# Informational Fragility of Dynamic Rational Expectations Equilibria*

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## Abstract

We study the stability properties of Rational Expectations equilibria in dynamic models with incomplete information when the information set of agents is slightly perturbed. We show that full-information equilibria can be *informationally fragile*, in the sense that a slight perturbation in the endogenous information set of the agents along the equilibrium path can lead to a deviation from that path. We then construct a class of dynamic rational expectations equilibria that are informationally stable for the same parameter space where other equilibria are informationally fragile. We show that an equilibrium that is informationally fragile is not least-squares learnable, while an equilibrium that is informationally stable always is. We provide two prominent examples of our concept. First, we show that a dynamic model with "news" shocks is informationally fragile, while removing news from the agents' information set leads to a stable equilibrium. We also present an application to a model with productivity shocks and nominal rigidities under incomplete information and demonstrate that both informationally fragile and stable equilibria can be obtained, albeit with very different shock propagation properties.

Keywords: Informational Stability, Rational Expectations, Dynamic Stability, Incomplete Information, Least Squares Learning, E-Stability

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# 1 Introduction

In an environment with incomplete information, the equilibrium outcome of a dynamic model provides valuable information that reduces information incompleteness. The information transmitted in equilibrium, in turn, affects the equilibrium process itself, requiring the search for a fixed point in *information* when solving for a dynamic rational expectations equilibrium. As in any dynamic process, the equilibrium process generated by the fixed point solution requires initial conditions. In presence of incomplete information the initial conditions include the initial information available to the agents. In this paper we study the stability of dynamic rational expectations equilibria with incomplete information with respect to small perturbations to the initial condition of information. Within the context of a stylized but quite general model, we show that rational expectations equilibria in which the endogenous variables reveal perfectly the unobservable state of the world are not robust to small perturbations in the information set of the agents.

More precisely, we show that if the information set of the agents is initialized in a neighborhood around the equilibrium value, the equilibrium dynamics become explosive and any initial perturbation in information accumulates over time without bound. We identify the cause of instability in the way the history of the equilibrium variable is used to learn about the unobservable state. The linear combination of past equilibrium outcomes that reveal the state is chosen so to exactly cancel the dynamics that is responsible for the partial information revelation. While such cancellation works exactly along the equilibrium path, it also injects what is known as a hidden instability in the autoregressive dynamics of the system. When information is perturbed the hidden instability emerges and the process becomes explosive. Our first message is therefore a negative one: rational expectations equilibria that reveal perfectly the underlying state are fragile to information perturbations. This type of instability is, to our knowledge, novel to the rational expectations literature. We term such equilibria *informationally fragile*, to emphasize the idea that the explosive dynamics are due to the information extrapolation that is performed in equilibrium.

We then move to characterize rational expectations equilibria that are robust to information perturbations. Our second message is a positive one: when a fully revealing rational expectations equilibrium is informationally fragile there always exists a rational expectations equilibrium whose dynamics are stable when information is perturbed. We term such equilibria *informationally stable*. Interestingly, the source of stability in this equilibria is the incomplete learning that agents achieve from taking linear combinations of the equilibrium outcome.

The notion of informational stability is closely related to the learnability of a rational expectations equilibrium. We show that an equilibrium that is informationally fragile cannot be learned by a least squares learning algorithm. On the other hand, an informationally stable equilibrium can always be learned by the least squares algorithm, and it is also E-stable.

Finally, we present two applications of our results. We first examine the theoretical properties of the burgeoning news-driven business cycle literature.[1] We demonstrate that news shocks

[1]Recent theoretical and empirical contributions includeBeaudry and Portier (2006, 2004), Jaimovich and Rebelo

are informationally fragile while their incomplete information counterpart are informationally stable. Second, we analyze a stylized macroeconomic model with nominal rigidities and permanent productivity shocks as presented in Lorenzoni (2009). We show that the fully revealing rational expectations equilibrium of that model is informationally fragile when the productivity process takes the form of an S-shaped diffusion process. We then solve for an incomplete information rational expectations equilibrium of the model and show that it is informationally stable. We analyze the impulse response of output and inflation to productivity innovations and we show that in the incomplete information equilibrium a positive productivity innovation leads to alternating oscillations of output, employment and inflation around their respective trends. More specifically, both output and inflation are above trend and below trend at the same time, so that, conditional on the productivity innovation output and inflation display a positive correlation over the cycle. In other words, in the informationally stable equilibrium of the model a positive supply shock (in productivity) generates a dynamic response that looks like alternating positive and negative demand shocks.

The focus on informational stability is important for two reasons: first, equilibria that are informationally stable have desirable "learnability" properties compared to the informationally fragile ones; second, informational stability has important consequences for the dynamic properties of the equilibrium path. We study the connection to learning in Section 5 and we show the consequences for the equilibrium dynamics in a macroeconomic context in Section 6.

## 2 Model and Equilibrium

We perform the analysis in the simplest possible setting to make clear the concept of informational stability.[2] In particular, we focus on a univariate model with a representative agent facing incomplete information about the state of the economy. The equilibrium equation is specified as

$$y_t = \kappa \mathbb{E}\big[y_{t+1}|\Omega_t\big] + \varphi a_t \tag{2.1}$$

with $|\kappa| \leq 1$ and $\varphi \in \mathbb{R}$. $\Omega_t$ denotes the information set of the representative agent at time $t$, to be specified shortly, and $a_t$ is an exogenous stochastic process given by

$$a_t - \rho a_{t-1} = A(L)u_t \tag{2.2}$$

where $|\rho| \leq 1$, $A(L)$ is a square summable lag polynomial in non-negative powers of $L$ and $u_t \sim N(0, \sigma_u^2)$. The immediate consequence of working in a linear-quadratic Gaussian setting is that the expectational operator corresponds to the linear projection operator, a property that will apply to all results in the paper.

(2009), Christiano, Ilut, Motto, and Rostagno (2010), Fujiwara, Hirose, and Shintani (2011), Barsky and Sims (2011), Leeper and Walker (2011), Mertens and Ravn (2010, 2011), Schmitt-Grohé and Uribe (2012), Khan and Tsoukalas (2012).

[2] Appendix B provides conditions under which the results derived below extend to more complex models.

An important benchmark for the following analysis is the rational expectations equilibrium of (2.1) when the representative agent is (exogenously) endowed with the knowledge of the entire history of shocks up to time $t$. We refer to this equilibrium as the Full Information equilibrium. Let $\mathbb{V}_t(x)$ denote the smallest closed linear subspace spanned by the infinite history of the random variable $x_t$ up to time $t$, namely $x^t \equiv \{x_t, x_{t-1}, x_{t-2}, ....\}$. In a full information equilibrium it is assumed that $\Omega_t = \mathbb{V}_t(u)$. The following proposition states the solution for the full information equilibrium.

**Proposition 1.** *The Full Information equilibrium of* (2.1) *always exists, is unique and is given by*

$$y_t - \rho y_{t-1} = \frac{\varphi}{L-\kappa}\left(LA(L) - \kappa A(\kappa)\frac{(1-\rho L)}{(1-\rho\kappa)}\right)u_t \tag{2.3}$$

We would like to work with a notion of rational expectations equilibrium that allows both exogenous and endogenous information. For this purpose we introduce notation to represent the exogenous information with which agents are endowed in a model by $\mathbb{U}_t$, for $t \in \mathbb{Z}$. Rationality implies that information is not forgotten, formally $\mathbb{U}_{t-1} \subseteq \mathbb{U}_t$. The endogenous information is instead represented by whatever information is conveyed by the history of the equilibrium variables, in our case $y_t$. In addition to the time series properties of $y_t$ which are summarize in the equilibrium autocovariance generating function for $y_t$, rational agents also recognize that $y_t$ obeys the equilibrium equation (2.1), which essentially provides a cross-equation restriction, or a structural perspective on the covariogram of $y_t$. The notation $\mathbb{V}_t(y)$ does not adequately capture such cross equation restrictions and so we follow Rondina and Walker (2012a) in denoting the information coming from the model by $\mathbb{M}_t(y)$. It follows that the combined information in the history of the equilibrium variable can be expressed as $\mathbb{V}_t(y) \vee \mathbb{M}_t(y)$ where $\vee$ denotes the smaller subspace containing all the elements in both spaces. We are now ready to define the notion of rational expectations equilibrium that we will employ throughout the paper.

