# FISCAL LIMITS AND SOVEREIGN CREDIT SPREADS

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ABSTRACT. This paper presents a novel sovereign credit risk model aimed at extracting information from the term structure of credit spreads. At the heart of the model lies the fiscal limit, defined as the maximum outstanding debt that can credibly be covered by future primary budget surpluses. By predicting how sovereign credit default swaps (CDS) react to changes in fiscal limit expectations, our model allows to back out such expectations from market data. The empirical analysis pertains to four large advanced economies. The resulting fiscal limit estimates feature substantial time-variation. Moreover, we obtain sizeable estimates of sovereign credit risk premiums – the components of sovereign spreads that would not exist if agents were risk-neutral.

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## 1. INTRODUCTION

Before the last financial crisis, it was widely assumed that developed-countries sovereign bonds were perfectly safe, in the sense that they were believed to provide the same payoff at any point in time, and in any state of the world. This belief has, however, been seriously undermined over the past decade, and in particular since the inception of the euro-area sovereign debt crisis in the early 2010s. Because sovereign defaults have severe economic implications (e.g. Panizza, Sturzenegger, and Zettelmeyer, 2009; Reinhart and Rogoff, 2011; Mendoza and Yue, 2012), it is crucial to improve our knowledge of the economic forces leading to such events.

Unsustainable fiscal paths lead to sovereign defaults. Accordingly, sovereign bond prices depend on investors' perception of public debt sustainability. Data on sovereign bond prices

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are abundant: they are available at high frequency and for large cross-sections of maturities. For several countries, we even observe the market prices of different types of sovereign debt instruments, namely nominal and inflation-indexed bonds. The richness of these data is however underused in the literature investigating sovereign debt sustainability. Arguably, this underuse can be accounted for by the lack of a modeling framework that explicitly incorporates the debt dynamics while being flexible enough to capture the time- and cross-section variability of sovereign-bond prices. The objective of this paper is to fill this gap.

In this paper, we develop a small-scale macroeconomic model that entails the debt accumulation process and where risk-averse investors take sovereign default into account when it comes to price financial assets exposed to government credit risk, such as credit default swaps (CDS). A sovereign default takes place when the level of debt exceeds the *fiscal limit*, that is the maximum outstanding debt that can credibly be covered by future primary budget surpluses. Because agents' expectations regarding maximum future surpluses are state-dependent, the fiscal limit is subject to time variation. This limit is not directly observed. However, since the model predicts how financial prices depend on current expectations regarding the fiscal limit, it can be inverted to retrieve fiscal limit expectations from observed market data.

Our model strikes a unique balance between macroeconomic structure and ability to fit observed credit derivative prices. This fitting performance hinges on the existence of approximate credit derivative pricing formulas that, in turn, depends on econometric modelling choices. In our case, the structure of the model dictates the type of relationships between default probabilities and macroeconomic variables – the latter being typically real-valued, i.e. with positive or negative values. The usual (affine) processes used in credit-risk models based on the specification of default intensities (see e.g. Duffie and Singleton, 1999) are not consistent with this situation. Indeed, to accommodate (non-negative) default-intensities, most of these settings typically entail only non-negative pricing factors. To address this issue, we build on the recent literature on the modelling of the term structure of credit-risk-free interest rates and, more particularly, on those studies exploring the shadow-rate approach.[1] In

[1]See e.g. Christensen and Rudebusch (2013), Kim and Singleton (2012), Kim and Priebsch (2013), Krippner (2013), Wu and Xia (2016). Coroneo and Pastorello (2017) also employ the shadow-rate approach to price sovereign bonds issued by different countries. However, contrary to the present paper, sovereign default probabilities (or default intensities) are not explicitly modelled in their reduced-form framework. Their framework does therefore not allow to recover sovereign probabilities of default, and cannot preclude negative default probabilities.

the resulting econometric framework, default-intensities can positively/negatively depend on positive/negative macroeconomic variables.

Our empirical analysis focuses on four large advanced economies: the United States, the United Kingdom, Germany and Japan. The results indicate that fiscal limits of these countries feature a substantial degree of time variation and comove with macroeconomic variables. The German and British fiscal limits respectively fall in 2011–2012 and 2016, amid the European sovereign debt crisis and in the context of the Brexit vote, respectively.

The estimated models predict a non-linear influence of fiscal space on spreads, in line with the findings of numerous regression-based studies that reveal a non-linear relationship between sovereign credit spreads and debt-to-GDP ratios (see e.g. Haugh, Ollivaud, and Turner, 2009; Caggiano and Greco, 2012; Di Cesare, Grande, Manna, and Taboga, 2012; Hördahl and Tristani, 2013).

Our approach also allows us to compute sovereign risk premiums, that are those components of sovereign credit spreads that would not exist if investors were not risk averse. If agents were risk-neutral, CDS spreads would be approximately equal to expected credit losses, i.e. to the products of the loss-given default multiplied by the probability of default. However, if agents are risk-averse and if sovereign defaults tend to take place in "bad states" – i.e. states of high marginal utility states – then protection sellers are willing to enter the credit swap only if the CDS spread is larger than the expected credit loss (see e.g. Chen, Collin-Dufresne, and Goldstein, 2009; Gabaix, 2012, Subsection III.D). In our model, three channels imply that sovereign defaults are expected to be more frequent in bad states, i.e. during recessions: first, the debt-to-GDP ratio mechanically soars as GDP falls; second, the fiscal limit tends to decrease in recessions because of the estimated positive relationship between the maximum surplus and GDP growth ; third, the model accounts for the recessionary effect of the default event itself (in a manner akin to catastrophe modelling in the disaster-risk literature). Together, these three mechanisms underly the substantial size of our risk premium estimates: we find that more than half of the sovereign credit spreads correspond to risk premiums, broadly in line with findings based on reduced-form intensity approaches (see e.g. Pan and Singleton, 2008; Longstaff, Pan, Pedersen, and Singleton, 2011; Monfort and Renne, 2014). Such high premiums translate into substantial differences between risk-premium-adjusted sovereign probabilities of default and the unadjusted ones stemming from basic models like in Litterman and Iben (1991). The latter are extensively used by market practitioners, who refer to them as *market-implied default probabilities* (see e.g. Hull, Predescu, and White, 2005).

The remaining of this paper is organized as follows. Section 2 reviews related literature. The model is developed in Section 3. Section 4 presents the estimation results. Section 5 summarizes our findings and makes concluding remarks. The appendix gathers technical results; an online appendix provides additional details or proofs.

## 2. RELATED LITERATURE

This paper contributes to the growing literature on sovereign default and the pricing of sovereign default risk. Over the last decades, sovereign credit risk has been studied both from the macroeconomic and the financial points of view, but in somewhat separate ways.

2.1. **Reduced-form approaches and sovereign risk premiums.** In finance, different models entailing realistic default probabilities and default risk premiums have been proposed.[2] In most of these studies, the basic ingredient is a reduced-form default intensity (Duffie and Singleton, 1999). Duffie, Pedersen, and Singleton (2003) employ such a reduced-form approach to model the term structure of Russian credit spreads. Pan and Singleton (2008) estimate intensity-based models using sovereign CDSs. Ang and Longstaff (2013) consider multi-factor affine models allowing for both systemic and sovereign-specific credit shocks to price the term structures of U.S. states and Eurozone Member States. Longstaff, Pan, Pedersen, and Singleton (2011) estimate default intensities for 26 countries; they find that, on average, the risk premium represents about a third of credit spreads. Allowing for both credit and liquidity effects – modeled by means of credit and liquidity intensities – Monfort and Renne (2014) find substantial sovereign risk premiums in euro-area sovereign spreads.

These studies generally present a close fit of sovereign bond yields and spreads; they also provide useful estimates of sovereign risk premiums. However, they are silent about the economic forces that drive the movements of the sovereign default probabilities. Borgy, Laubach, Mésonnier, and Renne (2011) propose a sovereign credit risk model where default intensities explicitly depend on fiscal variables. They show that expectations regarding the fiscal environment can capture part of the fluctuations of the term structures of euro-area sovereign CDS

[2] "Risk premium" refers here to the part of a credit spread that would not exist if investors were risk-neutral. This premium corresponds to the excess return (beyond expected credit losses) asked by the investors to be compensated for the fact that defaults tend to take place in "bad states" of the world, i.e. in states of high marginal utility (see e.g. Chen, Collin-Dufresne, and Goldstein, 2009; Gabaix, 2012, Subsection III.D). Such risk premiums are often seen as explanations to the so-called credit spread puzzle (D'Amato and Remolona, 2003).

spreads. However, in their framework, their is no structural model motivating the (linear) relationship between the default intensity and fiscal variables. Furthermore, their estimation approach does not preclude negative spreads and the estimation period includes only the first part of the Euro-area sovereign debt crisis. Augustin and Tedongap (2016) provide a more structural approach and value Eurozone CDSs from the perspective of an Epstein-Zin agent. Nevertheless, they posit a reduced-form function connecting the sovereign's default probability to expected consumption growth and macro volatility. Their model therefore remains silent about the influence of the fiscal context on sovereign credit risk.

To the best of our knowledge, all studies providing time-varying estimates of sovereign credit risk premiums rely on (reduced-form) intensity-based approach. The present study is the first to provide time-varying semi-structural estimates of sovereign credit risk premiums.

2.2. **Theory of sovereign defaults and fiscal limits.** Early studies on sovereign credit risk focus on the strategic aspect of such defaults. Following the seminal contribution of Eaton and Gersovitz (1981), different recent papers (Aguiar and Gopinath, 2006; Arellano, 2008; Arellano and Ramanarayanan, 2012; Mendoza and Yue, 2012) model sovereign defaults as a strategic decision of a government balancing the gains from stopping repaying debt against the costs of exclusion from international credit markets and (exogenous) output losses. A prediction of these models is that the probability of default increases in debt level. In most instances, these models are solved in the context of risk-neutral investors, ruling out the existence of risk premiums in credit spreads.[3] This is in contradiction to numerous empirical work pointing to the existence of sizeable risk premium components in sovereign spreads (see Subsection 2.1). In this literature, models are highly stylized and are therefore expected to make qualitative predictions only.

Recently, another line of work, that we will refer to as the *fiscal limit* literature, has emerged. This literature relates to Bohn (1998), who provides evidence of fiscal corrective action. More precisely, Bohn (1998) finds that the U.S. primary surplus is an increasing function of the debt-to-GDP ratio (see Mendoza and Ostry, 2008; Ghosh, Ostry, and Qureshi, 2013, for more recent evidence). If the government is committed to raising fiscal surplus in response to rising debt levels, then this government can guarantee intertemporal solvency as long as (i) tax rates are below the revenue-maximizing level and (ii) tax rates can be freely adjusted. Papers

[3]An exception is Verdelhan and Borri (2010) who consider risk-sensitive lenders buying emerging-market sovereign bonds.

of the fiscal limit literature depart from that intertemporal solvency situation by assuming that the government is not – or cannot be – committed to such a policy. In Bi (2012); Leeper (2013); Bi and Leeper (2013); Bi and Traum (2012, 2014); Juessen, Linnemann, and Schabert (2016), the fiscal limit corresponds to the discounted present value of future maximum primary surpluses. These maximum surpluses can be seen as the peak points of the Laffer curve (Trabandt and Uhlig, 2011). Ghosh, Kim, Mendoza, Ostry, and Qureshi (2013) estimate the responses of primary surpluses to debt levels for 23 advanced economies and observe that the responses are weaker at higher levels of debt – a phenomenon the authors dub "fiscal fatigue". After having introduced their estimated parametric reaction function in a model of debt accumulation, Ghosh et al. (2013) show that there is a point – akin to the fiscal limit– where the primary balance cannot realistically keep pace with the rising interest burden as debt increases. Beyond this point, debt dynamics becomes explosive and the government becomes unable to fully meet its obligations.

For tractability, the models used in the fiscal limit literature generally assume that the government issues one-period bonds only, which may alter the assessment of sovereign credit risk (Arellano and Ramanarayanan, 2012). An exception is the study by Chernov, Schmid, and Schneider (2016), where the government issues both short- and long-term bonds. In Chernov et al. (2016)'s model, increases in tax rates have a negative effect on output, which also implies that there is a point where taxes cannot be raised further without reducing future tax revenues, in the spirit of the Laffer curve. In this context, the default probability tends to be higher in recessions, translating into large sovereign risk premiums.[4]

Solving these fiscal-limit models is challenging as soon as several shocks and state variables are considered.[5] Typically, in most of these models, the risk-free rates are considered to be constant and the term-structure of credit spreads is not discussed. Because of the challenging solution procedures, the models are essentially calibrated – i.e. the parameters are not estimated econometrically – and the ability of the model to capture the time variation of the data is not examined.

## 3. Model

3.1. **Overview.** We consider a closed economy where a representative risk-averse investor prices credit derivatives, that are financial instruments whose payoffs are exposed to the default event of the government. We denote by $x_t$, a $n$-dimensional vector describing the state

[4]See Footnote 2 for the precise meaning of "risk premium".

[5]See notably Appendix A of Chernov, Schmid, and Schneider (2016).

of the economy on date $t$ and by $\mathcal{D}_t$ the government default status on date $t$, with $\mathcal{D}_t = 1$ if the government has defaulted at or before $t$ and $\mathcal{D}_t = 0$ otherwise. We assume that $\mathcal{D}_t$ does not Granger-cause $x_t$. On date $t$, the information available to the investor is $\mathcal{I}_t = \{X_t, X_{t-1}, \dots\}$, with $X_t = [x_t', \mathcal{D}_t]'$. In what follows, we denote by $\mathbb{E}_t$ the expectation conditional on $\mathcal{I}_t$.

Let us first highlight the modelling of two (related) key ingredients of the present framework: the stochastic default probability of the government (Subsection 3.2) and the fiscal limit (Subsection 3.3). Next, before turning to the specification of the different components of $x_t$ (Subsections 3.5 to 3.10), Subsection 3.4 introduces Credit Default Swaps (CDSs), whose pricing constitutes an instrumental aspect of our estimation approach.

3.2. **Sovereign default probability.** The probability of observing a sovereign default on date $t$ is of the form:

$$\mathbb{P}(\mathcal{D}_{t+1} = 1 | \mathcal{D}_t = 0, \mathcal{I}_t, x_{t+1}) = \mathcal{F}(d_{t+1} - \ell_{t+1}), \tag{1}$$

where $\mathcal{F}$ is a function valued in $[0,1]$, $d_t$ is the logarithm of the debt-to-GDP ratio and $\ell_t$ is the logarithm of the *fiscal-limit*-to-GDP. The distance $\ell_t - d_t$ can be interpreted as a measure of "fiscal space". The dynamics of $d_t$ and $\ell_t$, and therefore of the fiscal space, is developed in the following subsections.

Function $\mathcal{F}$ is assumed to be such that $\mathcal{F}(x) = 0$ for $x \leq 0$. Eq. (1) then implies that the default probability is equal to zero as long as the debt-to-GDP ratio is lower than the fiscal limit, i.e. when $d_t \leq \ell_t$. Moreover, function $\mathcal{F}$ is increasing. Hence, the larger the distance between the debt-to-GDP ratio and the fiscal limit, the higher the probability of default. In the following, we employ the following specification for $\mathcal{F}$:

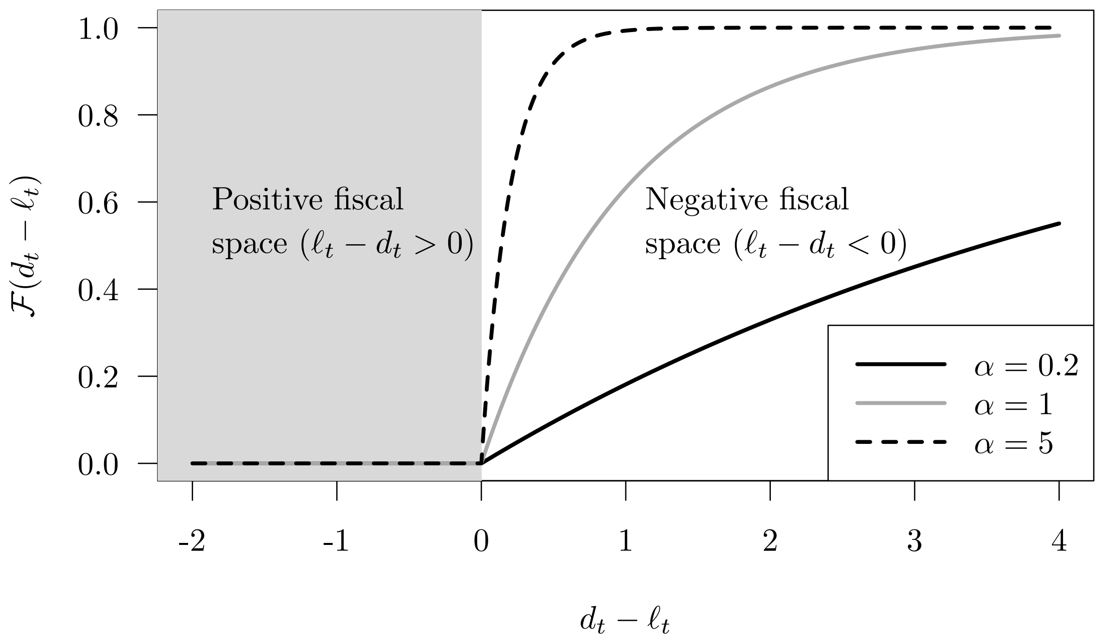
$$\mathcal{F}(d_t - \ell_t) = 1 - \exp(-\underbrace{\max[0, \alpha(d_t - \ell_t)]}_{=\underline{\lambda}_t}), \tag{2}$$

with $\alpha > 0$. Hereinafter, we refer to $\underline{\lambda}_t \equiv \alpha \max(0, d_t - \ell_t)$ as the default intensity. According to Eqs. (1) and (2), when it is small, the default intensity $\underline{\lambda}_t$ is close to the conditional probability of default $\mathbb{P}(\mathcal{D}_{t+1} = 1 | \mathcal{D}_t = 0, \mathcal{I}_t, x_{t+1})$.

Figure 1 displays the shapes of this function for different values of $\alpha$. If $\alpha$ is large, the fiscal limit is strict, in the sense that default is likely as soon as $d_t > \ell_t$. By contrast, if $\alpha$ is moderate, the fiscal limit is softer, in the sense that when $d_t > \ell_t$, a sovereign default is likely, but not certain.

3.3. **Fiscal limit.** On each date $t$, there exists a maximum primary budget surplus $s_t^* Y_t P_t$, where $Y_t$ is the real GDP and $P_t$ is the GDP deflator. In other words, $s_t^*$ is the maximum

FIGURE 1. Probabilities of default with respect to fiscal space ($\ell_t - d_t$)




This figure shows the parametric function used to account for the sensitivity of the default probability to the fiscal space ($\ell_t - d_t$), see Eq. (1). The higher $\alpha$, the more likely the occurrence of default when fiscal space is negative (i.e. when $d_t - \ell_t > 0$).

primary budget surplus expressed as a fraction of GDP. This maximal surplus can be interpreted as the peak point of the Laffer curve. In the spirit of Bi (2012), the fiscal limit can then be defined as the sum of present values of maximum future budget surplus, i.e.:

$$\begin{aligned}\exp(\ell_t) &= \frac{1}{Y_t P_t}\mathbb{E}_t\left(\sum_{h=1}^{+\infty} M^n_{t,t+h} s^*_{t+h} Y_{t+h} P_{t+h} | \mathcal{D} \equiv 0\right)\\ &= \mathbb{E}_t\left(\sum_{h=1}^{+\infty} M_{t,t+h} s^*_{t+h} \exp(\Delta y_{t+1} + \cdots + \Delta y_{t+h}) | \mathcal{D} \equiv 0\right),\end{aligned}$$

or

$$\ell_t = \log\left(\sum_{h=1}^{+\infty} \mathbb{E}_t\left[s^*_{t+h} M_{t,t+h} \exp(\Delta y_{t+1} + \cdots + \Delta y_{t+h}) | \mathcal{D} \equiv 0\right]\right), \qquad (3)$$

where $\Delta y_t$ and $\pi_t$ are the (log) growth rates of the real GDP and of the deflator, respectively, and where $M_{t,t+h}$ (respectively $M^n_{t,t+h}$) denotes the real (resp. nominal) stochastic discount factor, or s.d.f., between dates $t$ and $t+h$. (Note that we have $M^n_{t,t+k} = M_{t,t+h}\exp(-\pi_{t+1} - \cdots - \pi_{t+k})$.)

To get insights into Eq. (3), consider the deterministic case where $M_{t,t+1} = \exp(-r_t)$, $r_t$ being the real risk-free rate. Assume further that $\Delta y_{t+1} - r_t$ is constant and equal to $c$ (say). In this case, if $c > 0$ and if the maximal surplus rate ($s^*_{t+h}$) does not converge to zero, then $\ell_t$ is infinite. Indeed, we then have that real GDP growth is larger than the risk-free short term rate. As a result, even if future maximum surpluses are small fractions of future GDPs, the sum of these expected surpluses is large enough to cover any level of current debt.

Eq. (3) shows that the computation of the fiscal limit is deduced from that of conditional expectations of affine transformations of $[\Delta c_t, \Delta y_t, \pi_t, s^*_t]'$. In our framework, conditional on $\mathcal{D}_t \equiv 0$, these four variables will linearly depend on an affine multivariate process (Eqs. 5 and 7), implying the existence of closed-form formulas for these conditional expectations (see Appendix D).

3.4. **Credit Default Swaps.** A Credit Default Swap (CDS) is an agreement between a protection buyer and a protection seller, whereby the buyer pays a periodic fee in return for a contingent payment by the seller upon a credit event – such as bankruptcy or failure to pay – of a reference entity. The contingent payment usually replicates the loss incurred by a creditor of the reference entity in the event of its default (see e.g. Duffie, 1999).

More specifically, a CDS works as follows: the protection buyer pays a regular premium to the so-called protection seller. These payments end either after a given period of time (the maturity of the CDS), or at default of the reference entity. In the case of default of this debtor, the protection seller compensates the protection buyer for the loss the latter would incur upon default of the reference entity (assuming that the latter effectively holds a bond issued by the reference entity). By definition, the CDS spread is the regular payment paid by the protection buyer (expressed in percentage of the notional). Let us denote by $S^{cds}_{t,h}$ the maturity-$h$ CDS spread and by $RR$ the recovery rate.

At inception of the CDS contract (date $t$), there is no cash-flow exchanged between both parties and the CDS spread $S^{cds}_{t,h}$ is determined so as to equalize the present discounted values of the payments promised by each of them. Assuming the reference entity has not defaulted before date $t$, such that $\mathcal{D}_t = 0$, we have:

$$\underbrace{(1-RR)\mathbb{E}_t\left\{\sum_{k=1}^{h} M^n_{t,t+k}(\mathcal{D}_{t+k} - \mathcal{D}_{t+k-1})\right\}}_{\text{Protection leg}} = \underbrace{S^{cds}_{t,h}\mathbb{E}_t\left\{\sum_{k=1}^{h} M^n_{t,t+k}(1-\mathcal{D}_{t+k})\right\}}_{\text{Premium leg}}. \tag{4}$$

The CDS spread $S^{cds}_{t,h}$ is easily deduced from Eq. (4) once the following conditional expectations are known for $k \in \{1, \dots, h\}$: $\mathbb{E}_t[M^n_{t,t+k}(1-\mathcal{D}_{t+k})]$ and $\mathbb{E}_t[M^n_{t,t+k}(1-\mathcal{D}_{t+k-1})]$. The former (respectively latter) expectation corresponds to the date-$t$ price of a zero-coupon bond that provides a unit payoff on date $t+k$ if the reference entity has not defaulted before date $t+k$ (respectively before date $t+k-1$), and zero otherwise.

