Linear-Quadratic Approximation to Optimal Policy: An Algorithm and Two Applications*

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Abstract

This paper aims at bridging the gap between recent theoretical progress in welfare-based optimal policy and its application to models suitable for policy analysis. With this purpose, the framework of Benigno and Woodford (2005b) with a linear-quadratic (LQ) approximation (in a timeless perspective) to the optimal policy problem is applied to monetary policy in two relatively standard models. Furthermore the applications are performed through a standardised algorithm that is suitable for being implemented for a broad class of models.

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1 Introduction

In recent years there has been considerable theoretical progress in deriving welfare-based optimal monetary policy in DSGE models. There have been however relatively few applications of those results, probably due to computational difficulties in applying those results, in particular to models which are to some extent suitable for policy analysis. This paper aims at partly filling this gap in two ways. First, the paper presents an explicit algorithm and a Matlab function built to implement the theoretical results in Benigno and Woodford (2005b).[1] This function is general enough so that it can be readily implemented to a broad range of models.[2] The paper provides two applications of an approximation to optimal monetary policy, based on a linear-quadratic (LQ) approximation to the optimal policy programme in a timeless perspective. In doing this, we provide the description of the algorithm followed and show that, in our applications, LQ optimal monetary policy can be computationally solved for in a relatively simple way.[3]

The first of our two application is to a stylised closed economy model with monetary frictions. In this case the analysis of LQ optimal monetary policy takes three (related) dimensions. The first one is concerned with the steady state, the second entails the comparison to the cashless limiting case and the third is the evaluation of optimal policy vis-à-vis a standard Taylor rule.

In what concerns the steady state, under the cashless limiting case the inflation is always optimally set to zero. However, when monetary frictions are present there is always some deflation in the optimal steady state, at the same time that the nominal interest rate falls short of the discount rate of the households. Under the presence of monetary frictions, the Friedman rule is always valid if prices are flexible. It is moreover also valid under nominal rigidity, if taxes are not set to eliminate the monopolistic competition distortions and the level of price stickiness is very low.

Under the second dimension of analysis, the responses to the different shocks are very similar in the cashless limiting case when compared to the monetary frictions case, with exactly the same pattern in both cases and only minor differences in magnitude.

Finally, the comparison between the simple interest rate rule to the optimal policy in this economy reveals that, under increases to the tax rate or the price markup, the Taylor rule is not aggressive enough in reacting to inflation, compared to the optimal policy. Under a shock to the government expenditures the Taylor rule appears to be not as aggressive in reacting to the output changes as the optimal policy is. The responses to an output shock are however very different when we consider optimal policy and a simple Taylor rule. The optimal policy keeps the inflation close to the steady state values by switching the aggregate demand in the direction of the change in the aggregate supply (hence in the considered case

[1] The Matlab function uses the symbolic toolbox and therefore can be used only if that toolbox is installed.

[2] In each application we provide a small section detailing the transformations of the basic models that are needed in order to fit the structure for the code.

[3] For an additional application of the proposed algorithm to the linear-quadratic approximation of optimal policy to a model of a currency union, see Altissimo, Benigno, and Rodriguez-Palenzuela (2004).

of a positive shock the optimal monetary policy is further expansionary). The Taylor rule instead will react to correct the increase in the output vis-à-vis the steady state (the Taylor rule is not defined in terms of output gap) and therefore it will be a contractionary policy, further deepening the output gap in existence.

Our second application of LQ optimal policy in a timeless prespective is to a version of the standard Neo-Keynesian model of a closed economy with various structural shocks, which follows closely Smets and Wouters (2003). This class of models is increasingly popular in empirical studies as well as policy applications. One feature that is very typical of this class of estimated models is the assumption of Taylor rules. It then becomes especially important to investigate the extent to which a Taylor rule diverges from optimal policy. The results presented here show that the differences between the two are not negligible both qualitatively and quantitatively. In sum, the results suggest that LQ policy differentiates to a larger extent its impact taking into account the supply or demand nature of the shocks, in general exacerbating the effects of the former and dampening the effects of the latter relative to the Taylor case.

While results obtained are preliminary, the work indicates that the relatively light computational burden for calculating LQ optimal policy puts a premium to this approach when using it in the context of policy simulations. Moreover, the method proposed by Benigno and Woodford (2005b) and used in our algorithm is robust to any model that can be put in the form presented here, yielding a correct first-order approximation to the optimal policy problem. These properties promote the future implementation of the algorithm in models with high empirical content, useful for policy analysis.

The remainder of the paper is organized as follows. Section 2 presents the structure of the problem and the LQ solution. The code used to implement this solution is then discussed in section 3. The two applications, with model description, implementation of the code and results, follow in sections 4 and 5. Section 6 concludes.

2 The problem

Following the framework of Benigno and Woodford (2005b), consider a general maximization problem of the form:

$$V_{t_0} \equiv E \sum_{t=t_0}^{\infty} \beta^t U(x_t, u_t, X_t, \xi_t), \tag{2.1}$$

where $U(\cdot)$ is a functional, $0 < \beta < 1$; X_t is a vector of predetermined endogenous state variables of $\dim(X_t) = n_X$; x_t is a vector of non-predetermined endogenous variables of $\dim(x_t) = n_x$ and u_t is a vector of control variables of $\dim(u_t) = n_u$; ξ_t is a vector of stochastic disturbances of dimension $\dim(x_t) = n_e$. The maximization is subject to the vector of law of motion of dimension n_X

$$X_{t+1} = F(x_t, u_t, X_t, \xi_t) \tag{2.2}$$

for the predetermined state variables and to structural equations that define the set of possible rational-expectations equilibria and that include a set of forward-looking relations of the form

$$E_t G(x_t, u_t, X_t, \xi_t; x_{t+1}) = 0 \tag{2.3}$$

where set of constraints is of dimension n_G. In particular $n_G < n_x + n_u$. We also assume that the shocks follow

$$\xi_{t+1} = S\xi_t + \varepsilon_{t+1} \tag{2.4}$$

where ε_t should be a white noise with mean zero.

It is well known that the solution of the problem in (2.1) subject to the constrains in (2.2, 2.3 and 2.4) implies that the solution is generally not time-consistent. However if the policy is selected so that it fulfils some additional constrain on the value of the non-predetermined endogenous variables in the initial period, then the resulting problem has a well defined recursive structure and the implied policy are time invariant. Therefore the optimization problem is further subject to the constrain that the economy's initial evolution be the one associated with the implementation of the policy in question. In the terminology of Woodford (2003), we are looking for the *optimal from a timeless perspective.* Any such optimal policy will have to satisfy the additional constraint that the forward looking variables, in the initial period, need to be equal to their precommitted values, $x_{t_0} = \bar{x}_{t_0}$.

For future purposes, let us also define:

$$y_t \equiv \begin{bmatrix} x_t \\ u_t \\ X_t \end{bmatrix} \qquad\qquad z_t \equiv \begin{bmatrix} x_t \\ u_t \\ X_t \\ \xi_t \end{bmatrix}$$

where $\dim(y_t) = n_y = n_x + n_u + n_X$ and $\dim(z_t) = n_z = n_y + n_e$.

2.1 The linear-quadratic approximation of the problem

While general solution of the problem of interest involves the solution of non-linear rational expectation models associate to the fist order condition of the optimization problem in (2.1) under the constrains (2.2, 2.3 and 2.4) and the proper initial conditions, the aim here is to provide a proper ranking of policies which is correct in terms of welfare up to the second order in a proper neighbourhood of the deterministic steady state. To this end, Benigno and Woodford (2005b) construct a second order approximation of the objective function by a discounted sum of purely quadratic term in deviation from the steady state, up to terms that are independent for the policy. Their approximation begins by defining steady-state values ($\bar{x}$, $\bar{u}$, $\bar{X}$) of the endogenous variables, and the steady-state values of the Lagrange multipliers ($\bar{\lambda}$, $\bar{\varphi}$), associated with the problem of maximizing (2.1) under the constrains (2.2) and (2.3). They then consider second-order Taylor approximations of the U, F, and G functions, for the values of the endogenous variables near the steady-state values. For example, for the I element of the F function, they obtain

$$\begin{aligned} X^I_{t+1} &= F^I(x_t, u_t, X_t, \xi_t) \\ &= \bar{X} + DF^I(z_t - \bar{z}) + \frac{1}{2}(z_t - \bar{z})' D^2 F^I (z_t - \bar{z}) + O(\|\xi\|^3), \end{aligned} \tag{2.5}$$

where DF^I and D^2F^I refer to the Jacobian and Hessian of the I-th element of F with respect to z_t, respectively.

By manipulating these local expansions and the FOCs of the constrained maximization problem at the steady state values, they obtain a local approximation of the objective function (2.1) of the form

$$\begin{aligned} V_0 \equiv{} & \frac{1}{2} E \sum_{t=t_0}^{\infty} \beta^t \left[(z_t - \bar{z})' Q (z_t - \bar{z}) + (z_t - \bar{z})' R (z_{t-1} - \bar{z}) \right] \\ & - J(x_{t_0}, z_{-1}) + tip + O(\|\xi\|^3) \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} Q &\equiv D^2U + H, \\ H &\equiv \bar{\lambda}_J D^2 F^J + \bar{\varphi}_j \left[D^2 G^i + \beta^{-1} I_x' \hat{D}^2 G^i I_x \right], \\ R &\equiv \beta^{-1} \bar{\varphi}_j I_x' \hat{D} D G^j, \end{aligned}$$

and $\bar{\lambda}_J$ and $\bar{\varphi}_j$ are steady state values of the Lagrange multipliers associated with the J-th constraint (2.2) and the j-th constraint (2.3).[4]

The second order approximation of the law of motion for the predetermined variables, F, and the forward looking relations, G, as well as their values in steady states were used to eliminate first order terms from the second order approximation of the objective function. Furthermore if we consider allocations which satisfy the initial commitment that $x_{t_0} = \bar{x}_{t_0}$, then the value of the J term is nil and does not differ across policies considered.

Therefore the approximated objective function delivers a proper ranking in terms of welfare, up to second order, of different policies which satisfy the proper initial commitments ("*timeless perspective*"). In the class of policies satisfying those constrains, it follows that a correct linear approximation to optimal policy can be obtained by maximizing the quadratic function of deviation of variables from targets as:

$$E \sum_{t=t_0}^{\infty} \beta^t \left[(z_t - \bar{z})' Q (z_t - \bar{z}) + (z_t - \bar{z})' R (z_{t-1} - \bar{z}) \right] \tag{2.7}$$

[4] Further notice that D^2G^i refers to the Hessian of the i-th element of G with respect to z_t, $\hat{D}^2G^i$ to the Hessian of the i-th element of G with respect to x_{t+1} and $\hat{D}DG^j$ to the matrix of cross derivatives between z_t and x_{t+1} of the j-th element of G, and I_x is notation for the selection matrix such that $I_x z_t = x_t$

subject to

$$\begin{aligned} X^I_{t+1} &= \bar{X} + DF^I(z_t - \bar{z}) \\ 0 &= DG^i(z_t - \bar{z}) + \hat{D}G^i(x_{t+1} - \bar{x}). \end{aligned}$$

The problem (2.7) has the standard form of a linear-quadratic optimization and can be easily solved by standard tools; furthermore this provides easy to check second order conditions.

3 The LQ code

While the solution of the model in (2.7) is standard all the intermediate steps necessary in order to transform the general problem in (2.1) into its linear-quadratic equivalent are quite involved and lengthy, in particular in the case of models of medium size. To this end, we designed the following Matlab routine:

```
[A,B,C,Q,R,nr,nw,nd,labels] = LQ(x_t,u_t,X_t,csi_t,x_tt,X_tt,U,F,...
                              G,BETA,S,x_ss,u_ss,X_ss,FLM_ss,...
                              GLM_ss,Gcheck)
```

which, utilizing the symbolic tools of Matlab, finds for the linear-quadratic form associated to a generic optimization problem as in (2.1) under the constrains (2.2 and 2.3), solves for the optimal policy of the linear-quadratic problem and checks the second order conditions of such a problem.

3.1 Input in the programme

The inputs needed for the routine:

- list of the endogenous variables x_t, X_t and u_t: `[x_t,u_t,X_t];`
- list of the exogenous variables ξ_t (shocks) that satisfy (2.4) with $E_t[\varepsilon_t] = 0$: `[csi_t];`
- list of the endogenous variables next period x_{t+1} and X_{t+1}: `[x_tt,X_tt];`
- the functional forms $U(\cdot)$, $F(\cdot)$ and $G(\cdot)$: `[U,F,G]`
- the parameter values β (intertemporal discount factor) and S (autocorrelation of shocks): `[BETA,S];`
- steady state values $(\bar{x}, \bar{u}, \bar{X}, \bar{\lambda}, \bar{\varphi})$: `[x_ss,u_ss,X_ss,FLM_ss,GLM_ss]`;
- `Gcheck`, which is a binary variable flagging the optional testing for existence and uniqueness of the solution: `[true, false]`.

The routine requires as inputs the steady state values of the endogenous variables as well as of the multipliers associated with the optimal policy problem.[5] To this end the main routine, before calling the LQ function, is required to solve the non-linear system of equations as described in Benigno and Woodford (2005b). However solving such a system is not a trivial task, in particular for large dimensional system and it requires careful attention in the choice of initial conditions of the solution. Therefore we preferred to keep the steady state computation as separate and to treat steady state values as an input with respect to the LQ routine.It is important to notice that it is not advisable to use numerical minimization algorithms for the solution of the system due to their significant inaccuracy. Instead methods aimed at the solution of systems of non-linear equations yield much more precise solutions for the steady state values. This is all the more important given that these values will enter the matrices presented in (2.7).[6]

In what follows, the steps performed by the code are described.

1. Variables in log deviations with respect to the steady states
   The first step of the routine expresses all variables z in log terms performing the following transformation:

$$\hat{z}_{j,t} : \begin{cases} \hat{z}_{j,t} = \log z_{j,t}) & \text{if } \bar{z}_j \neq 0 \\ \hat{z}_{j,t} = z_{j,t} & \text{if } \bar{z}_j = 0 \end{cases} .$$

2. Quadratic objective function
   This first step forms the quadratic objective function of the above policy problem which has a form

$$V_{t_0} = \frac{1}{2} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[(\widehat{z}_t - \widehat{\bar{z}})' Q (\widehat{z}_t - \widehat{\bar{z}}) + 2(\widehat{z}_t - \widehat{\bar{z}})' R (\widehat{z}_t - \widehat{\bar{z}}) \right] \tag{3.1}$$

   where

$$\begin{aligned} \widehat{\bar{z}} &= \begin{cases} \log \bar{z}_j & \text{if } \bar{z}_j \neq 0 \\ \bar{z}_j & \text{if } \bar{z}_j = 0 \end{cases}, \\ Q &\equiv D^2 U + H, \\ H &\equiv \bar{\lambda}_J D^2 \tilde{F}^J + \bar{\varphi}_j [D^2 G^j + \beta^{-1} I'_{x,z} \hat{D}^2 G^j I_{x,z}], \\ R &\equiv \beta^{-1} \bar{\varphi}_j I'_{x,z} \hat{D} D G^j, \end{aligned}$$

   where D^2U, $D^2\tilde{F}^J$ and D^2G^j are the Hessian of U, $\tilde{F}^J$, G^j with respect to $\hat{z}_t$; $\hat{D}^2G^j$ is the Hessian of G^j with respect to $\hat{x}_{t+1}$; and $\hat{D}DG^j$ is a matrix of cross derivatives

[5] For further details on the exact system of equations that needs to be solved refer to Benigno and Woodford (2005b).