**Definition IE.** *Given the exogenous information specification* $\{\mathbb{U}_t\}, t \in \mathbb{Z}$*, a dynamic rational expectations Information Equilibrium (IE) is a stochastic process for* $\{y_t, t \in \mathbb{Z}\}$ *and a stochastic process for the information set* $\{\Omega_t = \mathbb{U}_t \vee \mathbb{V}_t(y) \vee \mathbb{M}_t(y), t \in \mathbb{Z}\}$ *such that the equilibrium condition* (2.1) *holds.*

The endogeneity of the information set does not make the Definition IE immediately helpful in the construction of an equilibrium. One way to proceed is to consider rational expectations equilibria obtained under different exogenous information assumptions and check whether such equilibria satisfy the requirements of an information equilibrium. We therefore ask when the Full Information equilibrium in (2.3) is an Information Equilibrium. It turns out that the answer to this question depends on the properties of $A(L)$, the exogenous stochastic process, and the structure of the exogenous information $\{\mathbb{U}_t\}, t \in \mathbb{Z}$. In terms of $A(L)$, the key feature will be the invertibility of the process, that is to say whether the observation of the history of $a_t$ is able to perfectly reveal the history of the innovations $u_t$. This property is related to the moving average roots of $A(L)$. If the roots are all outside the unit circle, then $A(L)$ is said to be invertible in current and past $a_t$,

and the history of $a_t$ reveals perfectly the history of $u_t$. On the other hand, if the at least one root is inside the unit circle, then $A(L)$ is non-invertible in current and past $a_t$, and the history of $a_t$ is able to reveal only an imperfect measure of $u_t$.
When $A(L)$ is non-invertible, $a_t$ can always be written as

$$a_t - \rho a_{t-1} = \tilde{A}(L) \prod_{i=1}^{m} (1 + \theta_i L) u_t, \ |\theta_i| > 1 \text{ for } i = 1, .., m, \tag{2.4}$$

where $m > 0$ is the number of roots inside the unit circle and $\tilde{A}(L)$ is the invertible portion of $A(L)$. The information contained in the history of $a_t$ in presence of non-invertibility can be summarized by its Wold fundamental representation, which is

$$a_t - \rho a_{t-1} = \tilde{A}(L) \prod_{i=1}^{m} (\theta_i + L) \tilde{u}_t, \text{ where } \tilde{u}_t \equiv \prod_{i=1}^{m} \left( \frac{1 + \theta_i L}{\theta_i + L} \right) u_t. \tag{2.5}$$

The innovation process $\tilde{u}_t$ is the measure that minimizes the mean squared forecast error in predicting $u_t$, and thus $a_{t+1}$. The variance of the forecast error under the information measured by $\tilde{u}_t$ is bigger than the variance of the forecast error under $u_t$. The particular dynamics of $a_t$ is said to be "confounding" [cf. Rondina and Walker (2012a)] in the sense that it unfolds in a way that prevents the full revelation of the innovation process. To clarify the role of confounding dynamics we consider the evolution of the process $a_t$ from some initial time $t = 1$, arbitrarily distant in the past. To keep things simple in terms of notation we restrict our attention to the case of $m = 1$, with $\theta \equiv \theta_1$ and we set $\rho = 0$ [3]. Equation (2.4) can be written as

$$a_t + \theta a_{t-1} + \theta^2 a_{t-2} + .... + \theta^{t-1} a_1 = u_t + \theta^t u_0 \tag{2.6}$$

The process $a_t$ is invertible when $|\theta| < 1$, which implies that as $t$ grows large (as the initial time becomes arbitrarily distant in the past), the summation on the left hand side of (2.6) remains well defined and it becomes exactly equal to $u_t$ since the term $\theta^t u_0$ tends to zero for any finite $u_0$. When $a_t$ is not invertible $|\theta| > 1$ and so as $t$ grows larger and larger the summation on the left hand side does not have a well defined limit, which means that it provides a very imprecise measure for $u_t$. Essentially, if $u_0$ is not observed, which is the assumption under which (2.6) is valid, such ignorance prevents the exact knowledge of the state $u_t$. In other words, in presence of confounding dynamics the ignorance about some initial state never disappears because the dynamics of the process are not able to exactly recover it. In this case, the best prediction is suggested by (2.5) which results in

$$a_t + \theta^{-1} a_{t-1} + \theta^{-2} a_{t-2} + .... + \theta^{-t+1} a_1 = \tilde{u}_t + \theta^{-t} u_0 \tag{2.7}$$

In this case, as $t$ becomes arbitrarily large the sum on the left hand side remains well defined and

[3] Allowing for an arbitrary $\rho$ is immediate, just substitute any instance of $a_t$ by the quasi-difference $a_t - \rho a_{t-1}$.

the direct incidence of the initial state $u_0$ disappears. When $m > 1$ the above argument still applies, but the initial state—the ignorance upon which matters for prediction—becomes the $m$-dimensional vector $\{u_0, u_{-1}, ...., u_{-m+1}\}$.

The relationship between the invertibility of $a_t$ and the question of whether the full information equilibrium is a rational expectations equilibrium is crucial because, in any equilibrium, the knowledge that $y_t$ is generated by (2.1) results in the entire history of $a_t$ always being part of the equilibrium information of the representative agent. This is immediately evident by considering that, along any equilibrium $y_t$, the representative agent also observes her own prediction $\mathbb{E}\big[y_{t+1}|\mathbb{U}_t \vee \mathbb{V}_t(y) \vee \mathbb{M}_t(y)\big]$ and so she must be able to compute

$$a_t = \frac{1}{\varphi}\bigg(y_t - \kappa\mathbb{E}\big[y_{t+1}|\mathbb{U}_t \vee \mathbb{V}_t(y) \vee \mathbb{M}_t(y)\big]\bigg). \tag{2.8}$$

The following proposition formalizes the implication of the discussion above.

**Proposition 2.** *If $m = 0$ ($A(L)$ is invertible), the Full Information equilibrium in (2.3) is always an Information Equilibrium, independent of the exogenous information structure $\mathbb{U}_t$.*
*If $m > 0$ ($A(L)$ is not invertible), the Full Information equilibrium (2.3) is an Information Equilibrium from time $t > 0$ onward if $\{u_{t-j}, u_{t-j-1}, ..., u_{t-j-m+1}\} \in \mathbb{U}_t$ with $0 < j < \infty$. If $j = \infty$ the Full Information equilibrium is always an Information Equilibrium.*

*Proof.* See Appendix A. □

The statement of the proposition distinguishes the case of an arbitrary long yet finite history, from the case of a non-finite history. The distinction is relevant because, technically, the argument that we have used to model confounding dynamics in terms of the ignorance about the initial state cannot be applied if the initial state does not exist.[4] The distinction of whether the Information Equilibrium is an Information Equilibrium because the initial state is exogenously revealed or because it is assumed not to exist, however, will not play a substantive role in our analysis of information fragility and stability, to which we now turn.

## 3 Informational Fragility

The result in Proposition 2 ensures that a rational expectations equilibrium takes the form of the Full Information equilibrium (2.3), provided that some mild conditions on $A(L)$ and $\mathbb{U}_t$ are met. For instance, if one were willing to assume that $A(L)$ is always invertible, then one could directly work under the full information equilibrium characterization. However, the conditions on the dynamics of $A(L)$ have a structural significance, in the sense that as a modeler one would like not to restrict a-priori the possible form that the exogenous dynamics of $a_t$ can take. For instance, in the context of a dynamic macroeconomic model, if $a_t$ represents aggregate productivity, by

[4]We are thankful to Marios Angeletos and Venky Venkateswaran for pointing our attention upon this technical issue.

assuming invertibility of $A(L)$ one would prevent the dynamics of productivity from taking an S-shape form that is typical of diffusion processes, and possibly of how new knowledge is transformed into increased productivity. More generally, any process that diffuses with a rate distribution that is non-monotonic can be represented by a non-invertible specification of $A(L)$. In addition, in a more structural setting the process $a_t$ might be the outcome of some decision making that takes place in a different market or technology in the economy. For instance, the one-period time-to-build assumption in a model of capital accumulation by firms with demand shocks results in a process for capital that is non-invertible, as in Townsend (1983). To the extent that the process for capital affects the equilibrium interest rate and this, in turn, the Euler equation for consumption, the resulting relationship between consumption and interest rate might take the form of (2.1) with a non-invertible $a_t$.[5] Finally as we demonstrate in Section 6, the burgeoning "news" shock literature is predicated on the assumption that agents have foresight about future technology, government spending, tax rates, etc., which relies heavily on the non-invertibility of $a_t$.

Proposition 2 is also useful for thinking about cases in which the state was exactly observed at some distant point in the past. Interestingly, this corresponds to the approach of "truncating" the informational incompleteness by assuming the the current state is revealed to the agents at some point in the future, an assumption that simplifies the characterization of the rational expectations solution and that is widely used in the incomplete information literature since Townsend (1983) first proposed it.[6]

Provided that one does not want to restrict $a_t$ to be invertible a-priori, but one is willing to assume that the state is eventually revealed, does Proposition 2 provide a result for safely focusing on the full information solution (2.3) as the relevant rational expectations equilibrium of model (2.1)? The answer, remarkably, is no. The reason lies in the inherent fragility of the dynamic rational expectations equilibrium under the form of the full information solution (2.3) to slight perturbations in the agents' information set.

To substantiate this point let us consider a particular example, before stating a more general result. Suppose that $a_t$ is specified as

$$a_t = u_t + \theta u_{t-1}, \text{ with } |\theta| > 1, \tag{3.1}$$

which corresponds to setting $\rho = 0$, $m = 1$ and $\tilde{A}(L) = 1$ in (2.4). Let us assume that the representative agent is endowed with the knowledge of some state $u_0$, so that $u_0 \in \mathbb{U}_t$ for any $t \geq 0$. Proposition 2 ensures that the full information equilibrium is a rational expectations information equilibrium. Such equilibrium is therefore obtained from (2.3) as

$$y_t = \varphi(1 + \theta\kappa)u_t + \varphi\theta u_{t-1}. \tag{3.2}$$

Definition IE requires that the rational expectations equilibrium representation of (3.2) is a lin-

[5] For more on this example see Rondina and Walker (2012b).