Appendix A shows that these two conditional expectations can be rewritten as expectations of (exponential) linear combinations of future values of $[x_t', \underline{\lambda}_t]'$, with $\underline{\lambda}_t = \max(0, \alpha(d_t - \ell_t))$. The model developed below, in Subsections 3.5 to 3.9, is such that $d_t$ and $\ell_t$ are affine combinations of $x_t$. Therefore, in our framework, pricing CDSs amounts to computing conditional expectations of (exponential) linear combinations of $[x_t', \max(0, \lambda_t)]'$, where $\lambda_t$ is itself affine in $x_t$. This problem is reminiscent of that arising in the context of shadow-rate models *à la* Black (1995).[6] Shadow-rate models have attracted a lot of interest over the last decade. The reason is that these models accommodate the existence of a lower bound for nominal interest rates, a welcome feature in a context of extremely low yields. Though simple and intuitive, this framework does not offer closed-form bond pricing formulas because of the non-linearity stemming from the max operator. Different approaches have however been proposed to approximate bond prices in shadow-rate models. Wu and Xia (2016) have notably proposed a particularly simple and accurate approximation. An adaptation of this approach to the present context is detailed in Online Appendix I.2.[7]

3.5. **Exogenous macroeconomic block.** Consumption growth, output growth, inflation and the maximum budget surplus jointly depend on (a) a $n_w$-dimensional vector of persistent latent variables $w_t$, (b) the government default and (c) a $n_\eta$-dimensional vector of volatile shocks $\eta_t$. That is:

$$[\Delta c_t, \Delta y_t, \pi_t, s^*_t]' = \mu + \Lambda w_t - b(\mathcal{D}_t - \mathcal{D}_{t-1}) + \Sigma_\eta \eta_t, \tag{5}$$

where $\eta_t \sim i.i.d.\,\mathcal{N}(0, Id)$. Denoting by $\mu_c$, $\mu_y$, $\mu_\pi$, $\mu_{s^*}$ the components of $\mu$, by $\sigma_c'$, $\sigma_y'$, $\sigma_\pi'$, $\sigma_{s^*}'$ the line vectors of $\Sigma_\eta$, by $\Lambda_c$, $\Lambda_y$, $\Lambda_\pi$, $\Lambda_{s^*}$ the line vectors of $\Lambda'$ and by $b_c$, $b_y$, $b_\pi$, $b_{s^*}$ the

[6] In shadow-rate models, credit-risk-free bond prices are given by $\mathbb{E}(\exp(-i_t - \dots - i_{t+h-1}))$, where $i_t = \max(0, s_t)$, $s_t$ ($\in \mathbb{R}$) being the shadow rate and $i_t$ ($\in \mathbb{R}^+$) is the effective short-term rate.

[7] Our approach shares some similarities with the Black-Scholes-Merton model (Black and Scholes, 1973; Merton, 1974) (and its numerous extensions) in that it also features a default threshold. As noted by Duffie and Singleton (2003, Subsection 3.2.2), the tractability of the Black-Scholes-Merton model rapidly declines as one allows for a time-varying default threshold. By contrast, although our framework features a a time-varying debt threshold, tractability is preserved.

components of $b$, we have, for instance:

$$\Delta c_t \quad = \quad \mu_c + \Lambda_c' w_t - b_c(\mathcal{D}_t - \mathcal{D}_{t-1}) + \sigma_c' \eta_t. \tag{6}$$

The components of $b$ reflect the influence of a government default on the macroeconomy. Eq. (6) is inspired by studies concerned with the asset-pricing influence of disasters (e.g. Barro, 2006, Eq. 7, Arellano, 2008, Eq. 3, Barro and Jin, 2011, Eq. 1, Gabaix, 2012, Eq. 1, Arellano and Ramanarayanan, 2012, last equation of Section III, Wachter, 2013, Eq. 1).

Moreover, we posit an exogenous Gaussian vector auto-regressive (VAR) process for $w_t$.[8] Specifically:

$$w_t = \Phi w_{t-1} + \varepsilon_t, \tag{7}$$

where $\varepsilon_t \sim i.i.d. \mathcal{N}(0, Id)$. The exogenous vectors $w_t$ and $\eta_t$ are components of the state vector $x_t$ (see Subsection 3.1).

3.6. **Investors' preferences and s.d.f.** The investor's utility is based on a CRRA function and, accordingly, the stochastic discount factor between dates $t$ and $t+1$ is given by:

$$M_{t,t+1} = \delta \exp(-\gamma \Delta c_{t+1}), \tag{8}$$

where $\gamma \geq 0$ is the coefficient of relative risk aversion and $\delta \geq 0$ is the rate of time preference.

Because consumption growth is affected by changes in $\mathcal{D}_t$ (see Eq. 6), the s.d.f. jumps upon sovereign default. This has important implications in terms of pricing, by giving rise to specific risk premiums – called credit-event premiums – in the prices of financial instruments, such as CDSs, whose payoffs depend on the government default status (Driessen, 2005; Gouriéroux, Monfort, and Renne, 2014; Bai, Collin-Dufresne, Goldstein, and Helwege, 2015).[9]

[8]Because it follows a Gaussian VAR model, $w_t$ is an affine process. This implies in particular that the conditional expectations of exponential affine transformations of future values of $w_t$ can be computed in closed-form (see Eq. a.5 in Appendix B.1). This property will be largely exploited in the present context.

[9]These risk premiums notably help to fit short-term credit spreads by allowing the "risk-neutral" default intensity of the considered entity to deviate from its historical default intensity. Specifically, Online Appendix V (Eq. V.4) shows that the relationship between the physical ($\mathbb{P}$) and risk-neutral ($\mathbb{Q}$) default intensities is:

$$\underline{\lambda}_t^{\mathbb{Q}} = \underline{\lambda}_t + \log(\exp(\gamma b_c)\{1 - \exp(-\underline{\lambda}_t)\} + \exp(-\underline{\lambda}_t)),$$

where $\underline{\lambda}_t^{\mathbb{Q}}$ is defined through $\exp(-\underline{\lambda}_t^{\mathbb{Q}}) \equiv \mathbb{Q}(\mathcal{D}_t = 0 | \mathcal{D}_t = 0, x_t, \mathcal{I}_{t-1})$ and where $\mathbb{Q}$ is a measure equivalent to the physical one ($\mathbb{P}$) defined through the Radon-Nikodym derivatives $d\mathbb{Q}/d\mathbb{P}|_{t,t+1} = M_{t,t+1}/\mathbb{E}_t(M_{t,t+1})$ (Subsection 4.4 elaborates further on the risk-neutral measure). In particular, if $\gamma b_c > 0$, then $\underline{\lambda}_t^{\mathbb{Q}} > \underline{\lambda}_t$. Moreover, in this context, the price of a sovereign maturity-$h$ zero-recovery-rate bond is not given by the standard formula $\mathbb{E}^{\mathbb{Q}}(\exp(-r_t - \underline{\lambda}_{t+1}^{\mathbb{Q}} - \cdots - r_{t+h-1} - \underline{\lambda}_{t+h}^{\mathbb{Q}}))$ because the "no-jump" condition is not verified for this bond (the

3.7. **Government debt issuances.** Following, among others Hatchondo and Martinez (2009), Arellano and Ramanarayanan (2012) or Bhattarai, Eggertsson, and Gafarov (2015), we adopt the simplifying assumption that the government issues perpetuity contracts with nominal coupon payments that decay geometrically at rate $\chi$. The closer $\chi$ to one, the larger the duration of the debt instrument.[10,11]

In the present context, perpetuities feature credit risk. Denoting by $RR$ the recovery rate, an investor having purchased the perpetuity on date $t$ receives the following nominal amount on date $t+h$:

$$\chi^{h-1}[1 \times (1-\mathcal{D}_{t+h}) + RR \times \mathcal{D}_{t+h}].$$

The price of the perpetuity therefore is:

$$\mathcal{P}_t = \sum_{h=1}^{\infty} \chi^{h-1} B_{t,h}, \tag{9}$$

where the $B_{t,h}$'s are prices of binary zero-coupon providing the payoffs $(1-\mathcal{D}_{t+h}) + RR\mathcal{D}_{t+h}$ on date $t+h$. Appendix B.1 shows that, under the convenient assumption that $RR = \exp(-\gamma b_c - b_\pi)$, we have:

$$B_{t,h} = \exp(B_h + A_h' w_t), \tag{10}$$

where the $B_h$'s and the $A_h$'s derive from simple Riccati's equations (see Eqs. a.4 and a.5 in Appendix B.1). Though *ad hoc*, the recovery-rate assumption brings essential simplification in our model. Importantly, it is satisfied for reasonable parameter values (see Subsection 4.2).

The perpetuity's yield-to-maturity will be needed to derive the debt accumulation process (next subsection). By definition, the latter is the rate $q_t$ satisfying:

$$\mathcal{P}_t = \sum_{h=1}^{\infty} \frac{\chi^{h-1}}{(1+q_t)^h} = \frac{1}{1+q_t-\chi}. \tag{11}$$

Together, Eqs. (9) and (11) determine $q_t$. Thanks to Eq. (10), the solution for $q_t$ is explicit. It is not affine in $x_t$; but let us explain why it is close to affine. As shown by Eq. (9) the perpetuity

previous expectation jumps on the default date; see Duffie, Schroder, and Skiadas, 1996, Kusuoka, 1999, Collin-Dufresne, Goldstein, and Hugonnier, 2004).

[10] As noted by Hatchondo and Martinez (2009), this coupon structure can be interpreted as if the debt issued by the government consisted of a portfolio of zero-coupon bonds of different maturities, where the portfolio weights decline geometrically with maturity. It has the advantage of making it possible to synthesize the repayment schedule, on any date, by means of a single number (the sum of future repayments, for instance). When this is not the case, all past issuances have to enter the state vector, which substantially complicates the model solution.

[11] The modified duration of such an instrument has the specificity to be equal to its price. Indeed, we have $\mathcal{P} = 1/(1+q-\chi)$, where $q$ is the perpetuity's yield-to-maturity (Eq. 11), which implies $(\partial\mathcal{P})/\partial q)\mathcal{P} = -\mathcal{P}$.

can be seen as a weighted sum of zero-coupon bonds whose yields-to-maturity are affine in $w_t$: denoting by $r_{t,h}$ the continuously-compounded yield-to-maturity associated with the zero-coupon of price $B_{t,h}$, it comes from Eq. (10) that $r_{t,h} = -\frac{1}{h}(B_h + A_h' w_t)$. We therefore expect $q_t$ to be particularly close to one of these $r_{t,h}$'s and, therefore, to be approximately affine in $w_t$. Online Appendix IV.2 details how we proceed to select a maturity $h^*$ satisfying:

$$q_t \approx -\frac{1}{h^*}(B_{h^*} + A_{h^*}' w_t). \tag{12}$$

3.8. **Debt dynamics.** At this stage, we have defined the dynamics of consumption, output, inflation and of the maximum budget surplus (through Eqs. 5 and 7). The law of motion of the fiscal limit $\ell_t$ (Eq. 3) further results from the previous equations and from the s.d.f. specification (Eq. 8). To close the model, we need to specify the debt accumulation process ($d_t$).

We consider a simplified (or approximated) debt dynamics for the approximate CDS pricing formulas presented in Subsection 3.4 – and detailed in Appendix A – requires $\ell_t - d_t$ to linearly depend on the state vector $x_t$, and the latter to follow a Gaussian VAR process. To achieve this, several simplifications have to be resorted to.

One of these simplifications pertains to the type of instruments issued by the government. When the government issues the perpetual bonds presented in the previous subsection, Appendix B shows that the apparent interest rate approximately takes the form of a weighted sum of the past perpetuity's yields-to-maturity, with weights decaying geometrically at rate $\chi$. That is:

$$\frac{R_{t+1}}{D_t} \approx (1-\chi)q_t + \chi \frac{R_t}{D_{t-1}}, \tag{13}$$

where $R_t$ denotes the date-$t$ nominal debt service and where $D_t$ denotes the face value of the debt on date $t$. Consistently with international debt accounting standards, the concept of debt valuation we opt for is that of "nominal valuation of debt securities", where the debt outstanding reflects the sum of funds originally advanced, plus any subsequent advances, less any repayments, plus any accrued interest.[12,13]

[12] See International Monetary Fund, Bank for International Settlements and European Central Bank (2015). Although such a precision is innocuous in the context of models considering only short-term issuances, it is not in the present context, where the government issues long-dated debt instruments. This definition is used in Eq. (a.6) of Appendix B.2.

[13] Regarding the recording of accrued interest, we follow international statistical standards and apply the so-called "debtor approach" (see Handbook of Securities Statistics (2015) document, 5.58 p.40). See Appendix B.2.

Let us introduce the following notations:

$$rd_t = \frac{R_t}{D_{t-1}} - \overline{q} \qquad sd_t = \frac{S_t}{D_{t-1}} - \overline{sd}, \tag{14}$$

where $\overline{q}$ and $\overline{sd}$ are the respective unconditional means of the perpetuity yield and of the surplus-to-debt ratio ($S_t/D_{t-1}$), where $S_t$ denotes the nominal values of the date-$t$ primary deficit, whose dynamics is discussed in the next subsection. With this notation, Eq. (13) rewrites:

$$rd_{t+1} \approx (1-\chi)(q_t - \bar{q}) + \chi rd_t. \tag{15}$$

If $rd_t - sd_t$ is small, and in the absence of default, Appendix B.2 shows that we get the following approximated law of motion for $d_t$:[14]

$$d_t \approx d_{t-1} - \Delta y_t - \pi_t + \log\left(1 + \overline{q} - \overline{sd}\right) + \frac{1}{1+\overline{q}-\overline{sd}}(rd_t - sd_t). \tag{16}$$

The previous equation is notably consistent with the fact that an increase in nominal GDP growth – coming either from higher real growth $\Delta y_t$ or from higher inflation $\pi_t$ – results in a decrease in the debt-to-GDP ratio.

3.9. **Primary deficit.** We assume that the deviation between the surplus-to-debt ratio and its unconditional mean (denoted by $sd_t$) follows:

$$sd_t = \rho sd_{t-1} + \gamma_y(\Delta y_{t-1} - \mu_y) + \gamma_d(d_{t-1} - \bar{d}) + \sigma_s \varepsilon_{s,t}, \tag{17}$$

where $\bar{d}$ denotes the unconditional mean of $d_t$. The term $\gamma_y(\Delta y_{t-1} - \mu_y)$ is aimed at capturing the cyclicality of primary deficit, which may stem from the partial indexation of fiscal policy to economic activity (see e.g. Blanchard and Perotti, 2002; Gali and Perotti, 2003). Moreover, with $\gamma_d > 0$, this specification allows for a reaction of the primary surplus to the debt level, consistently with the empirical evidence that governments typically tend to reduce the primary deficit in response to rising public debt (Bohn, 1998; Mendoza and Ostry, 2008).

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[14]This equation is internally inconsistent on the default date: it indeed implies that $d_t$ is contemporaneously affected by a sovereign default ($\mathcal{D}_t - \mathcal{D}_{t-1} = 1$) through the nominal growth $\Delta y_t + \pi_t$ (see Eq. 5); meanwhile, from Eq. (1), the probability of occurence of $\mathcal{D}_t - \mathcal{D}_{t-1} = 1$ is itself conditional on $d_t$. This inconsistency is solved by reckoning that Eq. (16) holds as long as $\mathcal{D}_t - \mathcal{D}_{t-1} = 0$. When a default happens, i.e. when $\mathcal{D}_t - \mathcal{D}_{t-1} = 1$, we assume for convenience that – as regards the $d_t$'s dynamics – the nominal growth $\Delta y_t + \pi_t = (\mu_y + \mu_\pi) + (\Lambda_y + \Lambda_\pi)'w_t + (\sigma_y + \sigma_\pi)'\eta_t - (b_y + b_\pi)(\mathcal{D}_t - \mathcal{D}_{t-1})$ is replaced by the same expression without the dependency in $\mathcal{D}_t - \mathcal{D}_{t-1}$. That amounts to saying that the measure of debt-to-GDP ratio that affects the sovereign default probability (that is the "$d_t$" appearing in Eq. 1) is based on a counterfactual GDP that does not incorporate sovereign-default feedback effects.

3.10. **The dynamics of the full state vector.** Let us denote by $x_t$ the extended state vector $x_t = [w_t', sd_t, \eta_t', d_t, rd_t, q_t]'$. The model described above is such that the dynamics of $x_t$ writes

$$x_t \approx \mu_x + \Phi_x x_{t-1} + \Sigma_x \left[\varepsilon_t', \varepsilon_{s,t}, \eta_t'\right]', \tag{18}$$

where matrices $\mu_x$, $\Phi_x$ and $\Sigma_x$ are deduced from Eqs. (7), (12), (15), (16) and (17). These matrices are detailed in Online Appendix VII.

## 4. Estimation and results

Bringing the model to the data amounts to determining two types of objects: model parameters and latent variables, i.e. the components of $w_t$. To lower the number of free parameters, some of them are calibrated (Subsection 4.2). The estimation of the remaining parameters and of the latent variables is based on maximum-likelihood techniques (Subsection 4.3). The next subsection provides a brief data description, additional details are contained in Appendix E.

4.1. **Data.** We consider four countries: the United States of America, the United Kingdom, Germany and Japan. The data are quarterly and span the period from 2007$Q$1 to 2018$Q$3 for the United Kingdom, Japan and Germany and to 2018$Q$4 for the U.S..

CDS spreads and bond yields are extracted from Thomson Reuters Datastream. For the U.S., the original source of the yields is the Federal Reserve. For the other three countries, we take zero-coupon bond yields bootstrapped by Thomson Reuters Datastream from government bond prices.

For the U.S., macroeconomic variables are drawn from the FRED database (Federal Reserve of St. Louis) and from the Bureau of Economic Analysis.[15] For the U.K., GDP, consumption and gross government debt are collected from the British Office for National Statistics. The series for gross and net government interest payments and primary surplus/deficit are gathered from the OECD Economic Outlook; the same holds true for Japan. For the latter country, GDP and consumption variables are acquired from the Cabinet Office database (Government of Japan). Gross government debt for Japan is drawn from the Bank of Japan. As regards German data, real and nominal GDP series are extracted from the German Federal Statistical Office database. Consumption series, gross government debt and primary surplus/deficit are collected from the Eurostat ESA2010 database. Lastly, the series of German government interest payments is acquired from the ECB Statistical Data Warehouse.

[15]Consumption is the sum of services and non-durables consumptions.

As shown by Kim and Orphanides (2012), introducing survey-based information among the observed variables helps to estimate term-structure models with latent factors, especially when the latter are persistent. In the present case, the set of measurement equations is augmented with equations stating that, up to measurement errors, model-implied macroeconomic forecasts are equal to (observed) professional forecasters' predictions. This approach is informative for the estimation of both the model parameters and the latent variables. For all countries, we make use of forecasts of debt-to-GDP ratios. For the U.S., the forecasts come from the Budget and Economic Outlook of the Congressional Budget Office. For the other three countries, we heavily rely on the IMF World Economic Outlooks (WEOs). Because debt-to-GDP forecasts were not available in WEOs between 2007 and 2009, missing observations have been replaced by economic forecast data produced by the European Commission for Japan, and by the European Commission Stability and Convergence Programme data for the U.K. and Germany.[16]

4.2. **Calibrated and constrained parameters.** Calibrated parameters are reported in the upper part of Table 1. We set the annual rate of time preference to 0.99 and the coefficient of relative risk aversion to 4 – a value used for instance by Barro (2006) and Gabaix (2012). We assume that a sovereign default results in a consumption drop of 20% (i.e. $b_c = 0.2$ in Eq. 6), which broadly corresponds to the average disaster magnitude documented by e.g. Barro and Ursua (2011). Though larger than the average recessionary effect associated with sovereign defaults documented by Mendoza and Yue (2012) (5%), or by Reinhart and Rogoff (2011) (7%), an output fall of 20% is commensurate to estimated losses resulting from a sovereign default combined with a banking crisis (De Paoli, Hoggarth, and Saporta, 2006) or from "hard defaults" – defined by Trebesch and Zabel (2017) as those defaults resulting in large losses for investors. As regards the inflationary effect of a default, we use $b_\pi = -2.1\%$, which is the average inflationary effect of a disaster used by Gabaix (2012).[17]

As explained in Subsection 3.7, the tractability of the model notably hinges on the assumption according to which $RR = \exp(-\gamma b_c - b_\pi)$. Given the calibrated values of $\gamma$, $b_c$ and $b_\pi$, this gives $RR = 45\%$, which turns out to be a reasonable value: According to sovereign defaults data collected by Moody's (2019, Exhibit 20), the value-weighted (respectively issuer-weighted) average recovery rate is of 41% (resp. 55%) over the last 35 years.

[16]The model frequency is quarterly but most of our forecast data are published twice a year (see Tables 2 to 5). In order to convert the raw forecasts into quarterly figures, we rely on cubic spline interpolations.

[17]In our framework, $b_\pi$ is only used to calibrate the recovery rate, equal to $\exp(-\gamma b_c - b_\pi)$ (see Subsection 3.7).

The decay rate of the perpetuity's coupons $\chi$ is set to 0.90. Implementing the approach mentioned at the end of Subsection 3.7, and detailed in Online Appendix IV.2, this choice leads to perpetuitys' average durations ($h^*$) varying between 4 and 6 years across countries (ninth line of Table 1). These values are in line with typical average maturities of sovereign issuances.

Parameters $\rho$, $\gamma_y$ and $\sigma_c$, which define the dynamics of $sd_t$ (Eq. 17), come from preliminary OLS regressions of $sd_t$ on its first lag and on $\Delta y_t$. Introducing $d_t$ in the regression provides non-statistically-significant estimates of $\gamma_d$. However, when $\gamma_d$ is set to zero, the optimization of the likelihood function – that is the next estimation step (Subsection 4.3) – becomes numerically unstable. Indeed debt then turns out to be explosive for many sets of parameters considered by the numerical routines used to optimize the likelihood function. To address this issue, we set $\gamma_d/(1-\rho)$ – that is the long-run impact of a change in debt on $sd_t$ – equal to an arbitrary low value of 0.001. This approach, which appears to have only mild effects on the results, suffices to solve the numerical problem.

For parsimony, inflation is assumed not to depend on $w_t$ (i.e. $\Lambda_\pi = 0$ in Eq. 5). Moreover, the unconditional variances of inflation, consumption and GDP growth rates are constrained to equate their respective sample values.[18] For all countries but Japan, $\mu_\pi$ is set to the sample mean of inflation; for Japan, where the sample mean of inflation is negative, we take $\mu_\pi = 1\%/4$. We assume that $\mu_y = \mu_c$ and impose that the components of $\Lambda_y$ and $\lambda_c$ are proportional to the sample standard deviations of $\Delta y_t$ and $\Delta c_t$. Although $\mu_c$ is included among the parameters to be estimated by Maximum Likelihood, we constrain the estimate to be larger than 1% per annum. We also impose an upper bound, of 2%, on the unconditional standard deviation of the maximum budget surplus and a maximum value of 0.995 for the diagonal entries of $\Phi$, the auto-regressive matrix of $w_t$ (Eq. 7); the other entries of $\Phi$ are set to zero.

Finally, we posit that the unconditional mean of $d_t$, denoted by $\bar{d}$ (Eq. 17), is equal to its sample mean. An internal consistency constraint weighs on $\bar{d}$ and $\overline{sd}$, the latter being the unconditional average of $sd_t$ (defined through Eq. 14). Indeed, as can be seen from the expanded expression of $\mu_x$ (Online Appendix VII), this vector depends on both $\bar{d}$ and $\overline{sd}$. As a result, for an arbitrary pair $(\bar{d}, \overline{sd})$, the third to last component of $\mathbb{E}(x_t) = (I - \Phi_x)\mu_x$ does

[18] The model-implied unconditional variances are actually those of the components $\Delta c_t$, $\Delta y_t$ and $\pi_t$ that are accounted for by $w_t$ and $\varepsilon_t$, i.e. excluding potential effects of $\mathcal{D}_t$ (given no default of the considered countries has been observed over the estimation period).

not coincide with $\bar{d}$. In other words, $\bar{d}$ and $\overline{sd}$ cannot be set independently. We address this issue by numerically determining $\overline{sd}$ for a given value of $\bar{d}$.[19]

4.3. **Maximum Likelihood estimation strategy.** For each country, the model is estimated via Maximum Likelihood (ML) techniques. The computation of the likelihood function is based on an extension of the inversion technique *à la* Chen and Scott (1993). The standard inversion technique consists in estimating the latent factors by inverting a non-singular system; this system results from the assumption that some of the observed variables – whose number is equal to that of the latent factors – are modelled without errors.[20] It is straightforward to extend this approach to the case where the model is supposed to perfectly fit combinations of the observed variables (instead of an arbitrary subset of them). In particular, when these combinations are based on the ordinary Least Squares formula, the approach can deliver a better fit of the whole dataset than in the standard case (Renne, 2017). Appendix C provides an exhaustive description of this estimation approach. It details, in particular, the computation of the likelihood function. The dimension of $w_t$, as that of $\eta_t$, is set to three. Estimated parameters and their standard deviations across countries are reported in Table 1.