[6] A method for solving the system is provided in the codes used to generate the results of this paper, for the two applications. The algorithm used is based on the csolve.m Matlab function created by Christopher Sims. These codes can be obtained, upon request.

between $\hat{x}_{t+1}$ and $\hat{z}_t$. The code uses the steady-state values for the Lagrangian multipliers; computes analytically the above Hessians and evaluates them imposing the steady-state condition and using the steady-state values obtained as input. It then computes the matrices Q, H and R. Furthermore, we needed to use the log of F because the constraint must be satisfied given that we log transformed the variables and so we define:

$$\tilde{F}(\hat{z}_t) : \begin{cases} \tilde{F}^J(\hat{z}_t) = \log\left(F^J(\hat{z}_t)\right) & \text{if } \bar{X}_J \neq 0 \\ \tilde{F}^J(\hat{z}_t) = F^J(\hat{z}_t) & \text{if } \bar{X}_J = 0 \end{cases}.$$

Finally note that we use the notation $I_{x,z}$ for the selection matrix such that $I_{x,z}\hat{z}_t = \hat{x}_t$.

3. Solve the LQ problem
This third step maximizes (3.1) under the log-linear approximation to the constraints

$$\hat{X}_{t+1} - \widehat{\overline{X}} = D\tilde{F} \cdot (\widehat{z}_t - \widehat{\overline{z}})$$

$$DG \cdot (\widehat{z}_t - \widehat{\overline{z}}) + E_t\left[\hat{D}G(\widehat{x}_{t+1} - \widehat{\overline{x}})\right] = 0$$

where again the derivatives $D\tilde{F}$, DG, $\hat{D}G$ are with respect to $\hat{z}_t$, $\hat{z}_t$, $\hat{x}_{t+1}$, respectively. The code computes analytically these derivatives that are evaluated imposing the steady-state condition and at steady-state values.
By assuming a process for the shocks as $\xi_{t+1} = S\xi_t + \varepsilon_{t+1}$ where ε_{t+1} is a vector of white-noise shocks, the set of conditions that is used to determine the optimal path is

$$\begin{aligned} 0 \;=\; & I_{u,z}QI'_{y,z} \cdot \left(\hat{y}_t - \widehat{\overline{y}}\right) + I_{u,z}QI'_{\xi,z} \cdot \xi_t + I_{u,z}RI'_{y,z} \cdot \left(\widehat{w}_t - \widehat{\overline{w}}\right) + \\ & \beta I_{u,z}R'I'_{y,z} \cdot \left(E_t\hat{y}_{t+1} - \widehat{\overline{y}}\right) + I_{u,z}RI'_{\xi,z} \cdot \xi_{t-1} + \beta I_{u,z}R'I'_{\xi,z}S \cdot \xi_t \\ & + I_{u,z}D\tilde{F}' \cdot \lambda_t + I_{u,z}DG' \cdot \varphi_t \end{aligned} \tag{3.2}$$

$$\begin{aligned} 0 \;=\; & I_{x,z}QI'_{y,z} \cdot \left(\hat{y}_t - \widehat{\overline{y}}\right) + I_{x,z}QI'_{\xi,z} \cdot \xi_t + I_{x,z}RI'_{y,z} \cdot \left(\widehat{w}_t - \widehat{\overline{w}}\right) \\ & + \beta I_{x,z}R'I'_{y,z} \cdot \left(E_t\hat{y}_{t+1} - \widehat{\overline{y}}\right) + I_{x,z}RI'_{\xi,z} \cdot \xi_{t-1} + \\ & \beta I_{x,z}R'I'_{\xi,z}S \cdot \xi_t + I_{x,z}D\tilde{F}' \cdot \lambda_t + I_{x,z}DG' \cdot \varphi_t + \beta^{-1}\hat{D}G' \cdot d_t \end{aligned} \tag{3.3}$$

$$\begin{aligned} 0 \;=\; & I_{X,z}QI'_{y,z} \cdot \left(E_t\hat{y}_{t+1} - \widehat{\overline{y}}\right) + I_{X,z}QI'_{\xi,z}S \cdot \xi_t + I_{X,z}RI'_{y,z} \cdot \left(\hat{y}_t - \widehat{\overline{y}}\right) \\ & + \beta I_{X,z}R'I'_{y,z} \cdot \left(E_t r_{t+1} - \widehat{\overline{y}}\right) + I_{X,z}RI'_{\xi,z} \cdot \xi_t + \beta I_{X,z}R'I'_{\xi,z}S^2 \cdot \xi_t \\ & - \beta^{-1}\lambda_t + I_{X,z}D\tilde{F}' \cdot E_t\lambda_{t+1} + I_{X,z}DG' \cdot E_t\varphi_{t+1} \end{aligned} \tag{3.4}$$

$$0 = -I_{X,y}\left(E_t\hat{y}_{t+1} - \widehat{\overline{y}}\right) + D\tilde{F}I'_{y,z} \cdot \left(\hat{y}_t - \widehat{\overline{y}}\right) + D\tilde{F}I'_{\xi,z} \cdot \xi_t \tag{3.5}$$

$$0 = DGI'_{y,z} \cdot \left(\hat{y}_t - \widehat{\overline{y}}\right) + DGI'_{\xi,z} \cdot \xi_t + \hat{D}GI_{x,y}\left(E_t\hat{y}_{t+1} - \widehat{\overline{y}}\right) \tag{3.6}$$

$$0 = w_{t+1} - \hat{y}_t \tag{3.7}$$

$$0 = d_{t+1} - \varphi_t \tag{3.8}$$

$$0 = E_t\hat{y}_{t+1} - r_t \tag{3.9}$$

$$0 = \xi_t - S\xi_{t-1} - \varepsilon_t \tag{3.10}$$

We can write more compactly the above system as

$$AE_tk_{t+1} = Bk_t + C\varepsilon_t \tag{3.11}$$

where

$$k_t \equiv \begin{bmatrix} r_t - \widehat{\overline{y}} \\ \lambda_t \\ \varphi_t \\ \hat{y}_t - \widehat{\overline{y}} \\ w_t - \widehat{\overline{y}} \\ d_t \\ \xi_{t-1} \end{bmatrix}$$

The code inputs the above ten conditions and finds representation (3.11). Then standard packages as REDS-SOLDS can be used to find the optimal path.[7]

4. Check for Second Order Conditions
   It is possible to evaluate second order conditions of the solution of the linear-quadratic problem and the code prints a message with a confirmation of their verification or, instead, a warning, if these conditions are not satisfied.

3.2 Output of the programme

The code provides the following outputs:

1. after solving the LQ problem the code yields matrices A, B and C for the solution in (3.11);

2. provides the Q and R matrices characterizing the approximated objective function;

3. prints a message about the status of the second order conditions;

4. if `Gcheck` is `true`, prints a message regarding the existence and uniqueness of the solution;

5. returns the number of artificial variables created and the entire list of labels for the k_t vector.

[7] The setup is not immediately in the form of the GENSYS routine but can be adapted as the code itself does for the Gcheck option.

4 Optimal policy in a model with monetary frictions

In this section we present a model that follows closely Benigno and Woodford (2005a) with the change that we introduce monetary frictions as suggested in Schmitt-Grohé and Uribe (2004). Given that the model is no different in any other respect, here we present only its main characteristics.

4.1 The model

The representative household maximizes

$$U_{t_0} = \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[\tilde{u}\left(C_t;\xi_t\right) - \int_0^1 \tilde{v}\left(H_t\left(j\right);\xi_t\right) dj \right],$$

where C_t is a Dixit-Stiglitz aggregate of consumption of each of a continuum of differentiated goods:

$$C_t \equiv \left(\int_0^1 c_t\left(i\right)^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}},$$

with the elasticity of substitution equal to $\theta > 1$, and $h_t\left(j\right)$ is the supply of labour of type j. The functional forms for $\tilde{u}$ and $\tilde{v}$ are the following ones:

$$\begin{aligned} \tilde{u}\left(C_t;\xi_t\right) &\equiv \frac{C_t^{1-\tilde{\sigma}^{-1}} \bar{C}_t^{1-\tilde{\sigma}^{-1}}}{1-\tilde{\sigma}^{-1}}, \\ \tilde{v}\left(H_t;\xi_t\right) &\equiv \frac{\lambda}{1+\nu} H_t^{1+\nu} \bar{H}_t^{1+\nu}. \end{aligned}$$

We follow Schmitt-Grohé and Uribe (2004) in modelling monetary frictions by assuming that money facilitates consumption purchases. In particular, we impose a proportional transaction cost to consumption purchases, $s\left(n_t\right)$, that depends on households consumption-based money velocity, $n_t \equiv P_t C_t / M_t$. The exact specification for $s\left(\cdot\right)$ is given by

$$s\left(n\right) = an + \frac{b}{n} - 2\sqrt{ab}.$$

The budget constraint is therefore

$$\begin{aligned} \left(1+\chi_t s\left(n_t\right)\right) P_t C_t + M_t + B_t &= \left(1+i_{t-1}^m\right) M_{t-1} + \left(1+i_{t-1}\right) B_{t-1} \\ &\quad + \int_0^1 W_t\left(j\right) H_t\left(j\right) dj + \int_0^1 \Pi_t\left(i\right) di + T_t, \end{aligned} \tag{4.1}$$

where χ_t is a shock to the transaction costs with mean one, i_t is the interest rate on bonds and i_t^m is the interest rate paid on money balances and set by the monetary authorities. We further have

$$P_t C_t \equiv \int_0^1 p_t\left(i\right) c_t\left(i\right) di,$$

with $p_t(i)$ as the price of good i and P_t is the Dixit-Stiglitz aggregate price index,

$$P_t \equiv \left[\int_0^1 p_t(i)^{1-\theta} \, di \right]^{\frac{1}{1-\theta}}. \tag{4.2}$$

Using λ_t to denote the Lagrangian multiplier of the budget constraint in the consumer's problem, we get the following equations:

$$\begin{aligned}
\frac{\tilde{u}'(C_t;\xi_t)}{P_t} &= \lambda_t \left[1 + \chi_t s(n_t) + \chi_t s'(n_t) n_t\right], &(4.3)\\
c_t(i) &= C_t \left(\frac{p_t(i)}{P_t}\right)^{-\theta}, &(4.4)\\
\lambda_t &= (1+i_t)\beta E_t \lambda_{t+1}, &(4.5)\\
\chi_t s'(n_t) n_t^2 &= \frac{i_t - i_t^m}{1+i_t}. &(4.6)
\end{aligned}$$

This last equation conveys the money demand of this economy. Notice that it depends on the spread between the interest rate on bonds relative to the interest rate on money balances. This is crucial because it allows us to distinguish between the monetary frictions model and the cashless limiting case as in Woodford (1998). In the case with monetary frictions, which will be our baseline case, the monetary authority simply sets the interest rate on money balances to zero and then uses the money supply to manage the money market so as to influence the interest rate in the bonds market. Therefore we can think of the operational target as i_t and the way to achieve that left to operations department of the monetary authority. In the cashless limiting case then we assume that money is not an issue in the economy and therefore money satiation is satisfied, meaning that we must have the two interest rates equal. And so again we can proceed as if the central bank actually controls i_t. In terms of modelling this and allowing the model to flow smoothly into the computer codes that we are proposing in this paper we set

$$i_t^m = \psi i_t, \tag{4.7}$$

with $0 \leq \psi \leq 1$ and (4.6) becomes

$$\chi_t s'(n_t) n_t^2 = (1-\psi) \frac{i_t}{1+i_t}. \tag{4.8}$$

We then assume that $\psi = 0$ as the benchmark case and, in the opposite case, $\psi = 1$, we recover the cashless limiting case, in which $n_t = \underline{n}$ and $s(n_t) = 0$, for all t.

Labour market conditions allow the households to charge an exogenous markup so that we have the wage determination by

$$w_t(j) = \mu_t \frac{\tilde{v}'(H_t(j);\xi_t)}{\lambda_t}. \tag{4.9}$$

Each differentiated good is supplied by a monopoly. The technology is common for all goods:

$$y_t(i) = A_t f(h_t(i)) = A_t h_t(i)^{1/\phi}, \tag{4.10}$$

where $\phi > 1$. In equilibrium output is equal to spending by the households and by the government hence

$$Y_t = (1 + \chi_t s(n_t)) C_t + G_t, \tag{4.11}$$

where the real government expenditures are exogenous. Hence we can write consumption as

$$C_t = \frac{Y_t - G_t}{1 + \chi_t s(n_t)}. \tag{4.12}$$

Notice that the real money balances are determined by

$$m_t \equiv \frac{Y_t - G_t}{n_t (1 + \chi_t s(n_t))}, \tag{4.13}$$

which is a residual equation with the only purpose of determining $m_t \equiv M_t/P_t$.

Each firm sets prices at a given period with probability $(1-\alpha)$, with $0 \leq \alpha < 1$. Hence there is a fraction α of prices (by the law of large numbers) that remain unchanged. The stochastic discount factor is in equilibrium equal to

$$\Lambda_{t,T} = \beta^{T-t} \frac{\lambda_T}{\lambda_t}, \tag{4.14}$$

so that the firms maximize the net present value of their profits:

$$E_t \sum_{T=t}^{\infty} \alpha^{T-t} \Lambda_{t,T} \Pi\left(p_t(i), p_T^j, P_T; Y_T, \xi_T\right),$$

where the after-tax profit function is

$$\Pi\left(p_t(i), p_t^j, P_t; Y_t, \xi_t\right) \equiv (1 - \tau_t) p_t(i) Y_t \left(\frac{p_t(i)}{P_t}\right)^{-\theta} - h_t(i) w_t(i).$$

The optimal pricing can be expressed as

$$(p_t^*)^{1+\theta\omega} = \frac{E_t \sum_{T=t}^{\infty} \alpha^{T-t} \Lambda_{t,T} \phi \frac{\theta}{\theta-1} \lambda \mu_T Y_T^{1+\omega} P_T^{\theta(1+\omega)} A_T^{-(1+\omega)} \bar{H}_T^{-\nu} / \lambda_T}{E_t \sum_{T=t}^{\infty} \alpha^{T-t} \Lambda_{t,T} (1 - \tau_T) Y_T P_T^{\theta}}.$$

But, in order to be able to apply the computer routines presented with this paper we need to have all expressions in a recursive way. In order to get that we can write then

$$\frac{p_t^*}{P_t} = \left(\frac{K_t}{F_t}\right)^{\frac{1}{1+\omega\theta}}, \tag{4.15}$$

and define

$$F_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} (1-\tau_T) \lambda_T P_T Y_T \left(\frac{P_T}{P_t}\right)^{\theta-1}, \tag{4.16}$$

$$K_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \frac{\theta}{\theta-1} \mu_T \upsilon_y (Y_T, \Delta_T; \xi_T) \frac{Y_T}{\Delta_T} \left(\frac{P_T}{P_t}\right)^{\theta(1+\omega)}. \tag{4.17}$$

The price index evolves through the following law of motion:

$$P_t = \left[(1-\alpha)(p_t^*)^{1-\theta} + \alpha P_{t-1}^{1-\theta}\right]^{\frac{1}{1-\theta}}. \tag{4.18}$$

Now insert (4.15) into this equation to get

$$P_t = \left[(1-\alpha)\left(\frac{F_t}{K_t}\right)^{\frac{\theta-1}{1+\omega\theta}} P_t^{1-\theta} + \alpha P_{t-1}^{1-\theta}\right]^{\frac{1}{1-\theta}},$$

hence

$$\frac{F_t}{K_t} = \left(\frac{1-\alpha\Pi_t^{\theta-1}}{1-\alpha}\right)^{\frac{1+\omega\theta}{\theta-1}}, \tag{4.19}$$

where $\Pi_t \equiv P_t/P_{t-1}$.

