,T}$	28162.48	n.a.	n.a.	n.a.
Case II: diagonal $K^{P,T}$	28142.43	6	40.10	< 0.00001
Case III: upper-triangular $K^{P,T}$	28153.83	3	17.30	0.00061
Case IV: lower-triangular $K^{P,T}$	28146.35	3	32.26	< 0.00001
Case V: S^T interaction only	28161.41	4	2.14	0.71003

Table 5: **The maximum log likelihood values for the AFDNS Treasury bond yield models.**
The maximum log likelihood values obtained for the five different AFDNS Treasury bond yield models. The data set used consists of weekly data on Treasury zero-coupon bond yields covering the period from January 6, 1995 to August 4, 2006.

We refer to these two specifications as "Case III: upper-triangular $K^{P,T}$" and "Case IV: lower-triangular $K^{P,T}$", respectively.

Given that all models considered are nested by the specification with full $K^{P,T}$-matrix we can perform likelihood ratio tests for the significance of the parameter restrictions imposed. First, we start by testing the independent-factors model against the model with full K^P-matrix, i.e. "Case I" vs. "Case II". In this case, the likelihood ratio test is given by

$$LR = 2[28162.48 - 28142.43] = 40.1 \sim \chi^2(6).$$

The probability of observing 40.1 in the χ^2 distribution with 6 degrees of freedom is less than 0.0001 and clearly rejected. Next, we test the independent-factors model against the model with lower-triangular K^P-matrix, i.e. "Case I" vs. "Case IV". The likelihood ratio test is given by

$$LR = 2[28146.35 - 28142.43] = 7.84 \sim \chi^2(3).$$

The probability of observing 7.84 in the χ^2 distribution with 3 degrees of freedom is exactly 0.05. Thus, the restrictions imposed on the independent-factors are only weakly rejected relative to the lower-trinagular specification of $K^{P,T}$. Since the model in "Case III" with the upper-triangular $K^{P,T}$-matrix has an even higher likelihood value, it immediately follows that the independent-factors model is also rejected relative to the latter model.

Finally, we test the model with upper-triangular $K^{P,T}$-matrix against the model with full $K^{P,T}$-matrix. This gives the following likelihood ratio test

$$LR = 2[28162.48 - 28153.83] = 17.3 \sim \chi^2(3).$$

This test is also clearly rejected.

The conclusion from this preliminary investigation is that both the lower- and the upper-triangular specification of $K^{P,T}$ add some significant explanatory power where the key link appears to be the slope factor. For this reason we try the following specification of the mean-reversion matrix

$$K^{P,T} = \begin{pmatrix} \kappa_{11}^{P,T} & 0 & 0 \\ \kappa_{12}^{P,T} & \kappa_{22}^{P,T} & \kappa_{23}^{P,T} \\ 0 & 0 & \kappa_{33}^{P,T} \end{pmatrix},$$

$K^{P,T}$	$K^{P,T}_{\cdot,1}$	$K^{P,T}_{\cdot,2}$	$K^{P,T}_{\cdot,3}$	$\mu^{P,T}$	Σ^T	$\Sigma^T_{\cdot,1}$	$\Sigma^T_{\cdot,2}$	$\Sigma^T_{\cdot,3}$
$K^{P,T}_{1,\cdot}$	0.1343	0	0	0.06288	$\Sigma^T_{1,\cdot}$	0.004679	0	0
	(0.200)			(0.00809)		(0.000149)		
$K^{P,T}_{2,\cdot}$	1.308	0.6809	-0.8203	-0.01780	$\Sigma^T_{2,\cdot}$	0	0.007526	0
	(0.517)	(0.163)	(0.147)	(0.0217)			(0.000224)	
$K^{P,T}_{3,\cdot}$	0	0	0.941629	-0.008832	$\Sigma^T_{3,\cdot}$	0	0	0.02852
			(0.418)	(0.00887)				(0.000578)

Table 6: **Parameter estimates for the preferred AFDNS Treasury bond yield model.** The estimated parameters of the $K^{P,T}$-matrix, the $\mu^{P,T}$-vector, and the Σ^T-matrix for the preferred arbitrage-free three-factor Nelson-Siegel model. The estimated value of λ^T is 0.5313 with a standard deviation of 0.00619. The maximum log-likelihood value is 28161.41. The Treasury bond yields used in the estimation cover the period from January 6, 1995 to August 4, 2006. The numbers in parentheses are the estimated standard deviations of the parameter estimates.

which we denote "Case V: S^T interaction only".

The likelihood ratio test for this restriction relative to the model with full $K^{P,T}$-matrix is

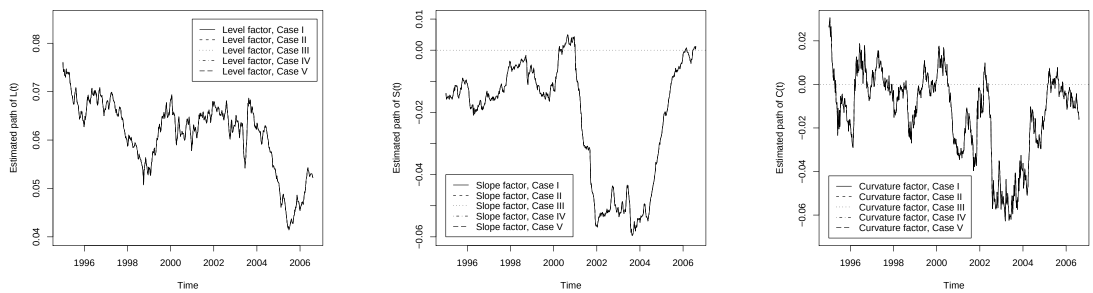
$$LR = 2[28162.48 - 28161.41] = 2.14 \sim \chi^2(4).$$

Thus, this restriction is overwhelmingly accepted.

Table 6 contains the estimated parameter values of the mean-reversion matrix, the mean vector, and the volatility matrix for the P-dynamics of the three state variables in the preferred Treasury bond yield model. From the table it is noted that the level factor is very persistent while the slope and curvature factor are much less so. From the table it also follows that the level factor has the smallest, and the curvature factor the largest, volatility. These results are in line with the results reported in other studies using the dynamic Nelson-Siegel model on different data sets covering other time periods, for examples see Christensen et al. (2007) and Diebold, Rudebusch, and Aruoba (2006). Furthermore, in line with the inference based on the likelihood ratio tests, the two off-diagonal elements in the $K^{P,T}$-matrix are highly significant. Thus, we should expect to maintain this dynamic structure between the three Treasury factors when we move to the joint estimation with credit spreads included in the data set.

Figure 3 illustrates the estimated paths from all five specifications of the Treasury bond yield model. The level factor exhibits a downward trend in line with the pattern observed for the long-term yields in Figure 2. The slope factor is highly correlated with the very short term yields, while the curvature factor is a more volatile, but quickly mean-reverting process in line with the estimated parameters for this factor. It is noted that there is practically perfect correlation across all five models. Naturally, this is driven by the fact that it is the same yield function that we fit to the observed yields independent of the specification of $K^{P,T}$. For the same reason, the summary statistics for the fitted errors reported in Table 7 are also indistinguishable across the five models. Thus, the specification of the mean-reversion matrix under the P-measure, $K^{P,T}$, has no detectable impact on the fit of the model. The fit of the model is determined by the imposition of the Nelson-Siegel factor loading structure. A similar lack of connection between the specification of the P-drift dynamics and model fit will be observed in the joint estimations analyzed later in this note.

The pairwise correlations between the estimated paths of the level factor, on one side, and the



(a) Treasury level factor. (b) Treasury slope factor. (c) Treasury curvature factor.

Figure 3: **The estimated Treasury level, slope, and curvature factor from the AFDNS Treasury bond yield models.**
The estimated Treasury level, slope, and curvature factor from the five AFDNS Treasury bond yield models. The data set used is weekly data covering the period from January 6, 1995 to August 4, 2006.

Maturity	Case I		Case II		Case III		Case IV		Case V	
	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE
3	-0.03	10.92	-0.05	10.91	-0.03	10.91	-0.03	10.91	-0.03	10.91
6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
12	1.68	6.41	1.70	6.40	1.68	6.40	1.68	6.41	1.68	6.40
24	2.29	4.21	2.31	4.22	2.30	4.21	2.30	4.22	2.30	4.21
36	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
60	-2.80	3.54	-2.84	3.60	-2.81	3.55	-2.82	3.58	-2.82	3.56
84	0.30	2.52	0.24	2.46	0.29	2.51	0.28	2.48	0.25	2.52
120	11.15	12.74	11.12	12.60	11.15	12.72	11.14	12.65	11.09	12.68

Table 7: **Summary statistics for the fitted errors from the AFDNS Treasury bond yield models.**
The mean and root mean squared error of the fitted errors along with the estimated measurement error standard deviation across the 8 different maturities in the Treasury bond yield data set. All numbers are measured in basis points. The data set used is weekly data covering the period from January 6, 1995 to August 4, 2006.

slope and the curvature on the other are -0.268 and 0.018, respectively. Given the low correlation between the level and the curvature factor, it is not surprising that we find the mean-reversion parameters $K_{13}^{P,T}$ and $K_{31}^{P,T}$ both to be zero. The correlation between the slope and the curvature factor is 0.602. Thus, the off-diagonal elements in the $K^{P,T}$-matrix in the preferred specification of the Treasury bond yield model reflects the observed dynamic interactions between the three state variables.

Finally, as for the fit of the model, Table 7 reports summary statistics for the measurement errors. With the exception of the 3-month and the 10-year yield, the errors are generally low indicating a very good fit to the cross-section of yields at any given point in time. The 3-month maturity is difficult to fit partly because of the structure of the model with only three factors, partly because the short end of the Treasury yield curve is known to be under influence of some short-term forces that are not apparent in the longer maturities, see Duffee (1996) for evidence on this. As for the 10-year maturity, it is located just about where the Nelson-Siegel term structure model gives up fitting (given our estimate of λ^T at 0.53). This can also be seen by the fact that the mean pricing error is close to the RMSE for this particular maturity.

6.2 Model specifications of the joint P-dynamics of the risk factors in the Treasury and corporate bond markets

Given the flexibility of the five-factor model we can estimate the joint dynamic interaction between the two credit risk factors and the three Treasury factors. In order to pin down the nature of any such interactions the only consistent way to proceed is to start by estimating the model with the maximum flexibility in terms of dynamic interaction between the five state variables. This specification is characterized by a complete 5×5 $K^{P,i}$-matrix

$$\begin{pmatrix} dL_t^S(i) \\ dS_t^S(i) \\ dL_t^T \\ dS_t^T \\ dC_t^T \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{P,i} & \kappa_{12}^{P,i} & \kappa_{13}^{P,i} & \kappa_{14}^{P,i} & \kappa_{15}^{P,i} \\ \kappa_{21}^{P,i} & \kappa_{22}^{P,i} & \kappa_{23}^{P,i} & \kappa_{24}^{P,i} & \kappa_{25}^{P,i} \\ \kappa_{31}^{P,i} & \kappa_{32}^{P,i} & \kappa_{33}^{P,i} & \kappa_{34}^{P,i} & \kappa_{35}^{P,i} \\ \kappa_{41}^{P,i} & \kappa_{42}^{P,i} & \kappa_{43}^{P,i} & \kappa_{44}^{P,i} & \kappa_{45}^{P,i} \\ \kappa_{51}^{P,i} & \kappa_{52}^{P,i} & \kappa_{53}^{P,i} & \kappa_{54}^{P,i} & \kappa_{55}^{P,i} \end{pmatrix} \left[\begin{pmatrix} \theta_{L^S}^{P,i} \\ \theta_{S^S}^{P,i} \\ \theta_{L^T}^{P,i} \\ \theta_{S^T}^{P,i} \\ \theta_{C^T}^{P,i} \end{pmatrix} - \begin{pmatrix} L_t^S(i) \\ S_t^S(i) \\ L_t^T \\ S_t^T \\ C_t^T \end{pmatrix} \right] dt$$
$$+ \begin{pmatrix} \sigma_{L^S}^i & 0 & 0 & 0 & 0 \\ 0 & \sigma_{S^S}^i & 0 & 0 & 0 \\ 0 & 0 & \sigma_{L^T} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{S^T} & 0 \\ 0 & 0 & 0 & 0 & \sigma_{C^T} \end{pmatrix} \begin{pmatrix} dW_t^{P,L^S(i)} \\ dW_t^{P,S^S(i)} \\ dW_t^{P,L^T} \\ dW_t^{P,S^T} \\ dW_t^{P,C^T} \end{pmatrix}.$$

We will denote this model by "Case I: Complete $K^{P,i}$".

Given the relatively weak connection between the Treasury level factor and all the other four factors, we would like to test whether the Treasury level factor is in fact independent. This can be done by restricting the mean-reversion matrix to take the following form

$$K^{P,i} = \begin{pmatrix} \kappa_{11}^{P,i} & \kappa_{12}^{P,i} & 0 & \kappa_{14}^{P,i} & \kappa_{15}^{P,i} \\ \kappa_{21}^{P,i} & \kappa_{22}^{P,i} & 0 & \kappa_{24}^{P,i} & \kappa_{25}^{P,i} \\ 0 & 0 & \kappa_{33}^{P,i} & 0 & 0 \\ \kappa_{41}^{P,i} & \kappa_{42}^{P,i} & 0 & \kappa_{44}^{P,i} & \kappa_{45}^{P,i} \\ \kappa_{51}^{P,i} & \kappa_{52}^{P,i} & 0 & \kappa_{54}^{P,i} & \kappa_{55}^{P,i} \end{pmatrix}.$$

We will denote this model by "Case II: Complete $K^{P,i}$, L^T indep.".