Figures 2, 3 and 4 show the fit of CDSs, expected changes in the debt-to-GDP, and sovereign bond yields, respectively. Overall, across countries, variables and maturities/horizons, the estimated models return a satisfactory fit of the data. Focusing on Figure 2, the model fit appears to be comparable with that found in those studies where the term structures of sovereign CDSs are fitted by means of reduced-form intensity-based approaches (see e.g. Pan and Singleton, 2008; Longstaff et al., 2011; Monfort and Renne, 2014; Ang and Longstaff, 2013; Augustin, 2018, and Subsection 2.1). The fact that the 5-year maturity CDS is perfectly fitted across countries is an implication of the inversion technique procedure (see Appendix C); the third component of $w_t$ is indeed determined so as to yield a perfect fit of this CDS. Moreover, the model-implied debt forecasts across countries and horizons manage to capture a substantial share of observed professional forecasts (Figures 3).

4.4. **Risk Premiums.** Risk premiums are defined as those components of asset returns that would not exist if investors were not risk averse. Consider a CDS contract. If agents were risk-neutral, the CDS spread would be approximately equal to the expected credit loss, i.e.

[19] This can be formulated as a fixed point problem, that is solved in a fast way by means of a Gauss-Newton algorithm.

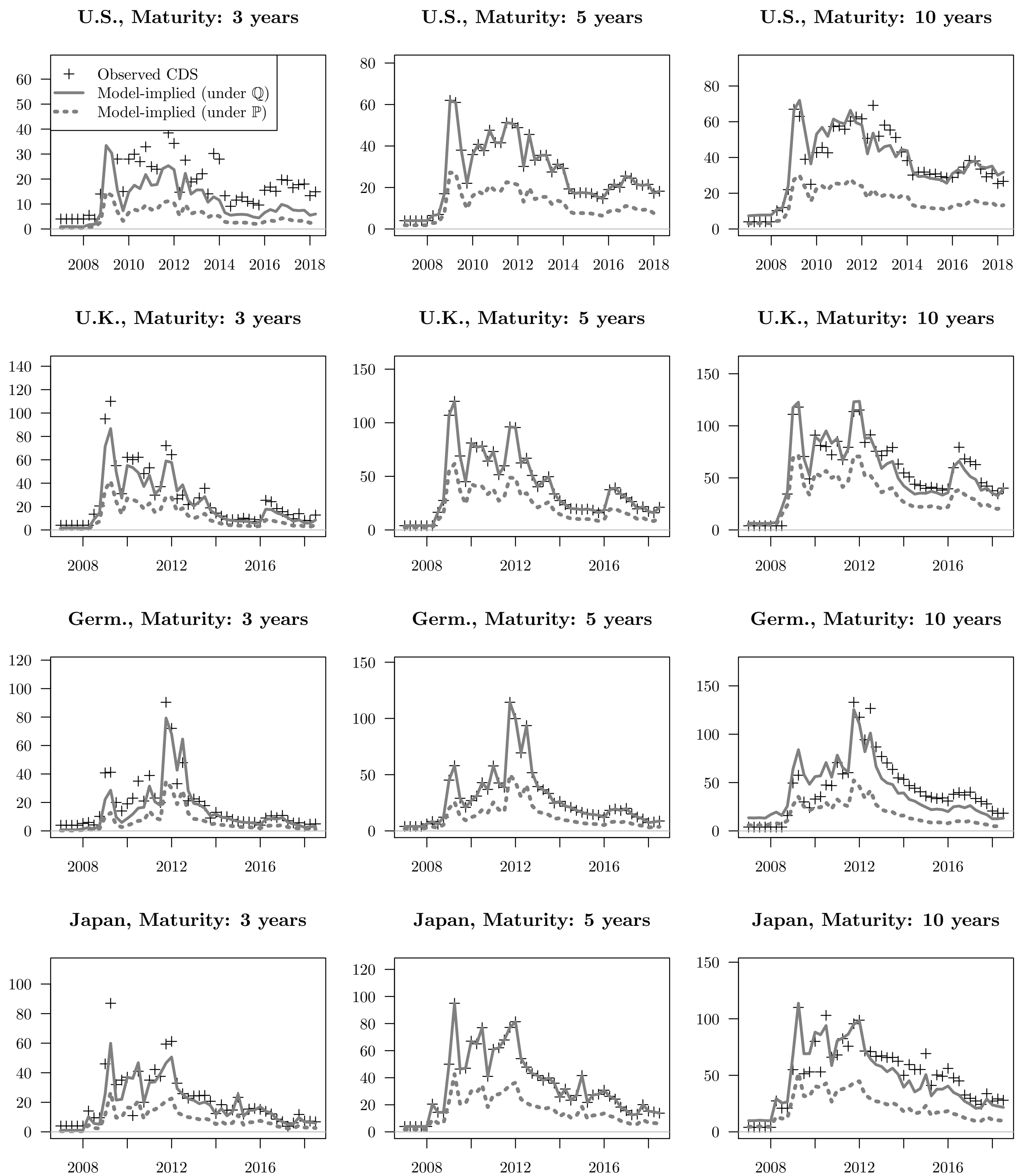
[20] The likelihood then involves an adjustment term corresponding to the determinant of the Jacobian matrix associated with the non-singular system; this adjustment results from the transformation of the observables to the latent components (see e.g. Ang and Piazzesi, 2003; Liu, Longstaff, and Mandell, 2006).

TABLE 1. Models' parameterization

| Designation | Notation | Multip. | U.S. | U.K. | Germ. | Japan |
|---|---|---|---|---|---|---|
| Recovery rate (Eq. 4) | $RR$ | $\times 10^2$ | 45 | 45 | 45 | 45 |
| Payoff decay rate (Sub. 3.8) | $\chi$ | $\times 10^2$ | 90 | 90 | 90 | 90 |
| Risk aversion parameter (Eq. 8) | $\gamma$ | | 4 | 4 | 4 | 4 |
| Consumption fall upon default (Eq. 5) | $b_c = b_y$ | $\times 10^2$ | 20 | 20 | 20 | 20 |
| Inflation increase upon default (Eq. 5) | $-b_\pi$ | $\times 10^2$ | 2.1 | 2.1 | 2.1 | 2.1 |
| Avg. debt-to-GDP ratio (Eq. 17) | $\exp(\bar{d})$ | $\times 10^2$ | 60 | 70 | 70 | 225 |
| Avg. surplus-to-debt ratio (Eq. 14) | $\overline{sd}$ | $\times 10^2$ | 1.00 | 0.81 | 1.08 | 0.97 |
| Elasticity of PD to $d_t - \ell_t$ (Eq. 2) | $\alpha$ | $\times 10^2$ | 12 | 10 | 10 | 10 |
| Duration of perpetuities, in yrs (Eq. 12) | $h^*$ | | 6.00 | 4.00 | 6.00 | 5.00 |
| Specification of $sd_t$ (Eq. 17) | $\gamma_y$ | | 0.28 | 0.45 | 0.07 | 0.02 |
| | $\gamma_d$ | $\times 10^4$ | 2.37 | 2.20 | 1.26 | 1.01 |
| | $\rho$ | | 0.76 | 0.77 | 0.87 | 0.89 |
| | $\sigma_s$ | $\times 10^2$ | 0.54 | 0.49 | 0.40 | 0.17 |
| Specification of $\Delta c_t$ (Eq. 5) | $\mu_c$ | $\times 10^2$ | 0.25 | 0.25 | 0.25 | 0.25 |
| | $\Lambda_{c,1}$ | $\times 10^3$ | 0.29 (0.00) | 1.69 (0.08) | 0.28 (0.00) | 0.17 (0.04) |
| | $\Lambda_{c,2}$ | $\times 10^3$ | 0.43 (0.01) | 1.80 (0.07) | 1.18 (0.04) | 0.44 (0.17) |
| | $\sigma_c$ | $\times 10^2$ | 0.32 (0.00) | 0.50 (0.01) | 0.49 (0.00) | 0.74 (0.00) |
| Specification of $\Delta y_t$ (Eq. 5) | $\mu_y$ | $\times 10^2$ | 0.25 | 0.25 | 0.25 | 0.25 |
| | $\Lambda_{y,1}$ | $\times 10^3$ | 0.41 (0.00) | 1.45 (0.07) | 0.45 (0.01) | 0.28 (0.07) |
| | $\Lambda_{y,2}$ | $\times 10^3$ | 0.60 (0.02) | 1.54 (0.06) | 1.86 (0.06) | 0.71 (0.27) |
| | $\sigma_y$ | $\times 10^2$ | 0.45 (0.00) | 0.43 (0.01) | 0.77 (0.00) | 1.20 (0.00) |
| Specification of $\pi_t$ (Eq. 5) | $\mu_\pi$ | $\times 10^2$ | 0.42 | 0.45 | 0.38 | 0.25 |
| | $\Lambda_{\pi,1}$ | $\times 10^3$ | – | – | – | – |
| | $\Lambda_{\pi,2}$ | $\times 10^3$ | – | – | – | – |
| | $\sigma_\pi$ | $\times 10^2$ | 0.24 | 0.49 | 0.22 | 0.52 |
| Specification of $s_t^*$ (Eq. VI.1) | $\mu_{s^*}$ | $\times 10^2$ | 3.51 (0.42) | 2.10 (0.21) | 3.04 (2.30) | 7.73 (0.11) |
| | $\Lambda_{s^*,1}$ | $\times 10^3$ | 2.78 (0.29) | 5.87 (1.07) | 2.17 (0.12) | 1.23 (0.33) |
| | $\Lambda_{s^*,2}$ | $\times 10^3$ | 4.08 (0.49) | 6.26 (1.10) | 8.91 (0.56) | 3.11 (1.61) |
| | $\Lambda_{s^*,3}$ | $\times 10^3$ | 1.99 | 1.12 | 1.54 | 1.99 |
| Specification of $w_t$ (Eq. 7) | $\phi_1$ | | 0.994 (0.000) | 0.944 (0.002) | 0.994 (0.000) | 0.988 (0.000) |
| | $\phi_2$ | | 0.932 (0.003) | 0.797 (0.026) | 0.800 (0.010) | 0.740 (0.052) |
| | $\phi_3$ | | 0.994 | 0.994 | 0.994 | 0.994 |

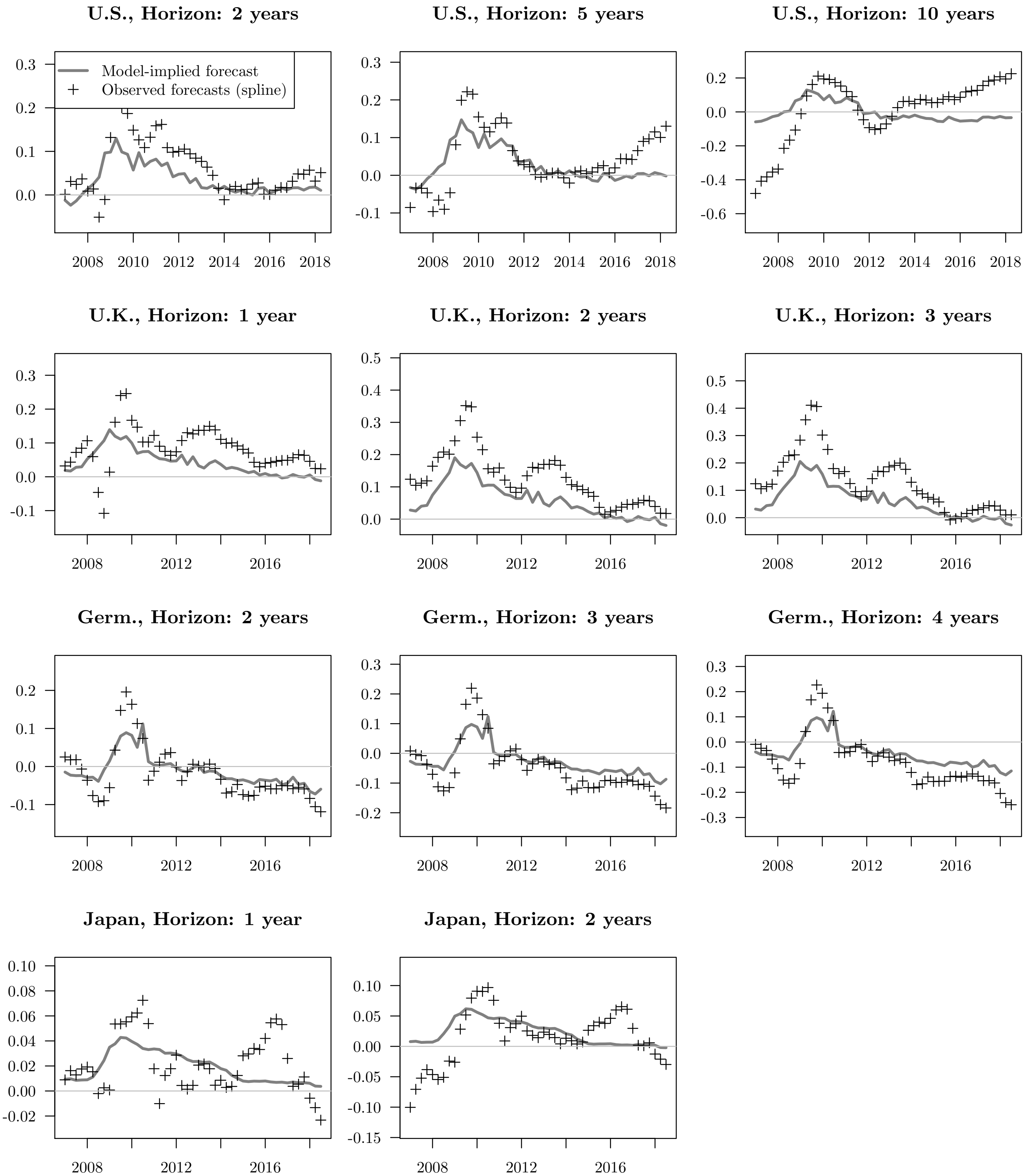
Note: This table presents the models' parameterizations. Part of the parameters are calibrated. The remaining ones are estimated by maximizing the log-likelihood associated with the model (see Appendix C). Standard deviations are reported in parentheses; the standard error estimates are based on the outer product of the score vectors. When no standard deviation is reported, it is either that the considered parameter is calibrated (i.e. not estimated) or that bounds on the parameter space are binding (see Subsection 4.2). The $\phi_i$'s denote the diagonal entries of matrix $\Phi$ (see Eq. 7).

FIGURE 2. Observed vs model-implied CDS



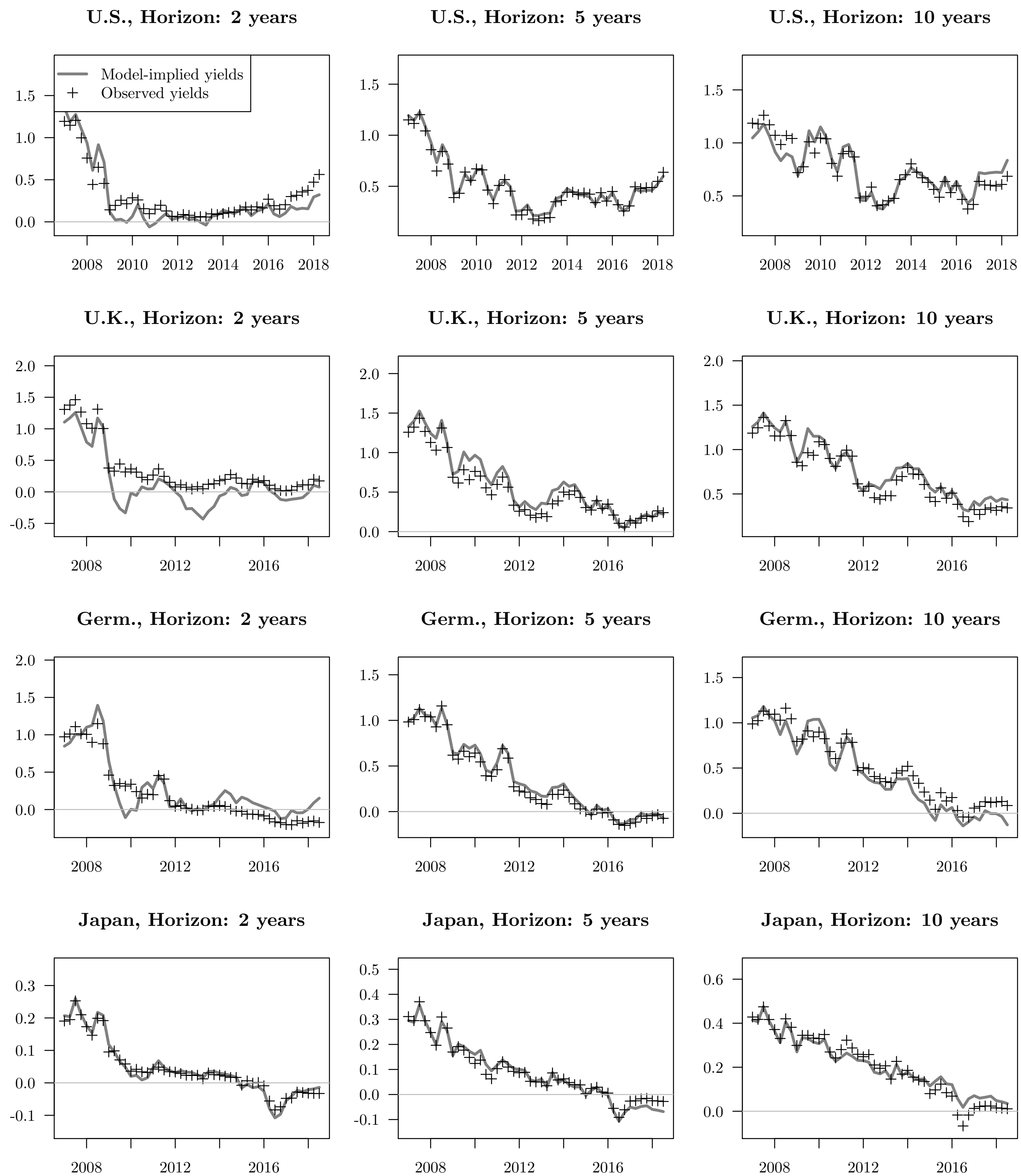

This figure compares observed CDS (crosses) with their model-implied counterparts (solid lines). Model-implied ($\mathbb{Q}$) CDS spreads result from Eq. (4). Dashed lines represent the CDS spreads that would be observed if agents were not risk averse; they are obtained by employing a formula similar to Eq. (4), but in a counterfactual model where the risk-aversion parameter $\gamma$ is set to 0. The differences between the two types of model-implied spreads ($\mathbb{Q}$ and $\mathbb{P}$) are credit-risk premiums.

FIGURE 3. Observed vs model-implied forecasted changes in debt




This figure compares model-implied expectations of changes in debt-to-GDP ratios (solid lines) to the cubic-spline-interpolated observed forecasts (crosses). The computation of model-implied forecasts is detailed in Online Appendix III. See Subsection 4.1 for details about observed forecasts.

FIGURE 4. Observed vs model-implied yields




This figure compares model-implied and observed quarterly yields of zero-coupon government yields. The computation of model-implied yields is based on Eq. (a.5) of Appendix B.1 (the maturity-$h$ yield is given by $-\frac{1}{h}\log B_{t,h}$, where $B_{t,h}$ is the date-$t$ price of a zero-coupon bond of maturity $h$).

the product of the loss-given default multiplied by the probability of default. However, if agents are risk-averse and if sovereign defaults tend to take place in bad states of nature – i.e. states of high marginal utility – then protection sellers are willing to enter the credit swap only if the CDS spread is larger than the expected credit loss.

In order to explore the model implications, it will prove convenient to introduce the risk-neutral measure $\mathbb{Q}$. This probability measure can be understood here as a convenient mathematical tool aimed at facilitating the presentation and interpretation of certain results – in particular those pertaining to risk premiums. It is defined through the following change of density, or *pricing kernel*:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{t,t+1} = \frac{M^n_{t,t+1}}{\mathbb{E}_t(M^n_{t,t+1})}. \tag{19}$$

This definition notably implies that the (forward) price, decided on date-$t$ but settled on date $t+h$, of a future nominal payoff $P_{t+h}$, is given by $\mathbb{E}^{\mathbb{Q}}_t(P_{t+h})$.[21] It is easily seen that if agents were risk-neutral ($\gamma = 0$) – that is if the *expectation hypothesis* held true – this forward price would be $\mathbb{E}_t(P_{t+h})$. The pricing kernel (19) reflects how physical probabilities are distorted when it comes to price uncertain future payoffs; it implies that assets that provide relatively higher payoffs when the s.d.f. is high – that is when consumption is low – have larger prices than under the expectation hypothesis.

To illustrate this notion in the present credit risk context, consider a forward contract providing $\mathcal{D}_{t+h} - \mathcal{D}_{t+h-1}$ on date $t+h$, with payment deferred to date $t+h$. If investors were risk-neutral, they would be willing to enter this contract if its (forward) price was equal to $\mathbb{E}_t(\mathcal{D}_{t+h} - \mathcal{D}_{t+h-1})$, that is to the physical probability that the government defaults on date $t+h$. But agents are risk averse, and the associated forward price is $\mathbb{E}^{\mathbb{Q}}_t(\mathcal{D}_{t+h} - \mathcal{D}_{t+h-1})$, which is the $\mathbb{Q}$-probability that the government will default on date $t+h$. By virtue of the pricing kernel definition (Eq. 19), it can be seen that the difference between the $\mathbb{P}$ and $\mathbb{Q}$ probabilities of default is:

$$\mathbb{E}^{\mathbb{Q}}_t(\mathcal{D}_{t+h} - \mathcal{D}_{t+h-1}) - \mathbb{E}_t\left(\mathcal{D}_{t+h} - \mathcal{D}_{t+h-1}\right) = \mathbb{C}ov_t\left(\frac{M^n_{t,t+h}}{\mathbb{E}_t(M^n_{t,t+h})}, \mathcal{D}_{t+h} - \mathcal{D}_{t+h-1}\right).$$

Hence the credit-risk premium, that is – by definition – the difference between the forward price and the price that would prevail under the expectation hypothesis, is equal to the covariance between the pricing kernel and the default event indicator. Because the pricing

[21] This price indeed is $\frac{1}{\mathbb{E}_t(M^n_{t,t+h})}\mathbb{E}_t\left(M^n_{t,t+h}P_{t+h}\right) = \mathbb{E}_t\left(\frac{M^n_{t,t+h}}{\mathbb{E}_t(M^n_{t,t+h})}P_{t+h}\right)$, which is equal to $\mathbb{E}^{\mathbb{Q}}_t(P_{t+h})$ given the change of density from $\mathbb{P}$ to $\mathbb{Q}$ (Eq. 19). More generally, the risk-neutral probability measure is such that the price of any asset is equal to the discounted $\mathbb{Q}$-expectation of this asset's payoffs.

kernel negatively depends on consumption and that defaults tend to happen in recessions (when consumption is low),[22] the covariance term is positive.