Finally the desired recursive representations of F_t and K_t are

$$F_t = (1-\tau_t) \lambda_t P_t Y_t + \alpha\beta E_t \left[\Pi_{t+1}^{\theta-1} F_{t+1}\right], \tag{4.20}$$

and

$$K_t = \frac{\theta}{\theta-1} \mu_t \upsilon_y (Y_t, \Delta_t; \xi_t) \frac{Y_t}{\Delta_t} + \alpha\beta E_t \left[\Pi_{t+1}^{\theta(1+\omega)} K_{t+1}\right], \tag{4.21}$$

where Δ_t is a measure of the price dispersion, defined as:

$$\Delta_t \equiv \int_0^1 \left(\frac{p_t(i)}{P_t}\right)^{-\theta(1+\omega)} di.$$

For the law of motion of the price dispersion, from its definition,

$$\Delta_t P_t^{-\theta(1+\omega)} = \alpha \Delta_{t-1} P_{t-1}^{-\theta(1+\omega)} + (1-\alpha)(p_t^*)^{-\theta(1+\omega)},$$

because we know that for the industries in which prices do not change their relative prices are still the same. With some more manipulation we obtain

$$\Delta_t = \alpha \Delta_{t-1} \Pi_t^{\theta(1+\omega)} + (1-\alpha)\left(\frac{1-\alpha\Pi_t^{\theta-1}}{1-\alpha}\right)^{\frac{\theta(1+\omega)}{\theta-1}}. \tag{4.22}$$

Finally, we rewrite the utility function in a more convenient way,[8]

$$U_{t_0} = \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[u\left(Y_t, n_t; \xi_t\right) - \upsilon\left(Y_t, \Delta_t; \xi_t\right) \right], \tag{4.23}$$

with

$$u\left(Y_t, n_t; \xi_t\right) = \frac{\bar{C}_t^{1-\tilde{\sigma}^{-1}}}{1-\tilde{\sigma}^{-1}} \left(\frac{Y_t - G_t}{1+\chi_t s\left(n_t\right)} \right)^{1-\tilde{\sigma}^{-1}}, \tag{4.24}$$

$$\upsilon\left(Y_t, \Delta_t; \xi_t\right) = \frac{\lambda}{1+\nu} \frac{Y_t^{1+\omega}}{A_t^{1+\omega} \bar{H}_t^{\nu}} \Delta_t, \tag{4.25}$$

where $\omega \equiv \phi\left(1+\nu\right) - 1$.[9]

Given that we will need to consider explicitly the value of the Lagrangian multiplier it is convenient to consider it normalised in such a way that it converges to a constant value in steady state. Therefore we define $\tilde{\lambda}_t \equiv \lambda_t P_t$. Taking this transformation into account we can summarise all the relevant equations in the economy as follows:

$$\tilde{\lambda}_t = \frac{1+\chi_t s\left(n_t\right)}{1+\chi_t s\left(n_t\right) + \chi_t s'\left(n_t\right) n_t} u_Y\left(Y_t, n_t; \xi_t\right), \tag{4.26}$$

$$\tilde{\lambda}_t = \left(1+i_t\right) \beta E_t \left[\frac{\tilde{\lambda}_{t+1}}{\Pi_{t+1}} \right], \tag{4.27}$$

$$\chi_t s'\left(n_t\right) n_t^2 = \left(1-\psi\right) \frac{i_t}{1+i_t}, \tag{4.28}$$

$$m_t \equiv \frac{Y_t - G_t}{\left(1+\chi_t s\left(n_t\right)\right) n_t}, \tag{4.29}$$

$$\frac{F_t}{K_t} = \left(\frac{1-\alpha \Pi_t^{\theta-1}}{1-\alpha} \right)^{\frac{1+\omega\theta}{\theta-1}}, \tag{4.30}$$

$$F_t = \left(1-\tau_t\right) \tilde{\lambda}_t Y_t + \alpha\beta E_t \left[\Pi_{t+1}^{\theta-1} F_{t+1} \right], \tag{4.31}$$

$$K_t = \frac{\theta}{\theta-1} \mu_t \upsilon_y\left(Y_t, \Delta_t; \xi_t\right) \frac{Y_t}{\Delta_t} + \alpha\beta E_t \left[\Pi_{t+1}^{\theta(1+\omega)} K_{t+1} \right], \tag{4.32}$$

$$\Delta_t = \alpha \Delta_{t-1} \Pi_t^{\theta(1+\omega)} + \left(1-\alpha\right) \left(\frac{1-\alpha \Pi_t^{\theta-1}}{1-\alpha} \right)^{\frac{\theta(1+\omega)}{\theta-1}}. \tag{4.33}$$

[8] For the derivation of these expressions, refer to Appendix A.2.

[9] As mentioned in Benigno and Woodford (2005a), defined as is, ω refers to the elasticity of real marginal cost with respect to a firm's own output.

Finally we can summarise all the shocks in this economy by making explicit the vector ξ_t. This vector includes the shocks already normalized so that they have zero mean,[10] the assumed value in steady state. We can then write

$$\xi_t \equiv \left[\begin{array}{ccccccc} \hat{\tau}_t & \hat{\mu}_t & \hat{G}_t & \hat{\bar{C}}_t & \hat{\bar{H}}_t & \hat{A}_t & \hat{\chi}_t \end{array} \right]'.$$

4.2 Model in the LQ form

In order to use the code described in section 3 we need to match the structure presented in section 2. In the current setup we can define

$$\begin{aligned} x_t &\equiv \left(F_t, K_t, \Pi_t, \tilde{\lambda}_t\right)', \\ u_t &\equiv \left(i_t, Y_t, n_t, m_t\right)', \\ X_t &\equiv \Delta_{t-1}, \end{aligned}$$

and the vector of constraints G is defined in (4.26-4.32) and the predetermined constraint, with the implicit definition of $F(\cdot)$ is presented in (4.33).

We must have the functional $U(\cdot)$ with the arguments $\{x_t, u_t, X_t; \xi_t\}$ but, in this model, inside it we have $\upsilon(Y_t, \Delta_t; \xi_t)$ hence $U(\cdot)$ depends on Δ_t which is one element of the X_{t+1} vector. In order to solve for this issue we can plug $F(\cdot)$ into $U(\cdot)$ so that it will depend on Δ_{t-1}, i.e. X_t. This substitution must be done only after the constraints are created, in particular (4.32). This shows that the framework necessary for applying the proposed computer routines is not so restrictive as it might look at a first glance.

4.3 Optimal policy

In this section we compare the impact of the monetary frictions in the model, starting with the impact in the optimal steady state. We then analyse how the optimal monetary policy differs in the case with monetary frictions and in the cashless limiting case. For the parameterization of the model we will follow the baseline parameters proposed by Benigno and Woodford (2005a) for easier comparison and for the monetary frictions we use the values estimated in Schmitt-Grohé and Uribe (2004). All of the parameters are shown in Table 1.

4.3.1 Optimal steady state

The optimal steady state is a focal point of the entire analysis as the approximations are all derived around these numbers.[11] Given its importance, we make here a thorough analysis of it in the most relevant scenarios considered.

[10] These normalizations are the following ones: $1-\tau_t = (1-\bar{\tau})\exp(-\hat{\tau}_t)$; $\mu_t = \bar{\mu}\exp(\hat{\mu}_t)$; $G_t = \hat{G}_t$ if $\bar{G} = 0$ and $G_t = \bar{G}\exp\left(\hat{G}_t\right)$ otherwise; $\bar{C}_t = \bar{C}\exp\left(\hat{\bar{C}}_t\right)$; $\bar{H}_t = \bar{H}\exp\left(\hat{\bar{H}}_t\right)$; $A_t = \bar{A}\exp\left(\hat{A}_t\right)$; and $\chi_t = \exp(\hat{\chi}_t)$, in which we assume $\bar{\chi} = 1$. Notice that $\bar{\tau}$, $\bar{\mu}$, $\bar{G}$, $\bar{C}$, $\bar{H}$, $\bar{A}$ and $\bar{\chi}$ refer to the steady state values of the respective variables.

[11] In the calculation of the steady state no approximation is made.

The first result worth mentioning is that under the cashless limiting case, $\psi = 1$, in the optimal steady state we find that inflation is always set to zero and therefore the nominal (and real) interest rate is then set to equal the discount rate of the households, roughly 4% in annual terms. When we consider the presence of monetary frictions, however, it is always the case that we get some deflation in the optimal steady state, at the same time that the nominal interest rate falls short of the discount rate of the households. The degree of optimal steady state deflation depends, among other things, on the price stickiness of the economy and on the tax rate. To help understand the optimal setting of both the nominal interest and inflation rates in steady state, Figure 1 presents these against different levels of price rigidity, and for two different levels of the tax rate. In panel A we have the baseline scenario, in which the tax rate is set to 0.2. In panel B we present the results under the level of taxes that would eliminate the distortions generated by the monopolistic competition.[12]

The first result is that the degree of optimal deflation in steady state decreases with price rigidity (α) and, in the limiting case, as the probability of not adjusting prices converges to unity the deflation level gradually converges to zero. Therefore, on this basis, the divergence between the monetary frictions and cashless limiting cases optimal deflation increases with price flexibility. This evolution of steady state inflation rate is also present in Khan, King, and Wolman (2003) and Schmitt-Grohé and Uribe (2004).

When commenting on the Friedman rule Khan et al. (2003) mention that due to the Keynesian frictions the nominal interest rate is kept above zero. Figure is in accordance with this indeed but if those frictions are weak enough then in our model the nominal interest rate begin to fall below zero. Because that is not operationally possible then the interest rate hits the so called "lower bound" and is set to be zero, as in the Friedman rule. Therefore the deflation takes place, in order to put the real interest rate in line with the time discount rate. However this happens only if the Calvo probability of not changing prices, α, falls to 16% or below, which are rather low numbers for this parameter. If, instead, the tax rate is set to eliminate the distortions created by the monopolistic competition then the figure shows that the nominal interest rate is always positive, converging to zero, but always from above. What happens in our model is that the monetary frictions are u-shaped as a function of the money velocity of consumption, not being restricted to be equal to zero after the minimum. Therefore if the distortions are eliminated the efficient optimum for output and real money balances can then be achieved. When, instead, the monopolistic distortions are not offset, reducing real money balances reduces output and then this is not the direction to go. So real money balances should be increased to the possible extent and this will raise steady state output but not up to the efficient level. Because the monetary frictions may have a negative slope, for values of the velocity below the efficient one, then the interest rate is pushed to the lower bound. Notice that Schmitt-Grohé and Uribe (2004) use exactly the same frictions in a model very similar to ours and they to not get this result (they never hit the lower bound of the nominal interest rate). The reason for that is that they are instead

[12] We actually should refer to it as subsidy rate: in order to eliminate the monopolistic competition distortions the tax rate must be set equal to $\bar{\tau} = -1/(\theta - 1)$ and considering $\theta > 1$ implies a negative tax rate.

optimizing on both monetary and fiscal policy simultaneously.

We can nevertheless reach the conclusion that under the existence of monetary frictions the Friedman rule is always valid in the flexible prices case. It is also valid if taxes are not set to eliminate the monopolistic competition distortions and the level of price stickiness is very low.

To conclude the optimal steady state analysis we report in Table 2 the values for some variables under different scenarios, in order to allow a clear comparison between the cashless limiting case and the monetary frictions one.

Regarding interest rates and inflation rate the most relevant conclusions were already mentioned before. The only thing that we can add is that the level of price stickiness will not influence the optimal steady state levels of the interest rate or inflation in the case of the cashless limiting case, precisely because as reported already in Benigno and Woodford (2005a), it is always optimal to set inflation to zero. More interesting, in the monetary frictions case, even though the nominal interest rate and inflation rate do change quite significantly across different levels of price rigidity, the level of price dispersion barely changes. Also notice that the velocity of money, n, does not change much. Moreover, the value of the velocity is about 6.3% above the satiation level, the one prevailing in the cashless limiting case. Only when we get very close to flexible prices does the velocity begin effective convergence to that level.

Finally it is important to verify the impact of the monetary frictions in the real side of the economy, namely, in the output level. In this respect we notice that the level of output is fairly constant across different levels of price stickiness and always lower under the case of monetary frictions relative to the cashless limiting case. However the difference in output is not very significant, about 0.6% of the cashless limiting level. So on this measure we can say that the monetary frictions do not have a significant impact on the real side of the economy.

4.3.2 Optimal responses to shocks

In this section we show the behaviour of optimal policy in face of the various shocks present in the economy. Notice however that we do not show the responses to shocks to the labour supply just because they are qualitatively exactly the same as a technology shock in an economy in which labour is the only input. The only differences are numerical due to different scales but that is not relevant in the current analysis.[13] In the following analysis the vertical axis in all the figures should be interpreted as percentage points and the responses of the interest rate and inflation are annualized. In the baseline case we use a persistence level corresponding to a 0.7 coefficient for AR(1) process describing the logs of the shocks.

We first compare the two main cases of our model: the frictions and cashless economies. The responses under the two cases are shown in Figures 2 through 7. The main results can be summarised as follows: first, the responses are very similar in the two scenarios considered, with exactly the same pattern in both cases and only minor differences in magnitude; second,

[13] Indeed the two shocks appear always together in the reduced set of equations for the economy with the form $A_t^{1+\omega}\bar{H}_t^{\nu}$ and that is why the qualitative effects are exactly the same and the scales different.

the consumption based velocity of money (n_t) stays constant in the cashless case and mimics the pattern of the interest rate in the frictions case and this is the case in all scenarios and shocks except for the shock to the transaction costs; and, third, the shock to the transaction costs can be considered negligible in their impact on the other variables. To be more specific about some of these issues we analyse now the responses to each of the shocks.