From a causality perspective it may seem little appealing to expect a feedback from the credit risk factors onto the three Treasury factors. To test whether this is in fact the case we restrict all these feedback effects to zero which leads to a mean-reversion matrix of the form

$$K^{P,i} = \begin{pmatrix} \kappa_{11}^{P,i} & \kappa_{12}^{P,i} & \kappa_{13}^{P,i} & \kappa_{14}^{P,i} & \kappa_{15}^{P,i} \\ \kappa_{21}^{P,i} & \kappa_{22}^{P,i} & \kappa_{23}^{P,i} & \kappa_{24}^{P,i} & \kappa_{25}^{P,i} \\ 0 & 0 & \kappa_{33}^{P,i} & \kappa_{34}^{P,i} & \kappa_{35}^{P,i} \\ 0 & 0 & \kappa_{43}^{P,i} & \kappa_{44}^{P,i} & \kappa_{45}^{P,i} \\ 0 & 0 & \kappa_{53}^{P,i} & \kappa_{54}^{P,i} & \kappa_{55}^{P,i} \end{pmatrix}.$$

This will be denoted "Case III: full $K^{P,T}$, zero credit risk feedback" as it allows for full interaction between the three Treasury factors while neglecting any feedback from the credit risk factors.

Finally, we estimate the specification where all five factors are assumed independent. Needless to say the five factors are correlated, but we know from the pure Treasury bond yield estimations that varying the specification of the $K^{P,i}$-matrix has little impact on the fit of the model and the decomposition of the observed yields. Thus, the estimation of this model can give us an impression, overall, of how important the interaction between the factors are as measured by the differences in likelihood values. The independent-factors specification is characterized by

$$K^{P,i} = \begin{pmatrix} \kappa_{11}^{P,i} & 0 & 0 & 0 & 0 \\ 0 & \kappa_{22}^{P,i} & 0 & 0 & 0 \\ 0 & 0 & \kappa_{33}^{P,i} & 0 & 0 \\ 0 & 0 & 0 & \kappa_{44}^{P,i} & 0 \\ 0 & 0 & 0 & 0 & \kappa_{55}^{P,i} \end{pmatrix}$$

and will be denoted "Case IV: indep. factors".

Based on the results from these four preliminary estimations we extract three preferred model specifications. Inspired by the structure in the preferred specification for the Treasury bond yield model, the first preferred specification for the five-factor credit spread model allows the Treasury level factor to impact the Treasury slope factor, while keeping all other interactions for the Treasury level factor at zero. In addition, there is no feedback from the Treasury slope factor to the Treasury curvature factor. Thus, the first preferred specification has a mean-reversion matrix given by

$$K^{P,i} = \begin{pmatrix} \kappa_{11}^{P,i} & \kappa_{12}^{P,i} & 0 & \kappa_{14}^{P,i} & \kappa_{15}^{P,i} \\ \kappa_{21}^{P,i} & \kappa_{22}^{P,i} & 0 & \kappa_{24}^{P,i} & \kappa_{25}^{P,i} \\ 0 & 0 & \kappa_{33}^{P,i} & 0 & 0 \\ \kappa_{41}^{P,i} & \kappa_{42}^{P,i} & \kappa_{43}^{P,i} & \kappa_{44}^{P,i} & \kappa_{45}^{P,i} \\ \kappa_{51}^{P,i} & \kappa_{52}^{P,i} & 0 & 0 & \kappa_{55}^{P,i} \end{pmatrix}.$$

This is denoted "Case V: preferred model I".

It turns out that this specification can be simplified even further by using the following mean-reversion matrix

$$K^{P,i} = \begin{pmatrix} \kappa_{11}^{P,i} & \kappa_{12}^{P,i} & 0 & \kappa_{14}^{P,i} & \kappa_{15}^{P,i} \\ \kappa_{21}^{P,i} & \kappa_{22}^{P,i} & 0 & \kappa_{24}^{P,i} & \kappa_{25}^{P,i} \\ 0 & 0 & \kappa_{33}^{P,i} & 0 & 0 \\ \kappa_{41}^{P,i} & 0 & \kappa_{43}^{P,i} & \kappa_{44}^{P,i} & \kappa_{45}^{P,i} \\ 0 & \kappa_{52}^{P,i} & 0 & 0 & \kappa_{55}^{P,i} \end{pmatrix}.$$

This is denoted "Case VI: preferred model II". The implication of the two additional parameter restrictions is that there is no feedback from the credit risk level factor on to the Treasury curvature factor ($\kappa_{51}^{P,i} = 0$) and there is no feedback from the credit slope factor on the Treasury slope factor ($\kappa_{42}^{P,i} = 0$).

Additional investigations lead us to a third preferred model with a mean-reversion matrix given by

$$K^{P,i} = \begin{pmatrix} 0 & 0 & 0 & \kappa_{14}^{P,i} & \kappa_{15}^{P,i} \\ \kappa_{21}^{P,i} & \kappa_{22}^{P,i} & 0 & \kappa_{24}^{P,i} & \kappa_{25}^{P,i} \\ 0 & 0 & \kappa_{33}^{P,i} & 0 & 0 \\ \kappa_{41}^{P,i} & 0 & \kappa_{43}^{P,i} & \kappa_{44}^{P,i} & \kappa_{45}^{P,i} \\ 0 & \kappa_{52}^{P,i} & 0 & 0 & \kappa_{55}^{P,i} \end{pmatrix}.$$

This is denoted "Case VII: preferred model III". The restriction $\kappa_{12}^{P,i} = 0$ indicates that there is no feedback from the credit risk slope factor on to the credit risk level factor, and the restriction $\kappa_{11}^{P,i} = 0$ indicates that the credit risk level factor has no own mean-reversion. Thus, the mean-reversion of the credit risk level factor has to be induced by changes in the Treasury slope and curvature factor.

In addition, we try and simplify the preferred model III both by fixing the factor loading of the Treasury level factor at zero, i.e. $\alpha_{L^T}^{BBB} = 0$, and by eliminating the feedback of the Treasury slope factor on the credit risk level factor, i.e. $\kappa_{14}^{P,i} = 0$. However, these additional restrictions do not find support across all the business sectors in the data.

Given the intriguing feedback effect from the credit risky factors to the Treasury factors represented by the parameters below the diagonal in the mean-reversion matrix (starting from the preferred II specification), we also estimate the two specifications of the mean-reversion matrix that cancel out either of these effects. The first of these specifications is given by

$$K^{P,i} = \begin{pmatrix} \kappa_{11}^{P,i} & \kappa_{12}^{P,i} & 0 & \kappa_{14}^{P,i} & \kappa_{15}^{P,i} \\ \kappa_{21}^{P,i} & \kappa_{22}^{P,i} & 0 & \kappa_{24}^{P,i} & \kappa_{25}^{P,i} \\ 0 & 0 & \kappa_{33}^{P,i} & 0 & 0 \\ 0 & 0 & \kappa_{43}^{P,i} & \kappa_{44}^{P,i} & \kappa_{45}^{P,i} \\ 0 & \kappa_{52}^{P,i} & 0 & 0 & \kappa_{55}^{P,i} \end{pmatrix}$$

and will be denoted "Misspecified model I". The other specification is given by

$$K^{P,i} = \begin{pmatrix} \kappa_{11}^{P,i} & \kappa_{12}^{P,i} & 0 & \kappa_{14}^{P,i} & \kappa_{15}^{P,i} \\ \kappa_{21}^{P,i} & \kappa_{22}^{P,i} & 0 & \kappa_{24}^{P,i} & \kappa_{25}^{P,i} \\ 0 & 0 & \kappa_{33}^{P,i} & 0 & 0 \\ \kappa_{41}^{P,i} & 0 & \kappa_{43}^{P,i} & \kappa_{44}^{P,i} & \kappa_{45}^{P,i} \\ 0 & 0 & 0 & 0 & \kappa_{55}^{P,i} \end{pmatrix}$$

and will be denoted "Misspecified model II".

6.3 Results for the credit risk models

This section will describe the results of the estimation of the joint five-factor model where Treasury bond yields are combined with credit spreads from one of the four business sectors in our data set. We present results from all four business sectors, and the purpose is ultimately to find a preferred specification for the dynamic interaction between the five state variables in the model.

Table 8 contains the maximum log likelihood values obtained for the various specifications considered in this paper when estimated using corporate bond credit spreads for US industrial firms.

Model	US industrials			
	Max log likelihood	D.f.	LR	p-value
Case I: Complete $K^{P,i}$	135031.69	n.a.	n.a.	n.a.
Case II: Complete $K^{P,i}$, L^T indep.	135016.51	8	30.36	0.00018
Case III: full $K^{P,T}$, zero credit risk feedback	135022.22	6	18.94	0.00427
Case IV: indep. factors	n.a.	20	n.a.	< 0.00001
Case V: preferred model I	135029.64	8	4.10	0.84799
Case VI: preferred model II	135028.77	10	5.74	0.83661
Case VII: preferred model III	135024.67	12	14.04	0.29816
Case VIII: preferred model III, $\alpha_{L^T}^{BBB} = 0$	n.a.	13	n.a.	n.a.
Case IX: preferred model III, $\kappa_{14}^{P,i} = 0$, $\alpha_{L^T}^{BBB} = 0$	135024.11	14	15.16	0.36731
Misspecified model I	135024.02	11	15.34	0.16745
Misspecified model II	135026.76	11	9.86	0.54302

Table 8: **The maximum log likelihood values for estimations based on Treasury bond yields and credit spreads for US industrial firms.**
The maximum log likelihood values obtained for different specifications of the joint estimation of the three Treasury bond yield risk factors and the two credit risk factors for US industrial firms. The data sets used contain weekly data covering the period from January 6, 1995 to August 4, 2006.

First, it is noted that, relative to the specification with complete $K^{P,i}$-matrix, none of the parameter restrictions in the first three specifications estimated initially are supported by the data. Thus, none of the factors are independent and, more importantly, there are significant feedback effects from the credit risk factors onto the three Treasury factors. By implication, if the dynamic interaction between all five state variables is modeled accurately, there could be benefits in the form of improvement of forecast performance by taking these feedback effects into consideration. Second, turning to the estimation results for the preferred models, we see that when the Treasury bond yield data is combined with credit spreads for US industrial firms both the preferred I and preferred II specification of $K^{P,i}$ is overwhelmingly supported by the data relative to the model with complete $K^{P,i}$-matrix. Thus, the treasury level factor, L_t^T, has no interaction with any of the other factors except for a feedback effect to the Treasury slope factor which is similar to the dynamic interaction in the preferred Treasury bond yield model obtained in Section 6.1. Furthermore, the feedback effects from the US industrial credit risk factors to the Treasury factors are limited to the two parameters $\kappa_{41}^{P,i}$ and $\kappa_{52}^{P,i}$.

Based on the results from the misspecified model I and II, none of these two parameters can be restricted to zero if we measure the marginal impact relative to the preferred II specification. Similarly, the additional restrictions in the preferred III specification are rejected relative to the preferred II specification. However, overall, the preferred III specification is supported by the data relative to the model with complete $K^{P,i}$-matrix, and we can even find support for the elimination of a few more parameters if we choose to use this 'absolute' measure when validating parameter restrictions.

In this situation, both the preferred II and the preferred III specification seem to strike a sensible balance, they eliminate 10 and 12 parameters, respectively, while preserving the ability of the model to represent all the key correlations between the five state variables.

Table 9 reports the maximum log likelihood values obtained when the Treasury bond yields are combined with credit spreads for US financial firms in the estimation of the joint five-factor

Model	US financials			
	Max log likelihood	D.f.	LR	p-value
Case I: Complete $K^{P,i}$	126083.16	n.a.	n.a.	n.a.
Case II: Complete $K^{P,i}$, L^T indep.	126068.19	8	29.94	0.00022
Case III: full $K^{P,T}$, zero credit risk feedback	126067.80	6	30.72	0.00003
Case IV: indep. factors	126023.34	20	119.64	< 0.00001
Case V: preferred model I	126076.02	8	14.28	0.07475
Case VI: preferred model II	126075.83	10	14.52	0.15057
Case VII: preferred model III	126074.39	12	17.54	0.13039
Case VIII: preferred model III, $\alpha_{L^T}^{BBB} = 0$	126073.54	13	19.24	0.11585
Case IX: preferred model III, $\kappa_{14}^{P,i} = 0$, $\alpha_{L^T}^{BBB} = 0$	126070.07	14	26.18	0.02456
Misspecified model I	126069.49	11	27.20	0.00429
Misspecified model II	126071.90	11	22.52	0.02064

Table 9: **The maximum log likelihood values for estimations based on Treasury bond yields and credit spreads for US financial firms.**
The maximum log likelihood values obtained for different specifications of the joint estimation of the three Treasury bond yield risk factors and the two credit risk factors for US financial firms. The data sets used contain weekly data covering the period from January 6, 1995 to August 4, 2006.

models.

For the US financial firms we see much the same pattern as was observed for US industrial firms. The results of the four preliminary estimations show that none of the five factors are independent and feedback effects from the US financial credit risk factors to the Treasury risk factors are an important part of their dynamic structure.

Relative to the model with complete $K^{P,i}$-matrix the preferred II model is the specification that finds the most support in the data with the preferred III specification a close second, and based on the results for the misspecified I and II estimations the elimination of $\kappa_{41}^{P,i}$ and $\kappa_{52}^{P,i}$ is not supported in this business sector either absolutely (i.e. relative to the complete $K^{P,i}$-model) or marginally (i.e. relative to the preferred II model). Thus, the interaction between the credit risk factors and the Treasury factors appears to be even more important for US financial firms than they were observed to be for US industrial firms. This might not be all that surprising given the differences in the nature of the business across these two sectors. Again, relative to the preferred III specification, we could get away with eliminating maybe one or two additional parameters, one of which is the factor loading for the BBB-rating category on the Treasury level factor in the credit spread function, i.e. $\alpha_{L^T}^{BBB} = 0$. We even tried to eliminate all the factor loadings on the Treasury level factor in the credit spread function, i.e. $\alpha_{L^T}^{BBB} = \alpha_{L^T}^{A} = \alpha_{L^T}^{AA} = \alpha_{L^T}^{AAA} = 0$. However, this was clearly rejected by the data. We interpret this rejection as an indication that the way to take any impact of the Treasury level factor on credit spreads into consideration is by letting it appear in the credit spread function with a separate factor loading for each rating category. Once this is done in the Nelson-Siegel based framework used here, there is no additional *direct* relationship between the Treasury level factor and the two credit risk factors as evidenced by the support observed so far for the preferred I-III specifications.