A CDS contract is more sophisticated than the basic forward contract considered above. Nevertheless, it is also, essentially, a contract whose payoffs jump upon sovereign default. As a result, CDS spreads include credit-risk premiums. Hereinafter, we refer to the model-implied CDS – including risk premiums – as the "$\mathbb{Q}$ CDS spread", to recall that it is obtained by computing conditional expectations under the $\mathbb{Q}$ measure.[23] By the same token, we refer to the CDS spread that would prevail if investors were not risk averse – i.e. excluding risk premiums – as the "$\mathbb{P}$ CDS spread". Figure 2 displays $\mathbb{Q}$ CDS spreads (solid lines) and $\mathbb{P}$ CDS spreads (dashed lines); credit risk premiums are the differences between the two curves. We find sizeable sovereign credit premiums: about 50% of the CDS spreads correspond to risk premiums. In other words, the $\mathbb{Q}$-over-$\mathbb{P}$ CDS ratio is about two, which is broadly comparable to the ones found in sovereign credit-risk studies based on reduced-form intensity approaches (see e.g. Pan and Singleton, 2008; Longstaff et al., 2011; Monfort and Renne, 2014).

Figure 5 aims at illustrating the relationship between the fiscal space ($\ell_t - d_t$) and CDS spreads. The plots show how the 10-year CDS spread increases as the fiscal space diminishes. This exercise is carried out *ceteris paribus*. That is, all variables, except $d_t$, are kept at their unconditional mean, or steady state value. The solid and dotted lines respectively correspond to the effects on the $\mathbb{Q}$ and $\mathbb{P}$ CDS spreads. These plots reveal a non-linear relationship between CDS spreads and the fiscal space.[24] This prediction is consistent with various empirical studies finding evidence of a non-linear relationship between CDSs and debt-to-GDP ratios (see e.g. Haugh et al., 2009; Caggiano and Greco, 2012; Di Cesare et al., 2012; Hördahl and Tristani, 2013).

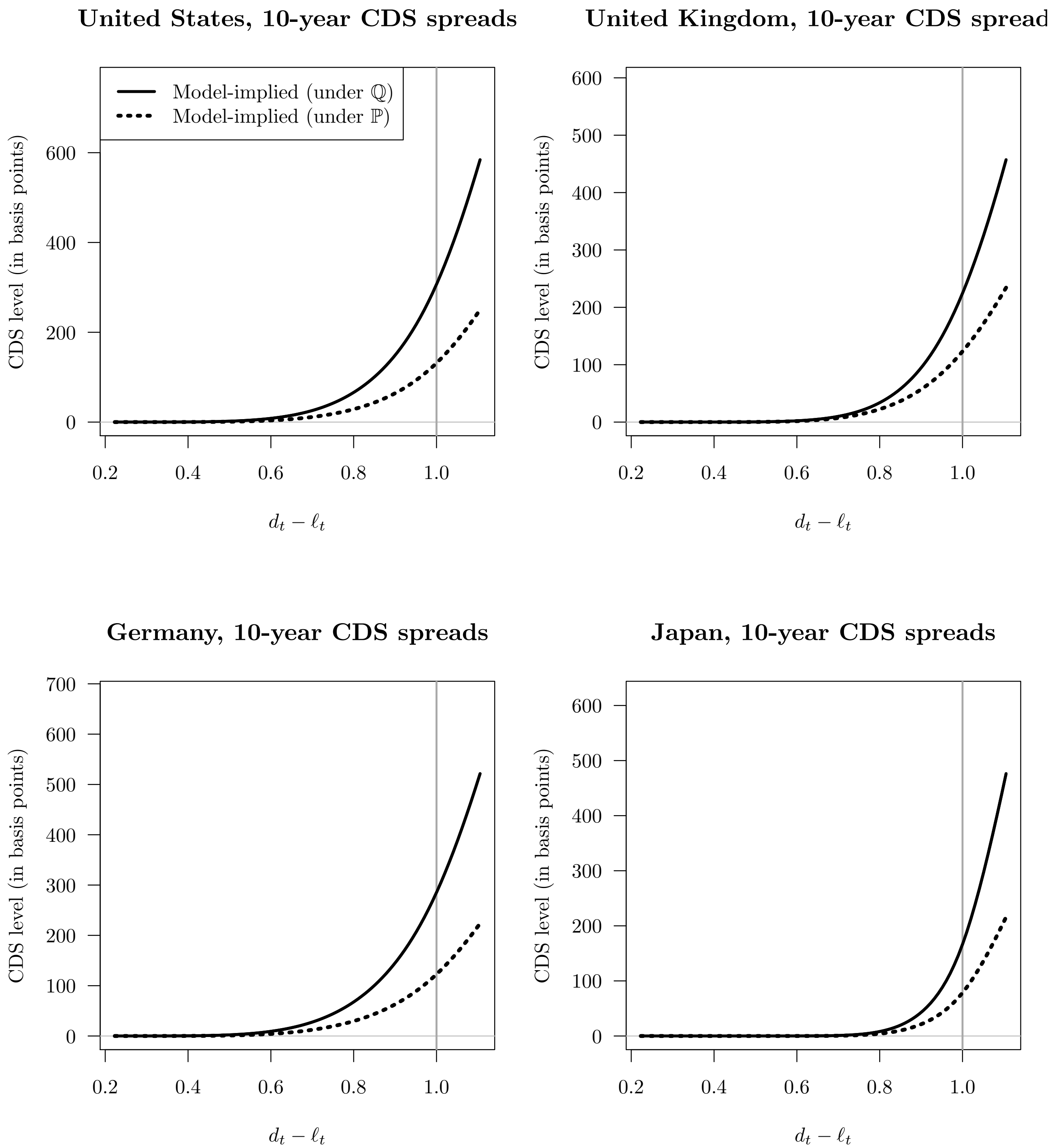
[22] Three channels account for the negative correlation between consumption and the default indicator in our model. Consider a recession. First, the debt-to-GDP ratio soars as GDP plunges. Second, because our estimated models imply that the maximum primary surplus ($s_t^*$) positively depends on GDP growth (Table 1), the fiscal limit tends to decrease in recessions. Therefore, amid recessions, both the rise in the debt-to-GDP ratio and the downsizing of the fiscal limit contribute to higher sovereign default probabilities. Third, upon default, consumption is expected to experiment a fall of magnitude $b_c$ (see Eq. 6).

[23] CDS spreads are given by Eq. (4), which can be rewritten:

$$S_{t,h}^{cds} = (1 - RR)\frac{\sum_{k=1}^{h} \mathbb{E}_t^{\mathbb{Q}}[\exp(-r_t - \cdots - r_{t+k-1})(\mathcal{D}_{t+k} - \mathcal{D}_{t+k-1})]}{\sum_{k=1}^{h} \mathbb{E}_t^{\mathbb{Q}}[\exp(-r_t - \cdots - r_{t+k-1})(1 - \mathcal{D}_{t+k})]}.$$

[24] in Figure 5, we observe CDS spreads plotted against $d_t - \ell_t$, that is the opposite of the fiscal space $\ell_t - d_t$.

FIGURE 5. Non-linear relationship between fiscal space and credit spreads




These plots illustrate the non-linear relationship between fiscal space ($\ell_t - d_t$) and CDS spreads. This exercise is carried out *ceteris paribus*. That is, all variables, except $d_t$, are kept at their unconditional mean – or steady state value. The dashed line represent "$\mathbb{P}$ CDS spreads", that are those counterfactual spreads that would be observed if agents were not risk-averse. Specifically, $\mathbb{P}$ CDS spreads are obtained by employing a formula similar to Eq. (4), but in a counterfactual model where the risk-aversion parameter $\gamma$ is set to 0.

4.5. **Fiscal limit estimates.** Even though policy makers have been increasingly focusing on the issue of fiscal sustainability and fiscal space, the literature building and studying measures of fiscal space/limits has been scarce. Kose, Kurlat, Ohnsorge, and Sugawara (2017) propose an extensive dataset collecting model-free proxies for fiscal sustainability for several countries, but these measures cannot be directly interpreted as fiscal limits/spaces. As explained in Subsection 2.2, Ostry, Ghosh, Kim, and Qureshi (2010), Ghosh, Kim, Mendoza, Ostry, and Qureshi (2013) and Ostry, Ghosh, and Espinoza (2015) compute debt limits based on the observation that the higher the levels of debt, the weaker the reaction of primary surpluses – a phenomenon they dub "fiscal fatigue". The latter approach however results in static debt limit estimates,[25] and, to the best of our knowledge, the present study is the first to deliver time-varying estimates of fiscal limits. These estimates are displayed on the left-hand-side plots of Figure 6. The right-hand-side plots display the dynamics of the actual Surplus to GDP ratio in solid lines and the estimates of the maximum Budget Surplus ($s_t^*$) in dashed lines.

In both the U.S. and the U.K., the fiscal limit has increased by more than 50 percentage points from 2008 to 2014. Since 2014, the fiscal limit has remained relatively stable in the U.S; at around 120% of GDP. In the U.K. the fiscal fell by around 10 percentage points in mid 2016, in coincidence with the Brexit vote. In Germany, the fiscal limit shows a slight downward trend over the estimation period. From 2010 to mid-2012, amid the European sovereign debt crisis, the German fiscal limit recorded a decrease of more than 20 percentage points, from 110% of GDP to less than 90%. The Japanese fiscal limit shows the same trend as the debt-to-GDP ratio, the former being on average 30 percentage point above the latter. In other words, the average Japanese fiscal space is of 30% of GDP.

Estimated fiscal space time series are shown on Figure 7. The vertical lines indicate announcements of key monetary-policy decisions involving large-scale bond purchases. In the U.S. and the U.K., the announcement and implementation of quantitative easing programs coincide with increases in the fiscal space. In August 2012, the ECB announcement of the Outright Monetary Transactions (OMT) – a mechanism aimed to "safeguard an appropriate monetary policy transmission and the singleness of the monetary policy" – was followed by a 10 percentage point jump in the German fiscal space.

In comparison with the debt limit point estimates[26] of Ostry et al. (2010) and Ghosh et al. (2013), our time-varying fiscal limit estimates (Fig. 6) seem to be more conservative: in these

[25] These studies consider two calibration periods: from 1970 to 2007 and from 1985 to 2007.
[26] The time span goes from 1985 until 2007.

alternative studies, the projected debt limits for U.S., U.K. and Germany are respectively of 160.5%, 166.5% and 175.8% of GDP (while Japan's debt limits failed to be computed). Moreover, updated point estimates of fiscal space in Ostry et al. (2015) and Moody's analytics[27] feature a much ampler manoeuvre capacity for the U.S., U.K. and Germany compared to our fiscal space estimates (Fig. 7); on the other hand, for Japan, their point estimate is equal to 0% of GDP, while our fiscal space for Japan fluctuates around 30% of GDP.

4.6. **Sovereign default probabilities.** Once estimated, our model also allows us to compute the default probabilities of the considered governments at any horizon. Conditional on the information available to the investors on date $t$, the probability that the government goes into defaut before date $t+h$ is (with $\mathcal{D}_t = 0$):

$$\mathbb{P}_t(\mathcal{D}_{t+h} = 1) \quad = \quad \mathbb{E}_t(\mathcal{D}_{t+h}) = 1 - \mathbb{E}_t(1 - \mathcal{D}_{t+h}). \tag{20}$$

Appendix A and Online Appendix I.1 discuss the computation of $\mathbb{E}_t[M_{t,t+h}(1-\mathcal{D}_{t+h})]$, whose knowledge is required to price CDSs (Subsection 3.4). The same type of formula can be used to compute $\mathbb{E}_t(1-\mathcal{D}_{t+h})$, that is the expectation appearing on the right-hand side of the previous equation.[28]

It is interesting to compare these default probabilities to the ones obtained under the risk-neutral – or pricing – measure $\mathbb{Q}$, that are (with $\mathcal{D}_t = 0$):

$$\mathbb{Q}_t(\mathcal{D}_{t+h} = 1) \quad = \quad 1 - \frac{\mathbb{E}_t\left[M_{t,t+h}(1-\mathcal{D}_{t+h})\right]}{\mathbb{E}_t\left(M_{t,t+h}\right)}. \tag{21}$$

As discussed in Subsection 4.4, the existence of risk premiums associated with sovereign credit risk implies that physical ($\mathbb{P}$) probabilities of default differ from their risk-neutral counterparts. Yet, the latter, derived from basic credit-risk models like in Litterman and Iben (1991), are extensively used by market practitioners, who refer to them as *market-implied default probabilities* (see e.g. Hull, Predescu, and White, 2005). Our approach makes it possible to assess the deviations between the two kinds of default probabilities: Physical, or "risk-adjusted", probabilities of default appear on the left-hand-side plots of Figure 8 (based on Eq. 20); the right-hand-side plots display associated $\mathbb{Q}$ default probabilities (based on Eq. 21).

[27] Moody's point estimates for fiscal space are based on the computation of Ostry et al. (2010) and Ghosh et al. (2013); see Moody's (2011) and https://www.economy.com/dismal/tools/global-fiscal-space-tracker.

[28] This is simply done by replacing the s.d.f. $M_{t,t+1h} = M_{t,t+1} \times \cdots \times M_{t+h-1,t+h}$ by 1, which amounts to taking $\delta = 1$ and $\gamma = 0$ in Eq. (8).

Consistently with the findings of the Subsection 4.4, the differences between the two types of probabilities, which reflect the existence of credit risk premiums, are sizeable.

## 5. Concluding remarks

The present research attempts at uncovering the structural dynamics of credit spreads, exploiting the time variation of fiscal limits. The fiscal limit – at the core of the sovereign credit-risk model we introduce – is defined as the maximum outstanding debt a government can sustain via budget surpluses in the future (as in Bi, 2012; Leeper, 2013). Compared to reduced-form default-intensity approaches, this model explicitly relates sovereign default probabilities to the macroeconomy. Specifically, it assumes that probabilities of default leave zero only when debt-to-GDP breaches the fiscal limit. Given that both the debt-to-GDP ratio and the fiscal limit depend on real activity and inflation, our model accounts for the dependency of credit spreads to macroeconomic fluctuations. The model notably predicts a non-linear relationship between credit risk spreads and the fiscal space, which is in line with, e.g., the regression-based approaches of Haugh, Ollivaud, and Turner (2009), Caggiano and Greco (2012), Di Cesare, Grande, Manna, and Taboga (2012), who find that spreads non-linearly depend on debt-to-GDP ratios.

We exploit the model-implied relationship between credit risk spreads – which are observed – and of the fiscal limit – which is not directly observed – to infer the levels of the fiscal limit. To the best of our knowledge, this paper is the first to provide time-varying estimates of fiscal limits. Our application considers four major advanced economies: the U.S., the U.K., Japan and Germany. Fiscal limit estimates show ample fluctuations over time. Moreover, the model succeeds in providing a good fit of the time series of sovereign Credit Default Swaps (CDSs).

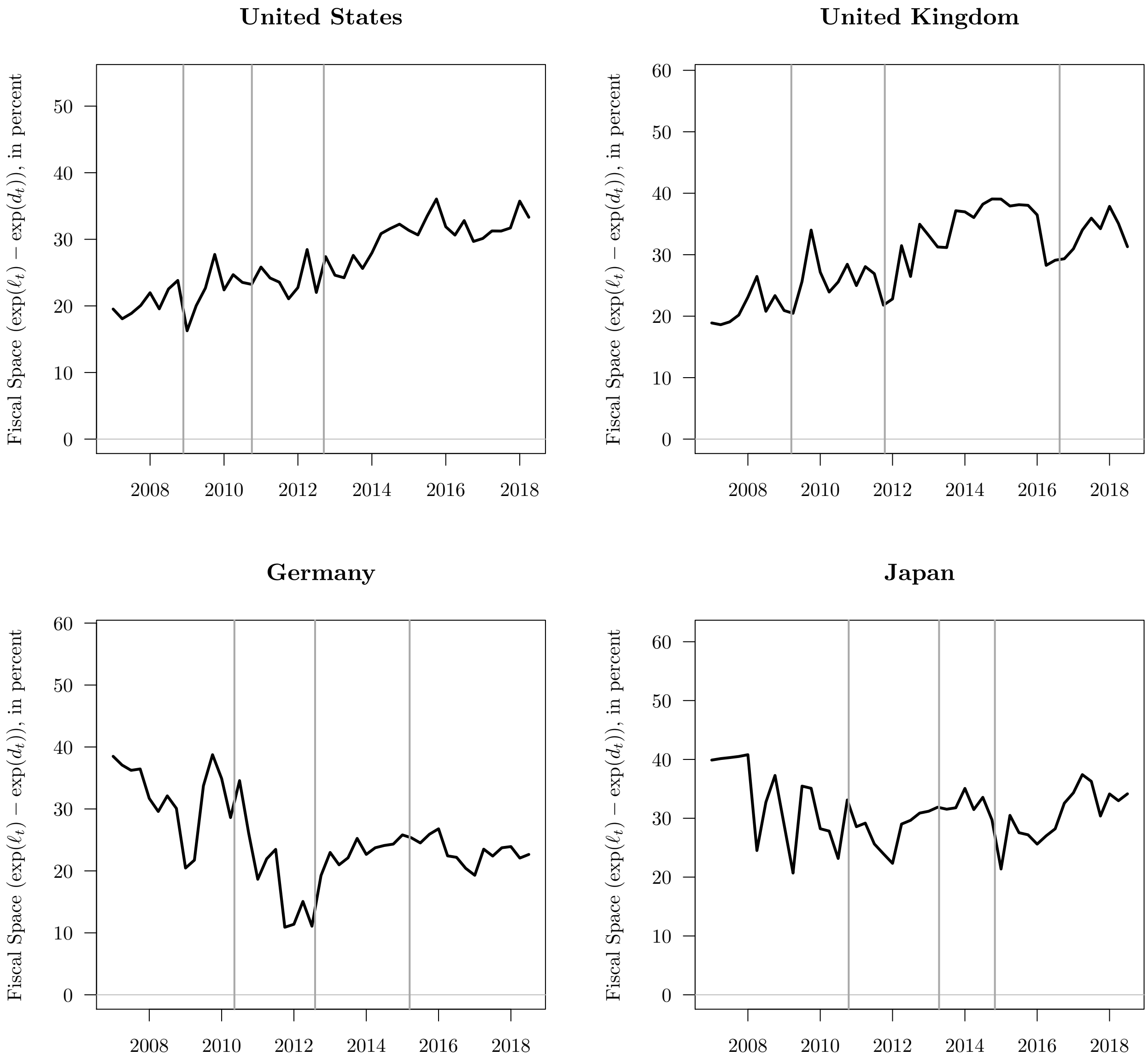
Because our model entails risk-averse investors, our approach also provides us with estimates of credit risk premiums. From a quantitative point of view, we observe that a substantial part of the credit risk spreads is accounted by credit risk premiums, in line with the findings of the purely-reduced-form default-intensity literature.[29] Such hefty risk premiums render in large discrepancies between the default intensities adjusted for risk and the ones that are not.

[29] see, e.g., Pan and Singleton (2008); Longstaff et al. (2011); Monfort and Renne (2014); Ang and Longstaff (2013); Augustin (2018).

FIGURE 6. Fiscal limits and maximum surplus estimates


The left-hand-side plots display estimated fiscal limits (exp($\ell_t$)). The right-hand-side plots shows the estimates of the maximum primary budget surplus $s_t^*$ (dashed lines) together with the actual series of surplus-to-GDP $\mathcal{S}_t/(\mathcal{Y}_t\mathcal{P}_t)$ (solid lines). On each plot, the vertical bars indicate key monetary-policy decisions involving large-scale purchases of sovereign bonds (see caption of Figure 7 for details regarding these dates).

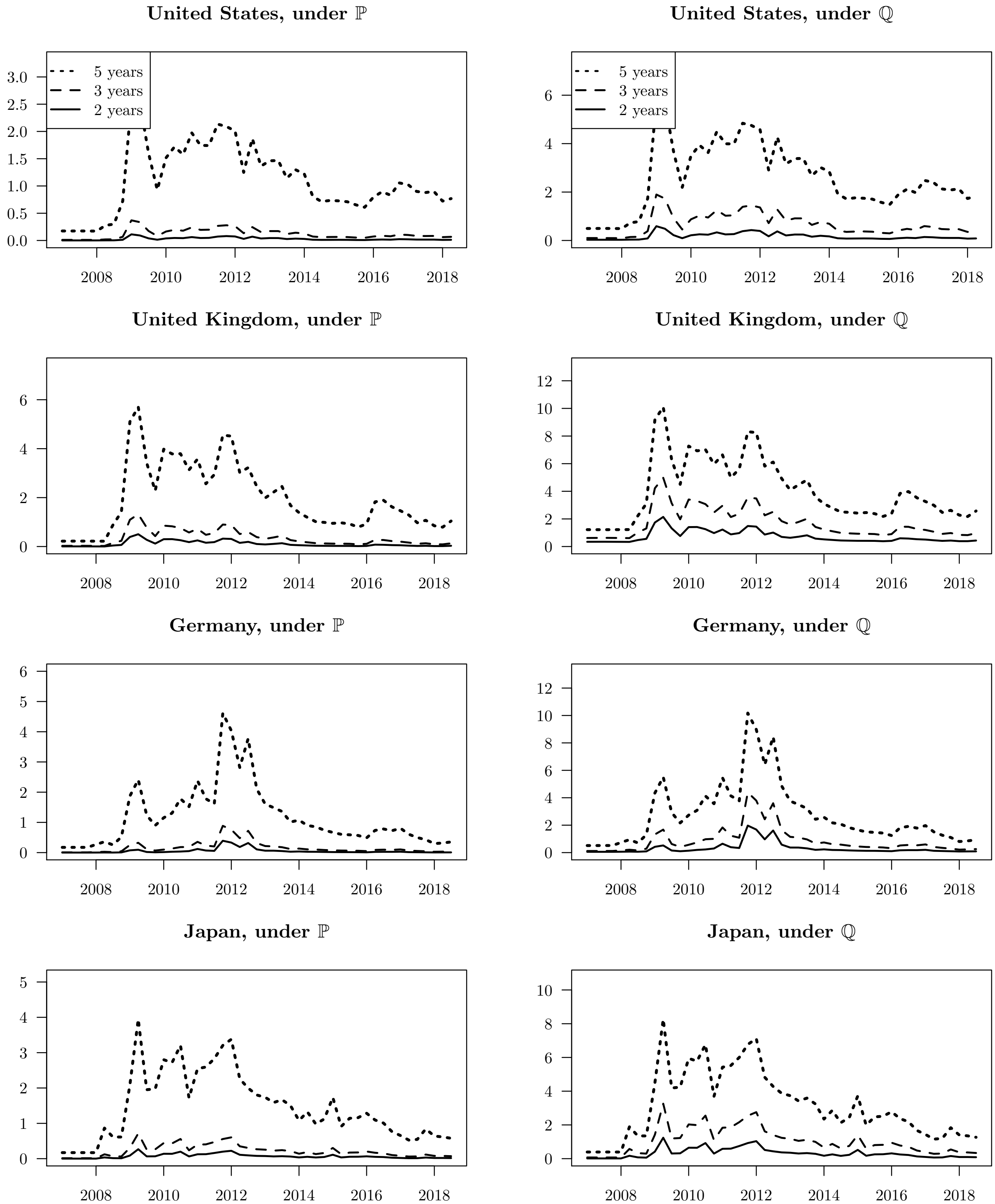
FIGURE 7. Fiscal space estimates




These plots show, for each country, the estimates of the fiscal space expressed in GDP percent, that is $100 \times (\exp(\ell_t) - \exp(d_t))$. On each plot, the vertical bars indicate key monetary-policy decisions involving large-scale purchases of sovereign bonds:

- U.S. – 25/11/2008: Announcement of QE1 purchase of direct obligations of GSEs and MBS; 3/11/2010: Announcement of QE2 (purchase of additional 600 bln \$ of longer-term Treasury securities); 13/09/2012: Announcement of QE3 (Additional purchases of longer-term securities).
- U.K. – 03/2009: Announcement of Asset Purchase Program (purchases of gilts and of high-quality debt issued by private companies); 10/2011: BoE announces new round of QE; 08/2016: BoE announces novel bond purchases to address uncertainty over Brexit.
- Germany – 10/05/2010: Announcement of Securities Market Program (purchases of sovereign bonds); 02/08/2012: ECB announces it may undertake outright transactions in sovereign bond markets (OMT); 04/03/2015: Announcement of the public sector purchase programme (PSPP).
- Japan – 10/2010: Announcement of QE; 04/2013: BoJ announces expansion of the asset purchase program; 31/10/2014: BoJ announces expansion of its bond buying program.