In Figure 2 we can observe that an increase of one percentage point in the tax rate leads firms to reduce production and increase prices. In response to the shock the monetary authority (optimally) sets higher interest rates to bring the inflation under control, so that the final impact on the inflation rate is only 0.08% in both scenarios and quickly is brought to levels very close to zero. Indeed the path of inflation is the same in the two cases. However there are slight differences in the policy that led to this outcome. Interest rates should increase 3 basis points more on impact in the monetary frictions case, which then leads to a minor worsening of the recession in that scenario.

The responses to an increase of the price markup of 1% is shown in Figure 3. Once more the effects in the variables are the expected ones. The increase in the price markup leads to inflationary pressures that lead the monetary authorities to use contractionary policy through the increase of the interest rate in 15 basis points. This in turn will slow down the economy, leading to a fall in output of the order of 0.8% and control inflationary pressures. This case is in fact very similar to the case of an increase in the tax rate. In the latter, however, the similarities between the two scenarios, are even more pressing, being hard to distinguish the two.

An increase in the government expenditures of the order of 1% of the GDP is shown in Figure 4. The increase in government expenditures is an aggregate demand (AD) expansion that generates inflationary gap, promptly closed by the monetary authorities with a substantial increase in the interest rate. The authorities react so strong that the inflation not only is brought under control but it actually goes to zero from below, on impact. Due to that policy the impact on output is rather small and the multiplier effect simply is erased, so that a 1% increase in government expenditures led to only 0.2% increase in output. Again the differences between the monetary frictions and the cashless limiting cases are very small, consisting of an increase in the interest rate of 20 basis points in the frictions scenario and about 22 basis points in the cashless case. On output there seems to be some difference in magnitude but not significant at all. A similar pattern ensues in the case of a positive shock to household preferences for consumption, which also entails an expansion in AD, as shown in Figure 5.

The responses to an increase in 1% in the productivity of labour are presented in Figure 6. As should be expected the output increases with the productivity. However the magnitude of the increase is remarkable, of the order of 2.5%, compared to a 1% shock. The reason for this is that as the output has some tendency to increase on impact, due to sluggish price adjustment, not all firms can adapt fully and so there is still a recessionary gap that is promptly closed by the monetary authorities with a very aggressive interest rate policy. Indeed the interest rate falls by 60 basis points in both scenarios considered. This closes the gap and prevents the prices from falling (they actually marginally increase on impact).

Again the responses are identical in the two scenarios. The inflation response seems a bit more different but given the scale that distinction can be considered to negligible.

Finally, the responses to the transactions cost shock are depicted in Figure 7. The only thing worth mentioning is that all the responses have scales that make the paths meaningless and basically we can say that all relevant variables are essentially set to zero. The exception is the velocity, n_t, which falls on impact under the frictions scenario, and then gradually recovers. This is the only departure from the the idea that the velocity is a scaled version of the path of the interest rate. This is also consistent with the fact that monetary frictions have very little impact in the economy. Actually it can be interpreted as the result that the transaction costs are very close to zero in the optimal steady state.

We now bring the analysis further and connect it to the discussion in the previous subsection relating the two cases for taxes: one in which the tax rate is simply set at 20% in steady state (and therefore the monopolistic distortions are not offset) and a second case in which the tax rate is set so as to offset these distortions, becoming actually a subsidy. The resulting responses in the case of monetary frictions under distorted steady state and no distortions are presented in Figures 8 through 12. The main conclusion that we can take is that the interest rate is less volatile in the non-distorted scenario relative to the distorted one. In particular, in response to an increase in the tax rate, an increase in the price markup or an increase in the government expenditures, the interest rate increases always less than in the distorted case, or even decreases on impact, allowing the inflation deviations from steady state to slightly be higher at the expense of higher output levels. In the case of a positive technological shock the interest rate is decreased in both cases but less so in the distorted one. This leaves inflation essentially at zero in both cases even if the paths are symmetric around the steady states. However output is slightly higher, relative to the respective steady state, in the distorted case. In the case of a consumption preferences shock, there is absolutely no change what so ever.

Finally, in order to close the analysis we compare the optimal policy to an alternative policy. That alternative policy could be defined in multiple ways but just as an illustrative example we use a simple Taylor rule with the following form:

$$1 + i_t = (1 + \bar{\imath}) \left(\frac{\Pi_t}{\bar{\Pi}} \right)^{\phi_\pi} \left(\frac{Y_t}{\bar{Y}} \right)^{\frac{\phi_y}{4}}$$

where ϕ_π and ϕ_y have the usual interpretations and the division by four of the latter is meant to keep the interpretation of ϕ_y in annual terms (given that each period in this model is considered to be a quarter). We use, just for illustrative purposes, coefficients equal to 1.5 and 0.5 respectively. Notice that these were not optimized in any respect. This example shows that it is also easy in the framework proposed to evaluate a simple rule and compare it to the optimal policy responses, as presented in Figures 13 through 18.

The comparison between the simple interest rate rule to the optimal policy in this economy reveals that under increases to the tax rate (Figure 13) or the price markup (Figure 14) output, inflation and nominal interest rates are higher on impact under the alternative Taylor rule compared to the optimal policy. The reason for this is that inflation is relatively higher than the interest rate so that the real interest rate is actually lower under the alternative policy. This implies that probably the reaction of the nominal interest rate to inflation is not as aggressive as it should optimally be.

In the case of the increase of government expenditures (Figure 15) we can observe under the alternative rule higher output level and lower interest rate, with the inflation rate very close to the steady state. This might imply that the interest rate is actually not aggressive enough to output changes. In the case of the consumption preferences shock nothing changes, though.

The responses to a positive productivity shock (Figure 17) under the alternative rule yields radically different responses than under the optimal policy. Now the output expansion is substantially trimmed down due a much higher interest rate. This leads to a significant level of deflation in the short run (0.25%). This shows that the interest rate falls in response to deflation but essentially real interest rates are left at the steady state level and so output does not expand as much as before. This shows that the monetary policy is rather passive, not reacting to the recessionary gap that formed (the potential output expanded much more than the output did). This is the result of the interest rate rule taking into account the output deviations from steady state instead of the actual output gap. In light of this interpretation we can review the other ones and conclude that this seems like a likely cause of the previous differences from optimal policy as well. So it could be interesting to incorporate an interest rate rule in terms of output gap, instead of just output deviations from steady state.

5 Optimal monetary policy in a Neo-Keynesian model (Smets-Wouters (2003))

In this section we discuss results on the Linear-Quadratic optimal monetary policy in a slightly simplified version of the closed economy model in Smets and Wouters (2003). We first lay out the model equations and the equilibrium conditions defining the LQ programme, as well as the calibration used. We then describe the impulse responses of selected variables to the shocks in the model, both under LQ optimal policy and under a standard monetary policy rule.

5.1 The model

5.1.1 Consumers

At time t, the utility function of the rapresentative agent is

$$U_t = \mathbb{E}_t \left[\sum_{s \geq t} \beta^{s-t} \left(U\left(C_{t+s}, C_{t+s-1}, E_t^B\right) - V\left(L_{t+s}^h, E_t^L, E_t^B\right)\right)\right], \tag{5.1}$$

with

$$U\left(C_{t+s},C_{t+s-1},E_t^B\right) \equiv E_t^B\frac{\left(C_{t+s}-\theta C_{t+s-1}\right)^{1-\sigma_C}}{1-\sigma_C},$$
$$V\left(L_{t+s}^h,E_t^L,E_t^B\right) \equiv E_t^B\int_0^1\overline{L}\frac{\left(L_{t+s}^h\right)^{1+\sigma_L}}{1+\sigma_L}E_t^L dh.$$

where it is understood that households obtain utility from consumption of an aggregate index C_t, relative to an internal habit depending on past aggregate consumption, while receiving disutility from labour L_t^h. Utility also incorporates a consumption preference shock E_t^B and a labour supply shock E_t^L. Each household h maximizes its utility function under the following budgetary constraint:

$$\frac{B_t}{P_t(1+i_t)}+I_t+C_t=\frac{B_{t-1}}{P_t}+\frac{\int_0^1\left(1-\tau_{W,t}\right)W_t^hL_t^h dh+A_t+TT_t}{P_t}+R_t^kCU_tK_t-\Phi\left(CU_t\right)K_t,$$

where B_t is a nominal bond, W_t^h is the wage, A_t^h is a stream of income coming from state contingent securities, TT_t^h and $\tau_{W,t}$ are government transfers and time-varying labour tax respectively, and $\left(R_t^kCU_tK_t-\Phi\left(CU_t\right)K_t\right)$ represents the return on the real capital stock minus the cost associated with variations in the degree of capital utilization. As in Christiano, Eichenbaum, and Evans (2005), the income from renting out capital services depends on the level of capital augmented for its utilization rate and the cost of capacity utilization is zero at full capacity ($\Phi(1)=0$). Separability of preferences and complete financial markets ensure that Households have identical consumption plans.

The first order condition related to consumption expenditures is given by

$$\Lambda_t=U_{C1,t}+\beta\mathbb{E}_tU_{C2,t+1} \tag{5.2}$$

where Λ_t is the Lagrangian multiplier associated with the budget constraint. The first order conditions corresponding to the quantity of contingent bond is:

$$\Lambda_t=(1+i_t)\beta\mathbb{E}_t\left[\Lambda_{t+1}\frac{P_t}{P_{t+1}}\right] \tag{5.3}$$

5.1.2 Labour supply and wage setting

Each household is a monopoly supplier of a differentiated labour service. For the sake of simplicity, we assume that he sells his services to a perfectly competitive firm (Labour Packers) which transforms it into an aggregate labour input using the following technology:

$$L_t=\left[\int_0^1 {L_t^h}^{\frac{\epsilon_W-1}{\epsilon_W}}dh\right]^{\frac{\epsilon_W}{\epsilon_W-1}}.$$

The household faces a labour demand curve with constant elasticity of substitution:

$$L_t^h=\left(\frac{W_t^h}{W_t}\right)^{-\epsilon_W}L_t,$$

where the aggregate wage is given by

$$W_t = \left(\int_0^1 W_t^{h\,1-\epsilon_W} dh \right)^{\frac{1}{1-\epsilon_W}} .$$

The real wage setting equations can be written in the following recursive form (see Appendix B.2 for a derivation):

$$(1-\alpha_W) \left(\frac{\widetilde{w}_t}{P_t} \right)^{\frac{1}{1-\mu_W}} = (1-\alpha_W) \left(\mu_w \frac{Z_{W1,t}}{Z_{W2,t}} \right)^{-\frac{1}{\mu_W(1+\sigma_L)-1}} \tag{5.4}$$

$$= W_{R,t}^{\frac{1}{1-\mu_W}} - \alpha_W W_{R,t-1}^{\frac{1}{1-\mu_W}} \left(\frac{\pi_t}{\pi_{t-1}^{\gamma_W} \overline{\pi}_t^{1-\gamma_W}} \right)^{\frac{-1}{1-\mu_W}} , \tag{5.5}$$

$$Z_{W1,t} = \overline{L} W_{R,t}^{\frac{(1+\sigma_L)\mu_W}{\mu_W-1}} L_t^{1+\sigma_L} E_t^L E_t^B + \beta\alpha_W \mathbb{E}_t \left(Z_{W1,t+1} \left[\frac{\pi_{t+1}}{\pi_t^{\gamma_W} \overline{\pi}^{1-\gamma_W}} \right]^{\frac{(1+\sigma_L)\mu_W}{\mu_W-1}} \right) , \tag{5.6}$$

$$Z_{W2,t} = (1-\tau_{W,t}) W_{R,t}^{\frac{\mu_W}{\mu_W-1}} L_t \Lambda_t + \beta\alpha_W \mathbb{E}_t \left(Z_{W2,t+1} \left[\frac{\pi_{t+1}}{\pi_t^{\gamma_W} \overline{\pi}^{1-\gamma_W}} \right]^{\frac{1}{\mu_W-1}} \right) . \tag{5.7}$$

5.1.3 Investment decisions

The capital is owned by households and rented out to the intermediate firms at a rental rate R_t^k. Households choose the capital stock, investment and the capacity utilisation rate in order to maximize their intertemporal utility function subject to the intertemporal budget constraint and the capital accumulation equation given by:

$$K_{t+1} = (1-\delta) K_t + E_t^I \left[1 - S\left(\frac{I_t}{I_{t-1}} \right) \right] I_t, \tag{5.8}$$

where δ is the depreciation rate and $S(\bullet)$ the adjustment cost function, where it is assumed that

$$S\left(\frac{I_t}{I_{t-1}} \right) = S(\frac{I_t}{I_{t-1}}) = \frac{\phi_i}{2} (\frac{I_t}{I_{t-1}} - 1)^2.$$

First-order conditions result in the following equations for the real value of capital, investment and the capacity utilisation rate:

$$Q_t = \mathbb{E}_t \left[\beta \frac{\Lambda_{t+1}}{\Lambda_t} \left(Q_{t+1}(1-\delta) + R_{t+1}^k CU_{t+1} - \Phi\left(CU_{t+1} \right) \right) \right] E_t^Q \tag{5.9}$$

$$1 = Q_t \left[1 - S\left(\frac{I_t}{I_{t-1}}\right) - \frac{I_t}{I_{t-1}} S'\left(\frac{I_t}{I_{t-1}}\right)\right] E_t^I + \beta \mathbb{E}_t \left[Q_{t+1} \frac{\Lambda_{t+1}}{\Lambda_t} \left(\frac{I_{t+1}}{I_t}\right)^2 S'\left(\frac{I_{t+1}}{I_t}\right) E_{t+1}^I\right] \quad (5.10)$$

$$R_t^k = \Phi'(CU_t) \quad (5.11)$$

5.1.4 Final goods sector

Final producers are in perfect competition and aggregate a continuum of differentiated intermediate products. The elementary differentiated goods are imperfect substitutes with elasticity of substitution denoted ϵ, such that

$$Y = \left[\int_0^1 Y(h)^{\frac{\varepsilon-1}{\varepsilon}} dh\right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

The aggregate price index is defined as

$$P = \left[\int_0^1 p(h)^{1-\varepsilon} dh\right]^{\frac{1}{1-\varepsilon}},$$

and domestic demand is allocated across the differentiated goods as follows

$$\forall h \in [0,1] \quad Y(h) = \left(\frac{p(h)}{P}\right)^{-\varepsilon} Y.$$

5.1.5 Intermediate firms

Intermediate goods are produced with a Cobb-Douglas technology as follows:

$$\forall h \in [0,1], \quad Y_t(h) = E_t^A \left(CU_t(h) K_t(h)\right)^{\alpha} L_t(h)^{1-\alpha} - \Omega,$$

where E_t^A is an exogenous technology shock and Ω is a fixed cost ensuring that profits are zero in the steady state.