Table 10 reports the maximum log likelihood values for all the model specifications considered in this paper when they are estimated using corporate bond credit spreads for US banks. It is noted that the preferred I specification is rejected by the data for US banks. However, given the

Model	US banks			
	Max log likelihood	D.f.	LR	p-value
Case I: Complete $K^{P,i}$	90314.83	n.a.	n.a.	n.a.
Case II: Complete $K^{P,i}$, L^T indep.	90297.52	8	34.62	0.00003
Case III: full $K^{P,T}$, zero credit risk feedback	90294.29	6	41.08	< 0.00001
Case IV: indep. factors	90241.93	20	145.80	< 0.00001
Case V: preferred model I	90303.73	8	22.20	0.00456
Case VI: preferred model II	90303.52	10	22.62	0.01224
Case VII: preferred model III	90303.07	12	23.54	0.02348
Case VIII: preferred model III, $\alpha^{BBB}_{L^T} = 0$	90301.50	13	26.66	0.01384
Case IX: preferred model III, $\kappa^{P,i}_{14} = 0$, $\alpha^{BBB}_{L^T} = 0$	90299.57	14	30.52	0.00647
Misspecified model I	90298.40	11	32.86	0.00055
Misspecified model II	90299.30	11	31.06	0.00108

Table 10: **The maximum log likelihood values for estimations based on Treasury bond yields and credit spreads for US banks firms.**
The maximum log likelihood values obtained for different specifications of the joint estimation of the three Treasury bond yield risk factors and the two credit risk factors for US banks. The data sets used are weekly data covering the period from January 6, 1995 to August 4, 2006.

very high persistence of the Treasury level factor it is hard to believe that this factor should have any stable connections to the four other factors beyond its feedback into the Treasury slope factor built into the preferred I specification. Furthermore, the pairwise correlations between the five state variables is no different when the model is estimated with credit spreads for US banks than what we observe for the other business sectors in the data. Thus, we will choose to overlook the rejection of the preferred I specification for US banks and continue the search for an even more parsimonious specification.

From Table 10 it follows that the preferred III specification is the specification that minimizes the probability of being rejected relative to the specification with complete K^P-matrix conditional on using the preferred I specification as the starting point for the search for a refinement. We take the very marginal loss in likelihood value when switching from the preferred I to the preferred III specification as evidence that the preferred III specification is supported by the data even for US banks despite the fact that the probability of the likelihood ratio test for this specification is only 2.35%. Relative to the preferred I specification the additional 4 parameter restrictions are well supported by the data.

From Table 10 it is also clear that eliminating one or both of the feedback parameters of the credit risk factors onto the Treasury factors, i.e. fixing $K^{P,i}_{41}$ or $K^{P,i}_{52}$ at zero, only deteriorates the fit of the model.[18] Thus, similar to the results we got for the US financial firms, the interaction between the credit risk factors and the Treasury factors appear to be more important for both of these sectors than what we found for the US industrial sector.

Finally, there does not appear to be room for any additional parameter restrictions beyond those contained in the preferred III specification for the US banking sector.

Finally, Table 11 shows the maximum likelihood values obtained for all the different specifications considered so far when applied to the credit spreads from US utility firms. In this sector the preferred III specification is supported by the data beyond any reasonable doubt. In addition, we

[18]This can be concluded by combining the results for Case III with the results from misspecified model I and II.

Model	US utilities			
	Max log likelihood	D.f.	LR	p-value
Case I: Complete $K^{P,i}$	80827.42	n.a.	n.a.	n.a.
Case II: Complete $K^{P,i}$, L^T indep.	80814.40	8	26.04	0.00103
Case III: full $K^{P,T}$, zero credit risk feedback	80815.63	6	23.58	0.00062
Case IV: indep. factors	n.a.	20	n.a.	< 0.00001
Case V: preferred model I	80826.72	8	1.40	0.99425
Case VI: preferred model II	80824.35	10	6.14	0.80337
Case VII: preferred model III	80823.99	12	6.86	0.86672
Case VIII: preferred model III, $\alpha_{L^T}^{BBB} = 0$	80823.98	13	6.88	0.90823
Case IX: preferred model III, $\kappa_{14}^{P,i} = 0$, $\alpha_{L^T}^{BBB} = 0$	80820.91	14	13.02	0.52495
Misspecified model I	80818.68	11	17.48	0.09446
Misspecified model II	80821.33	11	12.18	0.35027

Table 11: **The maximum log likelihood values for estimations based on Treasury bond yields and credit spreads for US utility firms.**
The maximum log likelihood values obtained for different specifications of the joint estimation of the three Treasury bond yield risk factors and the two credit risk factors for US utility firms. The data sets used are weekly data covering the period from January 6, 1995 to August 4, 2006.

could easily eliminate a few more parameters. However, most importantly, we cannot eliminate both the feedback parameters for the credit risk factors without running into a clear rejection as evidenced from the likelihood ratio test for Case III. Thus, even for this sector there are some intriguing feedback mechanisms between the two credit risk factors and the three Treasury factors. However, this connection appears to be much less strong in this sector than what we observed for US banks and US financial firms which makes sense intuitively as the activity in the utility sector should be less interest rate sensitive than those in the banking and financial sector.

Turning to the estimated parameters Table 12 contains the estimated mean-reversion rates, mean levels and volatility parameters for the preferred III specification of the five-factor credit spread models estimated with credit spreads from the four different business sectors in our data set.

The remarkable thing to take away from this table is how similar to estimated parameters in the mean-reversion matrix $K^{P,i}$ are across the four sectors despite the fact that these are the results of four independent estimations where the only common component is the Treasury bond yields.

As expected the interactions between the three Treasury factors have very similar parameter estimates across the four sectors and are relatively close to estimated parameters for the preferred Treasury bond yield model reported in Table 6. The largest deviation from the latter result is observed for the mean-reversion rate of the curvature factor, $\kappa_{55}^{P,i}$, this factor is much faster mean-reverting in the joint estimations which might be attributed to the fact that the credit risk slope factor has a feedback effect onto the Treasury curvature factor in the preferred III specification.

Even more surprisingly, the parameters that reflect the dynamic interaction between the Treasury factors and the two credit factors all have the same sign[19] and they are, with few exceptions, all significant. The estimated volatility parameters also tell the same story independent of the

[19]There is one exception to this pattern and that is $\kappa_{15}^{P,i}$ for US financials which is clearly insignificant and is positive while $\kappa_{15}^{P,i}$ in the other three sectors is negative, and significantly so.

US Industrials

$K^{P,i}$	$K^{P,i}_{\cdot,1}$	$K^{P,i}_{\cdot,2}$	$K^{P,i}_{\cdot,3}$	$K^{P,i}_{\cdot,4}$	$K^{P,i}_{\cdot,5}$	$\mu^{P,i}$	Σ^i
$K^{P,i}_{1,\cdot}$	0	0	0	-0.03630	-0.06448	0.002271	0.001565
				(0.0364)	(0.0333)	(0.0347)	(0.0000892)
$K^{P,i}_{2,\cdot}$	1.608	1.985	0	-0.1482	-0.1072	-0.004512	0.002681
	(0.497)	(0.603)		(0.0900)	(0.0575)	(0.0222)	(0.000173)
$K^{P,i}_{3,\cdot}$	0	0	5.38×10^{-8}	0	0	0.07657	0.004141
			(0.0835)			(0.123)	(0.000155)
$K^{P,i}_{4,\cdot}$	1.957	0	1.610	0.6489	-0.6633	-0.03945	0.006840
	(0.760)		(0.452)	(0.172)	(0.147)	(0.129)	(0.000238)
$K^{P,i}_{5,\cdot}$	0	-4.538	0	0	1.382	-0.005578	0.02648
		(2.30)			(0.465)	(0.0724)	(0.000531)

US Financials

$K^{P,i}$	$K^{P,i}_{\cdot,1}$	$K^{P,i}_{\cdot,2}$	$K^{P,i}_{\cdot,3}$	$K^{P,i}_{\cdot,4}$	$K^{P,i}_{\cdot,5}$	$\mu^{P,i}$	Σ^i
$K^{P,i}_{1,\cdot}$	0	0	0	-0.1233	0.001931	0.006013	0.002637
				(0.0868)	(0.0701)	(0.0428)	(0.0000578)
$K^{P,i}_{2,\cdot}$	3.395	3.271	0	-0.4002	-0.2643	-0.005234	0.003825
	(0.929)	(0.874)		(0.203)	(0.132)	(0.0558)	(0.000171)
$K^{P,i}_{3,\cdot}$	0	0	8.95×10^{-8}	0	0	0.07400	0.004417
			(0.103)			(0.112)	(0.000164)
$K^{P,i}_{4,\cdot}$	2.239	0	1.468	0.6599	-0.5508	-0.02368	0.007313
	(0.709)		(0.489)	(0.186)	(0.185)	(0.0154)	(0.000251)
$K^{P,i}_{5,\cdot}$	0	-5.408	0	0	2.247	-0.00005984	0.02725
		(1.80)			(0.627)	(0.134)	(0.000567)

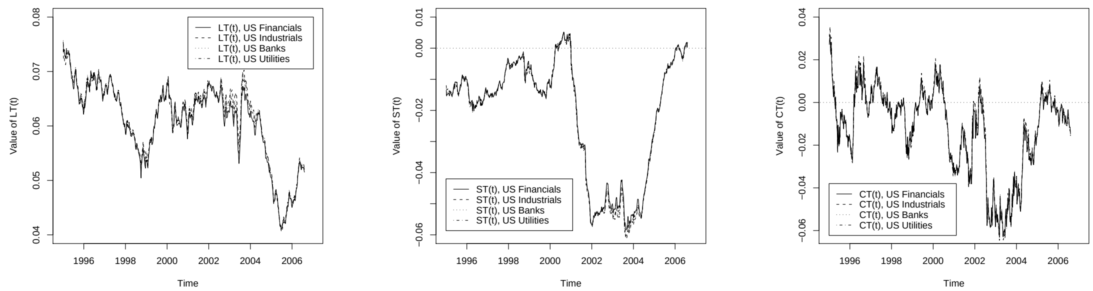
US Banks

$K^{P,i}$	$K^{P,i}_{\cdot,1}$	$K^{P,i}_{\cdot,2}$	$K^{P,i}_{\cdot,3}$	$K^{P,i}_{\cdot,4}$	$K^{P,i}_{\cdot,5}$	$\mu^{P,i}$	Σ^i
$K^{P,i}_{1,\cdot}$	0	0	0	-0.06058	-0.07803	0.006119	0.001687
				(0.0447)	(0.0422)	(0.0447)	(0.000130)
$K^{P}_{2,\cdot}$	2.942	2.721	0	-0.4741	-0.1739	-0.006896	0.002704
	(0.705)	(0.794)		(0.180)	(0.0971)	(0.0349)	(0.000228)
$K^{P}_{3,\cdot}$	0	0	3.71×10^{-7}	0	0	0.07683	0.004407
			(0.0877)			(0.139)	(0.000159)
$K^{P}_{4,\cdot}$	1.753	0	1.391	0.6090	-0.6044	-0.03391	0.007138
	(0.634)		(0.477)	(0.182)	(0.159)	(0.106)	(0.000228)
$K^{P}_{5,\cdot}$	0	-4.580	0	0	1.946	-0.004220	0.02759
		(1.73)			(0.585)	(0.0824)	(0.000560)

US Utilities

$K^{P,i}$	$K^{P,i}_{\cdot,1}$	$K^{P,i}_{\cdot,2}$	$K^{P,i}_{\cdot,3}$	$K^{P,i}_{\cdot,4}$	$K^{P,i}_{\cdot,5}$	$\mu^{P,i}$	Σ^i
$K^{P,i}_{1,\cdot}$	0	0	0	-0.06565	-0.06893	0.004722	0.001658
				(0.0412)	(0.0341)	(0.0336)	(0.0000942)
$K^{P,i}_{2,\cdot}$	1.220	1.502	0	-0.004154	-0.1005	-0.003676	0.002695
	(0.330)	(0.465)		(0.0849)	(0.0553)	(0.0324)	(0.000120)
$K^{P,i}_{3,\cdot}$	0	0	9.26×10^{-9}	0	0	0.07753	0.004538
			(0.195)			(0.144)	(0.000151)
$K^{P,i}_{4,\cdot}$	1.857	0	1.384	0.7133	-0.6503	-0.03411	0.007286
	(0.660)		(0.471)	(0.173)	(0.163)	(0.103)	(0.000242)
$K^{P,i}_{5,\cdot}$	0	-5.081	0	0	1.691	-0.003717	0.02792
		(2.33)			(0.523)	(0.0973)	(0.000570)

Table 12: **Parameter estimates for the joint five-factor credit spread model in the preferred III specification of $K^{P,i}$.**
The estimated parameters of the $K^{P,i}$-matrix, the Σ^i-matrix and the $\mu^{P,i}$-vector for the five-factor credit spread models with the preferred III specification of $K^{P,i}$. The estimated values of λ^T are: $\lambda^{T,Indu} = 0.4985$ (0.00530), $\lambda^{T,Fin} = 0.5458$ (0.00643), $\lambda^{T,Bank} = 0.5105$ (0.00582), and $\lambda^{T,Util} = 0.5189$ (0.00557). The estimated values of λ^S are: $\lambda^{S,Indu} = 0.4435$ (0.00749), $\lambda^{S,Fin} = 0.5290$ (0.00573), $\lambda^{S,Bank} = 0.5554$ (0.00840), and $\lambda^{S,Util} = 0.5215$ (0.00788). These values of λ requires maturity to be measured in years. The maximum log-likelihood values are $L^{Indu}_{max} = 135024.67$, $L^{Fin}_{max} = 126074.39$, $L^{Bank}_{max} = 90303.07$, and $L^{Util}_{max} = 80823.99$. The numbers in parentheses are the estimated standard deviations of the parameter estimates.