[6] Recent examples of papers that employ the truncation approach include Angeletos and La'O (2009), Hellwig (2002), Lorenzoni (2009).

ear combination of the history of $a_t$ and $y_t$, as both are elements of the information set of the representative agent in equilibrium. To obtain such representation we note that (3.2) implies $\mathbb{E}\big[y_{t+1}|\mathbb{U}_t \vee \mathbb{V}_t(y) \vee \mathbb{M}_t(y)\big] = \varphi^{-1}\mathbb{E}\big[\theta u_t|\mathbb{U}_t \vee \mathbb{V}_t(y) \vee \mathbb{M}_t(y)\big]$. To write the left hand side of this expression in terms of $a_t$'s and $y_t$'s we cannot use (2.8) because that relationship was already used to argue that $a_t$ should be part of the information set at time $t$. However, we can use the same relationship lagged once to obtain

$$\begin{aligned}\mathbb{E}\big[\theta u_t|\mathbb{U}_t \vee \mathbb{V}_t(y) \vee \mathbb{M}_t(y)\big] &= \mathbb{E}\big[\theta u_t + \theta^2 u_{t-1} - \theta^2 u_{t-1}|\mathbb{U}_t \vee \mathbb{V}_t(y) \vee \mathbb{M}_t(y)\big] \\ &= \theta a_t - \mathbb{E}\big[\theta^2 u_{t-1}|\mathbb{U}_t \vee \mathbb{V}_t(y) \vee \mathbb{M}_t(y)\big] \\ &= \theta a_t - \frac{\varphi}{\kappa}(y_{t-1} - a_{t-1}).\end{aligned}$$

Once substituted into the equilibrium equation one obtains

$$y_t = -\theta y_{t-1} + \varphi\bigg((1+\theta\kappa) + \theta L\bigg)a_t, \tag{3.3}$$

which must hold in the full information rational expectations equilibrium (3.2). Note that in this representation the equilibrium variable $y_t$ has an autoregressive root at $|\theta| > 1$ and this root is explosive, in the sense that any realization of the innovation different from zero would result in non-stationary behavior of the variable $y_t$. Despite the unstable autoregressive root, however, $y_t$ is still a stationary process, as equation (3.2) clearly shows. The reason that reconciles this apparent contradiction is that the autoregressive root of $y_t$ in (3.3) exactly cancels with the moving average root of $a_t$.: the potential instability that learning from the endogenous variable injects into the equilibrium representation is exactly defused by the dynamics of $a_t$. In the optimal control literature jargon the representation (3.3) is said to harbor a non-minimum phase zero which can be canceled with the equivalent pole. Mathematically such cancellation is legitimate, but for the optimal control problem it creates a "hidden instability": if the system is slightly misspecified, the cancellation fails and the hidden pole drives the system to instability.[7] To see how the instability can emerge consider equation (3.3) at time $t = 1$,

$$y_1 = \varphi\theta a_0 - \theta y_0 + \varphi(1+\theta\kappa)a_1. \tag{3.4}$$

The condition for unstable root to be exactly diffused by the root of the $a_t$ process is that $a_0$ and $y_0$ must be specified as

$$\varphi a_0 - y_0 = \frac{\kappa}{\theta}u_0. \tag{3.5}$$

In other words, because of the hidden instability, the dynamic equation (3.3) must be initialized at the initial condition (3.5) in order for it to be stationary. If the initialization is done at a different point, the unstable root would make the process explosive. For instance, suppose the

[7]See Skogestad and Postlethwaite (2005) for a textbook analysis of hidden instabilities in dynamic systems.

initial condition is specified as

$$\varphi a_0 - y_0 = \frac{\kappa}{\theta} u_0 + \varepsilon. \tag{3.6}$$

for $|\varepsilon| > 0$, then the dynamic solution to (3.3) would become

$$y_t = \varphi(1 + \theta\kappa)u_t + \varphi\theta u_{t-1} + \theta^t \varepsilon \tag{3.7}$$

which is non-stationary and diverging with respect to (3.2). The perturbation $\varepsilon$ can have at least two different sources. On the one hand, it could be a perturbation to $\kappa$, in which case the perturbation is to the knowledge of the structure of the model. On the other hand, it could be thought as a perturbation to $u_0$, in which case it corresponds to a perturbation in the information set along the equilibrium path. Because we assumed at the outset that the structure of the economy is perfectly known, the second interpretation applies to our setting. However, the first interpretation is suggestive of a relationship between the informational fragility of an equilibrium and the ability to learn such equilibrium. We will explore this parallel in Section 5.
The following proposition generalizes the example.

**Proposition 3.** *If $m > 0$ ($A(L)$ is non-invertible), the rational expectations Information Equilibrium represented by the Full Information equilibrium is fragile to small perturbations in the information set of the representative agent. If the information set is slightly perturbed the dynamics will become non-stationary and diverge away from the full information equilibrium.*

*Proof.* See Appendix A. ☐

In dynamic rational expectations models, learning from the endogenous variable is achieved by combining the available information in a way that can inject a hidden instability in the equilibrium representation. This observation suggests that any rational expectations equilibrium whose existence relies on learning from endogenous variables should be tested for informational fragility by looking for the presence of hidden instabilities.

## 4 Informational Stability

Proposition 3 casts a serious limit to the use of the closed form solution of the Full Information equilibrium as the rational expectations predicted outcome of model (2.1) when $A(L)$ is non-invertible. If the Full Information form is not robust to information perturbation, is there a rational expectations equilibrium that is informationally stable? In this section we show that the answer to this question, fortunately, is positive. We begin by stating a result that characterizes a class of rational expectations equilibria that obeys Definition 2. We then show that equilibria from this class are informationally stable.

**Proposition 4.** *Suppose the exogenous information is specified as $\mathbb{U}_t = \{\emptyset\}$ for $t \in \mathbb{Z}$, and suppose that $A(L)$ is specified as*

$$a_t - \rho a_{t-1} = \tilde{A}(L) \prod_{i=1}^{m} (1 + \theta_i L) u_t, \text{ where } |\theta_i| > 1 \text{ for } i = 1, ..., m. \tag{4.1}$$

*Then the stochastic process*

$$y_t - \rho y_{t-1} = \frac{\varphi}{L - \kappa} \left\{ LA(L) - \kappa A(\kappa) \frac{(1 - \rho L)}{(1 - \rho \kappa)} \frac{\prod_{i=1}^{m} \mathcal{B}_{\theta_i}(L)}{\prod_{i=1}^{m} \mathcal{B}_{\theta_i}(\kappa)} \right\} u_t \tag{4.2}$$

*where*

$$\mathcal{B}_{\theta_i}(L) \equiv \frac{1 + \theta_i L}{\theta_i + L},$$

*is a dynamic rational expectations Information Equilibrium of the model (2.1).*

*Proof.* See Appendix A. □

The next proposition describes the informational stability properties of the dynamic equation (4.2).

**Proposition 5.** *The dynamic rational expectations equilibrium represented by the stochastic process (4.2) is informationally stable, in the sense that, if the information set of the representative agent is slightly perturbed, the dynamics of the perturbed system revert back to the original dynamics.*

The proof of the proposition for the general case of $m > 0$ is reported in the Appendix, here we focus on a simple example as we did in Section 3. When $m = 1$ and $\rho = 0$ the equilibrium of Proposition 4 becomes

$$y_t = \varphi(\theta + \kappa)\tilde{u}_t + \varphi \tilde{u}_{t-1} \quad \text{where} \quad \tilde{u}_t \equiv \left( \frac{1 + \theta L}{\theta + L} \right) u_t. \tag{4.3}$$

The Definition 2 implies that there must exists a representation of this equilibrium in terms of the history of $y_t$ and $a_t$. Proceeding similarly to the previous section one can show that in this case such representation is

$$y_t = -\frac{1}{\theta} y_{t-1} + \frac{\varphi}{\theta} \left( \theta + \kappa + L \right) a_t \tag{4.4}$$

The equilibrium process $y_t$ has an auto-regressive representation, as in (3.3), but the autoregressive root is now stationary, since $|\frac{1}{\theta}| < 1$. In other words, the rational expectations equilibria that take the form (4.2) do not harbor hidden instabilities.

Consider the equation (4.4) at time $t = 1$,

$$y_1 = \frac{1}{\theta}(\varphi a_0 - y_0) + \varphi \left( \frac{\theta + \kappa}{\theta} \right) a_1 \tag{4.5}$$

In this case, to ensure that $y_1$ is exactly equal to the equilibrium level after an arbitrary history of innovations $u_t$'s the initial condition should be specified as

$$\varphi a_0 - y_0 = \varphi\kappa\tilde{u}_0 \tag{4.6}$$

The second equality in this expression is intuitive: in the same way as the initial condition of the Full Information equilibrium was a linear function of $u_0$, the initial condition for the equilibrium in Proposition 4 is a linear function of $\tilde{u}_0$, the innovation to the most precise information that the representative agent can learn from the equilibrium in Proposition 4. Suppose now that the initial condition is perturbed by $|\varepsilon| > 0$, which we interpret as a perturbation to the information $\tilde{u}_0$, so that

$$\varphi a_0 - y_0 = \varphi\kappa\tilde{u}_0 + \varepsilon, \tag{4.7}$$

then it can be showed that the dynamic solution to (4.4) is

$$y_t = \varphi\big(\theta + \kappa + L\big)\tilde{u}_t + \frac{1}{\theta^t}\varepsilon. \tag{4.8}$$

Since $|\frac{1}{\theta}| < 1$ the accumulated effect of the perturbation disappears as $t$ grows larger and the perturbed equilibrium converges back to the original equilibrium (4.3). Hence, the equilibrium in Proposition 4 is informationally stable.