Firms are monopolistic competitors and produce differentiated products. In each period, a firm h faces a constant probability, $1 - \alpha_P$, of being able to reoptimize its nominal price. This probability is independent across firms and time. The average duration of a rigidity period is $\frac{1}{1-\alpha_P}$. If a firm cannot reoptimize its price, the price evolves according to the following simple rule:

$$p_t(h) = \left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_H} \overline{\pi}^{1-\gamma_H} p_{t-1}(h).$$

Therefore, firm h chooses $\tilde{p}_t(h)$ to maximize its intertemporal profit

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha_p^j \Xi_{t,t+j} Y_{t+j}(h) \left((1-\tau_t)\tilde{p}_t(h) \left(\frac{P_{t-1+j}}{P_{t-1}}\right)^{\gamma} \left(\frac{\bar{P}_{t-1+j}}{\bar{P}_{t-1}}\right)^{1-\gamma} - MC_{t+j} P_{t+j}\right)\right],$$

where

$$\Xi_{t,t+j} = \beta^j \frac{\Lambda_{t+j} P_t}{\Lambda_t P_{t+j}}$$

is the marginal value of one unit of money to the household, MC_{t+j} is the real marginal cost, τ_t is a time-varying tax on firm's revenue. Due to our assumptions on the labour market and the rental rate of capital, the real marginal cost is identical across producers,

$$MC_t = \frac{W_t^{(1-\alpha)} \left(R_t^k\right)^\alpha}{E_t^A \alpha^\alpha (1-\alpha)^{(1-\alpha)}}.$$

In our model, all firms that can reoptimize their price at time t choose the same level. The first order condition associated with the firm's choice of $\tilde{P}_t(h)$ is

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \Xi_{t,t+j} Y_{t+j}(h) P_{t+j} \left((1-\tau_t) \frac{\tilde{p}_t(h)}{P_{t+j}} \left(\frac{P_{t-1+j}}{P_{t-1}} \right)^\gamma \left(\frac{\overline{P}_{t+j}}{P_t} \right)^{1-\gamma} - \frac{\varepsilon}{\varepsilon - 1} MC_{t+j} \right) \right] = 0.$$

When the probability of being able to change prices tends towards unity, this implies that the firm sets its price equal to a constant markup $\frac{\mu}{1-\tau}$ (with $\mu = \frac{\epsilon}{\epsilon-1}$) over marginal cost as in the flexible-price model. Otherwise the firm imposes this markup to the weighted-average of marginal costs over time.

Only a fraction $1-\alpha_p$ of producers can reoptimize its price, each period. So the aggregate producer-price-index has the following dynamic:

$$P_t^{1-\varepsilon} = \alpha_p \left(\left(\frac{P_{t-1}}{P_{t-2}} \right)^\gamma \overline{\pi}_t^{1-\gamma} P_{t-1} \right)^{1-\varepsilon} + (1-\alpha_p)\, \tilde{p}_t^{1-\varepsilon}(h).$$

This price setting scheme is easily rewritten in the following recursive form:

$$\left(\mu \frac{Z_{1,t}}{Z_{2,t}} \right)^{\frac{1}{1-\mu}} (1-\alpha_p) = 1 - \alpha_p \left(\frac{\pi_t}{\pi_{t-1}^\gamma \overline{\pi}^{1-\gamma}} \right)^{\frac{1}{\mu-1}}, \tag{5.12}$$

$$Z_{1,t} = \Lambda_t MC_t Y_t + \alpha_p \beta \mathbb{E}_t \left[\left(\frac{\pi_{t+1}}{\pi_t^\gamma \overline{\pi}^{1-\gamma}} \right)^{\frac{\mu}{\mu-1}} Z_{1,t+1} \right], \tag{5.13}$$

$$Z_{2,t} = E_t^P \Lambda_t Y_t + \alpha_p \beta \mathbb{E}_t \left[\left(\frac{\pi_{t+1}}{\pi_t^\gamma \overline{\pi}^{1-\gamma}} \right)^{\frac{1}{\mu-1}} Z_{2,t+1} \right]. \tag{5.14}$$

Capital labour ratio is equalized across firms and linked to the relative cost of factors:

$$\frac{W_t L_t}{R_t^k CU_t K_{t-1}} = \frac{1-\alpha}{\alpha}.$$

5.1.6 Government

Public expenditures are subject to random shocks E_t^G. The government finances public spending with labour tax, product tax and lump-sum transfers,

$$P_t E_t^G - \tau_{W,t} W_t L_t - \tau_t P_t Y_t - P_t TT_t = 0.$$

Specifying the interest rate rules followed by the monetary authorities finally closes the model. In the case of the Taylor rule the exact expression is, much like in the previous model,

$$\frac{1+i_t}{1+\bar{\imath}} = \left(\frac{1+i_{t-1}}{1+\bar{\imath}}\right)^{\rho_r} \left[\left(\frac{\pi_t}{\bar{\pi}}\right)^{\phi_\pi} \left(\frac{Y_t}{\overline{\overline{Y}}}\right)^{\frac{\phi_y}{4}}\right]^{1-\rho_r}.$$

This is not exactly the same policy rule that Smets and Wouters (2003) consider but it is one first approximation to it. In order to make it close to theirs we consider $\rho_r = 0.85$, $\phi_\pi = 1.5$ and $\phi_y = 0.8$ (hence $\phi_y/4 = 0.2$, similar to their estimates). One big difference is that we are not considering the output gap but only deviations from steady state output. This can have an important impact in policy and therefore shall be investigated in future research.

5.1.7 Market clearing conditions

Aggregate productions are obtained using the CES aggregator

$$\left[\int_0^1 \bullet^{\frac{\epsilon-1}{\epsilon}} dz\right]^{\frac{\epsilon}{\epsilon-1}},$$

and labour demands are given by the following relations:

$$Y_t D_t = E_t^A \left(CU_t K_t\right)^\alpha \left(L_t\right)^{1-\alpha} - (\mu - 1)\overline{\overline{Y}},$$

as $\Omega = (\mu - 1)\overline{\overline{Y}}$ implies that profits are zero in steady state.

Price dispersion is defined as $D_t = \int_0^1 \left(\frac{p_t(h)}{P_t}\right)^{-\frac{\mu}{\mu-1}} dh$, and follows the law of motion given by

$$D_t = (1-\alpha_p)\left(\mu \frac{Z_{1,t}}{Z_{2,t}}\right)^{-\frac{\mu}{\mu-1}} + \alpha_p D_{t-1}\left(\frac{\pi_t}{\pi_{t-1}^{\gamma}\bar{\pi}^{1-\gamma}}\right)^{\frac{\mu}{\mu-1}}, \tag{5.15}$$

while $D_{W,t} = \int_0^1 \left(\frac{W_t(h)}{W_t}\right)^{-\frac{(1+\sigma_L)\mu_W}{\mu_W - 1}} dh$ and the derivation of the related equations follows.

Aggregate demand is given by

$$Y_t = C_t + I_t + E_t^G + \Phi\left(CU_t\right) K_{t-1}$$

5.2 Summary of model equations

The set of structural equations is given by:

$$\Lambda_t = E_t^B \left(C_t - hC_{t-1}\right)^{-\sigma_C} - \beta\theta\mathbb{E}_t E_{t+1}^B \left(C_{t+1} - hC_t\right)^{-\sigma_C} \tag{g1}$$

$$\Lambda_t = (1+i_t)\beta\mathbb{E}_t\left[\Lambda_{t+1}\frac{P_t}{P_{t+1}}\right] \tag{g2}$$

$$Q_t = \mathbb{E}_t\left[\beta\frac{\Lambda_{t+1}}{\Lambda_t}\left(Q_{t+1}(1-\delta) + \Phi'\left(CU_{t+1}\right)CU_{t+1} - \Phi\left(CU_{t+1}\right)\right)\right]E_t^Q \tag{g3}$$

$$\begin{aligned} 1 &= Q_t\left[1 - S\left(\frac{I_t}{I_{t-1}}\right) - \frac{I_t}{I_{t-1}}S'\left(\frac{I_t}{I_{t-1}}\right)\right]E_t^I \\ &\quad +\beta\mathbb{E}_t\left[Q_{t+1}\frac{\Lambda_{t+1}}{\Lambda_t}\left(\frac{I_{t+1}}{I_t}\right)^2 S'\left(\frac{I_{t+1}}{I_t}\right)E_{t+1}^I\right]\end{aligned} \tag{g4}$$

$$R_t^k = \Phi'\left(CU_t\right) \tag{g5}$$

$$W_tL_t = \frac{1-\alpha}{\alpha}CU_t\Phi'\left(CU_t\right)K_{t-1} \tag{g6}$$

$$Y_t = C_t + I_t + E_t^G + \Phi\left(CU_t\right)K_{t-1} \tag{g7}$$

$$D_t\left(C_t + I_t + E_t^G + \Phi\left(CU_t\right)K_{t-1}\right) = E_t^A\left(CU_tK_t\right)^{\alpha}\left(L_t\right)^{1-\alpha} - (\mu-1)\overline{\overline{Y}} \tag{g8}$$

$$\left(\mu\frac{Z_{1,t}}{Z_{2,t}}\right)^{\frac{1}{1-\mu}}(1-\alpha_p) = 1 - \alpha_p\left(\frac{\pi_t}{\pi_{t-1}^{\gamma}\overline{\pi}_t^{1-\gamma}}\right)^{\frac{1}{\mu-1}} \tag{g9}$$

$$Z_{1,t} = \Lambda_t MC_t Y_t + \alpha_p\beta\mathbb{E}_t\left[\left(\frac{\pi_{t+1}}{\pi_t^{\gamma}\overline{\pi}_{t+1}^{1-\gamma}}\right)^{\frac{\mu}{\mu-1}}Z_{1,t+1}\right] \tag{g10}$$

$$Z_{2,t} = E_t^P\Lambda_tY_t + \alpha_p\beta\mathbb{E}_t\left[\left(\frac{\pi_{t+1}}{\pi_t^{\gamma}\overline{\pi}^{1-\gamma}}\right)^{\frac{1}{\mu-1}}Z_{2,t+1}\right] \tag{g11}$$

$$\begin{aligned} Z_{W1,t} &= E_t^L\overline{L}L_t^{1+\sigma_L}W_t^{\frac{(1+\sigma_L)\mu_W}{\mu_W-1}} \\ &\quad +\alpha_W\beta\mathbb{E}_t\left[\left(\frac{\pi_{t+1}}{\pi_t^{\gamma_W}\overline{\pi}^{1-\gamma_W}}\right)^{\frac{(1+\sigma_L)\mu_W}{\mu_W-1}}Z_{W1,t+1}\right]\end{aligned} \tag{g12}$$

$$\begin{aligned} Z_{W2,t} &= E_t^W\Lambda_tL_tW_t^{\frac{\mu_W}{\mu_W-1}} \\ &\quad +\alpha_W\beta\mathbb{E}_t\left[\left(\frac{\pi_{t+1}}{\pi_t^{\gamma_W}\overline{\pi}^{1-\gamma_W}}\right)^{\frac{1}{\mu_W-1}}Z_{W2,t+1}\right]\end{aligned} \tag{g13}$$

$$(1-\alpha_W)\left(\mu_w\frac{Z_{W1,t}}{Z_{W2,t}}\right)^{-\frac{1}{\mu_W(1+\sigma_L)-1}} = W_t^{\frac{1}{1-\mu_W}} - \alpha_W W_{t-1}^{\frac{1}{1-\mu_W}}\left(\frac{\pi_t}{\pi_{t-1}^{\gamma_W}\overline{\pi}^{1-\gamma_W}}\right)^{\frac{-1}{1-\mu_W}} \tag{g14}$$

$$K_t = (1-\delta)K_{t-1} + E_t^I\left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right]I_t \tag{f1}$$

$$D_t = (1-\alpha_p)\left(\mu\frac{Z_{1,t}}{Z_{2,t}}\right)^{-\frac{\mu}{\mu-1}} + \alpha_p D_{t-1}\left(\frac{\pi_t}{\pi_{t-1}^{\gamma}\overline{\pi}^{1-\gamma}}\right)^{\frac{\mu}{\mu-1}} \tag{f2}$$

$$\begin{aligned} D_{W,t} \quad &= \quad (1-\alpha_W)\,W_t^{\frac{(1+\sigma_L)\mu_W}{\mu_W-1}}\left(\mu_w\frac{Z_{W1,t}}{Z_{W2,t}}\right)^{-\frac{\mu_W(1+\sigma_L)}{\mu_W(1+\sigma_L)-1}} \\ &\quad +\alpha_W D_{W,t-1}\left(\frac{W_t}{W_{t-1}}\frac{\pi_t}{\pi_{t-1}^{\gamma_W}\overline{\pi}^{1-\gamma_W}}\right)^{\frac{(1+\sigma_L)\mu_W}{\mu_W-1}} \end{aligned} \tag{f3}$$

knowing that the following definition holds:

$$MC_t = \frac{W_t^{(1-\alpha)}\Phi'\left(CU_{t+1}\right)^{\alpha}}{E_t^A\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}.$$

and that

$$S(\frac{I_t}{I_{t-1}}) = \frac{\phi_i}{2}(\frac{I_t}{I_{t-1}}-1)^2$$

and

$$\Phi\left(CU_t\right) = \frac{1}{\overline{rc}}R_{ss}^k\exp(\overline{rc}*(CU_t-1)).$$

5.3 The model in LQ form

We now need to match the structure presented in section 2 one more time. First of all notice that in the model there are two elements of ξ_{t+1} showing up in the G-type constraints, which would violate the framework for the LQ function. In order to overcome this issue we define two new variables to include in the x_t vector, F_t^B and F_t^I. In this way they will show up in x_{t+1} and not violate the framework. In order to connect these to the true shocks we insert two extra equations to the G-type constraints,

$$E_t^B = F_t^B, \tag{g15}$$

and

$$E_t^I = F_t^I. \tag{g16}$$

We can then define:

$$\begin{aligned} x_t \quad &\equiv \quad \left(C_t, \pi_t, Q_t, I_t, CU_t, R_t^k, W_t, Z_{1,t}, Z_{2,t}, Z_{W1,t}, Z_{W2,t}, \Lambda_t, F_t^B, F_t^I\right)', \\ u_t \quad &\equiv \quad \left(Y_t, L_t, i_t\right)', \\ X_t \quad &\equiv \quad \left(K_t, D_t, D_{W,t}, I_{t-1}, C_{t-1}, W_{t-1}, \pi_{t-1}, i_{t-1}\right)', \end{aligned}$$

and the vector of constraints G is defined in (g1-g14), added by (g15-g16) and the predetermined constraint, with the implicit definition of $F\left(\cdot\right)$ is presented in (f1-f3).

5.4 Optimal responses to shocks

In this section we compare the behaviour of optimal policy in face of the various shocks present in the economy, using the behaviour under a standard Taylor rule as the comparative benchmark.[14] The calibration used for the parameters values is presented in Table 3.[15] Results are shown in Figures (19-24).[16]

In the case of productivity shocks (Figure 19) optimal monetary policy exacerbates the effect on output and consumption components of the productivity shock, relative to the Taylor rule benchmark. Under LQ optimal policy output, consumption and investment are thus significantly higher than under the Taylor rule case, in particular on impact. This leads to steadily higher path for inflation and real wages under LQ policy. Conversely, LQ policy implies lower nominal interest rates relative to the Taylor case during the first quarters. A possible interpretation of these results is that LQ policy aims at materisalising the efficiency gains that are hindered by nominal rigidities. By contrast, the Taylor rule, by muting the favourable impact of the positive productivity shock, is over-restrictive relative to LQ.