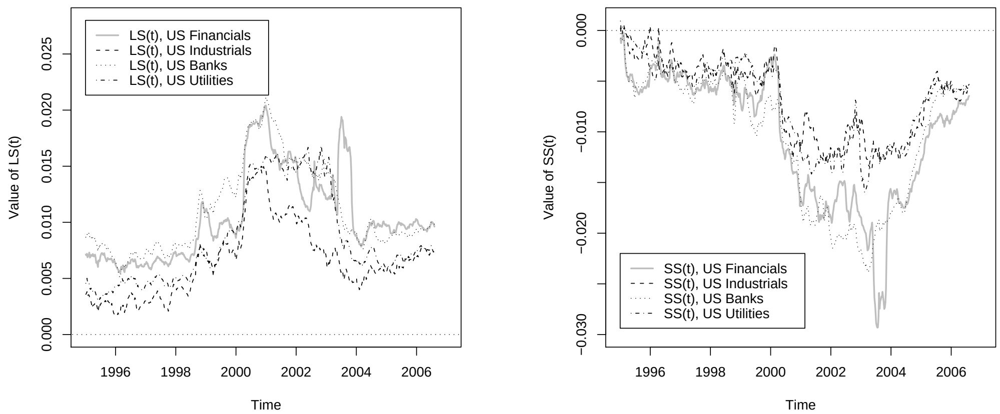
(a) Treasury level factor. (b) Treasury slope factor. (c) Treasury curvature factor.

Figure 4: **The estimated Treasury level, slope, and curvature factor from the five-factor credit spread model with the preferred III specification of $K^{P,i}$ from each business sector.**
The estimated Treasury level, slope, and curvature factor from the joint five-factor model for credit spreads estimated using Treasury zero-coupon bond yields and zero-coupon credit spreads covering the period from January 6, 1995 to August 4, 2006.

business sector included in the analysis. Thus, this leaves us with the impression that the two credit risk factors are economy-wide factors that impact the credit spreads of any business sector in much the same way and, at least as important, their interaction with the three Treasury factors share the same pattern across all four business sectors.

Figures 4 and 5 illustrate the estimated paths of the five state variables across the four business sectors using the preferred III specification. In terms of the Treasury factors there are only minor differences between these four estimation results, and relative to the results from the separate estimations using Treasury bond yields only reported in Section 6.2 the observed differences are also really small. Thus, one conclusion to take away from this is that, provided we are only interested in the dynamics of the three Treasury factors and not their interaction with the credit markets, a separate estimation will deliver perfectly satisfactory results.

As for the estimated paths of the two credit risk factors they are illustrated in Figure 5. The credit risk level factors exhibit a generally rising trend from the beginning of the sample to the beginning of 2001 where they stabilize and remain at a high level until the end of 2003 after which they drops fairly quickly to a level only little above that observed for the first few years of the sample period. The credit risk slope factors follow a similar, but negative, pattern with a generally declining trend up through 2003 and a rising trend until late 2005. The decomposition reveals that when credit spreads are high like in the 2001-2003 period, the spread curves tend to be steeper as indicated by the very negative values of the common slope factor during this period, and they tend to be more disperse across sectors as indicated by the larger variation across sectors for this particular period. The greater variation across sectors in the decomposition of the two credit risk factors relative to the very similar results for the Treasury factors can be explained by two things. First, the A-rating used as the benchmark rating with unit factor loadings on the two credit risk factors does not mean the same across business sectors, just for that reason we should not expect to see the same picture in all four business sectors. Second, there are some idiosyncracies specific to each sector that do not appear in the other sectors. Thus, taking these two things into consideration, we actually think that the obtained paths for the two credit risk factors are strikingly similar, and as we have said before we are lead to believe that these are two



(a) Estimated level factors.

(b) Estimated slope factors.

Figure 5: **The estimated level and slope factors from the joint five-factor credit spread model with the preferred III specification of $K^{P,i}$ from each business sector.** The estimated level and slope factors from the joint five-factor model. The credit spreads used in the estimation are weekly data covering the period from January 6, 1995 to August 4, 2006.

Level	$L_t^{S,Indu}$	$L_t^{S,Fin}$	$L_t^{S,Bank}$	$L_t^{S,Util}$
$L_t^{S,Indu}$	1	0.865	0.931	0.876
$L_t^{S,Fin}$		1	0.853	0.832
$L_t^{S,Bank}$			1	0.929
$L_t^{S,Util}$				1

Slope	$S_t^{S,Indu}$	$S_t^{S,Fin}$	$S_t^{S,Bank}$	$S_t^{S,Util}$
$S_t^{S,Indu}$	1	0.922	0.906	0.933
$S_t^{S,Fin}$		1	0.922	0.910
$S_t^{S,Bank}$			1	0.941
$S_t^{S,Util}$				1

Table 13: **Correlation coefficients in levels for the two common factors across sectors.** The correlation coefficients for the common level and slope factor, respectively, from four sectors: US Industrials, US Financials, US Banks, and US Utilities. In each case it is the estimated paths from the joint five-factor credit spread model with the preferred III specification of $K^{P,i}$ that are being used.

economy-wide factors that impact the credit spreads of all firms in the US economy independent of their rating or line of business.

Another way to study the connection between the credit risk factors across the various business sectors is by calculating the pairwise correlations of the estimated paths. This is done in Table 13 in levels, while Table 14 contains the pairwise correlations of the changes in the estimated paths. The numbers reported in these two tables support the view that the two credit risk factors are economy-wide factors. For the credit risk level factors the correlations in levels are all above 83%, while they are all larger than 90% for the credit risk slope factors, and even the weekly changes are surprisingly highly correlated, well above 30% with a single exception.

At this point it might be worthwhile to focus on the dynamic interactions between the five factors in the credit spread models as we observe them in the four sectors in our data. To that end

Level	$L_t^{S,Indu}$	$L_t^{S,Fin}$	$L_t^{S,Bank}$	$L_t^{S,Util}$
$L_t^{S,Indu}$	1	0.317	0.653	0.578
$L_t^{S,Fin}$		1	0.343	0.303
$L_t^{S,Bank}$			1	0.576
$L_t^{S,Util}$				1

Slope	$S_t^{S,Indu}$	$S_t^{S,Fin}$	$S_t^{S,Bank}$	$S_t^{S,Util}$
$S_t^{S,Indu}$	1	0.474	0.513	0.485
$S_t^{S,Fin}$		1	0.424	0.430
$S_t^{S,Bank}$			1	0.512
$S_t^{S,Util}$				1

Table 14: **Correlation coefficients in first-differences for the two common factors across sectors.**
The correlation coefficients for the common level and slope factor, respectively, from four sectors: US Industrials, US Financials, US Banks, and US Utilities. In each case it is the estimated paths from the joint five-factor credit spread model with the preferred III specification of $K^{P,i}$ that are being used.

US Industrials	L_t^T	S_t^T	C_t^T	L_t^S	S_t^S
L_t^T	1	-0.345	-0.084	-0.124	-0.003
S_t^T		1	0.603	-0.018	0.687
C_t^T			1	-0.185	0.640
L_t^S				1	-0.614
S_t^S					1

US Financials	L_t^T	S_t^T	C_t^T	L_t^S	S_t^S
L_t^T	1	-0.270	0.026	-0.056	0.039
S_t^T		1	0.619	-0.226	0.756
C_t^T			1	-0.466	0.777
L_t^S				1	-0.728
S_t^S					1

US Banks	L_t^T	S_t^T	C_t^T	L_t^S	S_t^S
L_t^T	1	-0.308	-0.034	0.098	-0.155
S_t^T		1	0.596	-0.117	0.814
C_t^T			1	-0.248	0.707
L_t^S				1	-0.606
S_t^S					1

US Utilities	L_t^T	S_t^T	C_t^T	L_t^S	S_t^S
L_t^T	1	-0.294	-0.014	0.077	-0.050
S_t^T		1	0.600	-0.341	0.729
C_t^T			1	-0.377	0.649
L_t^S				1	-0.743
S_t^S					1

Table 15: **Pairwise correlations between the common credit risk factors and the Treasury factors in the model with the preferred III specification of $K^{P,i}$.**
The pairwise correlation coefficients between the three Treasury bond yield factors, on one side, and the common credit spread level and slope factor, on the other side, for US industrial firms, US financial firms, US banks, and US utility firms, respectively.

Table 15 shows the pairwise correlations in levels between the estimated paths of the five state variables for each of the four business sectors. From the table we see that some correlations appear to be systematically small while other correlations are very high. A key question here is whether the estimated parameters for the preferred III specification reflect a dynamic relationship between the five state variables consistent with the pairwise correlations reported in Table 15. To provide an answer to that question we will briefly explain the connection between the sign of the estimated parameters in the mean-reversion matrix $K^{P,i}$ and the correlation of the corresponding factors that we should anticipate to observe. Given that the specification of the $K^{P,i}$-matrix plays no role in the determination of the filtered paths of the five state variables, we can base the discussion solely on the results for the preferred III specification without any loss of generality.

As an example, focus on the significant positive correlation between the Treasury slope and curvature factor (reported to be in the range from 0.596 to 0.619 in Table 15). If we neglect all other interactions for these two factors, the SDE for the Treasury slope factor reads

$$dS_t^T = K_{44}^{P,i}(\theta_{S^T}^{P,i} - S_t^T)dt + K_{45}^{P,i}(\theta_{C^T}^{P,i} - C_t^T)dt + \sigma_{S^T} dW_t^{P,S^T}.$$

Thus, the estimated values of $K_{45}^{P,i}$ in the range from -0.663 to -0.551 indicate that whenever the Treasury curvature factor is above its mean $\theta_{C^T}^{P,i}$ (and, by implication, $(\theta_{C^T}^{P,i} - C_t^T)$ is negative), it will tend to push up the Treasury slope factor. The positive pairwise correlation between these two factors reported in Table 15 indicates that this is exactly what is going on in the data. By similar arguments, the positive correlations between the Treasury slope and curvature factor, on one side, and the credit risk slope factor, on the other,[20] are consistent with the corresponding significant negative off-diagonal parameter estimates in the $K^{P,i}$-matrix.[21] The same holds for the strong negative correlation between the two credit risk factors and the large positive estimate of $K_{21}^{P,i}$. Finally, the negative correlation between the Treasury level factor and the Treasury slope factor is in line with the positive estimate for the $K_{23}^{P,i}$-parameter.

In summary, what we find when we look across business sectors is that the six key correlations

$$\rho(L_t^T, S_t^T), \rho(S_t^T, C_t^T), \rho(L_t^S, C_t^T), \rho(S_t^T, S_t^S), \rho(C_t^T, S_t^S), \rho(L_t^S, S_t^S)$$

all have not only the same sign, but also fairly close to the same size. To us this indicates that, qualitatively speaking, the dynamic interactions between the two credit risk factors, on one side, and the Treasury factors, on the other, are of the same nature across business sectors. One explanation for some of the variation we observe across sectors is tied to the fact that we are using the A-rating as the benchmark rating in each sector, and the results clearly show that an A-rating is not the same in all business sectors. The only way to bypass this problem would be to move to a full joint estimation across all business sectors. However, given the very large number of parameters in a full joint model we want to avoid going down that road for now.

If these factors turn out to be economy-wide factors, their interpretation will most naturally have to be tied to sentiments amongst bond investors rather than anything related to the economic developments in the respective business sectors, and an 'investor sentiment-based' explanation would fit well with the observed feedback effects from the credit risk factors on the Treasury factors, whereas a 'sector-specific, real economic-based' explanation would have to explain why these feedback effects are so similar across all four sectors given that the real economics of the four business sectors analyzed here is not the same.

[20] These correlations are in the range $(0.687, 0.814)$ and $(0.640, 0.771)$, respectively.

[21] This refers to the estimates of $K_{24}^{P,i}$, $K_{25}^{P,i}$ and $K_{52}^{P,i}$.

Rating	US Industrials				
	α_0^C	$\alpha_{L^T}^C$	$\alpha_{L^S}^C$	α_L^C	α_S^C
BBB	0.002856	-0.01062	-0.2740	1.492	1.530
	(0.000284)	(0.0344)	(0.00665)	(0.0123)	(0.0160)
A	0	0.02917	-0.1348	1	1
		(0.0231)	(0.00474)		
AA	0.001080	0.006656	-0.08676	0.6851	0.7489
	(0.000233)	(0.0161)	(0.00404)	(0.0110)	(0.0176)
AAA	0.001303	-0.0003272	-0.07147	0.6105	0.6982
	(0.000185)	(0.0141)	(0.00331)	(0.00822)	(0.0128)

Rating	US Financials				
	α_0^C	$\alpha_{L^T}^C$	$\alpha_{L^S}^C$	α_L^C	α_S^C
BBB+	-0.004112	-0.02191	-0.2164	2.012	1.268
	(0.000422)	(0.0662)	(0.00534)	(0.0132)	(0.0144)
A	0	0.002127	-0.1717	1	1
		(0.0331)	(0.00392)		
AA	0.002468	-0.03461	-0.1453	0.7951	0.8835
	(0.000346)	(0.0266)	(0.00439)	(0.0115)	(0.0206)
AAA	0.003664	-0.05191	-0.1387	0.6490	0.7899
	(0.000256)	(0.0213)	(0.00318)	(0.00813)	(0.0119)

Rating	US Banks				
	α_0^C	$\alpha_{L^T}^C$	$\alpha_{S^T}^C$	α_L^C	α_S^C
BBB	-0.0003142	-0.02581	-0.2658	1.210	1.165
	(0.000382)	(0.0318)	(0.00782)	(0.0128)	(0.0169)
A	0	-0.02290	-0.2048	1	1
		(0.0262)	(0.00782)		
AA	-0.0002333	-0.01377	-0.1750	0.8648	0.9005
	(0.000684)	(0.0244)	(0.00974)	(0.0207)	(0.0271)

Rating	US Utilities				
	α_0^C	$\alpha_{L^T}^C$	$\alpha_{S^T}^C$	α_L^C	α_S^C
BBB	0.001230	-0.003205	-0.2221	1.315	1.467
	(0.000218)		(0.00720)	(0.00547)	(0.0110)
A	0	0.01512	-0.1048	1	1
		(0.00278)	(0.00517)		

Table 16: **Estimated factors loadings in the credit spread function across rating categories in the model with the preferred III specification of $K^{P,i}$.**
The estimated factor loadings for each of the rating categories in the four sectors: US Industrial firms, US Financial firms, US Banks, and US Utility firms. In each case it is the five-factor credit spread model with the three Treasury factors included that has been estimated. The numbers in parentheses are the estimated standard deviations of the parameter estimates.