FIGURE 8. Risk-adjusted probabilities of default




This figures displays sovereign probabilities of default at different horizons. Formally, for each date, we compute $\mathbb{E}_t(\mathbb{1}_{\{\mathcal{D}_{t+h}=1\}})$ (left-hand-side plots) and $\mathbb{Q}_t(\mathbb{1}_{\{\mathcal{D}_{t+h}=1\}})$ (right-hand-side plots), for different values of $h$ (with $\mathcal{D}_t = 0$). See Subsection 4.6 for details regarding the computation of these probabilities.

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## APPENDIX A. CDS AND BOND PRICING

As explained in Subsection 3.4, pricing CDS (i.e. solving for $S^{cds}_{t,h}$ in Eq. 4) requires the computation of the following two conditional expectations: $\mathbb{E}_t[M^n_{t,t+k}(1-\mathcal{D}_{t+k})]$ and $\mathbb{E}_t[M^n_{t,t+k}(1-\mathcal{D}_{t+k-1})]$. Online Appendix I.1 shows that, in a model where: (i) the s.d.f. between dates $t$ and $t+1$ is if the form $M_{t,t+1} = \exp(\varphi_0 + \varphi_1' x_{t+1} + \varphi_2(\mathcal{D}_{t+1} - \mathcal{D}_t))$, where $\mathcal{D}_t$ does not Granger-cause $x_t$, and (ii) $\underline{\lambda}_t$ is the default intensity defined through Eqs. (1) and (2), we have:

$$\begin{aligned} P_{t,h} &\equiv \mathbb{E}_t\left[M_{t,t+1}\times\cdots\times M_{t+h-1,t+h}(1-\mathcal{D}_{t+h})\right] \\ &= (1-\mathcal{D}_t)\exp(h\varphi_0)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h}) - \underline{\lambda}_{t+1} - \cdots - \underline{\lambda}_{t+h}\}\right], \qquad \text{(a.1)}\\ P^*_{t,h} &\equiv \mathbb{E}_t\left[M_{t,t+1}\times\cdots\times M_{t+h-1,t+h}(1-\mathcal{D}_{t+h-1})\right] \\ &= (1-\mathcal{D}_t)\exp(h\varphi_0)(1-\exp(\varphi_2))\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h}) - \underline{\lambda}_{t+1} - \cdots - \underline{\lambda}_{t+h}\}\right] + \\ &\quad \exp(h\varphi_0+\varphi_2)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h}) - \underline{\lambda}_{t+1} - \cdots - \underline{\lambda}_{t+h-1}\}\right]. \qquad \text{(a.2)}\end{aligned}$$

Let us introduce the following notations:

$$\left\{\begin{array}{lcl} f_{n-1,n} &=& -\log P_{t,n} + \log P_{t,n-1} \\ f^*_{n-1,n} &=& -\log P^*_{t,n} + \log P^*_{t,n-1}, \end{array}\right.$$

that are such that

$$\left\{\begin{array}{lcl} P_{t,h} &=& (1-\mathcal{D}_t)\exp(f_{0,1}+\cdots+f_{h-1,h}) \\ P^*_{t,h} &=& (1-\mathcal{D}_t)\exp(f^*_{0,1}+\cdots+f^*_{h-1,h}). \end{array}\right. \qquad \text{(a.3)}$$

Online Appendix I.2 proposes approximations to $f_{n-1,n}$ and $f^*_{n-1,n}$ (building on Wu and Xia, 2016). These approximations further allow to approximate $P_{t,h}$ and $P^*_{t,h}$ (using Eq. a.3).

## APPENDIX B. PERPETUITIES AND DEBT ACCUMULATION PROCESS

B.1. **Perpetuities.** Consider the perpetuity described in Subsection 3.7. The date-$t$ price of this perpetuity is given by Eq. (9), that is: $P_t = \sum_{h=1}^{\infty}\chi^{h-1}B_{t,h}$, where $B_{t,h}$ is the date-$t$ price of a generic zero-coupon bond providing the nominal payoff $1-(1-RR)\mathcal{D}_{t+h}$ on date $t+h$. This price is given by:

$$B_{t,h} = \mathbb{E}_t(M^n_{t,t+1}\times\cdots\times M^n_{t+h-1,t+h}[(1-(1-RR)\mathcal{D}_{t+h}]),$$

where the nominal s.d.f. $M^n_{t,t+1} \equiv M_{t,t+1}\exp(-\pi_{t+1})$ is given by (using Eqs. 8 and 5):

$$\exp\left\{\log(\delta) - \gamma\mu_c - \mu_\pi - (\gamma\Lambda_c+\Lambda_\pi)'w_{t+1} + (\gamma b_c + b_\pi)(\mathcal{D}_{t+1}-\mathcal{D}_t) - (\gamma\sigma_c+\sigma_\pi)\eta_{t+1}\right\}.$$

Because the $\eta_t$'s are exogenous i.i.d. shocks of covariance matrix $Id$, and using the following notations:

$$\bar\varphi_0 = \log(\delta) - \gamma\mu_c - \mu_\pi + \frac{1}{2}(\gamma\sigma_c+\sigma_\pi)'(\gamma\sigma_c+\sigma_\pi) \quad \text{and} \quad \bar\varphi_1 = -(\gamma\Lambda_c+\Lambda_\pi),$$

we obtain:

$$\begin{aligned} B_{t,h} &= \mathbb{E}_t\{\exp[h\bar\varphi_0 + \bar\varphi_1'(w_{t+1}+\cdots+w_{t+h}) + (\gamma b_c+b_\pi)\mathcal{D}_{t+h}][(1-(1-RR)\mathcal{D}_{t+h}]\} \\ &= \mathbb{E}_t\{\exp[h\bar\varphi_0 + \bar\varphi_1'(w_{t+1}+\cdots+w_{t+h})]\}[(1-[1-RR\exp(\gamma b_c+b_\pi)]\mathcal{D}_{t+h}]).\end{aligned}$$

Therefore, under the convenient assumption that $RR = \exp[-(\gamma b_c + b_\pi)]$, we get:

$$B_{t,h} = \mathbb{E}_t\{\exp[h\bar\varphi_0 + \bar\varphi_1'(w_{t+1}+\cdots+w_{t+h})]\}. \qquad \text{(a.4)}$$

The conditional expectation appearing on the right-hand side of the previous equation is easily computed recursively. Indeed, as shown in the Online Appendix II, if $w_t$ follows a Gaussian VAR (as in Eq. 7), we have:

$$\mathbb{E}_t\left[\exp\{u'(w_{t+1}+\cdots+w_{t+h})\}\right] = \exp(b_h(u) + a_h(u)'w_t), \qquad \text{(a.5)}$$

where $a_h(u)$ and $b_h(u)$ recursively satisfy:

$$\left\{\begin{array}{lcl} a_h & = & \Phi'(a_{h-1}(u)+u) \\ b_h & = & b_{h-1}(u)+\frac{1}{2}(a_{h-1}(u)+u)'(a_{h-1}(u)+u), \end{array}\right.$$

with $a_0 = 0$ and $b_0 = 0$.

B.2. **Debt accumulation process.** Let us denote by $I_t$ the proceeds of date-$t$ issuances and by $X_t$ the resulting first payments (settled on date $t+1$). We have:

$$I_t = \sum_{j=1}^{\infty} \frac{\chi^{j-1} X_t}{(1+q_t)^j} = \frac{X_t}{1+q_t-\chi}.$$

Consider the date-$t$ (residual) face value, of those issuances that took place on date $t-h$. This face value is computed as the sum of (residual) future associated payoffs $\chi^{h+1}X_{t-h}$, $\chi^{h+2}X_{t-h}, \dots$, discounted using the issuance yield-to-maturity that materialized on date $t-h$, that is $q_{t-h}$. It is easily seen that it is equal to $\chi^h I_{t-h}$. As a consequence, and because current debt $D_t$ is the sum of the (residual) face values of all past issuances, we obtain:[30]

$$D_t \equiv I_t + \chi I_{t-1} + \chi^2 I_{t-2} + \dots = I_t + \chi D_{t-1}. \tag{a.6}$$

Using $X_t = (1+q_t-\chi)I_t = (1+q_t-\chi)(D_t - \chi D_{t-1})$, past debt issuances give rise to the following debt payments at date $t+1$:

$$\begin{array}{rl} & CF_{t+1} = X_t + \chi X_{t-1} + \chi^2 X_{t-2} + \dots \\ = & (1+q_t-\chi)(D_t-\chi D_{t-1}) + \\ & \chi(1+q_{t-1}-\chi)(D_{t-1}-\chi D_{t-2}) + \chi^2(1+q_{t-2}-\chi)(D_{t-2}-\chi D_{t-3}) + \dots \\ = & D_t - \chi D_t + q_t(D_t-\chi D_{t-1}) + \chi q_{t-1}(D_{t-1}-\chi D_{t-2}) + \chi^2 q_{t-2}(D_{t-2}-\chi D_{t-3}) + \dots \end{array} \tag{a.7}$$

On date $t$, the sum of the issuance proceeds ($I_t$) and of the primary budget surplus ($S_t$) has to equate date-$t$ payments associated with previous issuances ($CF_t$). That is: $I_t = CF_t - S_t$. Using Eq. (a.6), we get:

$$D_{t+1} - \chi D_t = CF_{t+1} - S_{t+1}. \tag{a.8}$$

Substituting for $CF_t$ (Eq. a.7) into Eq. (a.8), we have:

$$\begin{array}{rcl} D_{t+1} & = & D_t - S_{t+1} + \\ & & \underbrace{q_t(D_t-\chi D_{t-1}) + \chi q_{t-1}(D_{t-1}-\chi D_{t-2}) + \chi^2 q_{t-2}(D_{t-2}-\chi D_{t-3}) + \dots}_{\text{interest payments on date } t+1 \equiv R_{t+1}} \end{array} \tag{a.9}$$

Let us denote by $Y_t$ the real GDP and by $P_t$ the GDP price deflator. The previous equation rewrites:

$$\frac{D_t}{Y_t P_t} = \frac{D_{t-1}}{Y_{t-1}P_{t-1}} \frac{Y_{t-1}P_{t-1}}{Y_t P_t}\left(1 + \frac{R_t}{D_{t-1}} - \frac{S_t}{D_{t-1}}\right).$$

Introducing the log debt-to-GDP ratio $d_t$, we obtain:

$$d_t = d_{t-1} - y_t - \pi_t + \log\left(1 + \frac{R_t}{D_{t-1}} - \frac{S_t}{D_{t-1}}\right). \tag{a.10}$$

Appendix B.3 shows that the unconditional mean of the apparent debt interest rate $R_t/D_{t-1}$ is equal to that of $q_t$, that we denote by $\bar{q}$. Moreover, let us denote by $\overline{sd}$ the unconditional mean of $S_t/D_{t-1}$.

[30] This computation is based on the so-called "nominal valuation of debt securities", a standard international debt accounting principle (see Subsection 3.8).

The previous equation can be reformulated as:

$$d_t = d_{t-1} - \Delta y_t - \pi_t + \log\left(1 + \bar{q} - \overline{sd}\right) + \log\left(1 + \frac{rd_t - sd_t}{1 + \bar{q} - \overline{sd}}\right),$$

where

$$rd_t = \frac{R_t}{D_{t-1}} - \bar{q} \qquad sd_t = \frac{S_t}{D_{t-1}} - \overline{sd}.$$

Assuming that $rd_t - sd_t$ is small, we get the first-order approximation:

$$d_t \approx d_{t-1} - \Delta y_t - \pi_t + \log\left(1 + \bar{q} - \overline{sd}\right) + \frac{rd_t - sd_t}{1 + \bar{q} - \overline{sd}}. \tag{a.11}$$

B.3. **Interest payment dynamics.** Assuming $D_t \approx D_{t-1}$, we obtain the following recursive approximation for the interest payments (see Eq. a.9):

$$R_{t+1} \approx D_t(1-\chi)(q_t + \chi q_{t-1} + \chi^2 q_{t-2} + \dots), \tag{a.12}$$

which gives

$$\frac{R_{t+1}}{D_t} \approx (1-\chi)q_t + \chi \frac{R_t}{D_{t-1}}. \tag{a.13}$$

Hence, the apparent interest rate is given by an exponential smoothing of the yield-to-maturities associated with past debt issuances. This implies in particular that, when the apparent debt interest rate $R_t/D_{t-1}$ is stationary, then its unconditional mean is equal to that of $q_t$, i.e. $\mathbb{E}(R_t/D_{t-1}) = \mathbb{E}(q_t) = \bar{q}$. We therefore have:

$$rd_{t+1} \approx (1-\chi)(q_t - \bar{q}) + \chi rd_t. \tag{a.14}$$

## Appendix C. Inversion techniques and the likelihood function

The computation of the likelihood function is based on the so-called inversion technique, originally proposed by (Chen and Scott, 1993).

We denote by $n_w$ the dimension of vector $w_t$ – the vector of latent variables (see Eq. 5) – and by $\Theta$ the vector of model parameters to be estimated. We also denote by $\mathcal{Z}_t$ the $n_{\mathcal{Z}}$-dimensional vector of observed variables at date $t$ (with $n_{\mathcal{Z}} > n_w$). The inversion technique consists in assuming that a number $n_w$ of observed variables are modeled without measurement errors. In this context, the latent variables can be obtained by inverting a non-singular system of $n_w$ equations and $n_w$ unknowns. The likelihood function can then be derived; it has to be adjusted for the transformation of perfectly-measured variables to latent variables (this adjustment involves the determinant of the Jacobian matrix associated with the previous system, see e.g. Ang and Piazzesi, 2003, Appendix B, or Liu, Longstaff, and Mandell, 2006, Section IV, Eq. 19).

We assume that the last element of $w_t$ is a latent variable that affects $s_t^*$ only. More precisely, we assume that the last column of $\Lambda$ (in Eq. 5) is of the form: $[0, \dots, 0, \bullet]'$. Hence, the last component of $w_t$ does not affect $\Delta y_t$, $\Delta c_t$, $\pi_t$, nor the riskfree rates of any maturity. We denote by $\overline{w}_t$ the first $n_w - 1$ components of $w_t$. Alternatively put, we have $w_t = [\overline{w}_t', w_{n_w,t}]'$, where $n_w$ is the dimension of $w_t$.

The vector $\mathcal{Z}_t$ is decomposed as follows: $\mathcal{Z}_t = [\mathcal{Z}_t^{(1)\prime}, \mathcal{Z}_t^{(2)\prime}, \mathcal{Z}_t^{(3)\prime}, \mathcal{Z}_t^{(4)\prime}, \mathcal{Z}_t^{(5)\prime}]'$, where:

$$\left\{\begin{array}{lcl} \mathcal{Z}_t^{(1)} & : & \text{variables that linearly depend on } \overline{w}_t \text{ and } \mathcal{Z}_t^{(2)}, \\ \mathcal{Z}_t^{(2)} & = & [sd_t, d_t, rd_t, q_t]', \\ \mathcal{Z}_t^{(3)} & = & [\Delta c_t, \Delta y_t, \pi_t]', \\ \mathcal{Z}_t^{(4)} & : & \text{credit spreads (which nonlinearly depend on } x_t), \\ \mathcal{Z}_t^{(5)} & = & \mathcal{D}_t, \text{ Default indicator (0 if no default, 1 otherwise), see Eq. (1).} \end{array}\right.$$

The components of $\mathcal{Z}_t^{(1)}$ can for instance be zero-coupon sovereign yields (i.e. $r_{t,h} = -1/h \log B_{t,h} = -1/h(A_h' w_t + B_h)$, where $B_{t,h}$ is the zero-coupon bond price given by Eq. 10), or forecasts of macroeconomic variables, such as $\mathbb{E}_t(d_{t+h})$ (see Online Appendix III for the computation of such model-implied forecasts). Indeed, under the assumption on $w_{n_w,t}$ made *supra*, such variables do not depend on $w_{n_w,t}$.

Formally:

$$\begin{aligned} \mathcal{Z}_t^{(1)} &= A_{(1)} x_t + B_{(1)} + \varepsilon_t^{(1)} \\ &= \overline{A}_{(1)} \overline{w}_t + A_{(1,2)} \mathcal{Z}_t^{(2)} + B_{(1)} + \varepsilon_t^{(1)}. \end{aligned}$$

We assume that the measurement errors $\varepsilon_t^{(1)}$ are not independent; more precisely, a number $n_w - 1$ of linear combinations of $\mathcal{Z}_t^{(1)}$ are supposed to be modeled without errors. To optimize the model fit, these linear combinations are chosen to minimize the sum of the squared components of $\varepsilon_t^{(1)}$. This is achieved by assuming that $\overline{w}_t$ is given by the OLS formula:

$$\overline{w}_t = \underbrace{(\overline{A}_{(1)}' \overline{A}_{(1)})^{-1} \overline{A}_{(1)}'}_{P_{(1)}} \left( \mathcal{Z}_t^{(1)} - A_{(1,2)} \mathcal{Z}_t^{(2)} - B_{(1)} \right), \tag{a.15}$$

which *de facto* defines those linear combinations of $\mathcal{Z}_t^{(1)}$ and $\mathcal{Z}_t^{(2)}$ that are modeled without errors.

Let us decompose $\mathcal{Z}_t^{(1)}$ as follows: $\mathcal{Z}_t^{(1)} = [\overline{\mathcal{Z}}_t^{(1)\prime}, \underline{\mathcal{Z}}_t^{(1)\prime}]'$, where the dimension of $\overline{\mathcal{Z}}_t^{(1)}$ is $n_w - 1$, i.e. the same as that of $\overline{w}_t$. With these notations, we have:

$$\left[\begin{array}{c} \overline{w}_t \\ \underline{\mathcal{Z}}_t^{(1)} \\ \mathcal{Z}_t^{(2)} \end{array}\right] = cst + \left[\begin{array}{ccc} \overline{P}_{(1)} & \underline{P}_{(1)} & -A_{(1,2)} P_{(1)} \\ 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \left[\begin{array}{c} \overline{\mathcal{Z}}_t^{(1)} \\ \underline{\mathcal{Z}}_t^{(1)} \\ \mathcal{Z}_t^{(2)} \end{array}\right],$$

with $P_{(1)} = [\overline{P}_{(1)} \quad \underline{P}_{(1)}]$, where $\overline{P}_{(1)}$ is a matrix of dimension $(n_w - 1) \times (n_w - 1)$. Hence, the determinant of the Jacobian matrix associated with the transformation from $[\mathcal{Z}_t^{(1)\prime}, \mathcal{Z}_t^{(2)\prime}]'$ to $[\overline{w}_t', \underline{\mathcal{Z}}_t^{(1)\prime}, \mathcal{Z}_t^{(2)\prime}]'$ is $\det(\overline{P}_{(1)})$. (And it is naturally $|\overline{P}_{(1)}{}^{-1}|$for the inverse transformation.)

We also have:

$$\mathcal{Z}_t^{(3)} = \left[\begin{array}{c} \Delta c_t \\ \Delta y_t \\ \pi_t \end{array}\right] = \underbrace{\left[\begin{array}{c} \mu_c \\ \mu_y \\ \mu_\pi \end{array}\right]}_{=:\mu_{c,y,\pi}} + \underbrace{\left[\begin{array}{c} \Lambda_c' \\ \Lambda_y' \\ \Lambda_\pi' \end{array}\right]}_{=:\Lambda_{c,y,\pi}} w_t + \underbrace{\left[\begin{array}{c} \sigma_c' \\ \sigma_y' \\ \sigma_\pi' \end{array}\right]}_{=:\Sigma_{c,y,\pi}} \eta_t.$$

(Note that the last column of matrix $\Lambda_{c,y,\pi}$ is filled with zeros, i.e. $\Lambda_{c,y,\pi} w_t$ only depends on $\overline{w}_t$ and not on $w_{n_w,t}$.) With these notations, we have:

$$\eta_t = \Sigma_{c,y,\pi}^{-1} (\mathcal{Z}_t^{(3)} - \mu_{c,y,\pi} - \Lambda_{c,y,\pi} w_t). \tag{a.16}$$

Suppose that one of the credit spreads, say the first entry of $\mathcal{Z}_t^{(4)}$, is measured without model error. Let us denote by $g_h(x_t)$ the model-implied credit spread of maturity $h$. If the maturity of the perfectly fitted spread is $H$, we have:

$$\overline{\mathcal{Z}}_t^{(4)} = g_H(x_t), \tag{a.17}$$

where $\overline{\mathcal{Z}}_t^{(4)}$ is the first component of $\mathcal{Z}_t^{(4)}$. We will make use the notation $\mathcal{Z}_t^{(4)} = [\overline{\mathcal{Z}}_t^{(4)}, \underline{\mathcal{Z}}_t^{(4)'}]'$, where $\underline{\mathcal{Z}}_t^{(4)}$ is therefore a vector containing all observed credit spreads but the first one.

Recall that $x_t$ is equal to $[w_t', sd_t, \eta_t', d_t, rd_t, q_t]'$ (see Eq. 18). Therefore, Eqs (a.15), (a.16) and (a.17) imply that the information contained in $\mathcal{Z}_t^* := [x_t', \underline{\mathcal{Z}}_t^{(1)'}, \underline{\mathcal{Z}}_t^{(4)'}]'$ is the same as that contained in $\mathcal{Z}_t$. The determinant of the Jacobian matrix associated with the transformation from the former to the latter vector is given by:

$$\kappa_t := |\overline{P}_{(1)}^{-1}| \times |\Sigma_{c,y,\pi}| \times \left|\frac{\partial g_H}{\partial w_{n_w,t}}(x_t)\right|. \tag{a.18}$$

Define $\mathcal{Z}_{1:T} = \{\mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_T\}$. Under the assumption that the measurement errors associated with $\underline{\mathcal{Z}}_t^{(1)}$ and $\underline{\mathcal{Z}}_t^{(4)}$ are zero-mean Gaussian (with respective covariance matrices $\Omega_{(1)}$ and $\Omega_{(4)}$), the (conditional) log-likelihood associated with $\mathcal{Z}_{1:T}$ is given by:

$$\begin{aligned}
& \log \mathcal{L}\left(\mathcal{Z}_{1:T}; \Theta, x_0\right) \\
= & \sum_{t=1}^{T} \Big( \log f_{\underline{\mathcal{Z}}_t^{(1)}|x_t}(\underline{\mathcal{Z}}_t^{(1)}, x_t; \Theta) + \log f_{\underline{\mathcal{Z}}_t^{(4)}|x_t}(\underline{\mathcal{Z}}_t^{(4)}, x_t; \Theta) + \log f_{x_t|x_{t-1}}(x_t, x_{t-1}; \Theta) - \log(\kappa_t) \Big) \\
= & -\frac{T n_{\mathcal{Z}}}{2} \log(2\pi) - \frac{T}{2}\log|\Omega_{(1)}| - \frac{T}{2}\log|\Omega_{(4)}| - \frac{T}{2}\log|\Sigma_x\Sigma_x'| - \sum_{t=1}^{T}\log(\kappa_t) \\
& -\frac{1}{2}\sum_{t=1}^{T}[\underline{\mathcal{Z}}_t^{(1)} - F_{(1)}(x_t;\Theta)]'(\Omega_{(1)})^{-1}[\underline{\mathcal{Z}}_t^{(1)} - F_{(1)}(x_t;\Theta)] \\
& -\frac{1}{2}\sum_{t=1}^{T}[\underline{\mathcal{Z}}_t^{(4)} - F_{(4)}(x_t;\Theta)]'(\Omega_{(4)})^{-1}[\underline{\mathcal{Z}}_t^{(4)} - F_{(4)}(x_t;\Theta)] \\
& -\frac{1}{2}\sum_{t=1}^{T}[x_t - \mu_x - \Phi_x x_{t-1}]'(\Sigma_x\Sigma_x')^{-1}[x_t - \mu_x - \Phi_x x_{t-1}] \\
& +\sum_{t=1}^{T}\mathcal{D}_t \log(PD_t) + (1-\mathcal{D}_t)\log(1-PD_t),
\end{aligned} \tag{a.19}$$

where $PD_t$ is the date $t$ probability of default, that is $1 - \exp[-\alpha \max(0, \ell_{t-1} - d_{t-1})]$ (see Eq. 1), where $F_{(1)}(x_t;\Theta)$ and $F_{(4)}(x_t;\Theta)$ denote the respective model-implied equivalent of $\underline{\mathcal{Z}}_t^{(1)}$ and $\underline{\mathcal{Z}}_t^{(4)}$ and where the last term results from the relationship between $\mathcal{Z}_t$ and $\mathcal{Z}_t^*$.