For the preference shock (Figure 20), by contrast to the previous case, monetary policy largely counters the inflationary effects of the preference shock on output that takes place under the Taylor rule policy. In order to stabilise inflation LQ optimal policy counters the expansion in output and consumption that would take place, to the extent of actually reversing the effect and generating a contraction in these variables. This is achieved by a higher path for nominal interest rates in the first quarters. In exchange of this, LQ policy delivers a lower path for labour effort and more stable prices.

In the case of the government purchase shock (Figure 21), optimal monetary policy is again restrictive relative to the Taylor case. However, in this case the effect on output components is purely in composition, with the path for aggregate output being similar under LQ and Taylor policies. Indeed, the LQ policy diminishes overall the extent of investment crowding-out, while it has a more ambiguous effect in terms of consumption crowding-out relative to Taylor. The negative impact on wages is diminished under LQ compared to Taylor. These effects are, again, achieved with a relatively contractionary monetary policy up-front, relative to Taylor policy. In both cases inflation is stabilised around steady state levels under both policies.

Figure 22 depicts the case of a (negative) labour supply shock. In this case the departure between LQ and Taylor policies is relatively large compared to the other shocks. Under Taylor all variables except wages are affected to a rather limited extent, i.e. under Taylor the

[14]Unlike the case of the model with monetary frictions discussed above, in this case the calculation of the model's steady-sate is straightforward. In this case the steady-state is distorted as steady-state price and wage mark-ups are assumed to be above one and their effect on welfare is not compensated with lump-sum transfers. Therefore, in this case, the focus is the comparison of model dynamics under optimal and non-optimal monetary policies, around a distorted steady-state.

[15]The calibration used ensures that the impulse respond functions under this specification for the Taylor rule monetary policy case are qualitatively similar to the ones in Smets and Wouters (2003).

[16]The responses of the interest and inflation rates were annualized so that we can read them in terms of annual percentage points.

transmission mechanism takes place mainly through wage adjustments and is very confined. Under LQ policy, somewhat alike to the case of the productivity shock, monetary policy exacerbates the effect of the shock on output and output components, being restrictive relative to the Taylor policy. Consequently, output, labour and output components are well below steady state under LQ relative to Taylor policy. Inflation and wages are then also below in the LQ case. In this case, as in the productivity shock case, monetary policy could be seen as mimicking the adjustment process that would be seen under more flexible wages and prices.

Finally, the investment shock and the equity premium shock (Figures 23 and 24) are very similar, only with different scales. Much like in the productivity and labour supply shocks optimal policy exacerbates the impact on output at the same time that minimizes the effects on inflation. Again this can be understood as fostering the beneficial effects of the temporary positive shocks occuring. It is interesting to note that while the Taylor rule implies a negative initial path of consumption, implying that investment crowds out consumtpion, optimal policy through its expansionary stance allows the possitive effects on the economy to spread to cocnsumption without any delays.

In sum, the previous results suggest that LQ policy differentiates to a larger extent its impact taking into account the supply or demand nature of the shocks, in general exacerbating the effects of the former and dampening the effects of the latter relative to the Taylor case. These differences may however be influenced by the lack of a measure of potential output in the Taylor rule considered here. This is an issue that deserves future investigation.

6 Conclusion

This paper presents the evaluation of optimal policy in a timeless perspective for two applications. In the first application a standard closed economy model is compared under two alternatives cases: the cashless limiting case and the case with monetary frictions. In our second application, optimal monetary policy is evaluated within a standard Neo-Keynesian model of a closed economy with various structural shocks, close to a number of recently estimated models.

The results from the first application show that the consideration of monetary frictions or simply a cashless economy are not of significant importance when it comes to optimal policy. This is so because the responses of the economy to various shocks in the two cases are very similar. In that model, comparing the optimal policy from a timeless perspective to a simple Taylor rule yields substantially different results, with optimal policy stabilising inflation significantly more. Similar results are obtained in the case of the Neo-Keynesian model presented in the second application. In particular, the results suggest that LQ policy differentiates to a larger extent its impact taking into account the supply or demand nature of the shocks, in general exacerbating the effects of the former and dampening the effects of the latter relative to the Taylor case. This results should warn researchers about the use of Taylor rules as proxies to optimal policy, indeed they can be substantially different.

The method for computing and evaluating the LQ optimal policy in these two examples makes use of a standardised approximation algorithm which can be applied to a broad class of models, well beyond our two applications. In particular, the algorithm can be readily used for fairly complicated models, probably without a large amount of further programming involved beyond the adjustments required to fit the required structure, as exemplified in our two applications. The provided algorithm could thus help in bridging the gap between, on the one hand, theoretical model-based considerations on optimal policy – which has often been discussed on the basis of rather stylised models – and models used in practice for policy analysis, which are typically of large size and therefore less prone to deliver results on optimal policy. The algorithm proposed here, together with the companion Matlab function, is presented as one step in this direction.

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A Monetary-frictions model

A.1 Calibration

ω	0.473	elasticity of real marginal cost with respect to a firm's own output
β	0.99	time discount factor
α	0.75	Calvo probability of not adjusting prices
λ	0.7	scale of labour disutility
σ^{-1}	0.157	intertemporal elasticity of substitution
ϕ	1	production function parameter
θ	10	elasticity of substitution of among different varieties of goods
a	0.0111	parameter of the transaction frictions
b	0.07524	parameter of the transaction frictions that regulates the concavity
s_g	0.3	steady state ratio of government expenditures to output
$\bar{\tau}$	0.2	steady state tax rate

Table 1: Baseline parameters

A.2 Conversion of the utility expressions

First, inverting (4.10) we can write

$$h_t(i) = f^{-1}(y_t(i)/A_t) = \left(\frac{y_t(i)}{A_t}\right)^{\phi},$$

and, because the demand for labour by each firm in a given industry and the output are the same, we can also write

$$H_t(j) = f^{-1}\left(y_t^j/A_t\right),$$

and plug this into the utility function in the households problem:

$$v\left(y_t^j;\xi_t\right) = \tilde{v}\left(f^{-1}\left(y_t^j/A_t\right);\xi_t\right).$$

By using the equilibrium condition for the resources we can also write the utility in terms of Y instead of consumption:

$$u(Y_t, n_t; \xi_t) = \tilde{u}\left(\frac{Y_t - G_t}{1+\chi_t s(n_t)};\xi_t\right),$$

and now we can write the intertemporal utility function as

$$U_{t_0} = \sum_{t=t_0}^{\infty} \beta^{t-t_0}\left[u(Y_t, n_t; \xi_t) - \int_0^1 v\left(y_t^j;\xi_t\right)dj\right].$$

Notice that

$$\tilde{u}'(C_t;\xi_t) = u_Y(Y_t, n_t;\xi_t)(1+\chi_t s(n_t)),$$

and so λ_t is defined as

$$\lambda_t = \frac{u_Y(Y_t, n_t;\xi_t)}{P_t}\frac{1+\chi_t s(n_t)}{1+\chi_t s(n_t)+\chi_t s'(n_t)\,n_t}, \tag{A.1}$$

which replaces (4.3).

In particular we know that

$$v\left(y_t^j;\xi_t\right) = \frac{\lambda}{1+\nu}\left(y_t^j\right)^{\phi(1+\nu)}\bar{H}_t^{-\nu}A_t^{-\phi(1+\nu)},$$

but in industry j all goods are produced in the same quantity and the demand for good i is

$$y_t(i) = Y_t\left(\frac{p_t(i)}{P_t}\right)^{-\theta},$$

so that

$$v\left(y_t^j;\xi_t\right) = \frac{\lambda}{1+\nu}Y_t^{\phi(1+\nu)}\left(\frac{p_t(j)}{P_t}\right)^{-\theta\phi(1+\nu)}\bar{H}_t^{-\nu}A_t^{-\phi(1+\nu)}.$$

Integrating over all the industries j we get

$$\int_0^1 v\left(y_t^j;\xi_t\right)dj = \frac{\lambda}{1+\nu} Y_t^{\phi(1+\nu)} \bar{H}_t^{-\nu} A_t^{-\phi(1+\nu)} \int_0^1 \left(\frac{p_t\left(i\right)}{P_t}\right)^{-\theta\phi(1+\nu)} di,$$

and define $\omega \equiv \phi\left(1+\nu\right)-1$, so that we can write the intertemporal utility as

$$U_{t_0} = \sum_{t=t_0}^{\infty} \beta^{t-t_0}\left[u\left(Y_t, n_t;\xi_t\right) - v\left(Y_t, \Delta_t;\xi_t\right)\right], \tag{A.2}$$

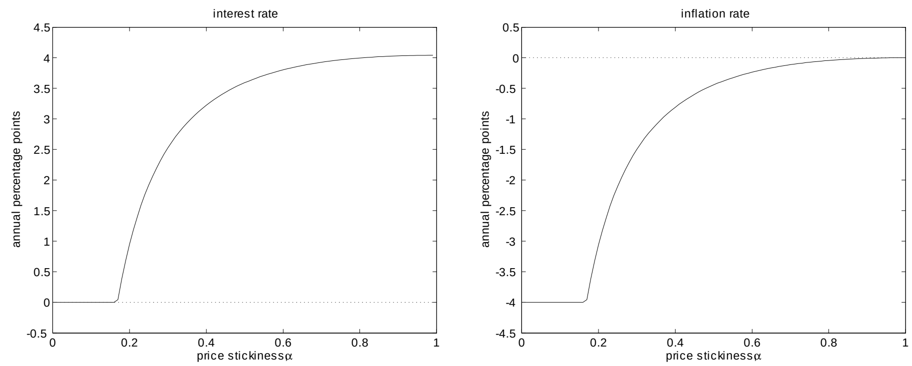
with

$$u\left(Y_t, n_t;\xi_t\right) = \frac{\bar{C}_t^{1-\tilde{\sigma}^{-1}}}{1-\tilde{\sigma}^{-1}}\left(\frac{Y_t - G_t}{1+\chi_t s\left(n_t\right)}\right)^{1-\tilde{\sigma}^{-1}}, \tag{A.3}$$

$$v\left(Y_t, \Delta_t;\xi_t\right) = \frac{\lambda}{1+\nu}\frac{Y_t^{1+\omega}}{A_t^{1+\omega}\bar{H}_t^{\nu}}\Delta_t. \tag{A.4}$$

A.3 Optimal steady state

Panel A: baseline



Panel B: tax rate set to eliminate distortions

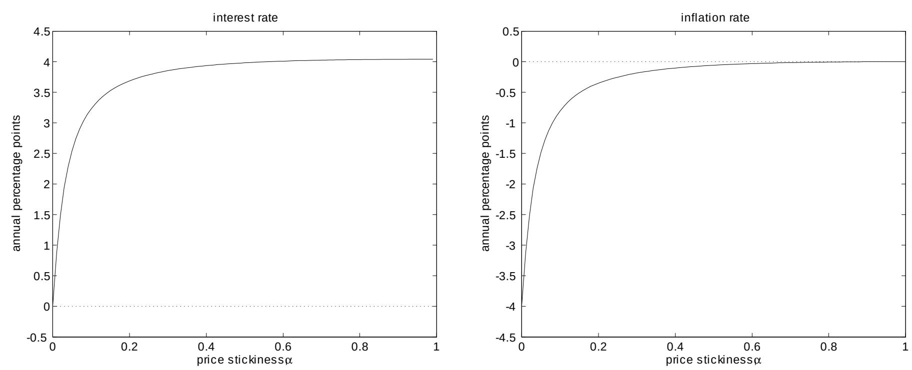


Figure 1: Steady state nominal interest and inflation rates for different levels of price stickiness

	Monetary frictions					Cashless limiting case				
	$\bar{\imath}$*	$\bar{\pi}$*	$\bar{Y}$	$\bar{n}$	$\bar{\Delta}$	$\bar{\imath}$*	$\bar{\pi}$*	$\bar{Y}$	$\bar{n}$	$\bar{\Delta}$
Baseline tax rate										
$\alpha = 0.3$	2.526	-1.500	1.1378	2.710	1.0003	4.040	0	1.1429	2.604	1
$\alpha = 0.6$	3.802	-0.236	1.1363	2.762	1.0001	4.040	0	1.1429	2.604	1
$\alpha = 0.7$	3.926	-0.114	1.1362	2.767	1.0000	4.040	0	1.1429	2.604	1
$\alpha = 0.75$	3.966	-0.074	1.1361	2.768	1.0000	4.040	0	1.1429	2.604	1
$\alpha = 0.8$	3.995	-0.045	1.1361	2.769	1.0000	4.040	0	1.1429	2.604	1
Tax rate set to eliminate distortions										
$\alpha = 0.3$	3.853	-0.185	1.9144	2.764	1.0000	4.040	0	1.9252	2.604	1
$\alpha = 0.6$	4.009	-0.031	1.9140	2.770	1.0000	4.040	0	1.9252	2.604	1
$\alpha = 0.7$	4.025	-0.015	1.9139	2.771	1.0000	4.040	0	1.9252	2.604	1
$\alpha = 0.75$	4.030	-0.010	1.9139	2.771	1.0000	4.040	0	1.9252	2.604	1
$\alpha = 0.8$	4.034	-0.006	1.9139	2.771	1.0000	4.040	0	1.9252	2.604	1