Table 16 presents the parameter estimates for the credit spread factor loadings for each rating category within the four different sectors. The estimates of the constant terms are significant for all of the ratings and industry categories. A clear pattern with regards to the magnitudes is not observed; while the constant decreases in value with decreases in credit rating for the financial category, it shows some increases for the other industry categories.

For the loadings on the two common factors we see a consistent pattern across sectors and rating categories. Within each sector, the loading on the two common risky factors increases monotonically as credit quality deteriorates. Thus, our modeling approach is supported by the data. Across sectors there are notable differences. For US financial and industrial firms the dispersion from the top to the bottom of the investment grade spectrum is about the same as measured by the ratios $\frac{\alpha_L^{BBB}}{\alpha_L^{AAA}}$ and $\frac{\alpha_S^{BBB}}{\alpha_S^{AAA}}$. However, for the intermediate ratings it is noted that for US financial firms all four ratings are fairly evenly distributed in terms of the distance between their factor loadings, whereas for US industrial firms the AA and AAA categories appear to be very close. For the US Banks, the differences between the AA and the BBB ratings are notably smaller than in the other sectors. Thus, the investors in the corporate bond market do not seem to differentiate between banks with respect to credit risk to the same extent as they do for non-bank borrowers; of course, this is assuming that ratings mean the same for banks and non-banks.

However, clear patterns are observed for the loadings on the two risky factors; in both cases, the loadings decrease in magnitude as credit quality improves. The loadings on the risky level factor decrease from a high range of between 1.2 to 2.0 (times the benchmark A category) for the BBB category to a low range of 0.61 to 0.65 for the AAA category. Across the industry categories, these estimated loadings show increasing divergence as the credit quality diminishes. Turning to the factor loadings for the risky slope factor, we find a similar pattern. The loadings on the risky slope factor decrease from a high range of between 1.2 to 1.5 (times the benchmark A category) for the BBB category to a low range of 0.7 to 0.8 for the AAA category. Again, across the industry categories, these estimated loadings show increasing divergence as the credit quality diminishes.

Turning to the Treasury level and slope factors included in the model, the level factor is found to be insignificant across all ratings and industry categories. However, the slope factor is statistically significant with a negative sign, such that an increase in the risk-free term structure would reduce credit spreads. The magnitudes of these factor loadings vary across the industry sectors. The loadings have a consistent pattern across ratings for all four sectors, it becomes increasingly negative as credit quality worsens. This implies that lower rated bond yields are more sensitive to changes in the slope of the risk-free yield curve.

Overall, the empirical results suggest the existence of common risky factors with factor loadings that decrease as credit quality increases and that exhibit more cross-industry dispersion as credit quality decreases. The greater responsiveness of lower quality bonds to common shocks is well established in the literature, as in for example Duffee (1998). Our finding that the relative magnitudes of these estimated loadings are similar across industry sectors provides support for the joint modeling of the risky factors across the rating categories and suggests that only minor adjustments need to be made for industrial sector, especially in the lower rating categories.

The principal component analysis of the credit spreads across 11 (sector, rating)-combinations performed in Section 3 indicated that the third principal component could explain an additional 6% of the variation in credit spreads. However, the analysis also indicated that this third component is likely to be of a different nature than that of a curvature factor in the Nelson-Siegel model, and controlling estimations performed with a third factor included in the form a curvature factor confirm that there is no third common credit factor with these properties.

6.4 Model fit

The overall fit of the credit spread models is illustrated based on the results for the five-factor credit spread models with the preferred III specification of $K^{P,i}$. Table 17 reports both the mean and root-mean-squared values of the fitted model errors; i.e., $e_{mijt}(\tau) = CS_{ijt}(\tau) - \widehat{CS}_{mijt}(\tau)$, where $CS_{ijt}(\tau)$ is the observed credit spread with maturity τ for rating category i and industry sector j at time t and $\widehat{CS}_{mijt}(\tau)$ is the fitted value of $CS_{ijt}(\tau)$ generated by model m. Overall,

Maturity in months	US Industrials							
	BBB		A		AA		AAA	
	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE
3	-1.24	9.99	0.00	9.48	-0.92	10.41	-1.04	10.54
6	3.95	8.66	5.60	8.99	5.59	9.95	5.49	10.29
12	-3.35	9.54	-1.03	8.49	-0.05	8.03	0.36	9.03
24	-5.72	9.82	-4.25	7.89	-2.62	7.47	-2.38	8.99
36	-2.87	8.63	-1.72	8.09	-1.59	5.87	-1.45	9.78
60	0.79	8.10	0.83	5.94	1.56	5.90	2.02	6.81
84	7.60	12.59	4.66	8.02	2.97	8.39	2.04	8.09
120	0.73	10.14	-4.13	10.76	-4.90	12.52	-4.96	12.27
No. obs	605		605		605		605	
Maturity in months	US Financials							
	BBB+		A		AA		AAA	
	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE
3	0.75	14.14	2.03	9.21	0.85	9.39	-0.73	10.98
6	4.76	14.82	5.35	9.63	4.28	8.96	3.71	9.66
12	-4.59	12.11	-1.84	10.48	-1.14	8.91	-1.25	7.66
24	-5.40	13.15	-6.34	11.13	-5.06	8.64	-2.99	7.62
36	-3.35	11.73	-3.44	12.78	-3.18	9.76	-0.74	9.31
60	1.89	13.86	-0.02	11.80	1.32	10.24	1.30	11.01
84	6.58	16.60	5.28	14.73	5.10	13.55	4.88	13.66
120	-0.27	18.44	-1.49	15.65	-2.43	12.96	-4.34	12.41
No. obs	515		605		605		605	
Maturity in months	US Banks							
	BBB		A		AA		AAA	
	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE
3	-1.40	8.70	-0.33	8.74	-1.57	8.92	n.a.	n.a.
6	6.29	10.42	5.79	9.73	0.86	9.81	n.a.	n.a.
12	-1.67	8.72	-2.05	7.91	3.59	9.81	n.a.	n.a.
24	-4.94	12.39	-5.78	8.67	-2.39	8.29	n.a.	n.a.
36	-3.78	11.29	-2.09	9.10	1.22	12.51	n.a.	n.a.
60	-0.76	10.97	-2.50	7.86	-7.00	10.93	n.a.	n.a.
84	10.56	16.63	8.79	13.04	14.86	19.64	n.a.	n.a.
120	-4.50	14.67	-1.98	14.28	-9.45	16.82	n.a.	n.a.
No. obs	605		605		255		n.a.	
Maturity in months	US Utilities							
	BBB		A		AA		AAA	
	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE
3	-0.05	10.16	-2.12	11.32	n.a.	n.a.	n.a.	n.a.
6	4.92	10.11	3.05	9.96	n.a.	n.a.	n.a.	n.a.
12	-0.61	7.47	-1.37	10.63	n.a.	n.a.	n.a.	n.a.
24	-3.50	8.90	-2.59	9.00	n.a.	n.a.	n.a.	n.a.
36	-4.23	8.45	-0.61	6.40	n.a.	n.a.	n.a.	n.a.
60	0.36	8.26	3.45	9.85	n.a.	n.a.	n.a.	n.a.
84	2.34	7.93	3.46	8.60	n.a.	n.a.	n.a.	n.a.
120	0.66	12.69	-3.22	12.59	n.a.	n.a.	n.a.	n.a.
No. obs	605		605		n.a.		n.a.	

Table 17: **Summary statistics for the fitted errors in the joint five-factor credit spread model with the preferred III specification of $K^{P,i}$.**
The mean and the root mean squared errors for the fit of the zero-coupon credit spreads across the 8 different maturities and the different rating categories in each of the four sectors: US Industrial firms, US Financial firms, US Banks, and US Utility firms. The model estimated is in each case the five-factor credit spread model with the three Treasury factors included. All numbers are measured in basis points.

the RMSE values are high relative to those reported for the risk-free model in Table 7.[22] This poorer fit could be due to a number of reasons; for example, the underling variances of credit spread curves could be higher than that of the risk-free yield curve. Another reason could be that the model is not well specified in that the two risk-free and two risky factors are not enough to capture the dynamics of credit spread curve. Aside from their relative magnitudes, the RMSE values present no clear patterns. While there does seem to be a U-shaped pattern across the maturity patterns, there are no clear patterns across industry sectors and rating categories.

6.5 Conclusion of the in-sample analysis

In this subsection we are going to briefly summarize the findings of our search for refinements of the joint dynamic specification of the three Treasury factors and the two common credit risk factors.

The first thing to take away is that 'Case III' which eliminated any feedback efffects from the credit risk factors onto the Treasury factors while allowing for maximum flexibility of all other parameters in the mean-reversion matrix $K^{P,i}$ was clearly rejected in all sectors. Thus, there are some significant feedback effects from the credit risk factors onto the three Treasury factors that need to be modeled.

In a trial-and-error type of approach we arrived at the preferred III specification where the mean-reversion matrix has the following structure

$$K^{P,i} = \begin{pmatrix} 0 & 0 & 0 & \kappa_{14}^{P,i} & \kappa_{15}^{P,i} \\ \kappa_{21}^{P,i} & \kappa_{22}^{P,i} & 0 & \kappa_{24}^{P,i} & \kappa_{25}^{P,i} \\ 0 & 0 & \kappa_{33}^{P,i} & 0 & 0 \\ \kappa_{41}^{P,i} & 0 & \kappa_{43}^{P,i} & \kappa_{44}^{P,i} & \kappa_{45}^{P,i} \\ 0 & \kappa_{52}^{P,i} & 0 & 0 & \kappa_{55}^{P,i} \end{pmatrix}.$$

This specification is well supported by the data for US industrial firms, US financial firms, and US utility firms, while it is only partially supported in the data for US banks. If this is close to the true specification for the joint dynamics of the five state variables, it has a number of implications for the workings of both the Treasury and the corporate bond markets.

First, the Treasury level factor is close to a unit root process that moves around without being impacted by changes in any of the other state variables, and its only interaction is to give a feedback to the Treasury slope factor.

Second, the Treasury curvature factor has three important roles. It acts as a stochastic mean for the Treasury slope factor, and it induces feedback effects into the credit risk level and slope factor.

Third, the Treasury slope factor is impacted by changes in the credit risk level factor, the Treasury curvature factor, and the Treasury level factor, while itself having an impact on both the credit risk level and slope factor.

[22] For the US Financial firms we can compare the performance to the one reported in Feldhütter and Lando (2006) who also fit a five-factor model to the zero-coupon yields for this particular sector. On average, the RMSEs reported here are 2-3 bps higher than the ones reported by Feldhütter and Lando (2006). However, in that study they only use the maturities from 1 year up to 10 years. Thus, the tricky short end of the spread curve is neglected. Furthermore, they take the possibility of rating transitions into consideration which may improve the performance of their model even further.

Fourth, the credit risk level factor has no own mean-reversion. This implies that it is moved back to steady state only by changes in the Treasury slope and curvature factors. Thus, once credit spreads have spiked up due to a significant weakening of the business cycle they will not come back down until there has been a significant change in these two Treasury factors, i.e. until the FED has responded by easing monetary policy. In addition, the credit risk level factor itself has a feedback effect on not only the credit risk slope factor (causing the spread curve to steepen when the general spread levels move up) but also on the Treasury slope factor - this indicates that the FED responds to elevated levels of credit spreads which was definitely the case at the FOMC meetings in the Fall of 2007 (an episode, by the way, outside the sample used in this paper!)

Fifth, the credit risk slope factor has a feedback effect on the Treasury curvature factor which could turn out to be useful in terms of the model's forecast ability. If the credit risk slope factor is a leading indicator of the turning points in the Treasury curvature factor, this model will exhibit superior out of sample forecast performance. However, only a forecast exercise with recursive re-estimations can validate this conjecture.

Finally, the visual inspection of the estimated paths as well as the pairwise correlations in Table 15 from the five-factor credit spread model with the Treasury factors included leave the impression that the two common credit risk factors from each industry appear merely to be sector specific representations of the same two economy-wide systematic risk factors. This result is in line with Driessen (2005) who also finds that there are two common credit risk factors in addition to two Treasury factors for his sample of 104 firms.

7 Forecast performance

First, the method of producing forecasts based on the arbitrage-free dynamic term structure models analyzed in this paper is described. Second, the results of the forecast exercise are analyzed.

7.1 Construction of out-of-sample forecasts

In the affine arbitrage-free models approximating the Nelson-Siegel Treasury yield function the τ-maturity yield in period $t+n$ is given by

$$y_{t+n}(\tau) = L_{t+n}^T + S_{t+n}^T\Big(\frac{1-e^{-\gamma\tau}}{\gamma\tau}\Big) + C_{t+n}^T\Big(\frac{1-e^{-\gamma\tau}}{\gamma\tau} - e^{-\gamma\tau}\Big) - \frac{C(\tau)}{\tau} + \varepsilon_{t+n}(\tau).$$

Given this model the best forecast of the yield based on the information available at time t is simply the conditional expectation

$$\widehat{y}_{t+n}^{AFDNS}(\tau) \equiv E_t^P[y_{t+n}(\tau)] = E_t^P[L_{t+n}^T] + E_t^P[S_{t+n}^T]\Big(\frac{1-e^{-\gamma\tau}}{\gamma\tau}\Big) + E_t^P[C_{t+n}^T]\Big(\frac{1-e^{-\gamma\tau}}{\gamma\tau} - e^{-\gamma\tau}\Big) - \frac{C(\tau)}{\tau}.$$

Thus, in order to be able to produce forecasts of Treasury yields based on this class of models all that is needed is to be able to calculate the conditional expectations $E_t^P[L_{t+n}^T]$, $E_t^P[S_{t+n}^T]$, and $E_t^P[C_{t+n}^T]$. In the literature on Ornstein-Uhlenbeck processes it is a well-known result that the conditional expectation is given by[23]

$$E_0^P[X_t] = (I - \exp(-K^P t))\theta^P + \exp(-K^P t)X_0,$$

where in our case $X_t = (L_t^S, S_t^S, L_t^T, S_t^T, C_t^T)$.