## 5 Informational Fragility and Learning

The notions of informational fragility and stability are relevant also because they have implications for the learnability of a rational expectations equilibrium. Within the context of the simple MA(1) example, we study whether the Full Information equilibrium is least-squares learnable under informational fragility. We will show that the equilibrium cannot be learned. On the other hand, the informationally stable equilibrium of Proposition 4 is always least-square learnable.

Our learning analysis follows the methods and the notation of Chapter 6 in Evans and Honkapohja (2001). The first step in the learning analysis is to recast the equilibrium relationships in terms of a bi-variate process for the equilibrium outcome $y_t$ and the implied forecast errors, which we will denote by $e_t$.[8] Formally, for any rational expectations information equilibrium define the forecast error process as

$$e_t \equiv y_t - \mathbb{E}\big[y_t|\mathbb{V}_{t-1}(p) \vee \mathbb{V}_{t-1}(p) \vee \mathbb{U}_{t-1}\big]. \tag{5.1}$$

Throughout the analysis we will focus on the case of $a_t = u_t + \theta u_{t-1}$ with $\theta > 1$; the case of negative $\theta$ follows the same steps but it would make notation more burdensome. Under this

[8] We are thankful to Pierre-Olivier Weill for suggesting to us to work within the bi-variate representation that includes the process of forecast errors.

specification for $a_t$ the rational expectations equilibrium can be then represented as

$$y_t = \kappa\eta e_t + \varphi a_t \tag{5.2}$$

$$e_t = y_t - \eta e_{t-1} \tag{5.3}$$

where the coefficient $\eta$ will take a different form depending on the equilibrium we are considering. More precisely, denote by $\hat{\eta}$ the coefficient for Full Information equilibrium and by $\eta^*$ the coefficient for the Information Equilibrium of proposition 4, with

$$\hat{\eta} = \frac{\kappa}{1+\kappa\theta} \qquad \text{and} \qquad \eta^* = \frac{1}{\kappa+\theta}. \tag{5.4}$$

The rational expectations equilibrium taking the form of the Full Information equilibrium (3.2) can be represented as the bivariate system (5.2)-(5.3) with $\eta = \hat{\eta}$, while the equilibrium (4.3) would specify $\eta = \eta^*$. We focus on least-squares learning of the parameter $\eta$, and so we specify the learning algorithm following Evans and Honkapohja (2001)

$$y_t = \kappa\eta_{t-1} e_t + \varphi a_t \tag{5.5}$$

$$e_t = y_t - \eta_{t-1} e_{t-1} \tag{5.6}$$

$$\eta_t = \frac{\frac{1}{t}\sum_{s=1}^{t} y_s e_{s-1}}{\frac{1}{t}\sum_{s=1}^{t} e_{s-1}^2}, \tag{5.7}$$

with $\eta_0$ and $e_0$ given. This learning system can be written in recursive form as follows

$$e_t = \frac{\eta_{t-1}}{1-\kappa\eta_{t-1}} e_{t-1} + \frac{\varphi}{1-\kappa\eta_{t-1}}\left(u_t + \theta u_{t-1}\right) \tag{5.8}$$

$$\eta_t = \eta_{t-1} + \frac{1}{t} S_{t-1}^{-1} e_t e_{t-1} \tag{5.9}$$

$$S_t = S_{t-1} + \frac{1}{t}\left(e_t^2 - S_{t-1}\right) + \frac{1}{t^2}\frac{-t}{t+1}\left(e_t^2 - S_{t-1}\right) \tag{5.10}$$

where $S_t$ represents the time-$t$ estimate of the variance-covariance matrix of the process for the forecast errors. To apply the formal results of Chapter 6 in Evans and Honkapohja (2001) it is useful to represent the system in compact notation, so we define

$$X_t = \begin{pmatrix} e_t \\ e_{t-1} \\ u_t \end{pmatrix} \qquad \text{and} \qquad \lambda_t = \begin{pmatrix} \eta_t \\ S_t \end{pmatrix}. \tag{5.11}$$

In the two equilibria that we are considering the vector $\lambda$ takes the form

$$\hat{\lambda} = \begin{pmatrix} \hat{\eta} \\ \varphi^2(1+\kappa\theta)^2 \end{pmatrix} \qquad \text{and} \qquad \lambda^* = \begin{pmatrix} \eta^* \\ \varphi^2(\kappa+\theta)^2 \end{pmatrix} \tag{5.12}$$

The learning algorithm can then be written as

$$X_t = G(\lambda_{t-1})X_{t-1} + F(\lambda_{t-1})u_t \tag{5.13}$$

$$\lambda_t = \lambda_{t-1} + \gamma_t H(\lambda_{t-1}, X_t) + \gamma_t^2 \rho_t(\lambda_{t-1}, X_t) \tag{5.14}$$

where

$$G(\lambda_{t-1}) \equiv \begin{pmatrix} -\frac{\eta_{t-1}}{1-\kappa\eta_{t-1}} & 0 & \frac{\theta\varphi}{1-\kappa\eta_{t-1}} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad F(\lambda_{t-1}) \equiv \begin{pmatrix} \frac{\varphi}{1-\kappa\eta_{t-1}} \\ 0 \\ 1 \end{pmatrix},$$

$$\gamma_t = \frac{1}{t}, \qquad H(\lambda_{t-1}, X_t) = \begin{pmatrix} S_{t-1}^{-1} e_t e_{t-1} \\ e_t^2 - S_{t-1} \end{pmatrix}, \qquad \rho_t(\lambda_{t-1}, X_t) = \begin{pmatrix} 0 \\ \frac{-t}{t+1}\left(e_t^2 - S_{t-1}\right) \end{pmatrix}.$$

An equilibrium $\lambda$ can be learned if the above dynamic system converges to $\lambda$ as $t \to \infty$ for $\lambda_0$ in a neighborhood of $\lambda$. For such convergence to happen several properties of the components of the dynamic system have to be checked. A necessary condition for convergence is that the process for $X_t$ is stationary in a neighborhood of $\lambda$, which corresponds to the Eigenvalues of $G(\lambda)$ to have modulus smaller than one. The following proposition shows that this necessary condition is not met by the Full Information equilibrium.

**Proposition 6.** *The Full Information rational expectations equilibrium $\hat{\lambda}$ cannot be learned by the least squares algorithm* (5.5)-(5.7) *for any $\lambda_0$ in a neighborhood of $\hat{\lambda}$.*

The proof of the proposition consists in computing the Eigenvalues of the matrix $G(\hat{\lambda})$. It is immediate to see that they are given by the vector $(-\theta, 0, 0)$. Since we have assumed that $\theta > 1$ and $G(\lambda)$ is continuous, it follows that the process $X_t$ is non-stationary in a neighborhood of $\hat{\lambda}$, hence the equilibrium cannot be learned by least squares methods. The reason behind this result lies in the structure of the dynamic process for the forecast errors $e_t$, which contain a hidden instability inherited from the equilibrium learning described in Section 3. At the equilibrium value $\hat{\eta}$ this process contains an explosive root which exactly cancels with the zero in the moving average of $a_t$. If the forecast error process is not initialized at the equilibrium value the explosive root will not cancel with the zero and the initial misalignment would grow indefinitely. In the learning algorithm the forecast error is not initialized at the equilibrium value, unless one happens to start learning exactly from the equilibrium point, and so the explosive root prevents the convergence of the learning algorithm. In other words, the informational fragility of the Full Information equilibrium affects the learning process by making it non-convergent.

We turn now to the analysis of the learning of the information equilibrium $\lambda^*$. The following proposition summarizes our result.

**Proposition 7.** *The rational expectations equilibrium $\lambda^*$ can always be learned by the least squares algorithm* (5.5)-(5.7) *for $\lambda_0$ in a neighborhood of $\lambda^*$.*

The first step in proving this result is to check for the stationarity of the process $X_t$ in a neighborhood of $\lambda^*$. The Eigenvalues of the matrix $G(\lambda^*)$ in this case are given by the vector $(-\frac{1}{\theta}, 0, 0)$, where $|-1/\theta| < 1$ which, together with the continuity of $G(\lambda)$, provides the desired result. The second and sufficient step in the proof is to show the E-stability of the learning algorithm. This is done by analyzing the limiting mapping $T(\lambda)$ defined as

$$T(\lambda) \equiv \lim_{t\to\infty} \mathbb{E}\big(H(\lambda, X_t(\lambda))\big) = \begin{pmatrix} S^{-1}\sigma_1 \\ \sigma_0 - S \end{pmatrix} \tag{5.15}$$

where $\sigma_i$ is the auto-covariance of order $i$ of the forecast errors process $e_t$ under $\lambda$. E-stability is verified by studying the stability of the differential equation

$$\frac{d\lambda}{d\tau} = T(\lambda) - \lambda \tag{5.16}$$

in the neighborhood of $\lambda^*$. It is possible to show that $\lambda^*$ is a stationary point of the above ordinary differential equation and that such stationary point is stable. This last property is obtained by deriving the partial derivatives matrix of $T(\lambda)$ evaluated at $\lambda^*$, which is

$$DT(\lambda^*) = \begin{pmatrix} -\frac{(\theta+\kappa)^2}{\theta} & 0 \\ \tau_{S\eta}(\theta) & -1 \end{pmatrix}. \tag{5.17}$$

where $\tau_{S\eta}(\theta)$ is a function whose form is not influential for the stability analysis. The Eigenvalues of $DT(\lambda^*)$ must be all negative for the stationary point to be stable. Note that this is verified under our maintained assumption $\theta > 1$. Hence, the Information Equilibrium $\lambda^*$ can be learned by the least square algorithm (5.5)-(5.7).