* in annualized percentage points

Table 2: Optimal steady state values under different scenarios

A.4 Responses to shocks

Monetary Frictions vs. Cashless Limiting Case

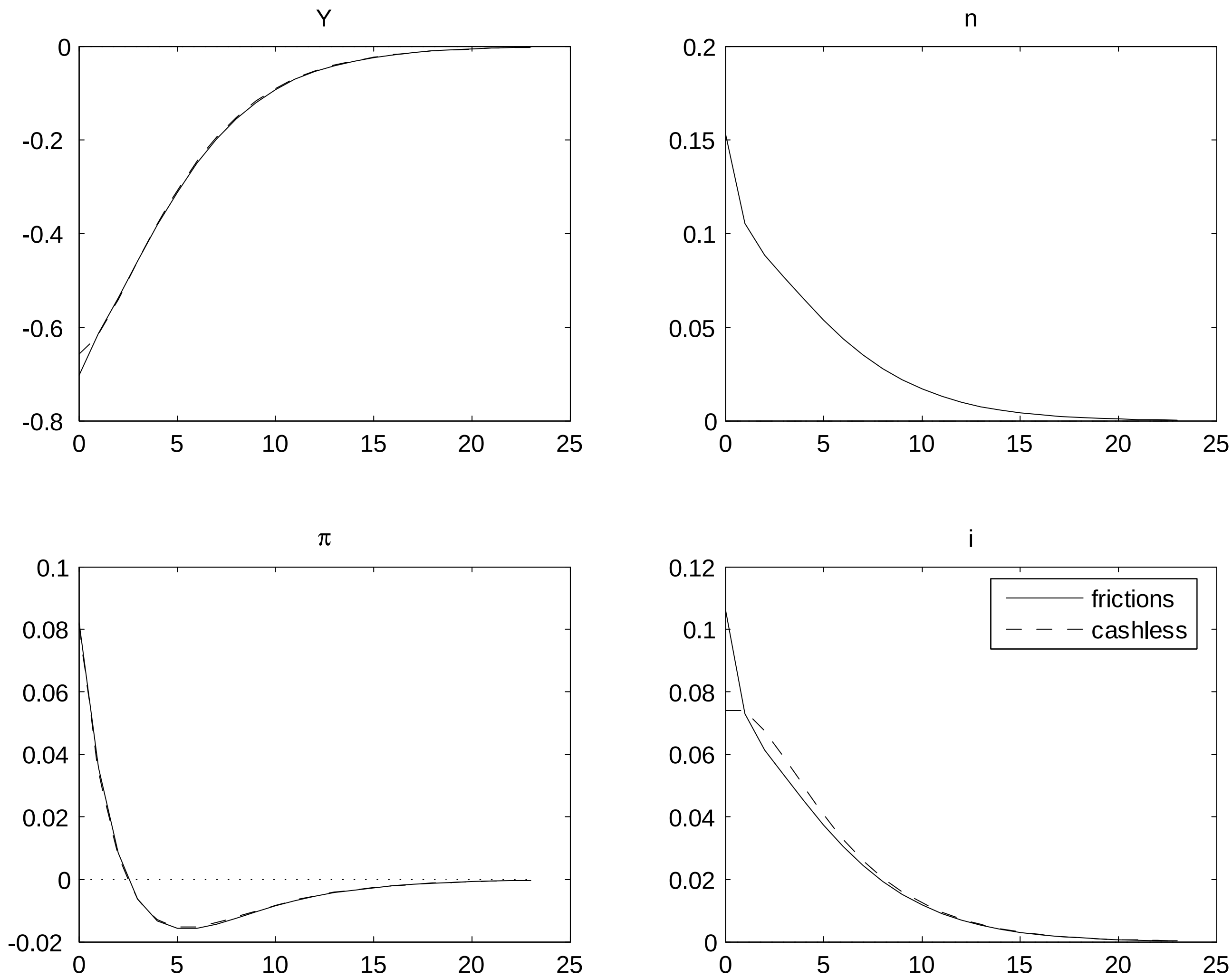


Figure 2: Responses to one percentage point increase in the tax rate (τ), under the case of monetary frictions compared to the cashless limiting case

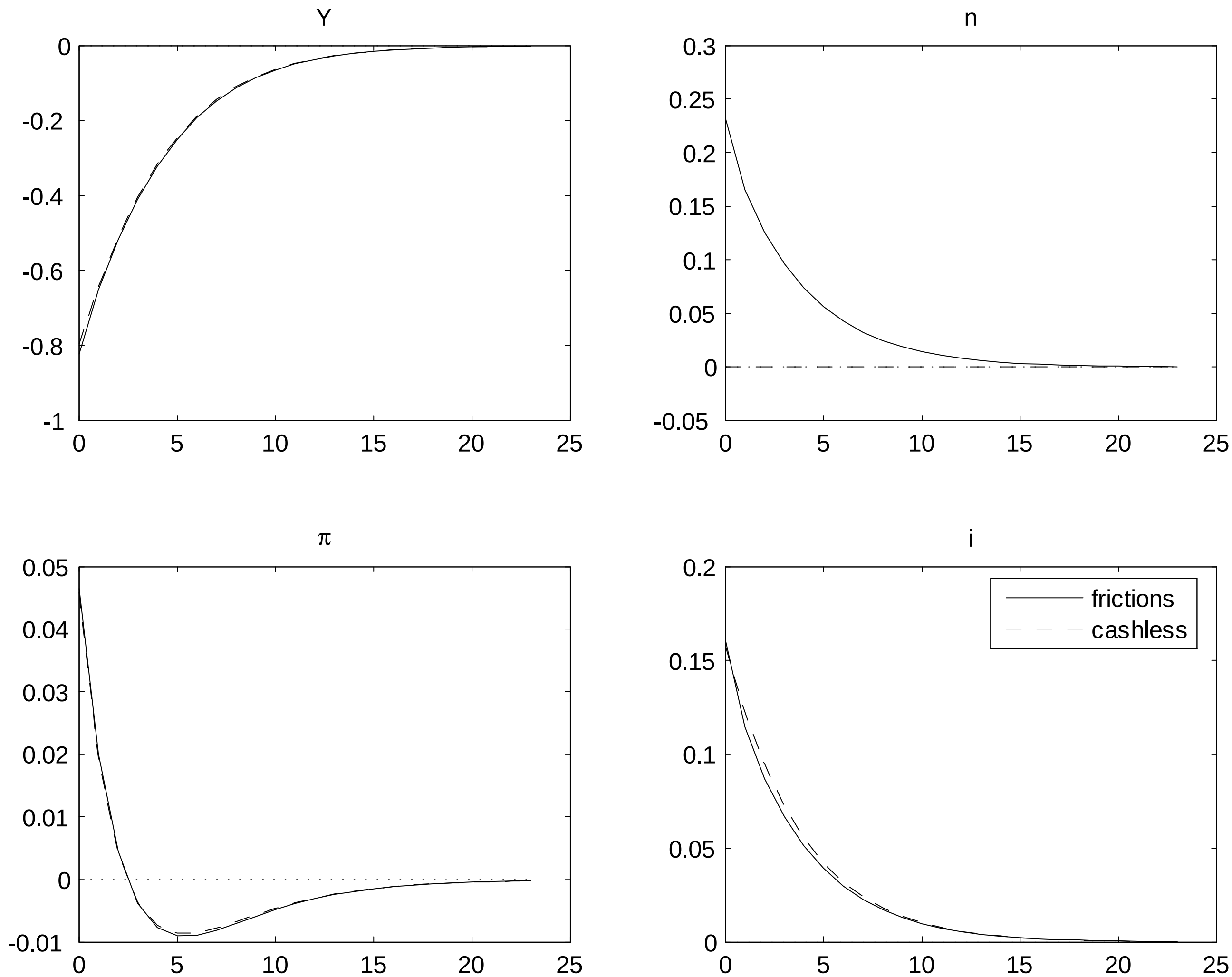


Figure 3: Responses to one percent increase in the price markup (μ), under the case of monetary frictions compared to the cashless limiting case

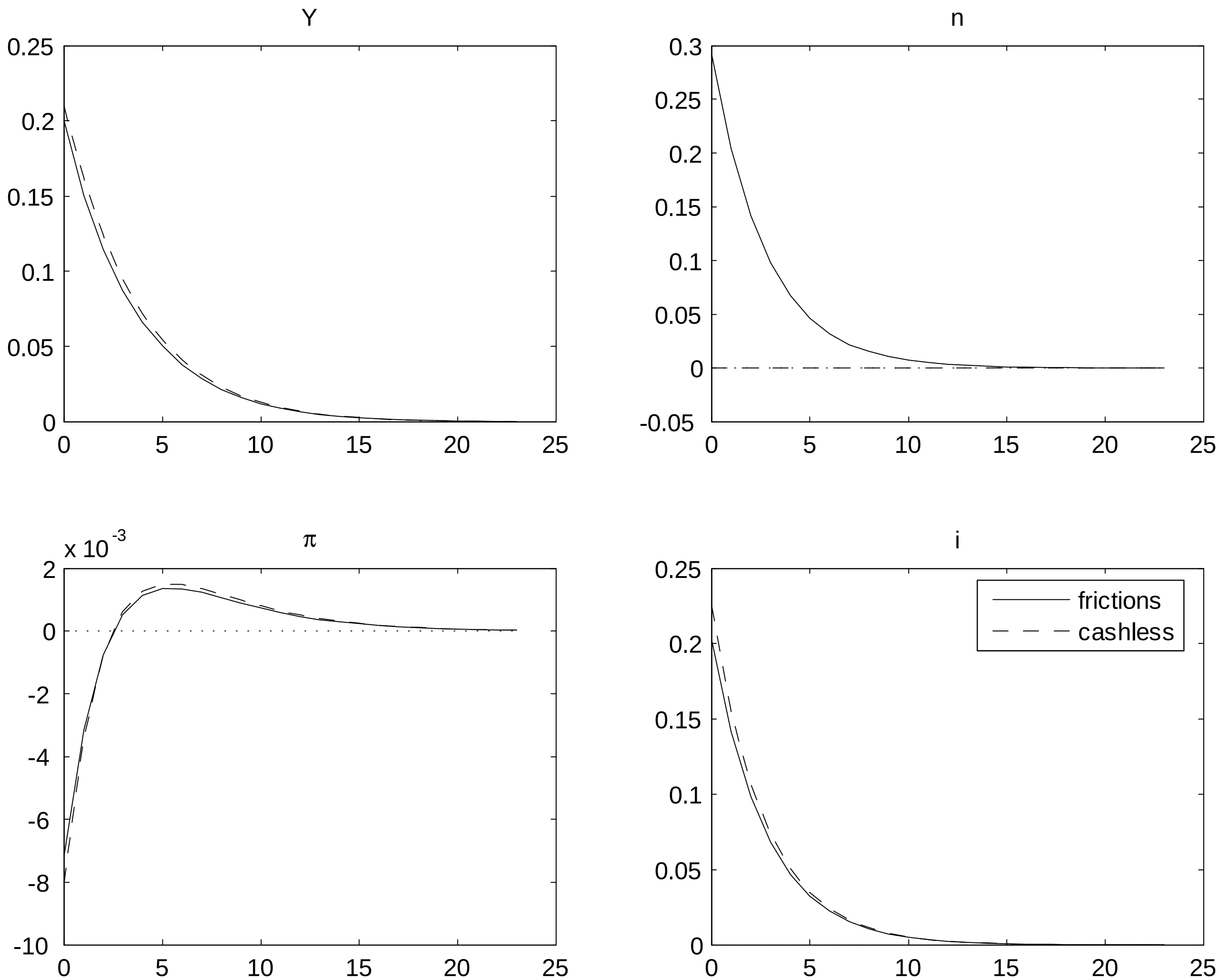


Figure 4: Responses to an increase in government expenditures (G) equivalent to one percent of the output, under the case of monetary frictions compared to the cashless limiting case

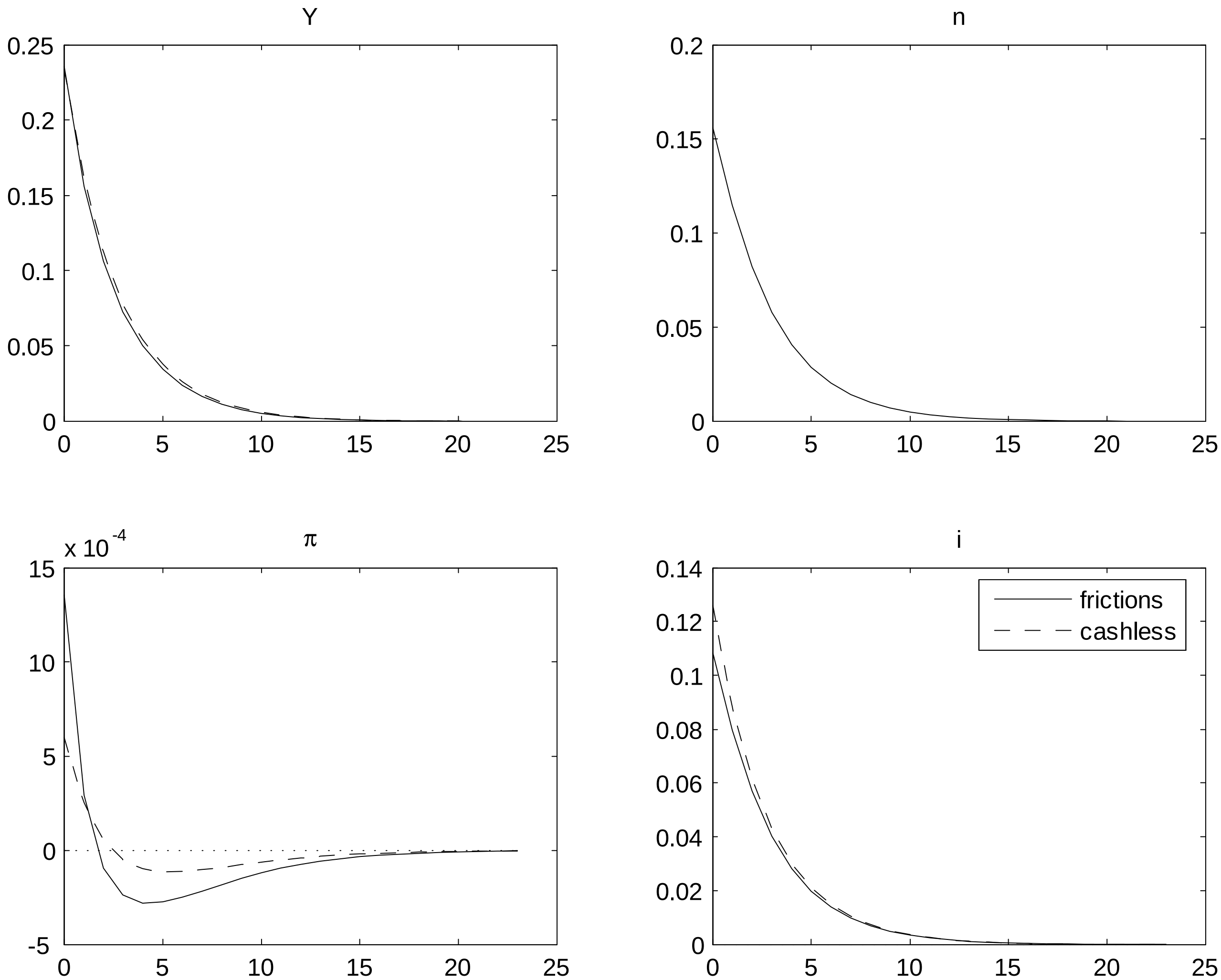


Figure 5: Responses to one percent increase in the shock to the marginal utility of consumption ($\bar{C}$), under the case of monetary frictions compared to the cashless limiting case