Thus, given the estimated parameters for the K^P-matrix and the θ^P-vector combined with the optimally filtered paths of the five state variables it is easy to calculate the conditional expected value for any time horizon. Once those conditional expected values have been calculated, it is an even easier task to derive the forecasted future yields as outlined above.

In this paper the focus is on the 4-week, 26-week, and 52-week forecast horizons, and the yields to be forecasted consist of the whole maturity range in the Treasury data set. Thus, the 3-month, 6-month, 1-year, 2-year, 3-year, 5-year, 7-year and 10-year Treasury yields are all forecasted.

To evaluate the performance of the joint five-factor model with complete K^P-matrix estimated based on credit spreads from the four different business sectors in our data set, the forecast errors for the Treasury bond yields are compared both to the forecast errors obtained from a random walk assumption and to the forecast errors from regular three-factor AFDNS models where we choose to focus on two specifications. The first specification is the AFDNS model with independent factors. This model has P-dynamics given by

$$\begin{pmatrix} dL_t^T \\ dS_t^T \\ dC_t^T \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{P,T} & 0 & 0 \\ 0 & \kappa_{22}^{P,T} & 0 \\ 0 & 0 & \kappa_{33}^{P,T} \end{pmatrix} \left[\begin{pmatrix} \theta_1^{P,T} \\ \theta_2^{P,T} \\ \theta_3^{P,T} \end{pmatrix} - \begin{pmatrix} L_t^T \\ S_t^T \\ C_t^T \end{pmatrix} \right] dt + \begin{pmatrix} \sigma_{L^T} & 0 & 0 \\ 0 & \sigma_{S^T} & 0 \\ 0 & 0 & \sigma_{C^T} \end{pmatrix} \begin{pmatrix} dW_t^{P,L^T} \\ dW_t^{P,S^T} \\ dW_t^{P,C^T} \end{pmatrix}.$$

This was the specification that performed the best in Christensen, Diebold, and Rudebusch (2007). However, from the in-sample analysis in Section 6.1 based on Treasury bond yields only we found significant support for the following specification

$$\begin{pmatrix} dL_t^T \\ dS_t^T \\ dC_t^T \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{P,T} & 0 & 0 \\ \kappa_{21}^{P,T} & \kappa_{22}^{P,T} & \kappa_{23}^{P,T} \\ 0 & 0 & \kappa_{33}^{P,T} \end{pmatrix} \left[\begin{pmatrix} \theta_1^{P,T} \\ \theta_2^{P,T} \\ \theta_3^{P,T} \end{pmatrix} - \begin{pmatrix} L_t^T \\ S_t^T \\ C_t^T \end{pmatrix} \right] dt + \begin{pmatrix} \sigma_{L^T} & 0 & 0 \\ 0 & \sigma_{S^T} & 0 \\ 0 & 0 & \sigma_{C^T} \end{pmatrix} \begin{pmatrix} dW_t^{P,L^T} \\ dW_t^{P,S^T} \\ dW_t^{P,C^T} \end{pmatrix}.$$

This specification allows for feedback effects from both the Treasury level and curvature factor to the Treasury slope factor in line with the pairwise correlations between the filtered paths of the three Treasury risk factors. We therefore term this specification 'AFDNS, S^T interaction only'.

The two regular three-factor AFDNS models are estimated using Treasury bond yields only and their yield forecasts are produced in the way described above.

Under the random walk assumption the forecasted yield n periods ahead is given by

$$\widehat{y}_{t+n}^{RW}(\tau) \equiv y_t(\tau).$$

The comparison of the different model forecasts is based on the forecast error simply defined as the forecasted yield minus the actually observed yield for that maturity n periods ahead, $\varepsilon_t(\tau, n) = y_{t+n}^{fore.}(\tau) - y_{t+n}(\tau)$, and the summary statistics to be used in the evaluation are the following

[23] In the following the industry identifier is suppressed in the notation.

- The mean $\mu(\tau, n) = \frac{1}{T}\sum_{i=1}^{T} \varepsilon_i(\tau, n)$, $\tau = 3, 6, 12, 24, 36, 60, 84, 120$, $n = 4, 26, 52$.

- The root mean squared error $RMSE(\tau, n) = \sqrt{\frac{1}{T}\sum_{i=1}^{T} \varepsilon_i(\tau, n)^2}$, $\tau = 3, 6, 12, 24, 36, 60, 84, 120$, $n = 4, 26, 52$.

- The standard deviation $\sigma(\tau, n) = \sqrt{\frac{1}{T-1}\sum_{i=1}^{T}(\varepsilon_i(\tau, n) - \mu(\tau, n))^2}$, $\tau = 3, 6, 12, 24, 36, 60, 84, 120$, $n = 4, 26, 52$.

For this exercise we updated the data set so that we have weekly observations from January 6, 1995 up to January 4, 2008. We use a recursive procedure. For the first set of forecasts, the model is estimated from January 6, 1995 to January 7, 2005; then, one week of data is added, the models are reestimated, and another set of forecasts is constructed. The largest estimation sample ends on January 5, 2007, leaving 104 forecasts at all three forecast horizons.

7.2 Out-of-sample forecast results for US banks

Table 18 contains the result of the out-of-sample forecast exercise using both separate three-factor models to fit just Treasury bond yields, and the result from the five-factor model with complete $K^{P,i}$-matrix estimated using Treasury bond yields and credit spreads for US banks.[24] The first important thing to take away from this table is that the preferred specification of the Treasury bond yield model, the 'AFDNS, S^T interaction only'-model, clearly outperforms the random walk assumption. This provides evidence that it is important to get the dynamics between the Treasury slope factor, on one side, and the Treasury level and curvature factor, on the other, right when it comes to forecasting Treasury yields.

Second, the results reported in Table 18 for the joint estimation of the five-factor model with credit spreads from US banks and $K^{P,i}$ specified by a complete matrix indicate that the interaction between the Treasury factors and the credit risk factors is so strong that even this very flexible and highly overidentified model cannot avoid picking up these effects and actually improve upon the already very good forecast performance of the preferred three-factor Treasury bond yield model.

The fact that the specification with a complete $K^{P,i}$-matrix is overidentified shows up at the 52-week forecast horizon where its flexibility with the 25 free parameters in the $K^{P,i}$-matrix start to confound the interactions between the five factors. Why is this the case? Since we use weekly data in the estimation we really only estimate the matrix, $\exp(-K^{P,i}\frac{1}{52})$, so when we generate the 52-week forecasts we are calculating this matrix to the power of 52. Thus, if there are just minor deviations in our estimates from the true parameters of the model in the one-week-ahead estimate of the dynamic interactions between the five factors, these deviations cumulate once we forecast at the 52-week horizon, and we think this is what drives the clearly weaker performance at the 52-week horizon compared with the good results at the 4- and 26-week horizons. This also encourages us to repeat the exercise with one or more of our preferred specifications, since a well-identified specification of the $K^{P,i}$-matrix should suffer less from such effects and thus be able to improve the forecast performance even further.

Finally, for the maturities up to one year the joint model shows a clear improvement relative to both the random walk and the best performing regular three-factor model. We take this as evidence that the information contained in the corporate bond markets has something important

[24] The remaining estimations of the complete $K^{P,i}$ and preferred specifications for the three sectors are available from the authors upon request.

Horizon	4 weeks			26 weeks			52 weeks		
Maturity: 3-month yield	Mean	Std.	RMSE	Mean	Std.	RMSE	Mean	Std.	RMSE
Random walk	10.18	11.08	15.01	55.89	48.23	73.67	64.12	117.69	133.53
AFDNS, indep. factors	5.50	9.92	11.30	51.78	48.27	70.63	60.56	117.59	131.77
AFDNS, S^T interaction only	-4.96	11.87	12.81	-6.22	24.42	25.09	-36.24	47.37	59.46
US banks, complete $K^{P,i}$	-6.01	10.80	12.32	-17.92	16.63	24.40	-56.23	48.19	73.90
Maturity: 6-month yield	Mean	Std.	RMSE	Mean	Std.	RMSE	Mean	Std.	RMSE
Random walk	9.37	11.35	14.67	49.83	45.03	67.02	52.94	110.95	122.45
AFDNS, indep. factors	9.53	11.71	15.05	50.70	48.37	69.91	54.39	115.29	126.98
AFDNS, S^T interaction only	-0.37	11.22	11.17	-4.00	22.45	22.70	-36.87	48.31	60.59
US banks, complete $K^{P,i}$	-1.45	10.52	10.57	-15.92	14.43	21.44	-57.03	49.85	75.59
Maturity: 1-year yield	Mean	Std.	RMSE	Mean	Std.	RMSE	Mean	Std.	RMSE
Random walk	8.07	14.09	16.18	40.74	41.55	58.04	37.37	100.79	107.04
AFDNS, indep. factors	12.84	15.13	19.79	46.36	48.02	66.59	43.59	110.35	118.16
AFDNS, S^T interaction only	3.98	13.22	13.74	-2.45	22.97	22.99	-37.81	49.95	62.46
US banks, complete $K^{P,i}$	2.97	13.12	13.39	-14.41	15.86	21.37	-57.87	52.60	78.03
Maturity: 2-year yield	Mean	Std.	RMSE	Mean	Std.	RMSE	Mean	Std.	RMSE
Random walk	6.36	17.22	18.28	30.21	39.45	49.54	22.03	87.04	89.38
AFDNS, indep. factors	10.61	17.78	20.64	35.30	46.21	57.97	27.51	99.75	103.02
AFDNS, S^T interaction only	3.44	16.48	16.76	-4.09	27.17	27.35	-38.16	50.66	63.23
US banks, complete $K^{P,i}$	2.82	16.72	16.88	-15.22	21.41	26.18	-56.97	54.81	78.87
Maturity: 3-year yield	Mean	Std.	RMSE	Mean	Std.	RMSE	Mean	Std.	RMSE
Random walk	5.29	18.39	19.05	24.90	38.96	46.08	16.89	77.10	78.57
AFDNS, indep. factors	5.48	18.53	19.24	25.70	44.35	51.08	17.76	89.38	90.70
AFDNS, S^T interaction only	-0.38	18.03	17.95	-6.65	30.26	30.84	-36.20	48.81	60.58
US banks, complete $K^{P,i}$	-0.53	18.33	18.25	-16.39	25.01	29.81	-53.17	54.07	75.65
Maturity: 5-year yield	Mean	Std.	RMSE	Mean	Std.	RMSE	Mean	Std.	RMSE
Random walk	3.92	18.60	18.93	19.69	37.96	42.60	16.17	61.71	63.51
AFDNS, indep. factors	0.13	18.87	18.78	15.89	42.14	44.85	11.75	72.41	73.02
AFDNS, S^T interaction only	-3.92	18.85	19.16	-7.05	33.60	34.17	-26.60	43.31	50.65
US banks, complete $K^{P,i}$	-3.31	19.19	19.38	-13.92	29.19	32.22	-39.85	50.39	64.06
Maturity: 7-year yield	Mean	Std.	RMSE	Mean	Std.	RMSE	Mean	Std.	RMSE
Random walk	2.95	18.28	18.43	16.44	37.11	40.43	17.24	50.50	53.13
AFDNS, indep. factors	3.29	18.66	18.86	16.18	41.52	44.38	15.82	61.50	63.22
AFDNS, S^T interaction only	0.36	18.43	18.35	-1.08	35.42	35.26	-13.14	39.69	41.62
US banks, complete $K^{P,i}$	1.42	18.95	18.91	-5.66	32.16	32.50	-23.47	48.25	53.45
Maturity: 10-year yield	Mean	Std.	RMSE	Mean	Std.	RMSE	Mean	Std.	RMSE
Random walk	1.87	17.98	17.99	12.59	36.83	38.75	16.93	40.74	43.94
AFDNS, indep. factors	15.60	18.80	24.36	25.10	41.72	48.52	27.68	53.61	60.11
AFDNS, S^T interaction only	13.63	18.27	22.72	12.79	37.26	39.23	6.92	38.53	38.97
US banks, complete $K^{P,i}$	14.95	19.29	24.33	10.42	35.79	37.11	-0.55	48.72	48.49

Table 18: Summary statistics for the out of sample forecast errors for US banks. In each segment, the best performing model is indicated by underlined numbers, while the best performing model overall including the random walk assumption is indicated by bold numbers. All numbers are measured in basis points. For all forecast horizons, the rolling re-estimation is performed over the period from January 7, 2005 to January 5, 2007. This provides a total of 104 forecasts for all forecast horizons.

to bear on the factors determining Treasury bond yields. In particular the Treasury slope and curvature factors that are the primary factors determining yields up to about the 3-year maturity appear to have interactions with these factors that are worthwhile modeling more precisely.

8 Conclusion

The vast majority of the term structure literature has focused on modeling the risk-free term structure as implied by Treasury bond yields. As fixed-income markets should be interconnected, we combine the modeling of Treasury yields with a modeling of the common factors present in representative risky credit spread term structures derived from Bloomberg corporate bond data. The question we address is whether we can improve our understanding of, and our ability to forecast, Treasury yields by incorporating information from corporate bond market. We use the arbitrage-free dynamic version of the Nelson-Siegel yield-curve model derived Christensen, Diebold and Rudebusch (2007) to model Treasury yields and corporate bond spreads across rating and industry categories. In addition to the three-factor Nelson-Siegel factors for Treasury yields, we find two common factors—a level and a slope factor—are required to capture the time series dynamics of aggregated credit spreads. We find that the preferred specifications of the joint dynamics of all five factors have feedback effects from the Treasury factors to the credit risk factors, but we also find feedback effects from the credit risk factors to the Treasury factors. To determine the significance of these feedback effects, we perform an out-of-sample forecast exercise. The results so far suggest that the preferred Treasury yield model can easily beat the random walk and that adding the information from the credit markets allows us to improve forecast performance even further for forecast horizons up to 26-weeks.