## Appendix D. Approximation to the fiscal limit

This appendix explains how $\ell_t$ (defined in Eq. 3) is approximated as an affine function of $w_t$ (and hence of $x_t$). Specifically, we look for the vector $a^\ell$ and the scalar $b^\ell$ that are such that $\exp(\ell_t) \approx$

$\exp(a^{\ell\prime} w_t + b^\ell)$. This is done by solving the following system:

$$\left\{\begin{array}{rcl} \mathbb{E}(\exp(\ell_t)) & = & \mathbb{E}(\exp(a^{\ell\prime} w_t + b^\ell)) \\ \mathbb{E}\left(\dfrac{\partial}{\partial w_{k,t}} \exp(\ell_t)\right) & = & \mathbb{E}\left(\dfrac{\partial}{\partial w_{k,t}} \exp(a^{\ell\prime} w_t + b^\ell)\right), \quad k \in \{1, \dots, n_w\}. \end{array}\right. \tag{a.20}$$

According to Eq. (5), and assuming that $\sigma_{s^*} = 0$, the maximum primary budget surplus (as a fraction of quarterly GDP) is given by:

$$s_t^* = \mu_{s^*} + \Lambda_{s^*} w_t.$$

In order to exploit the fact that we know how to compute conditional expectations of exponential affine functions of future values of $w_t$, it is computationally convenient to approximate $s_t^*$ as:

$$s_t^* \approx s^*(w_t) = \mu_s^* + \beta\left[1 - \exp\left(-\frac{1}{\beta}\Lambda_{s^*} w_t - \frac{1}{2\beta^2}\Lambda_{s^*}'\Omega_w\Lambda_{s^*}\right)\right], \tag{a.21}$$

where $\beta$ is a large scalar (we take $\beta = 10.000$ in our computations) and where $\Omega_w = \mathbb{V}ar(w_t)$ – implying notably $\mathbb{E}(s_t^*) = \mathbb{E}(s^*(w_t)) = \mu_s^*$.

Online Appendix VI further shows that System (a.20) is then approximately satisfied when, for all $k \in \{1, \dots, n_w\}$:

$$\begin{aligned} a_k^\ell = & \left\{(\mu_{s^*} + \beta)\sum_{h=1}^{+\infty} a_{1,h,k}^\ell \mathbb{E}\left(\exp\left(a_{1,h}^{\ell}{}' w_t + b_{1,h}^\ell\right)\right)\right. \\ & \left. -\beta\exp\left(-\frac{1}{2}\Lambda_{s^*}'\Omega_w\Lambda_{s^*}\right)\sum_{h=1}^{+\infty} a_{0,h,k}^\ell \mathbb{E}\left(\exp\left(a_{0,h}^{\ell}{}' w_t + b_{0,h}^\ell\right)\right)\right\} / \\ & \left\{(\mu_{s^*} + \beta)\sum_{h=1}^{+\infty} \mathbb{E}\left(\exp\left(a_{1,h}^{\ell}{}' w_t + b_{1,h}^\ell\right)\right)\right. \\ & \left. -\beta\exp\left(-\frac{1}{2}\Lambda_{s^*}'\Omega_w\Lambda_{s^*}\right)\sum_{h=1}^{+\infty} \mathbb{E}\left(\exp\left(a_{0,h}^{\ell}{}' w_t + b_{0,h}^\ell\right)\right)\right\}, \end{aligned} \tag{a.22}$$

where vectors $a_{0,h}^\ell$ and $a_{1,h}^\ell$ and scalars $b_{0,h}^\ell$ and $b_{1,h}^\ell$ satisfy:

$$\left\{\begin{array}{rcl} \exp\left(a_{0,h}^{\ell}{}' w_t + b_{0,h}^\ell\right) & = & \mathbb{E}_t\left[M_{t,t+h}\exp(\Delta y_{t+1} + \dots + \Delta y_{t+h} - \Lambda_{s^*} w_{t+h}) | \mathcal{D} \equiv 0\right] \\ \exp\left(a_{1,h}^{\ell}{}' w_t + b_{1,h}^\ell\right) & = & \mathbb{E}_t\left[M_{t,t+h}\exp(\Delta y_{t+1} + \dots + \Delta y_{t+h}) | \mathcal{D} \equiv 0\right]. \end{array}\right. \tag{a.23}$$

Online Appendix VI shows that these $a_{i,h}^\ell$'s and $b_{i,h}^\ell$'s ($i \in \{0,1\}$) are given by:

$$\begin{aligned} a_{i,h}^\ell &= \kappa_0 - \Phi'^h\left(\mathbb{1}_{\{i=0\}}\Lambda_{s^*} + \kappa_0\right) \\ b_{i,h}^\ell &= h\kappa_1 + (h-1)\kappa_0'(\Lambda_y - \gamma\Lambda_c) + \frac{1}{2}(h-1)\kappa_0'\kappa_0 + \\ & \quad - \left(\mathbb{1}_{\{i=0\}}\Lambda_{s^*} + \kappa_0\right)'(I - \Phi)^{-1}(\Phi - \Phi^h)(\Lambda_y - \gamma\Lambda_c + \kappa_0) + \\ & \quad + \frac{1}{2}\left(\mathbb{1}_{\{i=0\}}\Lambda_{s^*} + \kappa_0\right)'\left(\sum_{k=0}^{\infty}\Phi^k\Phi'^k - I - \Phi^h\left[\sum_{k=0}^{\infty}\Phi^k\Phi'^k\right]\Phi'^h\right)\left(\mathbb{1}_{\{i=0\}}\Lambda_{s^*} + \kappa_0\right), \end{aligned}$$

where

$$\left\{\begin{array}{rcl} \kappa_0 & = & (I - \Phi')^{-1}\Phi'(\Lambda_y - \gamma\Lambda_c) \\ \kappa_1 & = & \log(\delta) - \gamma\mu_c + \mu_y + \frac{1}{2}(\sigma_y - \gamma\sigma_c)'(\sigma_y - \gamma\sigma_c) + \frac{1}{2}(\Lambda_y - \gamma\Lambda_c)'(\Lambda_y - \gamma\Lambda_c). \end{array}\right.$$

and with $vec\left(\sum_{k=0}^{\infty}\Phi^k{\Phi'}^k\right) = \left(I_{n_w^2} - \Phi\otimes\Phi\right)^{-1} vec(I_{n_w^2})$.

Once the $a_k^\ell$'s, have been computed (using Eq. a.22), we compute $b^\ell$ as follows:

$$b^\ell = \log \mathbb{E}\left(\exp(\ell_t)\right) - \frac{1}{2}{a^\ell}'\Omega_w a^\ell,$$

where $\mathbb{E}\left(\exp(\ell_t)\right)$ is the denominator of the fraction appearing on the right-hand side of Eq. (a.22).

# Appendix E. Data sources

TABLE 2. Data Panel: United States of America

| **Variable** | **Horizon / Maturity** | **Source** | **Period** | **N. of Obs.** |
|---|---|---|---|---|
| Government Debt / Nominal GDP Forecasts | 2 Years | CBO - BEO[a] | 01/2007-04/2018 | 28 |
| | 3 Years | CBO - BEO | 01/2007-04/2018 | 28 |
| | 5 Years | CBO - BEO | 01/2007-04/2018 | 28 |
| | 10 Years | CBO - BEO | 01/2007-04/2018 | 28 |
| Senior CDS | 1 Year | Thomson Reuters Datastream | 2008Q2-2018Q4 | 43 |
| | 2 Years | Thomson Reuters Datastream | 2008Q2-2018Q4 | 43 |
| | 3 Years | Thomson Reuters Datastream | 2008Q2-2018Q4 | 43 |
| | 5 Years | Thomson Reuters Datastream | 2008Q2-2018Q4 | 43 |
| | 10 Years | Thomson Reuters Datastream | 2008Q2-2018Q4 | 43 |
| Yields | 1 Year | Federal Reserve, US | 2007Q1-2018Q4 | 48 |
| | 2 Years | Federal Reserve, US | 2007Q1-2018Q4 | 48 |
| | 3 Years | Federal Reserve, US | 2007Q1-2018Q4 | 48 |
| | 5 Years | Federal Reserve, US | 2007Q1-2018Q4 | 48 |
| | 10 Years | Federal Reserve, US | 2007Q1-2018Q4 | 48 |
| GDP, market constant prices (CHND 2012) | - | BEA[b] - US Dep. of Commerce | 2007Q1-2018Q4 | 48 |
| GDP, market current prices | - | BEA - US Dep. of Commerce | 2007Q1-2018Q4 | 48 |
| Final Consumption Expenditure, Services | - | FRED (ST. LOUIS FED) | 2007Q1-2018Q4 | 48 |
| Final Consumption Expenditure, Non-Durables | - | FRED (ST. LOUIS FED) | 2007Q1-2018Q4 | 48 |
| GDP Implicit Price Deflator (Index 2012=100) | - | BEA - US Dep. of Commerce | 2007Q1-2018Q4 | 48 |
| Gross Federal Government Debt, Current Prices | - | FRED (ST. LOUIS FED) | 2007Q1-2018Q4 | 48 |
| Net Government Interest Payments, Current Prices | - | FRED (ST. LOUIS FED) | 2007Q1-2018Q4 | 48 |
| Government Primary Surplus/Deficit, Current Prices | - | FRED (ST. LOUIS FED) | 2007Q1-2018Q4 | 48 |

[a] Congressional Budget Office - Budget and Economic Outlook

[b] Bureau of Economic Analysis

TABLE 3. Data Panel: United Kingdom

| Variable | Horizon / Maturity | Source | Period | N. of Obs. |
|---|---|---|---|---|
| Government Debt / Nominal GDP Forecasts | 1 Year | IMF - WEO[a] | 10/2007-10/2018 | 23 |
| | 2 Years | IMF - WEO | 10/2009-10/2018 | 23 |
| | 3 Years | IMF - WEO | 10/2009-10/2018 | 23 |
| | 4 Years | IMF - WEO | 10/2009-10/2018 | 23 |
| | 5 Years | IMF - WEO | 10/2009-10/2018 | 23 |
| | 2 Years | EC - Stab. and Conv. Prog.[b] | 12/2005-12/2008 | 4 |
| | 3 Years | EC - Stab. and Conv. Prog. | 12/2005-12/2008 | 4 |
| Senior CDS | 1 Year | Thomson Reuters Datastream | 2008Q3-2018Q4 | 42 |
| | 2 Years | Thomson Reuters Datastream | 2008Q3-2018Q4 | 42 |
| | 3 Years | Thomson Reuters Datastream | 2008Q3-2018Q4 | 42 |
| | 5 Years | Thomson Reuters Datastream | 2008Q3-2018Q4 | 42 |
| | 10 Years | Thomson Reuters Datastream | 2008Q4-2018Q4 | 41 |
| Yields | 1 Year | Thomson Reuters Datastream | 2007Q1-2018Q4 | 48 |
| | 2 Years | Thomson Reuters Datastream | 2007Q1-2018Q4 | 48 |
| | 3 Years | Thomson Reuters Datastream | 2007Q1-2018Q4 | 48 |
| | 5 Years | Thomson Reuters Datastream | 2007Q1-2018Q4 | 48 |
| | 10 Years | Thomson Reuters Datastream | 2007Q1-2018Q4 | 48 |
| GDP, market constant prices (CHND 2016) | - | Office for National Statistics | 2007Q1-2018Q3 | 47 |
| GDP, market current prices | - | Office for National Statistics | 2007Q1-2018Q3 | 47 |
| Final Consumption Expenditure, Services | - | Office for National Statistics | 2007Q1-2018Q3 | 47 |
| Final Consumption Expenditure, Non-Durables | - | Office for National Statistics | 2007Q1-2018Q3 | 47 |
| GDP Implicit Price Deflator (Index 2016=100) | - | Office for National Statistics | 2007Q1-2018Q3 | 47 |
| Gross Government Debt, Total, Current Prices | - | Office for National Statistics | 2007Q1-2018Q3 | 47 |
| Gross Government Interest Payments, Current Prices | - | OECD Economic Outlook | 2007Q1-2018Q4 | 48 |
| Net Government Interest Payments, Current Prices | - | OECD Economic Outlook | 2007Q1-2018Q4 | 48 |
| Government Primary Surplus/Deficit, Current Prices | - | OECD Economic Outlook | 2007Q1-2018Q4 | 48 |

[a] International Monetary Fund - World Economic Outlook

[b] European Commission - Stability and Convergence Programme

TABLE 4. Data Panel: Germany

| Variable | Horizon / Maturity | Source | Period | N. of Obs. |
|---|---|---|---|---|
| Government Debt / Nominal GDP Forecasts | 1 Year | IMF - WEO [a] | 10/2007-10/2018 | 23 |
| | 2 Years | IMF - WEO | 10/2009-10/2018 | 23 |
| | 3 Years | IMF - WEO | 10/2009-10/2018 | 23 |
| | 4 Years | IMF - WEO | 10/2009-10/2018 | 23 |
| | 5 Years | IMF - WEO | 10/2009-10/2018 | 23 |
| | 2 Years | EC - Stab. and Conv. Prog. [b] | 02/2006-12/2008 | 4 |
| | 3 Years | EC - Stab. and Conv. Prog. | 02/2006-12/2008 | 4 |
| | 4 Years | EC - Stab. and Conv. Prog. | 02/2006-12/2008 | 4 |
| Senior CDS | 1 Year | Thomson Reuters Datastream | 2008Q1-2018Q4 | 44 |
| | 2 Years | Thomson Reuters Datastream | 2008Q1-2018Q4 | 44 |
| | 3 Years | Thomson Reuters Datastream | 2008Q1-2018Q4 | 44 |
| | 5 Years | Thomson Reuters Datastream | 2008Q1-2018Q4 | 44 |
| | 10 Years | Thomson Reuters Datastream | 2008Q4-2018Q4 | 41 |
| Yields | 1 Year | Thomson Reuters Datastream | 2007Q1-2018Q4 | 48 |
| | 2 Years | Thomson Reuters Datastream | 2007Q1-2018Q4 | 48 |
| | 3 Years | Thomson Reuters Datastream | 2007Q1-2018Q4 | 48 |
| | 5 Years | Thomson Reuters Datastream | 2007Q1-2018Q4 | 48 |
| | 10 Years | Thomson Reuters Datastream | 2007Q1-2018Q4 | 48 |
| GDP, market constant prices (CHND 2010) | - | Federal Statistical Office | 2007Q1-2018Q3 | 47 |
| GDP, market current prices | - | Federal Statistical Office | 2007Q1-2018Q3 | 47 |
| Final Consumption Expenditure, Services | - | Eurostat (ESA2010 DB) | 2007Q1-2018Q3 | 47 |
| Final Consumption Expenditure, Non-Durables | - | Eurostat (ESA2010 DB) | 2007Q1-2018Q3 | 47 |
| GDP Implicit Price Deflator (Index 2010=100) | - | Thomson Reuters | 2007Q1-2018Q3 | 47 |
| Gross Government Debt, Total, Current Prices | - | Eurostat (ESA2010 DB) | 2007Q1-2018Q3 | 47 |
| Government Interest Payments, Current Prices | - | ECB - SDW | 2007Q1-2018Q3 | 47 |
| Government Primary Surplus/Deficit, Current Prices | - | Eurostat (ESA2010 DB) | 2007Q1-2018Q3 | 47 |

[a] International Monetary Fund - World Economic Outlook

[b] European Commission - Stability and Convergence Programme

TABLE 5. Data Panel: Japan

| Variable | Horizon / Maturity | Source | Period | N. degree of Obs. |
|---|---|---|---|---|
| Government Debt / Nominal GDP Forecasts | 1 Year | IMF - WEO [a] | 10/2007-10/2018 | 23 |
| | 2 Years | IMF - WEO | 10/2009-10/2018 | 23 |
| | 3 Years | IMF - WEO | 10/2009-10/2018 | 23 |
| | 4 Years | IMF - WEO | 10/2009-10/2018 | 23 |
| | 5 Years | IMF - WEO | 10/2009-10/2018 | 23 |
| | 2 Years | EC - Economic Forecast[b] | 10/2006-04/2009 | 6 |
| Senior CDS | 1 Year | Thomson Reuters Datastream | 2008Q2-2018Q4 | 43 |
| | 2 Years | Thomson Reuters Datastream | 2008Q2-2018Q4 | 43 |
| | 3 Years | Thomson Reuters Datastream | 2008Q2-2018Q4 | 43 |
| | 5 Years | Thomson Reuters Datastream | 2008Q2-2018Q4 | 43 |
| | 10 Years | Thomson Reuters Datastream | 2008Q2-2018Q4 | 43 |
| Yields | 1 Year | Thomson Reuters Datastream | 2007Q1-2018Q4 | 48 |
| | 2 Years | Thomson Reuters Datastream | 2007Q1-2018Q4 | 48 |
| | 3 Years | Thomson Reuters Datastream | 2007Q1-2018Q4 | 48 |
| | 5 Years | Thomson Reuters Datastream | 2007Q1-2018Q4 | 48 |
| | 10 Years | Thomson Reuters Datastream | 2007Q1-2018Q4 | 48 |
| GDP, market constant prices (CHND 2011) | - | Cabinet Office (Gov. of Japan) | 2007Q1-2018Q3 | 47 |
| GDP, market current prices | - | Cabinet Office (Gov. of Japan) | 2007Q1-2018Q3 | 47 |
| Final Consumption Expenditure, Services | - | Cabinet Office (Gov. of Japan) | 2007Q1-2018Q3 | 47 |
| Final Consumption Expenditure, Non-Durables | - | Cabinet Office (Gov. of Japan) | 2007Q1-2018Q3 | 47 |
| GDP Implicit Price Deflator (Index 2011=100) | - | Thomson Reuters | 2007Q1-2018Q3 | 47 |
| Gross Government Debt, Total, Current Prices | - | Bank of Japan | 2007Q1-2018Q4 | 48 |
| Gross Government Interest Payments, Current Prices | - | OECD Economic Outlook | 2007Q1-2018Q4 | 48 |
| Net Government Interest Payments, Current Prices | - | OECD Economic Outlook | 2007Q1-2018Q4 | 48 |
| Government Primary Surplus/Deficit, Current Prices | - | OECD Economic Outlook | 2007Q1-2018Q4 | 48 |

[a] International Monetary Fund - World Economic Outlook

[b] European Commission - Economic Forecast

# – Online Appendix –

# Fiscal Limits and Sovereign Credit Spreads

Kevin PALLARA and Jean-Paul RENNE

## APPENDIX I. APPROXIMATE CDS PRICING FORMULA

According to Eq. (4), we have:

$$S^{cds}_{t,h} = (1-RR)\frac{\mathbb{E}_t\left\{\sum_{k=1}^{h} M^n_{t,t+k}(\mathcal{D}_{t+k}-\mathcal{D}_{t+k-1})\right\}}{\mathbb{E}_t\left\{\sum_{k=1}^{h} M^n_{t,t+k}(1-\mathcal{D}_{t+k})\right\}}.$$

As a consequence, the computation of the CDS spread $S^{cds}_{t,h}$ necessitates the knowledge of the following two conditional expectations: $\mathbb{E}_t\left[M^n_{t,t+h}(1-\mathcal{D}_{t+h-1})\right]$ and $\mathbb{E}_t\left[M^n_{t,t+h}(1-\mathcal{D}_{t+h})\right]$, which can be seen as "binary CDSs" in the sense that they correspond to date-$t$ prices of instruments providing a binary payoff (0 and 1) depending on the default status of the government on dates $t+h-1$ and $t+h$, respectively.

Subsection I.1 shows that these two prices are given by combinations of conditional exponential expectations of future values of $(x_t', \underline{\lambda}_t)'$. Subsection I.2 and I.3 explain how to approximate these conditional expectations.

### I.1. Prices of binary CDSs.

We consider the following situation:

(a) the s.d.f. $M^n_{t,t+1}$ does not Granger-cause $x_t$ and is of the form:

$$M^n_{t,t+1} = \exp(\varphi_0 + \varphi_1' x_{t+1} + \varphi_2(\mathcal{D}_{t+1}-\mathcal{D}_t)), \tag{I.1}$$

which is consistent with Eqs. (6) and (8) of the paper, with

$$\varphi_0 = \log(\delta) - \gamma\mu_c - \mu_\pi, \quad \varphi_1 = [-\gamma\Lambda_c' - \Lambda_\pi, 0, -\sigma_c' - \sigma_\pi', 0, 0, 0]', \quad \text{and } \varphi_2 = \gamma b_c + \gamma_\pi \tag{I.2}$$

(recall that $x_t = [w_t', sd_t, \eta_t', d_t, rd_t, q_t]'$),

(b) $\underline{\lambda}_t$ is the default intensity defined through Eqs. (1) and (2).

Obviously, $\mathbb{E}_t\left[M^n_{t,t+h}(1-\mathcal{D}_{t+h-1})\right]$ and $\mathbb{E}_t\left[M^n_{t,t+h}(1-\mathcal{D}_{t+h})\right]$ are equal to zero if $\mathcal{D}_t = 1$. In the following, we proceed under the assumption that $\mathcal{D}_t = 0$.

- Computation of $\mathbb{E}_t\left[M^n_{t,t+h}(1-\mathcal{D}_{t+h})\right]$. We have:

$$\begin{aligned}
& \mathbb{E}_t\left[M^n_{t,t+1}\times\cdots\times M^n_{t+h-1,t+h}(1-\mathcal{D}_{t+h})\right] \\
= \;& \exp(h\varphi_0)\,\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})+\varphi_2\mathcal{D}_{t+h}\}(1-\mathcal{D}_{t+h})\right] \\
= \;& \exp(h\varphi_0)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})\}\mathbb{1}_{\{\mathcal{D}_{t+h}=0\}}\right] \\
= \;& \exp(h\varphi_0)\mathbb{E}_t\left[\mathbb{E}_t\left(\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})\}\mathbb{1}_{\{\mathcal{D}_{t+h}=0\}}|\underline{x_{t+h}}\right)\right] \\
= \;& \exp(h\varphi_0)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})-\underline{\lambda}_{t+1}-\cdots-\underline{\lambda}_{t+h}\}\right],
\end{aligned} \tag{I.3}$$

where the last equality results from the fact that $\mathcal{D}_t$ does not Granger-cause $x_t$ (and from the fact that Granger's ans Sims' types of causality are equivalent).