# 6 Applications

In this section we apply our results to two prominent examples. The first is a stylized model with news shocks. Many recent papers document the importance of news shocks from an empirical and theoretical viewpoint for understanding business cycle dynamics and changes in fiscal policy [Beaudry and Portier (2006, 2004), Jaimovich and Rebelo (2009), Christiano, Ilut, Motto, and Rostagno (2010), Fujiwara, Hirose, and Shintani (2011), Barsky and Sims (2011), Leeper and Walker (2011), Mertens and Ravn (2010, 2011), Schmitt-Grohé and Uribe (2012), Khan and Tsoukalas (2012)]. We demonstrate that news shocks, as they are typically modeled, result in equilibria that are informationally fragile. We also demonstrate this result in an incomplete information model with nominal rigidities and productivity shocks taken from Lorenzoni (2009). It represents an instance of the class of monetary models popularized by Clarida, Gali, and Gertler (1999) and Woodford (2003), among others.

**6.1 News Shocks** We examine the informational fragility of news shocks in the setup of Leeper and Walker (2011). While the model is quite stylized, the results readily extend to more complex settings [e.g., Jaimovich and Rebelo (2009), Christiano, Ilut, Motto, and Rostagno (2010), Mertens and Ravn (2011), Schmitt-Grohé and Uribe (2012), Khan and Tsoukalas (2012)].[9] Consider a standard growth model with a representative household that maximizes expected log utility, $E_0 \sum_{t=0}^{\infty} \beta^t \log(C_t)$, subject to $C_t + K_t \leq A_t K_{t-1}^{\alpha}$, where $C_t$, $K_t$, and $Y_t$ denote time-$t$ consumption, capital, and output, respectively, and $A_t$ is an exogenous technology shock. As usual, $0 < \alpha < 1$ and $0 < \beta < 1$. Labor is supplied inelastically. The equilibrium conditions are well known and given by

$$\frac{1}{C_t} = \alpha\beta E_t\left[\frac{1}{C_{t+1}}\frac{Y_{t+1}}{K_t}\right] \tag{6.1}$$

$$C_t + K_t = Y_t = A_t K_{t-1}^{\alpha}. \tag{6.2}$$

Let $A$ denote the steady state value of technology. The steady state capital stock is $K = [\alpha\beta A]^{1/(1-\alpha)}$. Let lower case letters denote percentage deviations from steady state values, $k_t = \log(K_t) - \log(K)$ and $a_t = \log(A_t) - \log(A)$. Log linearizing (6.1)–(6.2) yields an equilibrium that is characterized by a second-order difference equation in capital

$$\alpha\beta \mathbb{E}_t k_{t+1} - (1 + \alpha^2\beta)k_t + \alpha k_{t-1} = \alpha\beta\mathbb{E}_t(a_{t+1}) \tag{6.3}$$

If we assume that $a_t = A(L)\varepsilon_t$, where $A(\cdot)$ satisfies square summability, and $\{\varepsilon_{t-j}\}_{j=0}^{\infty} \in \mathbb{U}_t$ then by Proposition 2, the solution to (6.3) is the Full Information equilibrium and given by

$$k_t = \left[\frac{A(L) - A(\varpi)}{(1-\alpha L)(\varpi - L)}\right]\varepsilon_t \tag{6.4}$$

where $\varpi = \alpha\beta$.

Consider the following moving-average representation for the technology process

$$a_t = \varepsilon_t + \theta\varepsilon_{t-1} \tag{6.5}$$

which is identical to (2.4). In order for (6.5) to be labeled a "news shock", $\theta$ must be larger than one. This implies (6.5) is a non-fundamental moving-average representation, and the space spanned by current and past technology, $\{a_{t-j}\}_{j=0}^{\infty}$, is smaller than the space spanned by the structural innovations, $\{\varepsilon_{t-j}\}_{j=0}^{\infty}$.[10] The variance of the one-step-ahead forecast error for agents conditioning on structural innovations is smaller than the forecast error variance for agents conditioning *only* on current and past $a_t$. To show this analytically, we must derive the Wold representation of (6.5),

[9] Appendix B gives conditions under which a more complex model with news shocks would be informationally fragile.

[10] Other papers have assumed an i.i.d. process for news (e.g., $a_t = \varepsilon_{1,t-1} - \theta\varepsilon_{2,t}$, where $\varepsilon_{1,t-1}$ and $\varepsilon_{2,t}$ are orthogonal at all leads and lags). Leeper and Walker (2011) derive a mapping between i.i.d. news and the news structure given by (6.5). Once this mapping is taken into consideration, it is straightforward to extend to our results to models with i.i.d. news shocks.

which comes from flipping the root, $\theta^{-1}$, outside of the unit circle

$$a_t = \theta\tilde{\varepsilon}_t + \tilde{\varepsilon}_{t-1}, \qquad \tilde{\varepsilon}_t = \left[\frac{1+\theta L}{L+\theta}\right]\varepsilon_t \tag{6.6}$$

Representation (6.6) is the Wold representation where $\tilde{\varepsilon}_t$ is the one-step-ahead forecast error associated with forecasting $a_t$ conditional on $\{a_{t-j}\}_{j=1}^{\infty}$. This representation shows that current and past $a_t$ span an equivalent space to current and past $\tilde{\varepsilon}_t$, which is a strictly smaller space than $\varepsilon_t$. Optimal prediction formulas yield the variance of the one-step-ahead forecast error using (6.6)

$$\begin{aligned}\mathbb{E}\{a_{t+1} - \mathbb{E}[a_{t+1}|\{a_{t-j}\}_{j=0}^{\infty}]\}^2 &= \mathbb{E}\{(1-\theta L)\varepsilon_{t+1} - L^{-1}[(L-\theta)+\theta]\tilde{\varepsilon}_t\}^2 \\ &= \mathbb{E}\{(1-\theta L)\varepsilon_{t+1} - (1-\theta L)\varepsilon_{t+1} - \theta\tilde{\varepsilon}_{t+1}\}^2 \\ &= \theta^2 \text{var}(\tilde{\varepsilon}_{t+1}) = \theta^2\sigma_\varepsilon^2, \end{aligned} \tag{6.7}$$

where the last equality follows because the term, $(1+\theta L)/(L+\theta)$, known as a Blaschke factor, has a covariance generating function of one and hence var$(\varepsilon_t)$=var$(\tilde{\varepsilon}) = \sigma_\varepsilon^2$.

If agents condition on current and past structural innovations directly, then (6.5) can be used to forecast next period's technology process. The variance of the forecast error is given by

$$\begin{aligned}\mathbb{E}\{a_{t+1} - \mathbb{E}[a_{t+1}|\{\varepsilon_{t-j}\}_{j=0}^{\infty}]\}^2 &= \mathbb{E}\{(1-\theta L)\varepsilon_{t+1} - L^{-1}[(1-\theta L)-1]\varepsilon_t\}^2 \\ &= \mathbb{E}\{\varepsilon_{t+1}\}^2 = \sigma_\varepsilon^2 \end{aligned} \tag{6.8}$$

Comparing this forecast error variance with (6.7) shows that the moving-average coefficient, $\theta$, determines the degree to which agents conditioning on the structural shocks are better informed. As $\theta$ becomes "large", the news shock has a more significant impact and agents who observe the structural innovations have a much more precise one-step-ahead forecast. As $\theta \to 1$ from above, the information sets and the variance of forecast errors converge, which eliminates the news shock.

Equations (6.7) and (6.8) demonstrate that if the process for technology follows (6.5) with $|\theta| > 1$, the one-step-ahead forecast for technology and capital of (6.3) will be quite different for news shocks [i.e., $\{\varepsilon_{t-j}\}_{j=1}^{\infty} \in \mathbb{U}_t$] relative to the equilibrium that does not assume news shocks [i.e., $\{\emptyset\} \in \mathbb{U}_t$]. The news shock equilibrium corresponds to the Full Information Equilibrium and is given by

$$k_t = \alpha k_{t-1} + \theta\varepsilon_t = \left(\frac{\theta}{(1-\alpha L)(1+\theta L)}\right)a_t \tag{6.9}$$

In contrast to the "news" shocks, an alternative equilibrium assumes that agents must learn the underlying structural shocks. Using Proposition 4, the Information Equilibrium assuming $\{\emptyset\} \in \mathbb{U}_t$ is given by

$$k_t = \alpha k_{t-1} + \tilde{\varepsilon}_t = \left(\frac{\theta^{-1}}{(1-\alpha L)(1+\theta^{-1}L)}\right)a_t \tag{6.10}$$

Following the results in Sections 3 and 4, we have established the following corollary.