9 Appendix: The analytical solution of the corporate zero-coupon bond yield function

All models analyzed in this paper are nested versions of the six-factor model of corporate bond zero-coupon bond prices introduced in Section ??. The six state variables consist of a level, slope, and curvature factor related to the Treasury bond market, in addition to a level, slope, and curvature factor common to all the firms in sector i independent of their rating. That is, $X_t = (L_t^T, S_t^T, C_t^T, L_t^S(i), S_t^S(i), C_t^S(i))$. In order to obtain a Nelson-Siegel factor loading structure for these factors in the corporate zero-coupon bond yield function of the form described in Section ??, two assumptions must be imposed.

First, the instantaneous discount rate must be assumed to be

$$\begin{aligned} r_t^{i,c} &= \rho_0^{i,c} + (\rho_1^{i,c})' X_t \\ &= (1+\alpha_{L^T}^{i,c}) L_t^T + (1+\alpha_{S^T}^{i,c}) S_t^T + \alpha_{L^S}^{i,c} L_t^S(i) + \alpha_{S^S}^{i,c} S_t^S(i). \end{aligned}$$

This means that the vector of factor loadings in the discount rate are given by

$$\rho_0^{i,c} = 0 \quad \text{and} \quad \rho_1^{i,c} = \begin{pmatrix} 1+\alpha_{L^T}^{i,c} \\ 1+\alpha_{S^T}^{i,c} \\ 0 \\ \alpha_{L^S}^{i,c} \\ \alpha_{S^S}^{i,c} \\ 0 \end{pmatrix}.$$

Second, the dynamics of the six factors under the risk-neutral pricing measure must be given by the solution to the following SDE

$$\begin{pmatrix} dL_t^T \\ dS_t^T \\ dC_t^T \\ dL_t^S(i) \\ dS_t^S(i) \\ dC_t^S(i) \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^T & -\lambda^T & 0 & 0 & 0 \\ 0 & 0 & \lambda^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^S(i) & -\lambda^S(i) \\ 0 & 0 & 0 & 0 & 0 & \lambda^S(i) \end{pmatrix} \begin{pmatrix} L_t^T \\ S_t^T \\ C_t^T \\ L_t^S(i) \\ S_t^S(i) \\ C_t^S(i) \end{pmatrix} dt + \Sigma^i \begin{pmatrix} dW_t^{L^T,Q} \\ dW_t^{S^T,Q} \\ dW_t^{C^T,Q} \\ dW_t^{L^S,Q}(i) \\ dW_t^{S^S,Q}(i) \\ dW_t^{C^S,Q}(i) \end{pmatrix}.$$

Note here that all six factors are Gaussian processes with constant volatility matrix.

From Duffie and Kan (1996) it now follows that the price at time t of a representative zero-coupon bond from industry i with rating c and maturity at T is given by

$$\begin{aligned} P^{i,c}(t,T) &= E_t^Q[e^{-\int_t^T r_u^{i,c} du}] \\ &= \exp(B_{L^T}^{i,c}(t,T)L_t^T + B_{S^T}^{i,c}(t,T)S_t^T + B_{C^T}^{i,c}(t,T)C_t^T + B_{L^S}^{i,c}(t,T)L_t^S + B_{S^S}^{i,c}(t,T)S_t^S + B_{C^S}^{i,c}(t,T)C_t^S + A_y^{i,c}(t,T)), \end{aligned}$$

where $B^{i,c}(t,T) = (B_{L^T}^{i,c}(t,T), B_{S^T}^{i,c}(t,T), B_{C^T}^{i,c}(t,T), B_{L^S}^{i,c}(t,T), B_{S^S}^{i,c}(t,T), B_{C^S}^{i,c}(t,T))$ and $A_y^{i,c}(t,T)$

are the unique solutions to the following set of ODEs

$$\begin{aligned}
\frac{dB^{i,c}(t,T)}{dt} &= \rho_1^{i,c} + (K^Q(i))' B^{i,c}(t,T), \quad B^{i,c}(T,T) = 0, \\
\frac{dA_y^{i,c}(t,T)}{dt} &= \rho_0^{i,c} + -\frac{1}{2}\sum_{j=1}^{6}((\Sigma^i)' B^{i,c}(s,T) B^{i,c}(s,T)' \Sigma^i)_{j,j} \quad A_y^{i,c}(T,T) = 0.
\end{aligned}$$

Now, we solve these two systems of ODEs. We start with the system for $B^{i,c}(t,T)$. Since

$$\frac{d}{dt}\left[e^{(K^Q(i))'(T-t)} B^{i,c}(t,T)\right] = e^{(K^Q(i))'(T-t)} \frac{dB^{i,c}(t,T)}{dt} - (K^Q(i))' e^{(K^Q(i))'(T-t)} B^{i,c}(t,T),$$

we can use the system of ODEs for $B^{i,c}(t,T)$ to obtain

$$\int_t^T \frac{d}{ds}\left[e^{(K^Q(i))'(T-s)} B^{i,c}(s,T)\right] ds = \int_t^T e^{(K^Q(i))'(T-s)} \rho_1^{i,c} ds$$

or, equivalently, using the boundary conditions

$$B^{i,c}(t,T) = -e^{-(K^Q(i))(T-t)} \int_t^T e^{(K^Q(i))'(T-s)} \rho_1^{i,c} ds.$$

Here, it is easy to show that

$$e^{(K^Q(i))'(T-t)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{\lambda^T(T-t)} & 0 & 0 & 0 & 0 \\
0 & -\lambda^T(T-t)e^{\lambda^T(T-t)} & e^{\lambda^T(T-t)} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{\lambda^S(i)(T-t)} & 0 \\
0 & 0 & 0 & 0 & -\lambda^S(i)(T-t)e^{\lambda^S(i)(T-t)} & e^{\lambda^S(i)(T-t)}
\end{pmatrix}$$

and

$$e^{-(K^Q(i))'(T-t)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{-\lambda^T(T-t)} & 0 & 0 & 0 & 0 \\
0 & \lambda^T(T-t)e^{-\lambda^T(T-t)} & e^{-\lambda^T(T-t)} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{-\lambda^S(i)(T-t)} & 0 \\
0 & 0 & 0 & 0 & \lambda^S(i)(T-t)e^{-\lambda^S(i)(T-t)} & e^{-\lambda^S(i)(T-t)}
\end{pmatrix}.$$

Now, inserting this into the expression for $B^{i,c}(t,T)$ we obtain

$$B^{i,c}(t,T) = -\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\lambda^T(T-t)} & 0 & 0 & 0 & 0 \\ 0 & \lambda^T(T-t)e^{-\lambda^T(T-t)} & e^{-\lambda^T(T-t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\lambda^S(i)(T-t)} & 0 \\ 0 & 0 & 0 & 0 & \lambda^S(i)(T-t)e^{-\lambda^S(i)(T-t)} & e^{-\lambda^S(i)(T-t)} \end{pmatrix}$$
$$\times \int_t^T \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\lambda^T(T-s)} & 0 & 0 & 0 & 0 \\ 0 & -\lambda^T(T-s)e^{\lambda^T(T-s)} & e^{\lambda^T(T-s)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda^S(i)(T-s)} & 0 \\ 0 & 0 & 0 & 0 & -\lambda^S(i)(T-s)e^{\lambda^S(i)(T-s)} & e^{\lambda^S(i)(T-s)} \end{pmatrix} \begin{pmatrix} 1+\alpha^{i,c}_{L^T} \\ 1+\alpha^{i,c}_{S^T} \\ 0 \\ \alpha^{i,c}_{L^S} \\ \alpha^{i,c}_{S^S} \\ 0 \end{pmatrix} ds$$

This can be rewritten as

$$B^{i,c}(t,T) = -\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\lambda^T(T-t)} & 0 & 0 & 0 & 0 \\ 0 & \lambda^T(T-t)e^{-\lambda^T(T-t)} & e^{-\lambda^T(T-t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\lambda^S(i)(T-t)} & 0 \\ 0 & 0 & 0 & 0 & \lambda^S(i)(T-t)e^{-\lambda^S(i)(T-t)} & e^{-\lambda^S(i)(T-t)} \end{pmatrix}$$
$$\times \int_t^T \begin{pmatrix} 1+\alpha^{i,c}_{L^T} \\ (1+\alpha^{i,c}_{S^T})e^{\lambda^T(T-s)} \\ -(1+\alpha^{i,c}_{S^T})\lambda^T(T-s)e^{\lambda^T(T-s)} \\ \alpha^{i,c}_{L^S} \\ \alpha^{i,c}_{S^S}e^{\lambda^S(i)(T-s)} \\ -\alpha^{i,c}_{S^S}\lambda^S(i)(T-s)e^{\lambda^S(i)(T-s)} \end{pmatrix} ds$$

Since

$$\int_t^T ds = T-t,$$

and

$$\int_t^T e^{\lambda(T-s)}ds = \Big[-\frac{1}{\lambda}e^{\lambda(T-s)}\Big]_t^T = -\frac{1-e^{\lambda(T-t)}}{\lambda},$$

and

$$\begin{aligned} \int_t^T -\lambda(T-s)e^{\lambda(t-s)}ds &= \frac{1}{\lambda}\int_{\lambda(T-t)}^0 xe^x dx = \frac{1}{\lambda}[xe^x]^0_{\lambda(T-t)} - \frac{1}{\lambda}\int_{\lambda(T-t)}^0 e^x dx \\ &= -(T-t)e^{\lambda(T-t)} - \frac{1}{\lambda}[e^x]^0_{\lambda(T-t)} = -(T-t)e^{\lambda(T-t)} - \frac{1-e^{\lambda(T-t)}}{\lambda}, \end{aligned}$$

the system of ODEs can be reduced to

$$
B^{i,c}(t,T) = -\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\lambda^T(T-t)} & 0 & 0 & 0 & 0 \\ 0 & \lambda^T(T-t)e^{-\lambda^T(T-t)} & e^{-\lambda^T(T-t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\lambda^S(i)(T-t)} & 0 \\ 0 & 0 & 0 & 0 & \lambda^S(i)(T-t)e^{-\lambda^S(i)(T-t)} & e^{-\lambda^S(i)(T-t)} \end{pmatrix}
$$

$$
\times \begin{pmatrix} (1+\alpha^{i,c}_{L^T})(T-t) \\ -(1+\alpha^{i,c}_{S^T})\frac{1-e^{\lambda^T(T-t)}}{\lambda^T} \\ (1+\alpha^{i,c}_{S^T})\Big[-(T-t)e^{\lambda^T(T-t)} - \frac{1-e^{\lambda^T(T-t)}}{\lambda^T}\Big] \\ \alpha^{i,c}_{L^S}(T-t) \\ -\alpha^{i,c}_{S^S}\frac{1-e^{\lambda^S(i)(T-t)}}{\lambda^S(i)} \\ \alpha^{i,c}_{S^S}\Big[-(T-t)e^{\lambda^S(i)(T-t)} - \frac{1-e^{\lambda^S(i)(T-t)}}{\lambda^S(i)}\Big] \end{pmatrix}.
$$

This system can be reduced even further whereby the final solution is obtained as

$$
B^{i,c}(t,T) = \begin{pmatrix} -(1+\alpha^{i,c}_{L^T})(T-t) \\ -(1+\alpha^{i,c}_{S^T})\frac{1-e^{-\lambda^T(T-t)}}{\lambda^T} \\ (1+\alpha^{i,c}_{S^T})\Big[(T-t)e^{-\lambda^T(T-t)} - \frac{1-e^{-\lambda^T(T-t)}}{\lambda^T}\Big] \\ -\alpha^{i,c}_{L^S}(T-t) \\ -\alpha^{i,c}_{S^S}\frac{1-e^{-\lambda^S(i)(T-t)}}{\lambda^S(i)} \\ \alpha^{i,c}_{S^S}\Big[(T-t)e^{-\lambda^S(i)(T-t)} - \frac{1-e^{-\lambda^S(i)(T-t)}}{\lambda^S(i)}\Big] \end{pmatrix}.
$$

Thus, the yield-to-maturity of the corporate zero-coupon bond is given by

$$
\begin{aligned}
y^{i,c}(t,T) &= -\frac{1}{T-t}\ln P^{i,c}(t,T) \\
&= (1+\alpha^{i,c}_{L^T})L^T_t + (1+\alpha^{i,c}_{S^T})\frac{1-e^{-\lambda^T(T-t)}}{\lambda^T(T-t)}S^T_t + (1+\alpha^{i,c}_{S^T})\Big[\frac{1-e^{-\lambda^T(T-t)}}{\lambda^T} - e^{-\lambda^T(T-t)}\Big]C^T_t \\
&+ \alpha^{i,c}_{L^S}L^S_t(i) + \alpha^{i,c}_{S^S}\frac{1-e^{-\lambda^S(i)(T-t)}}{\lambda^S(i)(T-t)}S^S_t(i) + \alpha^{i,c}_{S^S}\Big[\frac{1-e^{-\lambda^S(i)(T-t)}}{\lambda^S(i)(T-t)} - e^{-\lambda^S(i)(T-t)}\Big]C^S_t(i) - \frac{A^{i,c}_y(t,T)}{T-t}.
\end{aligned}
$$

In order to obtain a full analytical formula for the corporate bond yields we need to look into the details of the maturity adjustment term $\frac{A^{i,c}_y(t,T)}{T-t}$. That is the focus of the following section.

9.1 The maturity adjustment term

The formula for the maturity adjustment term is given by

$$
\frac{A^{i,c}_y(t,T)}{T-t} = \frac{1}{2}\frac{1}{T-t}\int_t^T \sum_{j=1}^{6}\big((\Sigma^i)'B^{i,c}(s,T)B^{i,c}(s,T)'\Sigma^i\big)_{j,j}ds,
$$

where $B^{i,c}(t,T) = (B^{i,c}_{L^T}(t,T), B^{i,c}_{S^T}(t,T), B^{i,c}_{C^T}(t,T), B^{i,c}_{L^S}(t,T), B^{i,c}_{S^S}(t,T), B^{i,c}_{C^S}(t,T))$. Since all factors are assumed to be independent, the industry-specific volatility matrix Σ^i is diagonal

$$\Sigma^i = \begin{pmatrix} \sigma^T_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma^T_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma^T_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^i_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^i_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^i_{33} \end{pmatrix}.$$

Thus, the maturity adjustment term reduces to

$$\begin{aligned} \frac{A^{i,c}_y(t,T)}{T-t} &= \frac{1}{2}\frac{1}{T-t}\Big[(\sigma^T_{11})^2\int_t^T B^{i,c}_{L^T}(s,T)^2ds + (\sigma^T_{22})^2\int_t^T B^{i,c}_{S^T}(s,T)^2ds + (\sigma^T_{33})^2\int_t^T B^{i,c}_{C^T}(s,T)^2ds\Big] \\ &+ \frac{1}{2}\frac{1}{T-t}\Big[(\sigma^i_{11})^2\int_t^T B^{i,c}_{L^S}(s,T)^2ds + (\sigma^i_{22})^2\int_t^T B^{i,c}_{S^S}(s,T)^2ds + (\sigma^i_{33})^2\int_t^T B^{i,c}_{C^S}(s,T)^2ds\Big]. \end{aligned}$$

All these integrals have already been solved in Christensen, Diebold, and Rudebusch (2007), so we are able to immediately present the final result.