- Computation of $\mathbb{E}_t\left[M^n_{t,t+h}(1-\mathcal{D}_{t+h-1})\right]$. We have:

$$
\begin{aligned}
& \mathbb{E}_t\left[M^n_{t,t+1}\times\cdots\times M^n_{t+h-1,t+h}(1-\mathcal{D}_{t+h-1})\right]\\
= \;& \exp(h\varphi_0)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})+\varphi_2\mathcal{D}_{t+h}\}(1-\mathcal{D}_{t+h-1})\right]\\
= \;& \exp(h\varphi_0)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})\}\mathbb{1}_{\{\mathcal{D}_{t+h-1}=0\}}\left(\mathbb{1}_{\{\mathcal{D}_{t+h}=0\}}+\mathbb{1}_{\{\mathcal{D}_{t+h}=1\}}\exp(\varphi_2)\right)\right]\\
= \;& \exp(h\varphi_0)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})\}\mathbb{1}_{\{\mathcal{D}_{t+h}=0\}}\right]+\\
& \exp(h\varphi_0+\varphi_2)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})\}\mathbb{1}_{\{\mathcal{D}_{t+h-1}=0\}}\mathbb{1}_{\{\mathcal{D}_{t+h}=1\}}\right]\\
= \;& \exp(h\varphi_0)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})\}\mathbb{1}_{\{\mathcal{D}_{t+h}=0\}}\right]+\\
& \exp(h\varphi_0+\varphi_2)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})\}\mathbb{1}_{\{\mathcal{D}_{t+h-1}=0\}}\left(1-\mathbb{1}_{\{\mathcal{D}_{t+h}=0\}}\right)\right]\\
= \;& \exp(h\varphi_0)(1-\exp(\varphi_2))\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})\}\mathbb{1}_{\{\mathcal{D}_{t+h}=0\}}\right]+\\
& \exp(h\varphi_0+\varphi_2)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})\}\mathbb{1}_{\{\mathcal{D}_{t+h-1}=0\}}\right]\\
= \;& \exp(h\varphi_0)(1-\exp(\varphi_2))\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})-\underline{\lambda}_{t+1}-\cdots-\underline{\lambda}_{t+h}\}\right]+\\
& \exp(h\varphi_0+\varphi_2)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})-\underline{\lambda}_{t+1}-\cdots-\underline{\lambda}_{t+h-1}\}\right]. \qquad \text{(I.4)}
\end{aligned}
$$

**I.2. Approximating the conditional expectations appearing in Eqs. (I.3) and (I.4) – and in Eq. (IV.1).** Let us introduce the following notations:

$$
\begin{aligned}
P_{t,n} &\equiv \mathbb{E}_t\left[\exp(\varphi_1'(x_{t+1}+\cdots+x_{t+n})-(\underline{\lambda}_{t+1}+\cdots+\underline{\lambda}_{t+n})\right] && \text{(I.5)}\\
P^*_{t,n} &\equiv \mathbb{E}_t\left[\exp(\varphi_1'(x_{t+1}+\cdots+x_{t+n})-(\underline{\lambda}_{t+1}+\cdots+\underline{\lambda}_{t+n-1})\right] && \text{(I.6)}
\end{aligned}
$$

and, further:

$$
\begin{aligned}
f_{n-1,n} &= -\log P_{t,n}+\log P_{t,n-1}\\
f^*_{n-1,n} &= -\log P^*_{t,n}+\log P^*_{t,n-1}.
\end{aligned}
$$

Following Wu and Xia (2016), we approximate $P_{t,n}$ and $P^*_{t,n}$ by approximating, in the first place, $f_{n-1,n}$ and $f^*_{n-1,n}$. Using that, for any random variable $Z$, we have $\log\mathbb{E}[\exp(Z)]\approx\mathbb{E}(Z)+1/2\mathbb{V}ar(Z)$, we get:

$$
\begin{aligned}
f_{n-1,n} &= \mathbb{E}_t(-\varphi_1'x_{t+n}+\underline{\lambda}_{t+n})\\
&\quad -\frac{1}{2}\mathbb{V}ar_t\left(-\varphi_1'x_{t+n}+\underline{\lambda}_{t+n}\right)-\mathbb{C}ov_t\left(-\varphi_1'x_{t+n}+\underline{\lambda}_{t+n},\sum_{i=1}^{n-1}(-\varphi_1'x_{t+i}+\underline{\lambda}_{t+i})\right) && \text{(I.7)}\\
f^*_{n-1,n} &= \mathbb{E}_t(-\varphi_1'x_{t+n}+\underline{\lambda}_{t+n-1})-\frac{1}{2}\mathbb{V}ar_t\left(-\varphi_1'x_{t+n}+\underline{\lambda}_{t+n-1}\right)\\
&\quad -\mathbb{C}ov_t\left(-\varphi_1'x_{t+n}+\underline{\lambda}_{t+n-1},-\varphi_1'x_{t+1}+\sum_{i=2}^{n-1}(-\varphi_1'x_{t+i}+\underline{\lambda}_{t+i-1})\right). && \text{(I.8)}
\end{aligned}
$$

Let us introduce the following notations:

$$\lambda_t=\alpha(d_t-\ell_t)\quad\text{(implying } \underline{\lambda}_t=\max(0,\lambda_t)\text{, see Eq. 2) and}\quad p_{t,n}=\mathbb{P}_t(d_{t+n}>\ell_{t+n}).$$

In the spirit of Wu and Xia (2016), exploiting the fact that $\lambda_t$ is a persistent process, we have, for $0 < n$ and $0 \leq j \leq n$:

$$\begin{aligned} \mathbb{C}ov_t(-\varphi_1' x_{t+n}, \underline{\lambda}_{t+n-j}) &\approx p_{t,n-j}\mathbb{C}ov_t\left[-\varphi_1' x_{t+n}, \lambda_{t+n-j}\right] && \text{(I.9)}\\ \mathbb{C}ov_t(\underline{\lambda}_{t+n}, \underline{\lambda}_{t+n-j}) &\approx p_{t,n-j}\mathbb{C}ov_t\left[\lambda_{t+n}, \lambda_{t+n-j}\right]. && \text{(I.10)}\end{aligned}$$

Using the last two equations, approximations to Eqs. (I.7) and (I.8) are respectively given by:

$$\begin{aligned} f_{n-1,n,t} \approx\; & \mathbb{E}_t\left[-\varphi_1' x_{t+n} + \underline{\lambda}_{t+n}\right] && \text{(I.11)}\\ & -\frac{1}{2}\left(p_{t,n}\mathbb{V}ar_t\left[-\varphi_1' x_{t+n} + \lambda_{t+n}\right] + (1-p_{t,n})\mathbb{V}ar_t(-\varphi_1' x_{t+n})\right)\\ & -\sum_{j=1}^{n-1}\left\{p_{t,j}\mathbb{C}ov_t\left[-\varphi_1' x_{t+n} + \lambda_{t+n}, -\varphi_1' x_{t+j} + \lambda_{t+j}\right] + (1-p_{t,j})\mathbb{C}ov_t(-\varphi_1' x_{t+n}, -\varphi_1' x_{t+j})\right\},\end{aligned}$$

$$\begin{aligned} f^*_{n-1,n,t} \approx\; & \mathbb{E}_t\left[-\varphi_1' x_{t+n} + \underline{\lambda}_{t+n-1}\right] && \text{(I.12)}\\ & -\frac{1}{2}\left(p_{t,n}\mathbb{V}ar_t\left[-\varphi_1' x_{t+n} + \lambda_{t+n-1}\right] + (1-p_{t,n})\mathbb{V}ar_t(-\varphi_1' x_{t+n})\right)\\ & -\sum_{j=2}^{n-1}\left\{p_{t,j}\mathbb{C}ov_t\left[-\varphi_1' x_{t+n} + \lambda_{t+n-1}, i_{t+j} + \lambda_{t+j}\right] + (1-p_{t,j})\mathbb{C}ov_t(-\varphi_1' x_{t+n}, i_{t+j})\right\}\\ & -p_{t,1}\mathbb{C}ov_t\left[-\varphi_1' x_{t+n} + \lambda_{t+n-1}, -\varphi_1' x_{t+1}\right] + (1-p_{t,1})\mathbb{C}ov_t(-\varphi_1' x_{t+n}, -\varphi_1' x_{t+1}).\end{aligned}$$

Introducing the following notations:

$$\begin{aligned} \lambda_t = a + b' x_t \quad &\text{and} \quad \dot{b} = -\varphi_1, && \text{(I.13)}\\ \mu_{t,n} = \mathbb{E}_t(x_{t+n}), \quad &\text{and} \quad \Gamma_{n,j} = \mathbb{C}ov_t(x_{t+n}, x_{t+n-j}), && \text{(I.14)}\\ \mu_{\lambda,t,n} = \mathbb{E}_t(\lambda_{t+n}) = a + b'\mu_{t,n}, \quad &\text{and} \quad \sigma_{\lambda,n} = \sqrt{\mathbb{V}ar_t(\lambda_{t+n})} = \sqrt{b'\Gamma_{n,0} b}, && \text{(I.15)}\end{aligned}$$

Eqs. (I.11) and (I.12) respectively rewrite:

$$\begin{aligned} f_{n-1,n,t} \approx\; & \dot{b}'\mu_{t,n} + \Phi\left(\mu_{\lambda,t,n}/\sigma_{\lambda,n}\right)\mu_{\lambda,t,n} + \phi(-\mu_{\lambda,t,n}/\sigma_{\lambda,n})\sigma_{\lambda,n}\\ & -\frac{1}{2}\left(p_{t,n}\left[b+\dot{b}\right]'\Gamma_{n,0}\left[b+\dot{b}\right] + \left[1-p_{t,n}\right]\dot{b}'\Gamma_{n,0}\dot{b}\right)\\ & -\sum_{j=2}^{n-1}\left\{p_{t,n-j}\left[\dot{b}'\Gamma_{n-1,j-1}(b+\dot{b}) + b'\Gamma_{n,j}(b+\dot{b})\right] + \left[1-p_{t,n-j}\right]\dot{b}'\Gamma_{n-1,j-1}\dot{b}\right\}\\ & -p_{t,n-1}\left[\dot{b}'\Gamma_{n-1,0} b + b'\Gamma_{n,1} b\right], && \text{(I.16)}\end{aligned}$$

$$\begin{aligned} f^*_{n-1,n,t} \approx\; & \dot{a} + \dot{b}'\mu_{t,n-1} + \Phi\left(\mu_{\lambda,t,n-1}/\sigma_{\lambda,n-1}\right)\mu_{\lambda,t,n-1} + \phi(-\mu_{\lambda,t,n-1}/\sigma_{\lambda,n-1})\sigma_{\lambda,n-1}\\ & -\frac{1}{2}\left(p_{t,n}\left[b+\dot{b}\right]'\Gamma_{n-1,0}\left[b+\dot{b}\right] + \left[1-p_{t,n}\right]\dot{b}'\Gamma_{n-1,0}\dot{b}\right)\\ & -\sum_{j=2}^{n-1}\left\{p_{t,n-j}(b+\dot{b})'\Gamma_{n-1,j-1}(b+\dot{b}) + \left[1-p_{t,n-j}\right]\dot{b}'\Gamma_{n-1,j-1}\dot{b}\right\}. && \text{(I.17)}\end{aligned}$$

Appendix I.3 details the computation of $\mu_{t,n}$ and $\Gamma_{n,j}$. (Note that $\Gamma_{n,0} = \mathbb{V}ar_t(x_{t+n})$.)

I.3. **Computation of $\mu_{t,n}$ and $\Gamma_{n,j}$.** Recall $x_t$'s law of motion is (Eq. 18):

$$x_t = \mu_x + \Phi_x x_{t-1} + \Sigma_x \varepsilon_{x,t}, \quad \varepsilon_{x,t} = [\varepsilon_t', \varepsilon_{s,t}, \eta_t']' \sim i.i.d.\mathcal{N}(0, Id).$$

Using the notation $\Omega = \Sigma_x \Sigma_x'$, we have:

$$\left\{\begin{array}{lllll} \mu_{t,n} & = & \mathbb{E}_t(x_{t+n}) & = & (Id-\Phi_x)^{-1}(Id-\Phi_x^n)\mu_x + \Phi_x^n x_t, \\ \Gamma_{n,0} & = & \mathbb{V}ar_t(x_{t+n}) & = & \Omega + \Phi_x \Gamma_{n-1,0}\Phi_x', \quad \text{with} \quad \Gamma_{1,0} = \Omega \\ & & & = & \Omega + \Phi_x \Omega \Phi_x' + \dots + \Phi_x^{n-1}\Omega \Phi_x^{n-1\prime}, \\ \Gamma_{n,j} & = & \mathbb{C}ov_t(x_{t+n}, x_{t+n-j}) & = & \Phi_x^j \Gamma_{n-j,0} \quad \text{if} \quad n-j>0. \end{array}\right.$$

The estimation involves a large number of computations of the $\Gamma_{n,j}$'s. In order to speed up the computation, we employ the following matrix computation.

Consider a vector $\beta$ of dimension $n_x$, that is the dimension of $x_t$, and let us denote by $\xi_i^\beta$ the vector defined by $\xi_i^\beta = (\Phi_x^i)'\beta$ ($\beta$ will typically be $b$, or $(b+\dot{b})$, see Eqs. I.16 and I.17).

Because we have $\Gamma_{n,j} = \Phi_x^j\Omega + \Phi_x^{j+1}\Omega\Phi_x' + \dots + \Phi_x^{n-1}\Omega\Phi_x^{n-1-j\prime}$, it comes that:

$$\beta'\Gamma_{n,j}\beta = \xi_j^{\beta\prime}\Omega\xi_0^\beta + \xi_{j+1}^{\beta\;\prime}\Omega\xi_1^\beta + \dots + \xi_{n-1}^{\beta\;\prime}\Omega\xi_{n-1-j}^\beta. \tag{I.18}$$

Let us consider a maximal value for $n$, say $H$, and let us denote by $\Xi_\beta$ the $n_x \times (H+1)$ matrix whose $i^{th}$ column is $\xi_{i-1}^\beta$. It can then be seen that the $(j,k)$ entry of $\Psi^\beta := \Xi_\beta{}'\Omega\Xi_\beta$ – which is a matrix of dimension $(H+1)\times(H+1)$ – is equal to $\xi_{j-1}^{\beta\;\prime}\Omega\xi_{k-1}^\beta$. The sum of the entries $(j+1,1)$, $(j+2,2)$, $\dots$, $(j+k,k)$ of $\Psi^\beta$ therefore is

$$\xi_j^{\beta\prime}\Omega\xi_0^\beta + \xi_{j+1}^{\beta\;\prime}\Omega\xi_1^\beta + \dots + \xi_{j+k-1}^{\beta\;\prime}\Omega\xi_{k-1}^\beta,$$

which is equal to $\beta'\Gamma_{j+k,j}\beta$ according to (I.18). Equivalently, $\beta'\Gamma_{n,j}\beta$ is the sum of the entries $(j+1,1)$, $(j+2,2)$, $\dots$, $(n,n-j)$ of $\Psi^\beta$.

In particular, the entry $(1,1)$ of $\Psi^\beta$ is equal to $\beta'\Gamma_{1,0}\beta$, the sum of the entries $(1,1)$ and $(2,2)$ is equal to $\beta'\Omega\beta + \beta'\Phi_x\Omega\Phi_x'\beta = \beta'\Gamma_{2,0}\beta$, and, more generally, the sum of the entries $(1,1),\dots,(n-1,n-1)$ of $\Psi^\beta$ is equal to $\beta'\Gamma_{n,0}\beta$.

## Appendix II. Multi-horizon Laplace-transform in the context of a Gaussian VAR

If $w_t$ follows a Gaussian VAR, that is if

$$w_t = \Phi w_{t-1} + \varepsilon_t, \tag{II.1}$$

where $\varepsilon_t \sim i.i.d.\mathcal{N}(0, Id)$, then we have:

$$\mathbb{E}_t\left[\exp\{u'(w_{t+1}+\dots+w_{t+h})\}\right] = \exp(b_h(u) + a_h(u)'w_t),$$

where $a_h(u)$ and $b_h(u)$ recursively satisfy:

$$\left\{\begin{array}{lll} a_h & = & \Phi'(a_{h-1}(u)+u) \\ b_h & = & b_{h-1}(u) + \frac{1}{2}(a_{h-1}(u)+u)'(a_{h-1}(u)+u), \end{array}\right.$$

where $a_0 = 0$ and $b_0 = 0$.

**Proof.** If $\mathbb{E}_t\left[\exp\{u'(w_{t+1}+\cdots+w_{t+h-1})\}\right] = \exp(b_{h-1}(u)+a_{h-1}(u)'w_t)$ holds for any vector $u$, then:

$$\begin{aligned}
& \mathbb{E}_t\left[\exp\{u'(w_{t+1}+\cdots+w_{t+h})\}\right] \\
= \ & \mathbb{E}_t\left[\exp\{u'w_{t+1}\}\mathbb{E}_{t+1}\left[\exp\{u'(w_{t+2}+\cdots+w_{t+h})\}\right]\right] \\
= \ & \mathbb{E}_t[\exp\{u'w_{t+1}+b_{h-1}(u)+a_{h-1}(u)'w_{t+1}\}] \quad \text{(using the recursive assumption)} \\
= \ & \mathbb{E}_t[\exp\{(a_{h-1}(u)+u)'w_{t+1}+b_{h-1}(u)\}] \\
= \ & \mathbb{E}_t\left[\exp\left\{b_{h-1}(u)+[\Phi'(a_{h-1}(u)+u)]'w_t+\frac{1}{2}(a_{h-1}(u)+u)'(a_{h-1}(u)+u)\right\}\right],
\end{aligned}$$

where the last equality results from Eq. (II.1), that is $w_t$'s law of motion. □

## APPENDIX III. MODEL-IMPLIED FORECASTS

Consider a process $x_t$ following a Gaussian VAR(1) process (as in Eq. 18):

$$x_t = \mu_x + \Phi_x x_{t-1} + \Sigma_x \eta_t, \tag{III.1}$$

where $\eta_t \sim i.i.d.\mathcal{N}(0, Id)$.

Consider a variable, say $Z_t$, that is a linear combination of $x_t$, that is:

$$Z_t = a'_Z x_t + b_Z.$$

We have:

$$\mathbb{E}_t(Z_{t+h}) = b_Z + a'_Z \mathbb{E}_t(x_{t+h}), \tag{III.2}$$

with, given Eq. (III.1):

$$\begin{aligned}
\mathbb{E}_t(x_{t+h}) &= (I+\Phi_x+\cdots+\Phi_x^{h-1})\mu_x + \Phi_x^h x_t \\
&= (I-\Phi_x)^{-1}(I-\Phi_x^h)\mu_x + \Phi_x^h x_t.
\end{aligned}$$

Moreover:

$$\begin{aligned}
\mathbb{E}_t(Z_{t+1}+\cdots+Z_{t+h}) &= hb_Z + ha'_Z(I-\Phi_x)^{-1}\mu_x \\
&\quad -a'_Z(I-\Phi_x)^{-1}\left(I+\Phi_x+\cdots+\Phi_x^{h-1}\right)\Phi_x\mu_x \\
&\quad +a'_Z\left(I+\Phi_x+\cdots+\Phi_x^{h-1}\right)\Phi_x x_t \\
&= hb_Z + ha'_Z(I-\Phi_x)^{-1}\mu_x \\
&\quad -a'_Z\{(I-\Phi_x)^{-1}\}^2(I-\Phi_x^h)\Phi_x\mu_x + a'_Z(I-\Phi_x)^{-1}(I-\Phi_x^h)\Phi_x x_t.
\end{aligned}$$

## APPENDIX IV. RISK-FREE VERSUS PERPETUITIES YIELDS

IV.1. **Risk-free yields.** Considering the same situation as the one described in Appendix I.1, let us derive the date-$t$ price of a maturity-$h$ nominal risk-free zero-coupon bond:

$$\begin{aligned}
& \mathbb{E}_t\left[M^n_{t,t+1}\times\cdots\times M^n_{t+h-1,t+h}\right]\\
= \; & \exp(h\varphi_0)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})+\varphi_2\mathcal{D}_{t+h}\}\right]\\
= \; & \exp(h\varphi_0)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})\}\left(\mathbb{1}_{\{\mathcal{D}_{t+h}=0\}}+\mathbb{1}_{\{\mathcal{D}_{t+h}=1\}}\exp(\varphi_2)\right)\right]\\
= \; & \exp(h\varphi_0)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})\}\left(\exp(\varphi_2)+\mathbb{1}_{\{\mathcal{D}_{t+h}=0\}}(1-\exp(\varphi_2))\right)\right]\\
= \; & \exp(h\varphi_0+\varphi_2)\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})\}\right]+\\
& \exp(h\varphi_0)(1-\exp(\varphi_2))\mathbb{E}_t\left[\exp\{\varphi_1'(x_{t+1}+\cdots+x_{t+h})-\underline{\lambda}_{t+1}-\cdots-\underline{\lambda}_{t+h}\}\right]. \qquad \text{(IV.1)}
\end{aligned}$$

The price of a real risk-free zero-coupon bond is given by $\mathbb{E}_t\left[M_{t,t+1}\times\cdots\times M_{t+h-1,t+h})\right]$. Using Eq. (I.1) together with the specification of $\pi_t$ underlying Eq. (5), it is easily seen that the real s.d.f. is of the form $\exp(\bar\varphi_0+\bar\varphi_1'x_{t+1}+\bar\varphi_2(\mathcal{D}_{t+1}-\mathcal{D}_t))$, where the $\bar\varphi_i$'s are easily deduced from the $\varphi_i$'s (defined in Eq. I.2) and $\mu_\pi$, $\Lambda_\pi$ and $\sigma_\pi$ (since $M_{t,t+1}=M^n_{t,t+1}\exp(\pi_{t+1})$). Therefore, the price of a real risk-free zero-coupon bond is given by the same expression as in Eq. (IV.1), replacing the $\varphi_i$'s by the $\bar\varphi_i$'s.

The fact that $\underline{\lambda}_t$ appears in the risk-free bond pricing formula implies that, to be computed, risk-free yields have to rely on the approximate formula presented in Online Appendix I.2. As a result, (nominal and real) risk-free yields are non-linear functions of $x_t$.

IV.2. **Pricing the decaying-coupon perpetuity.** Subsection 3.7 (and more precisely Eqs. 9 and 10) shows that the price of the perpetuity is of the form:

$$\mathcal{P}_t \equiv \sum_{i=1}^{\infty}\chi^{i-1}\exp[B_i+A_i'w_t], \qquad \text{(IV.2)}$$

where $B_i = i\bar\varphi_0 + b_i(\bar\varphi_1)$ and $A_i'w_t = a_i(\bar\varphi_1)'w_t$, where $\bar\varphi_0$, $\bar\varphi_1$, as well as functions $a_i(\bullet)$ and $b_i(\bullet)$ are defined in Appendix B.1 (functions $a_i(\bullet)$ and $b_i(\bullet)$ are also given in Appendix II). By definition, the yield-to-maturity of the perpetuity, denoted by $q_t$, satisfies:

$$\mathcal{P}_t = \sum_{h=1}^{\infty}\frac{\chi^{h-1}}{(1+q_t)^h}.$$

The right-hand-side sum of the previous expression is equal to

$$\mathcal{P}(q_t) \equiv \frac{1}{1+q_t-\chi}.$$

Therefore, the yield-to-maturity $q_t$ of the perpetuity is the solution of the following equation $\mathcal{P}_t = \mathcal{P}(q_t)$, where $\mathcal{P}_t$ is given by Eq. (IV.2). Solving for $q_t$ is straightforward and leads to:

$$q_t = \frac{1}{\sum_{i=1}^{\infty}\chi^{i-1}\exp[B_i+A_i'w_t]}-(1-\chi),$$

which shows that $q_t$ is not an affine function of $w_t$ (and therefore of $x_t$). However, because the perpetuity is a collection of zero-coupons of price $B_{t,h}$ (with geometrically-decaying weights, Eq. **??**), the yield-to-maturity of the perpetuity is expected to be close to the yield of an "average" zero-coupon, that is to one of the $r_{t,h}$'s, where $r_{t,h} = -\frac{1}{h}B_h - \frac{1}{h}A_h'w_t$. Practically, we look for the maturity $h\in\mathbb{N}$ that minimizes the deviation between $\mathbb{V}ar(\mathcal{P}_t)$ and $\mathbb{V}ar(\mathcal{P}(r_{t,h}))$. Formally, we use the following approximation:

$$\begin{aligned} q_t &\approx a'_{h^*} x_t + b_{h^*}, \\ \text{where } h^* &= \underset{h \in \mathbb{N}}{\operatorname{argmin}} |\mathbb{V}ar(\mathcal{P}_t) - \mathbb{V}ar(\mathcal{P}(r_{t,h}))|. \end{aligned} \tag{IV.3}$$

It remains to explain how $\mathbb{V}ar(\mathcal{P}_t)$ and $\mathbb{V}ar(\mathcal{P}(r_{t,h}))$ are computed.