**Corollary 1.** *The "News Shock" equilibrium* (6.9) *is informationally fragile, while the Information Equilibrium* (6.10) *is informationally stable.*

*Proof.* Follows directly from Propositions 3 and 5. ☐

6.2 Productivity Shocks with Nominal Rigidities The model is an incomplete information version of a monetary equilibrium model with nominal rigidities driven by productivity shocks. The version of the model is taken from Lorenzoni (2009) and it represents an instance of the class of models popularized by Clarida, Gali, and Gertler (1999) and Woodford (2003), among others. The linearized economy is fully described by an output equation, an inflation equation and a monetary policy rule as

$$y_t = \mathbb{E}(y_{t+1}|\Omega_t) - i_t + \mathbb{E}(\pi_{t+1}|\Omega_t) \tag{6.11}$$

$$\pi_t = \alpha(y_t - a_t) + \beta\mathbb{E}(\pi_{t+1}|\Omega_t) \tag{6.12}$$

$$i_t = \phi\pi_t \tag{6.13}$$

where all the constants have been dropped for convenience and $\Omega_t$ represents the information set of the representative agent, to be specified shortly. The variable $a_t$ represents the exogenous process for aggregate productivity in the economy, the term $y_t - a_t$ is a measure of the real marginal costs in the economy along the labor market equilibrium. We follow Lorenzoni (2009) in specifying the process for aggregate productivity as having a stochastic trend, but we also allow a richer moving average structure, more specifically we let

$$a_t = a_{t-1} + u_t + \theta u_{t-1}, \tag{6.14}$$

with $u_t \sim \mathcal{N}(0, \sigma_u^2)$. For the moment we do not impose any restriction on the value of $\theta$, we only notice that the permanent component of productivity can be modeled as a diffusion process with a typical S-shape behavior if one allows $\theta > 1$. In such case the innovation to productivity at time $t$, $u_t$, can be interpreted as the amount of new knowledge available at that time, which results in higher productivity only gradually over time, with an initial low diffusion at time $t$, a subsequent steep increase in diffusion at time $t+1$, and a leveling-off phase afterwards, from $t+2$ onward. Arguably, such dynamic behavior may adequately capture episodes of innovations to productivity that have permanent effects (cf. Canova (2003)).

Our objective is to characterize the dynamic rational expectations Information Equilibrium as defined in Definition 2. To that end we assume, for simplicity, that the initial time is not specified, we let $\mathbb{U}_t = 0$ for all $t$'s and define the information set as

$$\Omega_t \equiv \mathbb{V}_t(y, \pi) \vee \mathbb{M}_t(y, \pi). \tag{6.15}$$

The knowledge of the model in this setting always reveals the realization for aggregate productivity

$a_t$, and so in any equilibrium we will have $\mathbb{V}_t(a) \subseteq \mathbb{M}_t(y,\pi)$. In addition, we follow Lorenzoni (2009) and conjecture that $\mathbb{E}(y_{t+1}|\Omega_t) = \mathbb{E}(a_{t+1}|\Omega_t)$, which implies $\mathbb{E}(\pi_{t+1}|\Omega_t) = 0$.[11] Substituting the interest rate rule into the output equation and the inflation equation under the above conjecture the equilibrium condition reduces to

$$y_t = \kappa \mathbb{E}\big[y_{t+1}|\Omega_t\big] + (1-\kappa)a_t, \tag{6.16}$$

where $\kappa \equiv \frac{1}{1+\alpha\phi} \in (0,1)$. The equilibrium equation we want to study is thus of the form (2.1) with $\varphi = (1-\kappa)$. Once the solution for $y_t$ is obtained, the solution for $\pi_t$ immediately follows. At this point one might think that there is really no fixed point problem to solve for in this setting since our conjecture already implies $\mathbb{E}(y_{t+1}|\Omega_t) = \mathbb{E}(a_{t+1}|\Omega_t)$. Note, however, that such conjecture does not pin down the equilibrium information set $\Omega_t$, which remains to be determined. Equation (6.16) imposes a fixed point condition through the specification of the information set $\Omega_t$ as containing the current and past equilibrium variables $y_t$ and $\pi_t$. To simplify matters we note that the knowledge of the variable $\pi_t$ corresponds to the knowledge of the variables $y_t$ and $a_t$, therefore in what follows we proceed by working with the information set $\Omega_t = \mathbb{V}_t(y) \vee \mathbb{V}_t(a)$ without loss of generality. The equilibrium equation we want to solve for is then

$$y_t = \kappa \mathbb{E}\big[x_{t+1}|\mathbb{V}_t(y) \vee \mathbb{V}_t(a)\big] + (1-\kappa)a_t. \tag{6.17}$$

The Full Information equilibrium of (6.17) is a straightforward application of Proposition 1 and it is given by the $ARMA(1,1)$ process

$$\tilde{y}_t - \tilde{y}_{t-1} = (1+\theta\kappa)u_t + \theta(1-\kappa)u_{t-1}. \tag{6.18}$$

We know from Proposition 2 that the Full Information equilibrium (6.18) is a legitimate rational expectations Information Equilibrium. The representation of the equilibrium in terms of the components of the information set is

$$\tilde{y}_t - \tilde{y}_{t-1} = -\theta(\tilde{y}_{t-1} - \tilde{y}_{t-2}) + \theta(1-\kappa)(a_t - a_{t-1}). \tag{6.19}$$

The autoregressive root $-\theta$, which is the result of learning from the endogenous variable, dictates the informational stability of the equilibrium. If $\theta \leq 1$ the equilibrium is informationally stable, while for $\theta > 1$ the equilibrium is informationally fragile. Therefore, if the modeler was interested in allowing the productivity process to display an S-shaped diffusion pattern, i.e. allowing $\theta > 1$, focusing on the equilibrium (6.18) would be a choice that is not robust to the information set initialization. In addition, we know from Section 5 that such equilibrium cannot possibly by justified by least squares learning convergence.

The key message underscoring this example is that when solving incomplete information rational

[11] To see this just lead the inflation equation forward one period and apply the expectational operator on both sides taking into account the conjecture for the expectations of output and aggregate productivity.

equilibrium models, the sole fact that a guess on the expectations in equilibrium (which corresponds to a guess of the equilibrium information) satisfies the fixed point equilibrium condition, does not guarantee that the resulting equilibrium is robust to information perturbation.

To conclude our application we compute the rational expectations equilibrium for (6.17) that is suggested by Proposition 4 when $\theta > 1$, and which is, therefore, informationally stable. Application of the proposition shows that the equilibrium is given by an $ARMA(2,2)$ of the form

$$y_t^* - \frac{\theta-1}{\theta} y_{t-1}^* - \frac{1}{\theta} y_{t-2}^* = \frac{\theta+\kappa}{\theta} u_t + \left(\frac{1-\kappa}{\theta} + \theta + \kappa\right) u_{t-1} + (1-\kappa) u_{t-2}. \tag{6.20}$$

The equilibrium process contains a unit root, but the learning from the endogenous variable $y_t$ is now stable, which corresponds to an additional stable auto-regressive root in the output process at $\frac{1}{\theta}$. This additional persistence combines with the additional moving average component and it results in a different dynamic response compared to the full information counterpart.

To see this we consider the dynamic response of the equilibrium variables to a unit (standard deviation) increase in the structural innovations $u_t$. The impulse response of productivity is given by

$$\frac{da_t}{du_t} = 1 \qquad \text{and} \qquad \frac{da_{t+\tau}}{du_t} = 1 + \theta, \qquad \text{for} \qquad \tau > 0. \tag{6.21}$$

A unit innovation of $u_t$ has a permanent effect on aggregate productivity of $1 + \theta$ which is reached in part at impact, and then in full from $t + 1$ onward. In the Full Information equilibrium the impulse response of output is given by

$$\frac{d\tilde{y}_t}{du_t} = 1 + \theta\kappa \qquad \text{and} \qquad \frac{d\tilde{y}_{t+\tau}}{du_t} = 1 + \theta, \qquad \text{for} \qquad \tau > 0, \tag{6.22}$$

and the impulse response of employment $\tilde{n}_t$ by

$$\frac{d\tilde{n}_t}{du_t} = \theta\kappa \qquad \text{and} \qquad \frac{d\tilde{n}_{t+\tau}}{du_t} = 0, \qquad \text{for} \qquad \tau > 0, \tag{6.23}$$

The permanent increase in productivity is not completely realized at $t$, but it is entirely anticipated and so output increases by more at impact. This generates a temporary boom in employment and a temporary spike in inflation, since $\tilde{\pi}_t = \kappa\tilde{n}_t$. Once productivity reaches the new permanent level at $t + 1$, employment is back to normal (or trend), and so is inflation. In the full information equilibrium a permanent increase in productivity generates a one-period above-trend reaction for employment and inflation as output climbs to a new higher permanent level.