9.1.1 The first integral

$$I_1 = \frac{1}{2}\frac{(\sigma^T_{11})^2}{T-t}\int_t^T B^{i,c}_{L^T}(s,T)^2ds = \frac{(\sigma^T_{33})^2(1+\alpha^{i,c}_{L^T})^2}{6}(T-t)^2.$$

9.1.2 The second integral

$$\begin{aligned} I_2 &= \frac{1}{2}\frac{(\sigma^T_{22})^2}{T-t}\int_t^T B^{i,c}_{S^T}(s,T)^2ds = \frac{1}{2}\frac{(\sigma^T_{22})^2(1+\alpha^{i,c}_{S^T})^2}{T-t}\int_t^T \Big[-\frac{1-e^{-\lambda^T(T-s)}}{\lambda^T}\Big]^2ds \\ &= (\sigma^T_{22})^2(1+\alpha^{i,c}_{S^T})^2\Big[\frac{1}{2(\lambda^T)^2} - \frac{1}{(\lambda^T)^3}\frac{1-e^{-\lambda^T(T-t)}}{T-t} + \frac{1}{4(\lambda^T)^3}\frac{1-e^{-2\lambda^T(T-t)}}{T-t}\Big]. \end{aligned}$$

9.1.3 The third integral

$$\begin{aligned} I_3 &= \frac{1}{2}\frac{(\sigma^T_{33})^2}{T-t}\int_t^T B^{i,c}_{C^T}(s,T)^2ds = \frac{1}{2}\frac{(\sigma^T_{33})^2(1+\alpha^{i,c}_{S^T})^2}{T-t}\int_t^T \Big\{(T-s)e^{-\lambda^T(T-s)} - \frac{1-e^{-\lambda^T(T-s)}}{\lambda^T}\Big\}^2ds \\ &= (\sigma^T_{33})^2(1+\alpha^{i,c}_{S^T})^2\Big[\frac{1}{2(\lambda^T)^2} + \frac{1}{(\lambda^T)^2}e^{-\lambda^T(T-t)} - \frac{1}{4\lambda^T}(T-t)e^{-2\lambda^T(T-t)} - \frac{3}{4(\lambda^T)^2}e^{-2\lambda^T(T-t)}\Big] \\ &+ (\sigma^T_{33})^2(1+\alpha^{i,c}_{S^T})^2\Big[\frac{5}{8(\lambda^T)^3}\frac{1-e^{-2\lambda^T(T-t)}}{T-t} - \frac{2}{(\lambda^T)^3}\frac{1-e^{-\lambda^T(T-t)}}{T-t}\Big]. \end{aligned}$$

9.1.4 The fourth integral

$$I_4 = \frac{1}{2}\frac{(\sigma^i_{11})^2}{T-t}\int_t^T B^{i,c}_{L^S}(s,T)^2 ds = \frac{(\sigma^i_{11})^2(\alpha^{i,c}_L)^2}{6}(T-t)^2.$$

9.1.5 The fifth integral

$$\begin{aligned} I_5 &= \frac{1}{2}\frac{(\sigma^i_{22})^2}{T-t}\int_t^T B^{i,c}_{S^S}(s,T)^2 ds = \frac{1}{2}\frac{(\sigma^i_{22})^2(\alpha^{i,c}_S)^2}{T-t}\int_t^T \Big[-\frac{1-e^{-\lambda^S(i)(T-s)}}{\lambda^S(i)}\Big]^2 ds \\ &= (\sigma^i_{22})^2(\alpha^{i,c}_S)^2\Big[\frac{1}{2(\lambda^S(i))^2} - \frac{1}{(\lambda^S(i))^3}\frac{1-e^{-\lambda^S(i)(T-t)}}{T-t} + \frac{1}{4(\lambda^S(i))^3}\frac{1-e^{-2\lambda^S(i)(T-t)}}{T-t}\Big]. \end{aligned}$$

9.1.6 The sixth integral

$$\begin{aligned} I_6 &= \frac{1}{2}\frac{(\sigma^i_{33})^2}{T-t}\int_t^T B^{i,c}_{C^S}(s,T)^2 ds = \frac{1}{2}\frac{(\sigma^i_{33})^2(\alpha^{i,c}_S)^2}{T-t}\int_t^T \Big\{(T-s)e^{-\lambda^S(i)(T-s)} - \frac{1-e^{-\lambda^S(i)(T-s)}}{\lambda^S(i)}\Big\}^2 ds \\ &= (\sigma^i_{33})^2(\alpha^{i,c}_S)^2\Big[\frac{1}{2(\lambda^S(i))^2} + \frac{1}{(\lambda^S(i))^2}e^{-\lambda^S(i)(T-t)} - \frac{1}{4\lambda^S(i)}(T-t)e^{-2\lambda^S(i)(T-t)} - \frac{3}{4(\lambda^S(i))^2}e^{-2\lambda^S(i)(T-t)}\Big] \\ &+ (\sigma^i_{33})^2(\alpha^{i,c}_S)^2\Big[\frac{5}{8(\lambda^S(i))^3}\frac{1-e^{-2\lambda^S(i)(T-t)}}{T-t} - \frac{2}{(\lambda^S(i))^3}\frac{1-e^{-\lambda^S(i)(T-t)}}{T-t}\Big]. \end{aligned}$$

9.2 Summary

Given the assumption of independent factors, the corporate zero-coupon yield function for a representative bond from industry i with rating c and τ years to maturity is equal to

$$\begin{aligned} y^{i,c}(t,T) &= \alpha^{i,c}_0 + (1+\alpha^{i,c}_{L^T})L^T_t + (1+\alpha^{i,c}_{S^T})\frac{1-e^{-\lambda^T(T-t)}}{\lambda^T(T-t)}S^T_t + (1+\alpha^{i,c}_{S^T})\Big[\frac{1-e^{-\lambda^T(T-t)}}{\lambda^T(T-t)} - e^{-\lambda^T(T-t)}\Big]C^T_t \\ &+ \alpha^{i,c}_L L^S_t(i) + \alpha^{i,c}_S\frac{1-e^{-\lambda^S(i)(T-t)}}{\lambda^S(i)(T-t)}S^S_t(i) + \alpha^{i,c}_S\Big[\frac{1-e^{-\lambda^S(i)(T-t)}}{\lambda^S(i)(T-t)} - e^{-\lambda^S(i)(T-t)}\Big]C^S_t(i) - \frac{A^{i,c}_y(t,T)}{T-t}, \end{aligned}$$

where

$$
\begin{aligned}
\frac{A_y^{i,c}(t,T)}{T-t} &= \frac{(\sigma_{33}^T)^2(1+\alpha_{L^T}^{i,c})^2}{6}(T-t)^2 \\
&+ (\sigma_{22}^T)^2(1+\alpha_{S^T}^{i,c})^2\Big[\frac{1}{2(\lambda^T)^2} - \frac{1}{(\lambda^T)^3}\frac{1-e^{-\lambda^T(T-t)}}{T-t} + \frac{1}{4(\lambda^T)^3}\frac{1-e^{-2\lambda^T(T-t)}}{T-t}\Big] \\
&+ (\sigma_{33}^T)^2(1+\alpha_{S^T}^{i,c})^2\Big[\frac{1}{2(\lambda^T)^2} + \frac{1}{(\lambda^T)^2}e^{-\lambda^T(T-t)} - \frac{1}{4\lambda^T}(T-t)e^{-2\lambda^T(T-t)} - \frac{3}{4(\lambda^T)^2}e^{-2\lambda^T(T-t)}\Big] \\
&+ (\sigma_{33}^T)^2(1+\alpha_{S^T}^{i,c})^2\Big[\frac{5}{8(\lambda^T)^3}\frac{1-e^{-2\lambda^T(T-t)}}{T-t} - \frac{2}{(\lambda^T)^3}\frac{1-e^{-\lambda^T(T-t)}}{T-t}\Big] \\
&+ \frac{(\sigma_{11}^i)^2(\alpha_L^{i,c})^2}{6}(T-t)^2 \\
&+ (\sigma_{22}^i)^2(\alpha_S^{i,c})^2\Big[\frac{1}{2(\lambda^S(i))^2} - \frac{1}{(\lambda^S(i))^3}\frac{1-e^{-\lambda^S(i)(T-t)}}{T-t} + \frac{1}{4(\lambda^S(i))^3}\frac{1-e^{-2\lambda^S(i)(T-t)}}{T-t}\Big] \\
&+ (\sigma_{33}^i)^2(\alpha_S^{i,c})^2\Big[\frac{1}{2(\lambda^S(i))^2} + \frac{1}{(\lambda^S(i))^2}e^{-\lambda^S(i)(T-t)} - \frac{1}{4\lambda^S(i)}(T-t)e^{-2\lambda^S(i)(T-t)} - \frac{3}{4(\lambda^S(i))^2}e^{-2\lambda^S(i)(T-t)}\Big] \\
&+ (\sigma_{33}^i)^2(\alpha_S^{i,c})^2\Big[\frac{5}{8(\lambda^S(i))^3}\frac{1-e^{-2\lambda^S(i)(T-t)}}{T-t} - \frac{2}{(\lambda^S(i))^3}\frac{1-e^{-\lambda^S(i)(T-t)}}{T-t}\Big].
\end{aligned}
$$

9.2.1 The credit spread function

By implication, the credit spread function becomes

$$
\begin{aligned}
s^{i,c}(t,T) &= y^{i,c}(t,T) - y^T(t,T) \\
&= \alpha_0^{i,c} + \alpha_{L^T}^{i,c}L_t^T + \alpha_{S^T}^{i,c}\frac{1-e^{-\lambda^T(T-t)}}{\lambda^T(T-t)}S_t^T + \alpha_{S^T}^{i,c}\Big[\frac{1-e^{-\lambda^T(T-t)}}{\lambda^T(T-t)} - e^{-\lambda^T(T-t)}\Big]C_t^T \\
&+ \alpha_L^{i,c}L_t^S(i) + \alpha_S^{i,c}\frac{1-e^{-\lambda^S(i)(T-t)}}{\lambda^S(i)(T-t)}S_t^S(i) + \alpha_S^{i,c}\Big[\frac{1-e^{-\lambda^S(i)(T-t)}}{\lambda^S(i)(T-t)} - e^{-\lambda^S(i)(T-t)}\Big]C_t^S(i) - \frac{A_s^{i,c}(t,T)}{T-t},
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{A_s^{i,c}(t,T)}{T-t} &= \frac{(\sigma_{33}^T)^2(\alpha_{L^T}^{i,c})^2}{6}(T-t)^2 \\
&+ (\sigma_{22}^T)^2(\alpha_{S^T}^{i,c})^2\Big[\frac{1}{2(\lambda^T)^2} - \frac{1}{(\lambda^T)^3}\frac{1-e^{-\lambda^T(T-t)}}{T-t} + \frac{1}{4(\lambda^T)^3}\frac{1-e^{-2\lambda^T(T-t)}}{T-t}\Big] \\
&+ (\sigma_{33}^T)^2(\alpha_{S^T}^{i,c})^2\Big[\frac{1}{2(\lambda^T)^2} + \frac{1}{(\lambda^T)^2}e^{-\lambda^T(T-t)} - \frac{1}{4\lambda^T}(T-t)e^{-2\lambda^T(T-t)} - \frac{3}{4(\lambda^T)^2}e^{-2\lambda^T(T-t)}\Big] \\
&+ (\sigma_{33}^T)^2(\alpha_{S^T}^{i,c})^2\Big[\frac{5}{8(\lambda^T)^3}\frac{1-e^{-2\lambda^T(T-t)}}{T-t} - \frac{2}{(\lambda^T)^3}\frac{1-e^{-\lambda^T(T-t)}}{T-t}\Big] \\
&+ \frac{(\sigma_{11}^i)^2(\alpha_L^{i,c})^2}{6}(T-t)^2 \\
&+ (\sigma_{22}^i)^2(\alpha_S^{i,c})^2\Big[\frac{1}{2(\lambda^S(i))^2} - \frac{1}{(\lambda^S(i))^3}\frac{1-e^{-\lambda^S(i)(T-t)}}{T-t} + \frac{1}{4(\lambda^S(i))^3}\frac{1-e^{-2\lambda^S(i)(T-t)}}{T-t}\Big] \\
&+ (\sigma_{33}^i)^2(\alpha_S^{i,c})^2\Big[\frac{1}{2(\lambda^S(i))^2} + \frac{1}{(\lambda^S(i))^2}e^{-\lambda^S(i)(T-t)} - \frac{1}{4\lambda^S(i)}(T-t)e^{-2\lambda^S(i)(T-t)} - \frac{3}{4(\lambda^S(i))^2}e^{-2\lambda^S(i)(T-t)}\Big] \\
&+ (\sigma_{33}^i)^2(\alpha_S^{i,c})^2\Big[\frac{5}{8(\lambda^S(i))^3}\frac{1-e^{-2\lambda^S(i)(T-t)}}{T-t} - \frac{2}{(\lambda^S(i))^3}\frac{1-e^{-\lambda^S(i)(T-t)}}{T-t}\Big]
\end{aligned}
$$

This is the credit spread function we use in the measurement equation of the Kalman filter when we estimate the common credit risk factors for each industry i.

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