- The approximation of $\mathbb{V}ar(\mathcal{P}(r_{t,h}))$ is based on Taylor expansions of $\mathcal{P}(q)$. Specifically, a fourth-order Taylor expansion of $q \mapsto \mathcal{P}(q) = \frac{1}{1+q-\chi}$ around $q_0$ gives $\mathcal{P}(q) = \sum_{i=0}^{4} \frac{(q-q_0)^i}{(1+q_0-\chi)^{i+1}} + o((q-q_0)^4)$, leading to the following approximation of $\mathbb{E}(\mathcal{P}(q))$:
$$\frac{1}{1+\mathbb{E}(q)-\chi} + \frac{\mathbb{V}ar(q)}{(1+\mathbb{E}(q)-\chi)^3} + \frac{Skew(q)\mathbb{V}ar(q)^{3/2}}{(1+\mathbb{E}(q)-\chi)^3} + \frac{Kurt(q)\mathbb{V}ar(q)^2}{(1+\mathbb{E}(q)-\chi)^5}.$$
By the same token, using a second-order Taylor expansion of $q \mapsto \mathcal{P}(q)^2 = \frac{1}{(1+q-\chi)^2}$, we get to the following approximation of $\mathbb{E}(\mathcal{P}(q)^2)$:
$$\frac{1}{(1+\mathbb{E}(q)-\chi)^2} + 3\frac{\mathbb{V}ar(q)}{(1+\mathbb{E}(q)-\chi)^4} - 4\frac{Skew(q)\mathbb{V}ar(q)^{3/2}}{(1+\mathbb{E}(q)-\chi)^5} + 5\frac{Kurt(q)\mathbb{V}ar(q)^2}{(1+\mathbb{E}(q)-\chi)^6}.$$
An approximation of $\mathbb{V}ar[\mathcal{P}(r_{t,h})] = \mathbb{E}[\mathcal{P}(r_{t,h})^2] - \mathbb{E}[\mathcal{P}(r_{t,h})]^2$ can then be obtained by employing the last two approximations of $\mathbb{E}[\mathcal{P}(r_{t,h})^2]$ and $\mathbb{E}[\mathcal{P}(r_{t,h})]$, replacing $\mathbb{E}(q)$ by $\mathbb{E}(r_{t,h}) = b_h$, and $\mathbb{V}ar(q)$ by $\mathbb{V}ar(r_{t,h}) = a'_h \Sigma_x a_h$ and – considering a Gaussian distribution for $r_{t,h}$ – using $Skew(q) = 0$ and $Kurt(q) = 3$ .
- Let us turn to the computation of $\mathbb{V}ar(\mathcal{P}_t)$, where $\mathcal{P}_t$ is given in Eq. (IV.2). We compute the variance in a recursive fashion. For this purpose, let us introduce the following notation:
$$\mathcal{P}_{t,h} \equiv \sum_{i=1}^{h} \chi^{i-1} \exp[B_i + A'_i w_t] \underset{h\to\infty}{\to} \mathcal{P}_t.$$
The variance of $\mathcal{P}_t$ can be approximated by $\mathbb{V}ar(\mathcal{P}_{t,H})$ for a sufficiently large $H$. The variance of $\mathcal{P}_{t,H}$ is computed recursively: We have $\mathbb{V}ar(\mathcal{P}_{t,0}) = 0$ and, $\mathbb{V}ar(\mathcal{P}_{t,h+1})$, $h \geq 1$, is given by:

$$\begin{aligned} & \mathbb{V}ar(\mathcal{P}_{t,h}) + \mathbb{V}ar(\chi^h \exp[B_{h+1} + A'_{h+1} w_t]) + 2\sum_{i=1}^{h} \mathbb{C}ov\left\{\chi^{i-1}\exp[B_i + A'_i w_t], \chi^h \exp[B_{h+1} + A'_{h+1} w_t]\right\} \\ = \; & \mathbb{V}ar(\mathcal{P}_{t,h}) + \chi^{2h}\exp(2B_{h+1})\left[\exp(2A'_{h+1}\mathbb{V}(w_t)A_{h+1}) - \exp(A'_{h+1}\mathbb{V}(w_t)A_{h+1})\right] + \\ & +2\chi^h \exp\left(B_{h+1} + \frac{1}{2}A'_{h+1}\mathbb{V}(w_t)A_{h+1}\right)\sum_{i=1}^{h}\chi^{i-1}\exp\left(B_i + \frac{1}{2}A'_i\mathbb{V}(w_t)A_i\right)\left[\exp\left(A'_i\mathbb{V}(w_t)A_{h+1}\right) - 1\right]. \end{aligned}$$

## Appendix V. Relationship between $\underline{\lambda}_t$ and $\underline{\lambda}_t^{\mathbb{Q}}$

By Bayes, we have:

$$f^{\mathbb{Q}}(\mathcal{D}_t | x_t, \mathcal{I}_{t-1}) = \frac{f^{\mathbb{Q}}(\mathcal{D}_t, x_t | \mathcal{I}_{t-1})}{f^{\mathbb{Q}}(x_t | \mathcal{I}_{t-1})}. \tag{V.1}$$

Assume $\mathcal{D}_{t-1} = 0$. We have:

$$\begin{aligned}
f^{\mathbb{Q}}(\mathcal{D}_t, x_t|\mathcal{I}_{t-1}) &= \frac{M_{t-1,t}}{\mathbb{E}(M_{t-1,t}|\mathcal{I}_{t-1})} f^{\mathbb{P}}(\mathcal{D}_t, x_t|\mathcal{I}_{t-1}) \\
&= \frac{\exp(\varphi_1' x_t + \varphi_2 \mathcal{D}_t)}{\mathbb{E}[\exp(\varphi_1' x_t + \varphi_2 \mathcal{D}_t)|\mathcal{I}_{t-1}]} f^{\mathbb{P}}(\mathcal{D}_t, x_t|\mathcal{I}_{t-1}) \\
&= \frac{\exp(\varphi_1' x_t + \varphi_2 \mathcal{D}_t)}{\mathbb{E}[\exp(\varphi_1' x_t + \varphi_2 \mathcal{D}_t)|\mathcal{I}_{t-1}]} f^{\mathbb{P}}(\mathcal{D}_t|x_t, \mathcal{I}_{t-1}) f^{\mathbb{P}}(x_t|x_{t-1}) \\
&= \frac{\exp(\varphi_1' x_t + \varphi_2 \mathcal{D}_t)}{\mathbb{E}[\exp(\varphi_1' x_t + \varphi_2 \mathcal{D}_t)|\mathcal{I}_{t-1}]} \times \\
&\quad (\mathcal{D}_t\{1 - \exp(-\underline{\lambda}_t)\} + (1 - \mathcal{D}_t)\{\exp(-\underline{\lambda}_t)\}) f^{\mathbb{P}}(x_t|x_{t-1}).
\end{aligned} \tag{V.2}$$

Integrating both sides w.r.t. $\mathcal{D}_t$, we get:

$$f^{\mathbb{Q}}(x_t|\mathcal{I}_{t-1}) = \exp(\varphi_1' x_t) \frac{\exp(\varphi_2)\{1 - \exp(-\underline{\lambda}_t)\} + \exp(-\underline{\lambda}_t)}{\mathbb{E}[\exp(\varphi_1' x_t + \varphi_2 \mathcal{D}_t)|\mathcal{I}_{t-1}]} f^{\mathbb{P}}(x_t|x_{t-1}). \tag{V.3}$$

Using Eqs. (V.2) and (V.3) in Eq. (V.1), we get:

$$f^{\mathbb{Q}}(\mathcal{D}_t|x_t, \mathcal{I}_{t-1}) = \frac{\exp(\varphi_2 \mathcal{D}_t)\,(\mathcal{D}_t\{1 - \exp(-\underline{\lambda}_t)\} + (1 - \mathcal{D}_t)\{\exp(-\underline{\lambda}_t)\})}{\exp(\varphi_2)\{1 - \exp(-\underline{\lambda}_t)\} + \exp(-\underline{\lambda}_t)},$$

which implies that:

$$\exp(-\underline{\lambda}_t^{\mathbb{Q}}) \equiv \mathbb{Q}(\mathcal{D}_t = 0|\mathcal{D}_t = 0, x_t, \mathcal{I}_{t-1}) = \frac{\exp(-\underline{\lambda}_t)}{\exp(\varphi_2)\{1 - \exp(-\underline{\lambda}_t)\} + \exp(-\underline{\lambda}_t)},$$

which can be rewritten as:

$$\underline{\lambda}_t^{\mathbb{Q}} = \underline{\lambda}_t + \log(\exp(\varphi_2)\{1 - \exp(-\underline{\lambda}_t)\} + \exp(-\underline{\lambda}_t)). \tag{V.4}$$

If $\varphi_2 > 0$, we have $\exp(\varphi_2)\{1 - \exp(-\underline{\lambda}_t)\} + \exp(-\underline{\lambda}_t) > 1$, and therefore $\underline{\lambda}_t^{\mathbb{Q}} > \underline{\lambda}_t$.

## Appendix VI. Approximation to the fiscal limit

In this appendix, we explain how Eq. (3) can be approximated. We consider the following specification of $s_t^*$:

$$s_t^* = \mu_s^* + \beta \left[1 - \exp\left(-\Lambda_{s^*} w_t - \frac{1}{2}\Lambda_{s^*}' \Omega_w \Lambda_{s^*}\right)\right], \tag{VI.1}$$

where $\Omega_w = \mathbb{V}ar(w_t)$, and therefore with $\mathbb{E}(s_t^*) = \mu_s^*$.

Let us denote by $a_{0,h}^{\ell}$, $a_{1,h}^{\ell}$, $b_{0,h}^{\ell}$, $b_{1,h}^{\ell}$ the vectors and scalars that are such that:

$$\left\{\begin{array}{lcl}
\exp\left({a_{0,h}^{\ell}}' w_t + b_{0,h}^{\ell}\right) & = & \mathbb{E}_t\left[M_{t,t+h} \exp(\Delta y_{t+1} + \dots + \Delta y_{t+h} - \Lambda_{s^*} w_{t+h})|\mathcal{D} \equiv 0\right] \\
\exp\left({a_{1,h}^{\ell}}' w_t + b_{1,h}^{\ell}\right) & = & \mathbb{E}_t\left[M_{t,t+h} \exp(\Delta y_{t+1} + \dots + \Delta y_{t+h})|\mathcal{D} \equiv 0\right].
\end{array}\right. \tag{VI.2}$$

With these notations, Eq. (3) becomes:

$$\exp(\ell_t) = (\mu_{s^*} + \beta) \sum_{h=1}^{+\infty} \exp\left({a_{1,h}^{\ell}}' w_t + b_{1,h}^{\ell}\right) - \beta \exp\left(-\frac{1}{2}\Lambda_{s^*}' \Omega_w \Lambda_{s^*}\right) \sum_{h=1}^{+\infty} \exp\left({a_{0,h}^{\ell}}' w_t + b_{0,h}^{\ell}\right), \tag{VI.3}$$

We want to find $a^\ell$ and $b^\ell$ that are such that $\exp(\ell_t) \approx \exp(a^{\ell\prime} w_t + b^\ell)$. This is done by solving the following system:

$$\left\{\begin{array}{rcl} \mathbb{E}(\exp(\ell_t)) & = & \mathbb{E}(\exp(a^{\ell\prime} w_t + b^\ell)) \\ \mathbb{E}\left(\dfrac{\partial}{\partial w_{k,t}}\exp(\ell_t)\right) & = & \mathbb{E}\left(\dfrac{\partial}{\partial w_{k,t}}\exp(a^{\ell\prime} w_t + b^\ell)\right), \quad k \in \{1,\dots,n_w\}. \end{array}\right. \tag{VI.4}$$

We have:

$$\mathbb{E}(\exp(a^{\ell\prime} w_t + b^\ell)) = \exp(a^{\ell\prime}\Omega_w a^\ell + b^\ell) \tag{VI.5}$$

$$\mathbb{E}\left(\frac{\partial}{\partial w_{k,t}}\exp(a^{\ell\prime} w_t + b^\ell)\right) = a_k^\ell \exp(a^{\ell\prime}\Omega_w a^\ell + b^\ell) \tag{VI.6}$$

Using Eq. (VI.3), we have:

$$\begin{aligned} \mathbb{E}\left(\exp(\ell_t)\right) &= (\mu_{s^*}+\beta)\sum_{h=1}^{+\infty}\mathbb{E}\left(\exp\left(a_{1,h}^{\ell}{}' w_t + b_{1,h}^\ell\right)\right) \\ &\quad -\beta\exp\left(-\frac{1}{2}\Lambda_{s^*}'\Omega_w\Lambda_{s^*}\right)\sum_{h=1}^{+\infty}\mathbb{E}\left(\exp\left(a_{0,h}^{\ell}{}' w_t + b_{0,h}^\ell\right)\right) \end{aligned} \tag{VI.7}$$

$$\begin{aligned} \mathbb{E}\left(\frac{\partial}{\partial w_{k,t}}\exp(\ell_t)\right) &= (\mu_{s^*}+\beta)\sum_{h=1}^{+\infty}a_{1,h,k}^\ell\mathbb{E}\left(\exp\left(a_{1,h}^{\ell}{}' w_t + b_{1,h}^\ell\right)\right) \\ &\quad -\beta\exp\left(-\frac{1}{2}\Lambda_{s^*}'\Omega_w\Lambda_{s^*}\right)\sum_{h=1}^{+\infty}a_{0,h,k}^\ell\mathbb{E}\left(\exp\left(a_{0,h}^{\ell}{}' w_t + b_{0,h}^\ell\right)\right). \end{aligned} \tag{VI.8}$$

System (VI.4) implies that the result of the division of (VI.6) by (VI.5) – that is $a_k^\ell$ – should be equal to that of (VI.8) by (VI.7). Once the $a_k^\ell$'s, are obtained (by VI.8/VI.7), we compute $b^\ell$ as follows:

$$b^\ell = \log\mathbb{E}\left(\exp(\ell_t)\right) - \frac{1}{2}a^{\ell\prime}\Omega_w a^\ell,$$

where $\mathbb{E}\left(\exp(\ell_t)\right)$ is given by Eq. (VI.7).

Let us now explain how to compute the $a_{i,h}^\ell$'s and $b_{i,h}^\ell$'s (for $i \in \{0,1\}$, as defined in Eq. VI.2). For $h = 1$, we have:

$$\left\{\begin{array}{rcl} a_{i,1}^\ell & = & \Phi'(\Lambda_y - \gamma\Lambda_c - \mathbb{1}_{\{i=0\}}\Lambda_{s^*}) \\ b_{i,1}^\ell & = & \log(\delta) - \gamma\mu_c + \mu_y + \frac{1}{2}(\sigma_y - \gamma\sigma_c)'(\sigma_y-\gamma\sigma_c) \\ && +\frac{1}{2}(\Lambda_y - \gamma\Lambda_c - \mathbb{1}_{\{i=0\}}\Lambda_{s^*})'(\Lambda_y - \gamma\Lambda_c - \mathbb{1}_{\{i=0\}}\Lambda_{s^*}). \end{array}\right. \tag{VI.9}$$

For $h > 0$, we have

$$\begin{aligned} & \exp\left(a_{i,h+1}^{\ell}{}' w_t + b_{i,h+1}^\ell\right) \\ = \; & \mathbb{E}_t\left[M_{t,t+1}\exp\left(\Delta y_{t+1} + a_{i,h}^{\ell}{}' w_{t+1} + b_{i,h}^\ell\right)\right] \quad \text{(by the law of iterated expectations)} \\ = \; & \exp\left(\log(\delta) - \gamma\mu_c + \mu_y + \frac{1}{2}(\sigma_y-\gamma\sigma_c)'(\sigma_y-\gamma\sigma_c) + b_{i,h}^\ell\right)\mathbb{E}_t\left[\exp\left(\{\Lambda_y - \gamma\Lambda_c + a_{i,h}^\ell\}' w_{t+1}\right)\right]. \end{aligned}$$

Hence, for $h > 0$, we have:

$$\left\{\begin{array}{lcl} a^{\ell}_{i,h+1} & = & \Phi'(\Lambda_y - \gamma\Lambda_c + a^{\ell}_{i,h}) \\ b^{\ell}_{i,h+1} & = & \log(\delta) - \gamma\mu_c + \mu_y + \frac{1}{2}(\sigma_y - \gamma\sigma_c)'(\sigma_y - \gamma\sigma_c) + b^{\ell}_{i,h} \\ & & +\frac{1}{2}(\Lambda_y - \gamma\Lambda_c + a^{\ell}_{i,h})'(\Lambda_y + \Lambda_\pi + a^{\ell}_{i,h}), \end{array}\right. \quad \text{(VI.10)}$$

with $a^{\ell}_0 = -\mathbb{1}_{\{i=0\}}\Lambda_{s^*}$ and $b^{\ell}_0 = 0$ (which results from Eq. VI.9).

By iterating, we obtain, for $h \geq 1$:

$$\begin{aligned} a^{\ell}_{i,h} &= \Phi'(\Lambda_y - \gamma\Lambda_c + a^{\ell}_{i,h-1}) \\ &= \Phi'(\Lambda_y - \gamma\Lambda_c) + \Phi'^2(\Lambda_y - \gamma\Lambda_c + a^{\ell}_{i,h-2}) \\ &= (I + \Phi' + \cdots + \Phi'^{h-1})\Phi'(\Lambda_y - \gamma\Lambda_c) + \Phi'^h a^{\ell}_{i,0} \\ &= (I - \Phi')^{-1}(I - \Phi'^h)\Phi'(\Lambda_y - \gamma\Lambda_c) + \Phi'^h a^{\ell}_{i,0} \\ &= \underbrace{(I - \Phi')^{-1}\Phi'(\Lambda_y - \gamma\Lambda_c)}_{=\kappa_0} + \Phi'^h\left(a^{\ell}_{i,0} - (I - \Phi')^{-1}\Phi'(\Lambda_y - \gamma\Lambda_c)\right) \\ &= \kappa_0 - \Phi'^h\left(\mathbb{1}_{\{i=0\}}\Lambda_{s^*} + \kappa_0\right). \end{aligned} \quad \text{(VI.11)}$$

Moreover, for $h > 0$:

$$\begin{aligned} b^{\ell}_{i,h} &= \underbrace{\log(\delta) - \gamma\mu_c + \mu_y + \frac{1}{2}(\sigma_y - \gamma\sigma_c)'(\sigma_y - \gamma\sigma_c) + \frac{1}{2}(\Lambda_y - \gamma\Lambda_c)'(\Lambda_y - \gamma\Lambda_c)}_{=\kappa_1} \\ &\quad + b^{\ell}_{i,h-1} + a^{\ell\,\prime}_{i,h-1}(\Lambda_y - \gamma\Lambda_c) + \frac{1}{2}a^{\ell\,\prime}_{i,h-1}a^{\ell}_{i,h-1} \\ &= \kappa_1 + a^{\ell\,\prime}_{i,h-1}(\Lambda_y - \gamma\Lambda_c) + \frac{1}{2}a^{\ell\,\prime}_{i,h-1}a^{\ell}_{i,h-1} + \\ &\quad \kappa_1 + a^{\ell\,\prime}_{i,h-2}(\Lambda_y - \gamma\Lambda_c) + \frac{1}{2}a^{\ell\,\prime}_{i,h-2}a^{\ell}_{i,h-2} + \ldots \\ &\quad \kappa_1 + a^{\ell\,\prime}_{i,0}(\Lambda_y - \gamma\Lambda_c) + \frac{1}{2}a^{\ell\,\prime}_{i,0}a^{\ell}_{i,0}. \end{aligned}$$

Using Eq. (VI.11) for $k > 0$, we obtain:

$$\begin{aligned}
b^{\ell}_{i,h} &= h\kappa_1 + \sum_{k=1}^{h-1} \left(\kappa_0 - \Phi'^{k}\left(\mathbb{1}_{\{i=0\}}\Lambda_{s^*} + \kappa_0\right)\right)' (\Lambda_y - \gamma\Lambda_c) + \\
&\quad \frac{1}{2}\sum_{k=1}^{h-1} \left(\kappa_0 - \Phi'^{k}\left(\mathbb{1}_{\{i=0\}}\Lambda_{s^*} + \kappa_0\right)\right)' \left(\kappa_0 - \Phi'^{k}\left(\mathbb{1}_{\{i=0\}}\Lambda_{s^*} + \kappa_0\right)\right) \\
&\quad -\mathbb{1}_{\{i=0\}}\Lambda'_{s^*}(\Lambda_y - \gamma\Lambda_c) + \frac{1}{2}\mathbb{1}_{\{i=0\}}\Lambda'_{s^*}\mathbb{1}_{\{i=0\}}\Lambda_{s^*} \qquad \text{(VI.12)} \\
&= h\kappa_1 + (h-1)\kappa'_0(\Lambda_y - \gamma\Lambda_c) + \frac{1}{2}(h-1)\kappa'_0\kappa_0 + \\
&\quad -\left(\mathbb{1}_{\{i=0\}}\Lambda_{s^*} + \kappa_0\right)' \sum_{k=1}^{h-1} \Phi^k(\Lambda_y - \gamma\Lambda_c + \kappa_0) + \\
&\quad \frac{1}{2}\left(\mathbb{1}_{\{i=0\}}\Lambda_{s^*} + \kappa_0\right)' \left(\sum_{k=1}^{h-1}\Phi^k\Phi'^k\right)\left(\mathbb{1}_{\{i=0\}}\Lambda_{s^*} + \kappa_0\right) \\
&= h\kappa_1 + (h-1)\kappa'_0(\Lambda_y - \gamma\Lambda_c) + \frac{1}{2}(h-1)\kappa'_0\kappa_0 + \\
&\quad -\left(\mathbb{1}_{\{i=0\}}\Lambda_{s^*} + \kappa_0\right)' (I - \Phi)^{-1}(\Phi - \Phi^h)(\Lambda_y - \gamma\Lambda_c + \kappa_0) + \\
&\quad +\frac{1}{2}\left(\mathbb{1}_{\{i=0\}}\Lambda_{s^*} + \kappa_0\right)' \left(\sum_{k=0}^{\infty}\Phi^k\Phi'^k - I - \Phi^h\left[\sum_{k=0}^{\infty}\Phi^k\Phi'^k\right]\Phi'^h\right)\left(\mathbb{1}_{\{i=0\}}\Lambda_{s^*} + \kappa_0\right),
\end{aligned}$$

with

$$vec\left(\sum_{k=0}^{\infty}\Phi^k\Phi'^k\right) = \left(I_{n_w^2} - \Phi\otimes\Phi\right)^{-1} vec(I_{n_w^2}).$$

## Appendix VII. VAR(1) dynamics of the state vector $x_t$

The dynamics of the state vector $x_t$ (Eq. (18) approximately is:

$$x_t = \begin{bmatrix} w_t \\ sd_t \\ \eta_t \\ d_t \\ rd_t \\ q_t \end{bmatrix} \approx \mu_x + \Phi_x x_{t-1} + \Sigma_x\left[\varepsilon'_t, \varepsilon_{s,t}, \eta'_t\right]',$$

with

$$\mu_x = \begin{bmatrix} 0 \\ -\gamma_d\bar{d} \\ 0 \\ (-\mu_y - \mu_\pi + \log(1 + \bar{q} - \overline{sd}) - \Psi(1-\chi)\bar{q} + \Psi\gamma_d\bar{d}) \\ -(1-\chi)\bar{q} \\ a_{h^*} \end{bmatrix},$$

$$\Phi_x = \begin{bmatrix} \Phi & 0 & 0 & 0 & 0 & 0 \\ (\gamma_y \Lambda_y') & (\rho) & (\gamma_y \sigma_y) & (\gamma_d) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -(\Lambda_y' \Phi + \Lambda_\pi' \Phi + \Psi \gamma_y \Lambda_y') & (-\Psi \rho) & (-\Psi \gamma_y \sigma_y') & (1 - \Psi \gamma_d) & (\Psi \chi) & (\Psi(1-\chi)) \\ 0 & 0 & 0 & 0 & (\chi) & (1-\chi) \\ (b_{h^*} \Phi) & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\Sigma_x = \begin{bmatrix} I & 0 & 0 \\ 0 & \sigma_s & 0 \\ 0 & 0 & I \\ (-(\Lambda_\pi + \Lambda_y)') & (-\Psi \sigma_s) & (-(\sigma_\pi + \sigma_y)') \\ 0 & 0 & 0 \\ b_{h^*} & 0 & 0 \end{bmatrix},$$

where $\Psi = 1/(1 + \bar{q} - \overline{sd})$.