The impulse responses in the Information Equilibrium (6.20) are quite different. At impact the representative agent does not know exactly the extent of the innovation $u_t$ and so she is unable to exactly forecast the permanent effect on productivity. This will result in an under-reaction of output and employment to the innovation, compared to the Full Information case. In turn, the under-reaction will make learning from the equilibrium variables imprecise and result in an overestimation of the innovation in the subsequent period. The overestimation generates a an over-reaction in output and employment, compared to the full information case. The over-reaction will then result in an underestimation of the innovation in the subsequent period, with an under-reaction of output and employment and so on. The impulse response for output in this case is given by

$$\frac{dy_t^*}{du_t} = 1 + \frac{\kappa}{\theta} \quad \text{and} \quad \frac{dy_{t+\tau}^*}{du_t} = 1 + \theta + \left(-\frac{1}{\theta}\right)^{\tau-1} \kappa \left(1 - \frac{1}{\theta^2}\right), \quad \text{for} \quad \tau > 0, \tag{6.24}$$

and the impulse response of employment $n_t^*$ by

$$\frac{dn_t^*}{du_t} = \frac{\kappa}{\theta} \quad \text{and} \quad \frac{dn_{t+\tau}^*}{du_t} = \left(-\frac{1}{\theta}\right)^{\tau-1} \kappa \left(1 - \frac{1}{\theta^2}\right) \quad \text{for} \quad \tau > 0. \tag{6.25}$$

The uncertainty about the extent of the permanent innovation at impact results in a smaller increase in output, the bigger is $\theta$. Note that $\theta$ was playing exactly the opposite role in the Full Information case. The increase in output is still higher than the increase in productivity at impact, which generates an increase in employment of $\frac{\kappa}{\theta}$, which is smaller than the Full Information case. Overall, the reaction at impact under $y_t^*$ is similar in quality to the reaction under $\tilde{y}_t$, but the overall magnitude is smaller because of the incomplete information about the extent of the permanent innovation. In the subsequent periods output oscillates around the permanent steady state increase $1 + \theta$ by a factor $k\left(1 - \frac{1}{\theta^2}\right)$. The second term is the unconditional forecast error due to the non-invertibility of the productivity process, the first term represents how relevant is the forecast of future output for today's decisions. As periods unfold there is also an additional effect due to $\theta$ that dampens the oscillations, the bigger is $\theta$. This effects comes from the signal extraction problem of the representative agent: the higher is $\theta$, the noisier is $y_t$, and thus the less useful to predict future output, the smaller the effect of noise on the prediction, the smaller the resulting misalignment of output with productivity.

Figure 1 plots the impulse responses for productivity $a_t$ (dotted line), the informationally fragile solution (black dashed line) and the informationally stable solution (blue solid line) for both output and employment, to a one time standard deviation increase in $u_t$. The parameters values are $\alpha = .05$, $\beta = .7$, $\phi = 1.5$, $\rho = 1$ and $\theta = 2$. Looking at the left side panel, in the fragile solution output increases at impact almost at the new permanent level as the extent of the innovation is fully anticipated. The increase in productivity is limited at impact because of the initial slow diffusion. This creates the need for higher employment in order to increase output at impact. In the subsequent period productivity reaches the new permanent level, so does output, and employment goes back to the long run trend. In the stable solution the initial reaction of output is positive, but is around only 50% of the new higher trend. This level is still higher than the increase in productivity and so employment increases at impact, but only by a 25% of the increase in the fragile case. One period from impact agents learn that the permanent increase in productivity might actually be higher than previously thought and output overshoots the trend by around a

20% margin. Productivity is now at the new higher level, but the overshooting in output requires a higher employment to be achieved, which means that employment is still above trend one period after impact, and even higher than the level at impact. The subsequent period agents realize that the innovation might have been smaller than they thought, which creates a drop in output below trend, and a drop in employment below trend. In other words, two periods after the onset of the productivity innovation and a gradual boom in output and employment, the economy experiences a recession. The same type of oscillation then repeats itself with a smaller and smaller magnitude, until the economy finally settles on the new higher trend.

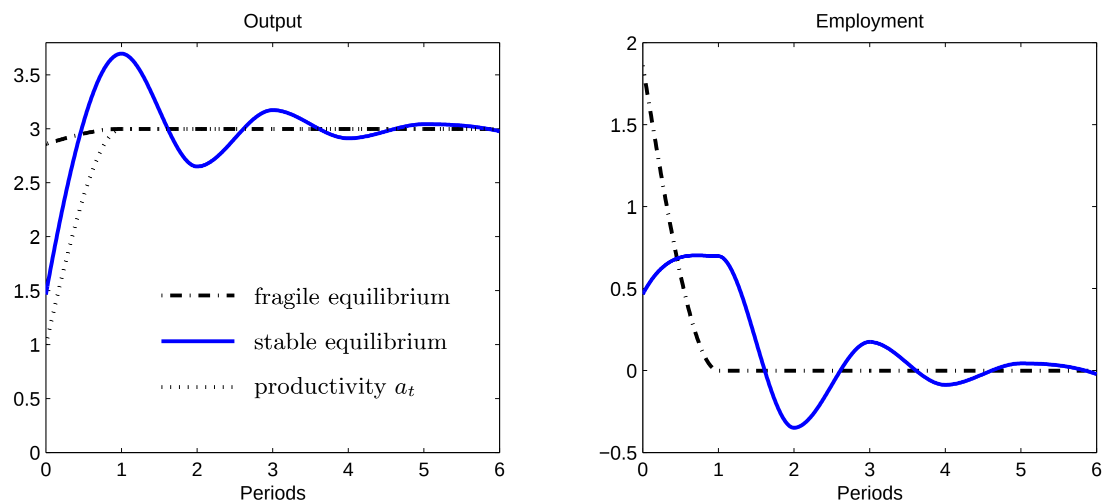


Figure 1: Impulse Response of Output and Employment to Innovation in Productivity

In summary, applying our stability analysis to the equilibrium of the stylized monetary model (6.11)-(6.13) when the productivity process $a_t$ displays a diffusion-type dynamics, results in a rational expectations equilibrium with several differences compared to the equilibrium on which a researcher unaware of our analysis would focus. From a normative perspective, the equilibrium (6.20) is robust to perturbations in the information set and is least-squares learnable, two conditions that would make it preferable to equilibrium (6.19). From a positive perspective, the equilibrium (6.20) displays a rich propagation dynamic that not only generates a demand-shock like response at impact, which is qualitatively similar to what happens in equilibrium (6.19), but also generates a longer expansion of employment over trend, and, remarkably, a subsequent recession in both output and employment (both below trend), followed by a smaller expansion, and so on. This all in response to a permanent positive productivity shock that diffuses only gradually in the economy.

## 7 Conclusion

In this paper we studied the stability properties of Rational Expectations equilibria in dynamic models with incomplete information when the information set of agents is slightly perturbed. We showed that equilibria where the endogenous variables resolve the information incompleteness can be informationally fragile, in the sense that a slight perturbation in the endogenous information set of the agents along the equilibrium path can lead to a break-down of the equilibrium dynamics. We then presented a class of dynamic rational expectations equilibria that are informationally stable for the same parameter space where other equilibria are informationally fragile. We also showed that an equilibrium that is informationally fragile is not least-squares learnable, while an equilibrium that is informationally stable always is. We concluded by presenting two applications from established literatures.

# A Appendix A: Proofs

Proof of Proposition 2 The Information Equilibrium will be a Full Information Equilibrium if the Hilbert space generated by $\Omega_t$, as defined in Definition 2, spans the space as the sequence of underlying structural shocks, $\{u_{t-j}\}_{j=0}^{\infty}$. The first two cases are straightforward: [i.] If $m = 0$, then knowledge of the model and (2.8) implies that $\{a_{t-j}\}_{j=0}^{\infty}$ spans the same space as $\{u_{t-j}\}_{j=0}^{\infty}$ because $a_t$ is fundamental for $u_t$. Therefore even if the exogenous information $\mathbb{U}_t$ is the empty set, the Information Equilibrium will be the Full Information Equilibrium. [ii.] If $m > 0$ and $\mathbb{U}_t = \{u_{t-j}\}_{j=0}^{\infty}$, then the Information Equilibrium will be a Full Information Equilibrium because the agents observe the entire sequence of current and past shocks by assumption. [iii.] If $m > 0$, then $\{a_{t-j}\}_{j=0}^{\infty}$ spans $\{u_{t-j}\}_{j=0}^{\infty}$ if $\dim \mathbb{U}_t \geq m$, which is the condition given in Proposition 2. As an example, suppose a moving average representation has two distinct zeros, $x_t = (L - \lambda_1)(L - \lambda_2)\varepsilon_t$ with $|\lambda_1| < 1$, $|\lambda_2| < 1$. If agents observe $\varepsilon_{t-1}$ and $\varepsilon_{t-2}$ directly, then we may write the moving average as $x_t - \varepsilon_{t-2} + (\lambda_1 + \lambda_2)\varepsilon_{t-1} = \lambda_1\lambda_2\varepsilon_t$, which is always invertible.

Proof of Proposition 3

The proof of the proposition consists in expressing the Full Information equilibrium in terms of the information set of the rational expectations equilibrium, similarly to (3.3) and then argue that the expression is fragile to the initialization of the information set. The Full Information equilibrium for $m > 0$ can be expressed as

$$(y_t - \rho y_{t-1}) \prod_{i=1}^{m} (1 + \theta_i L) = \varphi \frac{\Big(LA(L)(1 - \rho\kappa) - \kappa A(\kappa)(1 - \rho L)\Big)}{(L - \kappa)(1 - \rho\kappa)\tilde{A}(L)} (a_t - \rho a_{t-1}). \tag{A.1}$$

The lag polynomial on the right hand side has a zero at $\kappa$ by construction that cancels with the root at the denominator. It follows that the polynomial is stationary. On the other hand, the equilibrium representation for $y_t$ has an $AR(m+1)$ component, where $m$ of the roots are inside the unit circle. Unless the initial conditions on the information set $(y_0, y_{-1}, ....., y_{-m})$ and $(a_0, a_{-1}, ....., a_{-m})$ are chosen to exactly cancel the explosive roots, the equilibrium dynamics will diverge.

Proof of Proposition 4

We begin by guessing a functional form for the equilibrium price as

$$y_t = Q(L) \prod_{i=1}^{m} (1 - \lambda_i z) \tilde{u}_t \tag{A.2}$$

with

$$\tilde{u}_t = \prod_{i=1}^{m} \mathcal{B}_{\lambda_i}(L). \tag{A.3}$$

and

$$\mathcal{B}_{\lambda_i}(L) \equiv \frac{1-\lambda_i L}{\lambda_i - L}. \tag{A.4}$$

Under such guess one can derive the conditional expectation for future productivity and substitute it into the equilibrium equation (2.1) and get the following $z$-transform expression

$$\begin{aligned} Q(z)\prod_{i=1}^{m}(z-\lambda_i) &= \kappa z^{-1}[Q(z)\prod_{i=1}^{m}(1-\lambda_i z) - Q_0]\prod_{i=1}^{m}\mathcal{B}_{\lambda_i}(z) + \varphi A(z) \\ &= \kappa z^{-1}[Q(z)\prod_{i=1}^{m}(z-\lambda_i) - Q_0\prod_{i=1}^{m}\mathcal{B}_{\lambda_i}(z)] + \varphi A(z) \end{aligned}$$

Working out the algebra yields

$$Q(z)(z-\kappa)\prod_{i=1}^{m}(z-\lambda_i) = \varphi z A(z) - Q_0\prod_{i=1}^{m}\mathcal{B}_{\lambda_i}(z) \tag{A.5}$$

For $|\kappa| < 1$, stationarity requires the $Q(\cdot)$ process to be analytic inside the unit circle, which will not be the case unless the process vanishes at the poles $z = \{\lambda_i, \kappa\}$ for every $i$. For simplicity, we assume $\lambda_i \neq \lambda_j$ for any $i \neq j$, however this restriction can be relaxed [see, Whiteman (1983)]. Evaluating at $z = \lambda_i$ provides a restriction on the $A(\cdot)$ process,

$$A(\lambda_i) = 0 \quad \text{for} \quad i = 1, ..., m, \tag{A.6}$$

which implies that $\lambda_i = -1/\theta_i$ for all $i$. By Proposition 10.4 of Conway (1991), this restriction guarantees that the knowledge of the model does not reveal any additional information than the posited price sequence. Finally, evaluating (A.5) at $z = \kappa$ gives

$$Q_0 = \frac{\kappa A(\kappa)}{\prod_{i=1}^{m}\mathcal{B}_{\lambda_i}(\kappa)} \tag{A.7}$$

Substituting this into (A.5) and rearranging the algebra returns expression (4.2).

### Proof of Proposition 7

We need to evaluate the stability of the mapping

$$T\begin{pmatrix}\eta \\ S\end{pmatrix} = \begin{pmatrix}T_\eta(\eta, S) \\ T_S(\eta, S)\end{pmatrix} = \begin{pmatrix}S^{-1}\sigma_1(\eta) \\ \sigma_0(\eta) - S\end{pmatrix} \tag{A.8}$$

where the two covariances are given by

$$\sigma_0(\eta) = \frac{\varphi^2}{(1-\eta\kappa)^2 - \eta^2}\left(1 - \frac{2\eta}{1-\eta\kappa}\theta + \theta^2\right) \tag{A.9}$$

$$\sigma_1(\eta) = \frac{\varphi^2}{(1-\eta\kappa)^2 - \eta^2}\left(\theta - \frac{\eta}{1-\eta\kappa}\right)\left(1 - \frac{\eta\theta}{1-\eta\kappa}\right) \tag{A.10}$$

We use these expressions to compute the matrix of partial derivatives

$$DT\begin{pmatrix}\eta\\ S\end{pmatrix} = \begin{pmatrix}DT_\eta(\eta,S)\\ DT_\eta(\eta,S)\end{pmatrix} = \begin{pmatrix}\frac{dT_\eta}{d\eta}(\eta,S) & \frac{dT_\eta}{dS}(\eta,S)\\ \frac{dT_S}{d\eta}(\eta,S) & \frac{dT_S}{dS}(\eta,S)\end{pmatrix} \tag{A.11}$$

evaluated at $\eta = \eta^* = \frac{1}{\theta+\kappa}$ and $S = S^* = \varphi^2(\theta+\kappa)^2$. Proceeding with the algebra one obtains matrix (5.17), and the least squares learning convergence immediately follows.

# B Appendix B: Multivariate Extension

We now extend the results of Sections 3 and 4 to a more general setting and show that our results apply to a larger class of models. Consider the generic multivariate rational expectations model

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi z_t + \Pi\eta_t, \tag{B.1}$$

where $y_t$ is an $n\times 1$ vector of endogenous variables, $z_t$ is an $m\times 1$ vector of exogenous random shocks, $\eta$ is a $k\times 1$ vector of expectation errors, which satisfy $E_t\eta_{t+1} = 0$ for all $t$. $\Gamma_0$ and $\Gamma_1$ are $n\times n$ coefficient matrices, along with $\Psi$ $(n\times m)$ and $\Pi$ $(n\times k)$. The model collapses to the univariate setting of Section 2 when $z_t = u_t + \theta u_{t-1}$ and

Klein (2000) and Sims (2002) use a generalized Schur decomposition of $\Gamma_0$ and $\Gamma_1$ to show that there exist matrices such that $Q'\Lambda Z' = \Gamma_0$, $Q'\Omega Z' = \Gamma_1$, $Q'Q = Z'Z = I_{n\times n}$, where $\Lambda$ and $\Omega$ are upper-triangular. The ratios of the diagonal elements of $\Omega$ and $\Lambda$, $\omega_{ii}/\lambda_{ii}$, are the generalized eigenvalues. Defining $w_t = Z'y_t$ and pre-multiplying (B.1) by $Q$, yields the decomposition

$$\begin{bmatrix}\Lambda_{11} & \Lambda_{12}\\ 0 & \Lambda_{22}\end{bmatrix}\begin{bmatrix}w_{1,t}\\ w_{2,t}\end{bmatrix} = \begin{bmatrix}\Omega_{11} & \Omega_{12}\\ 0 & \Omega_{22}\end{bmatrix}\begin{bmatrix}w_{1,t-1}\\ w_{2,t-1}\end{bmatrix} + \begin{bmatrix}Q_1\\ Q_2\end{bmatrix}(\Psi z_t + \Pi\eta_t) \tag{B.2}$$

The system is partitioned so that the generalized eigenvalues imply an explosive path for $w_{2,t}$ and a stable path for $w_{1,t}$. To ensure stability of the system, $w_{2,t}$ must be solved forward. Sims shows that the forward solution of (B.1) is

$$y_t = \Theta_1 y_{t-1} + \Theta_0 z_t + \Theta_y\sum_{s=1}^{\infty}\Theta_f^{s-1}\Theta_z E_t z_{t+s} \tag{B.3}$$

$$\text{where } H = Z\begin{bmatrix}\Lambda_{11}^{-1} & -\Lambda_{11}^{-1}(\Lambda_{12}-\Phi\Lambda_{22})\\ 0 & I\end{bmatrix}, \qquad \Theta_0 = H\begin{bmatrix}Q_1-\Phi Q_2\\ 0\end{bmatrix}\Psi$$

and $\Theta_y = -H_2$, $\Theta_1 = Z_1\Lambda_{11}^{-1}[\Omega_{11}(\Omega_{12}-\Theta\Omega_{22})]Z$, $\Theta_f = \Omega_{22}^{-1}\Lambda_{22}$, and $\Theta_z = \Omega_{22}^{-1}Q_2\Psi$.[12]

The most basic informational assumption that will deliver non-invertibility is $z_t = \epsilon_{i,t-q}$ for some $i$, which is non-invertible because the moving average has a zero inside the unit circle at $L = 0$. If the agents do not observe the structural shocks, $\epsilon_{it}$, (i.e., $\mathbb{U}_t = \{0\}$ for $t\in\mathbb{Z}$) then the

[12] We assume that the conditions necessary for a unique solution to exist hold. Specifically that the row space of $Q_1\Pi$ be contained in that of $Q_2\Pi$ [See Sims (2002, Section 4)].

last term in (B.3) drops out of the solution. Under this information structure, the solution (B.3) emits a stable vector-autoregression representation in current and past observables. Thus there are no hidden instabilities in the model.

However, if the agents observe the structural shocks directly (i.e., $\mathbb{U}_t = \{\epsilon_{i,t-j}\}_{j=0}^{\infty}$ for $i = 1, ..., m$), the equilibrium is given by

$$y_t = \Theta_1 y_{t-1} + \Theta_0 \epsilon_{t-q} + \Theta_y \Theta_z [\epsilon_{t-q+1} + \Theta_f \epsilon_{t-q+2} + \cdots + \Theta_f^{q-1} \epsilon_t] \tag{B.4}$$

which is the multivariate analog of (3.2). The term $\Theta_f$ is the multivariate analog to $\theta^{-1}$ in Sections 3 and 4. In order to apply Proposition 3, we must show that the equilibrium is non-invertible in current and past $y_t$. Writing the equilibrium as a moving average, $y_t = \mathcal{A}(L)\epsilon_t$, a sufficient condition for non-inveritiblity is for $\det\mathcal{A}(L)$ to have a zero inside the unit circle. This will, of course, depend upon the model itself and parameterization of the model. In Section 6 we show that this condition holds in models with "news" shocks and in an incomplete information monetary model.

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