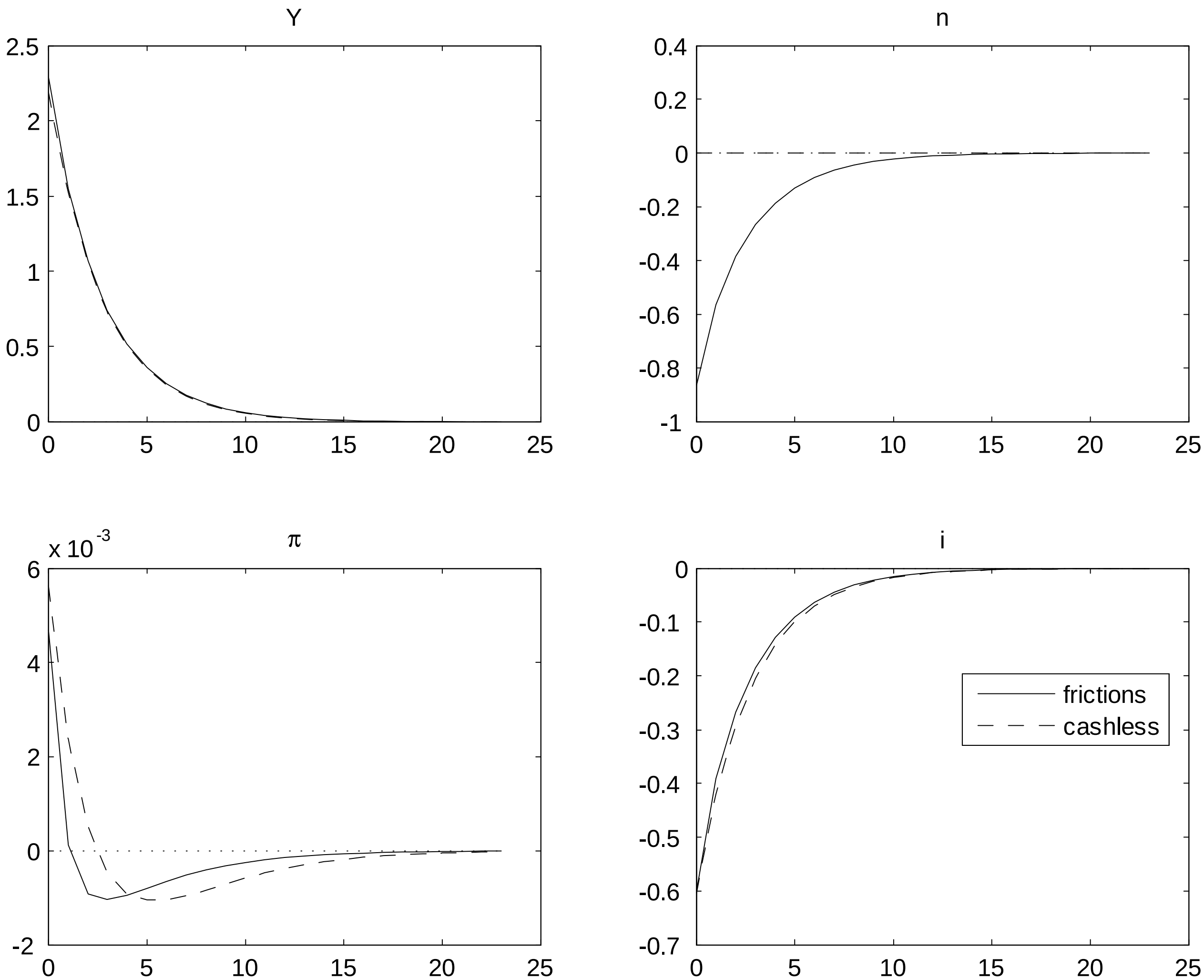


Figure 6: Responses to one percent increase in productivity (A), under the case of monetary frictions compared to the cashless limiting case

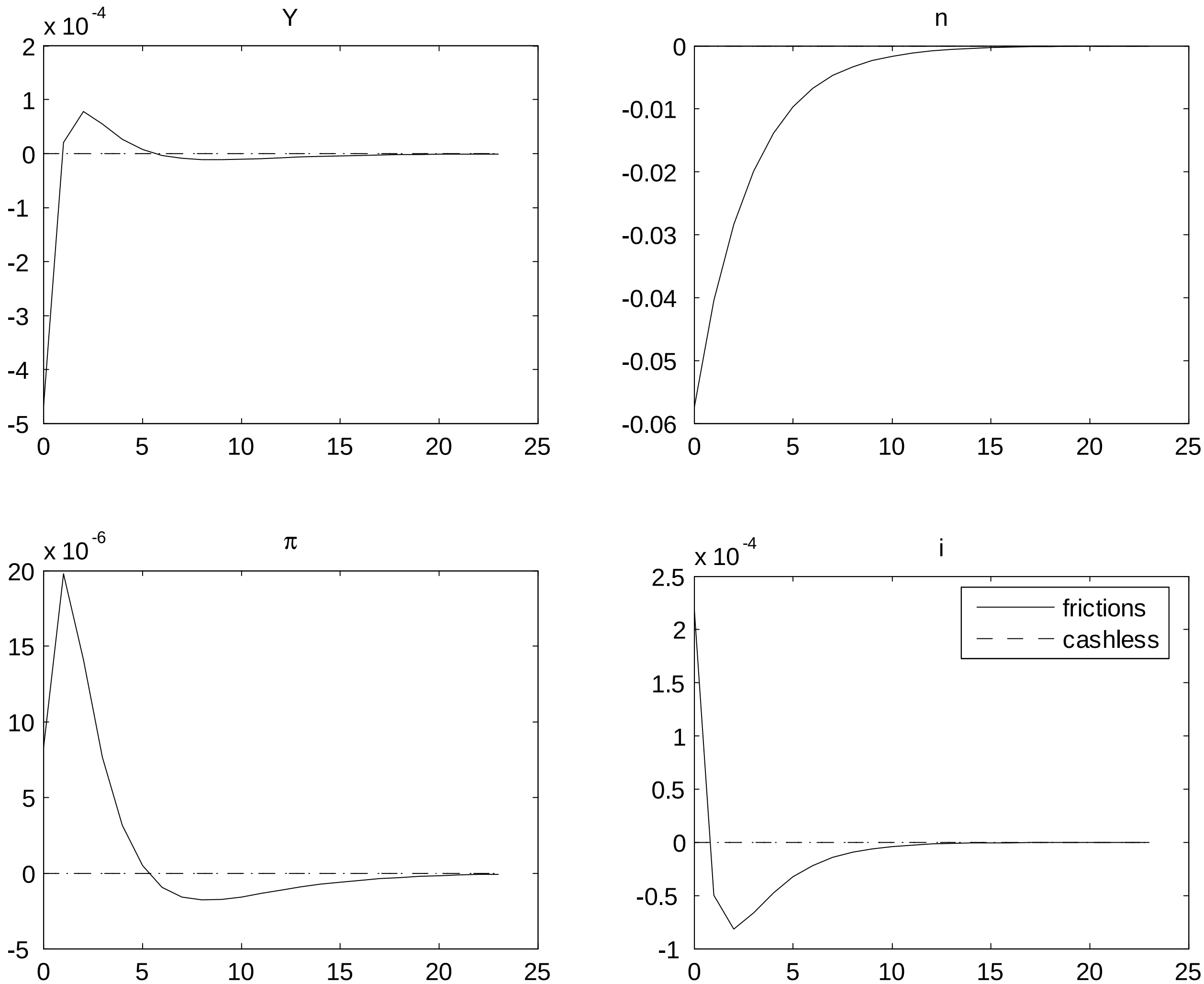


Figure 7: Responses to one percent increase in the shock to the transaction costs (χ), under the case of monetary frictions compared to the cashless limiting case

The Case of No Steady State Distortions

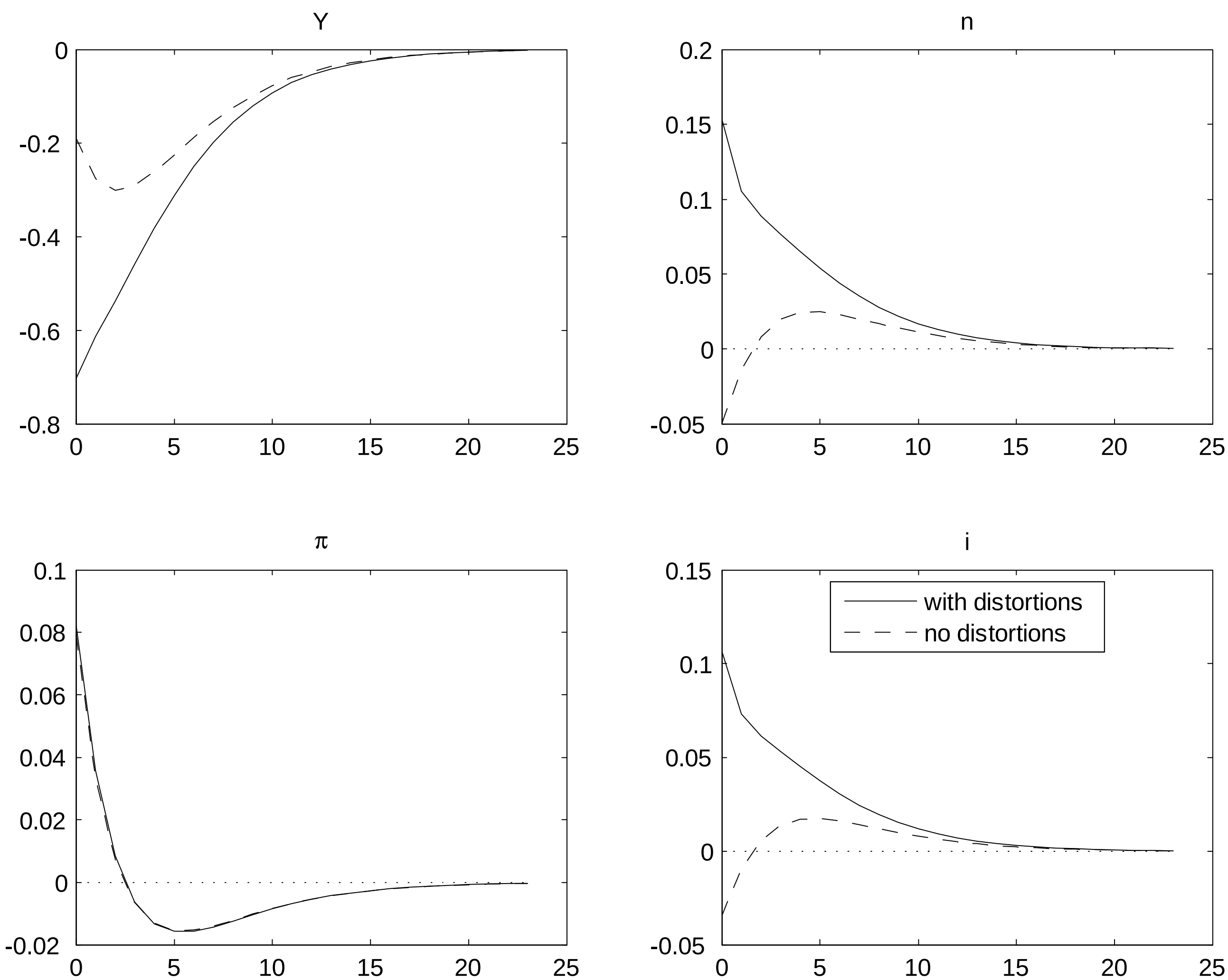


Figure 8: Responses to one percentage point increase in the tax rate (τ), under the case with steady state distortions in compared to the case of no steady state distortions

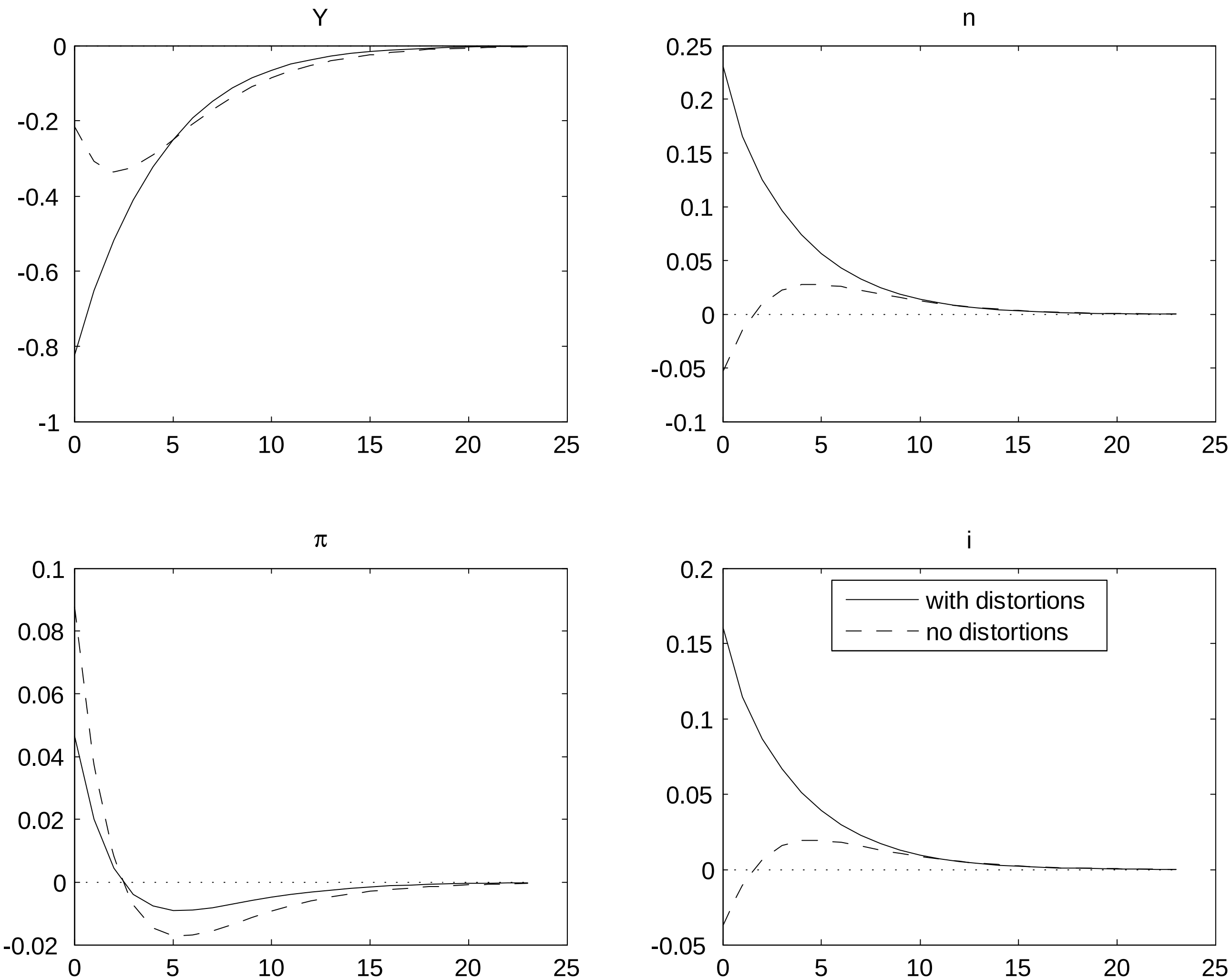


Figure 9: Responses to one percent increase in the makup (μ), under the case with steady state distortions in compared to the case of no steady state distortions

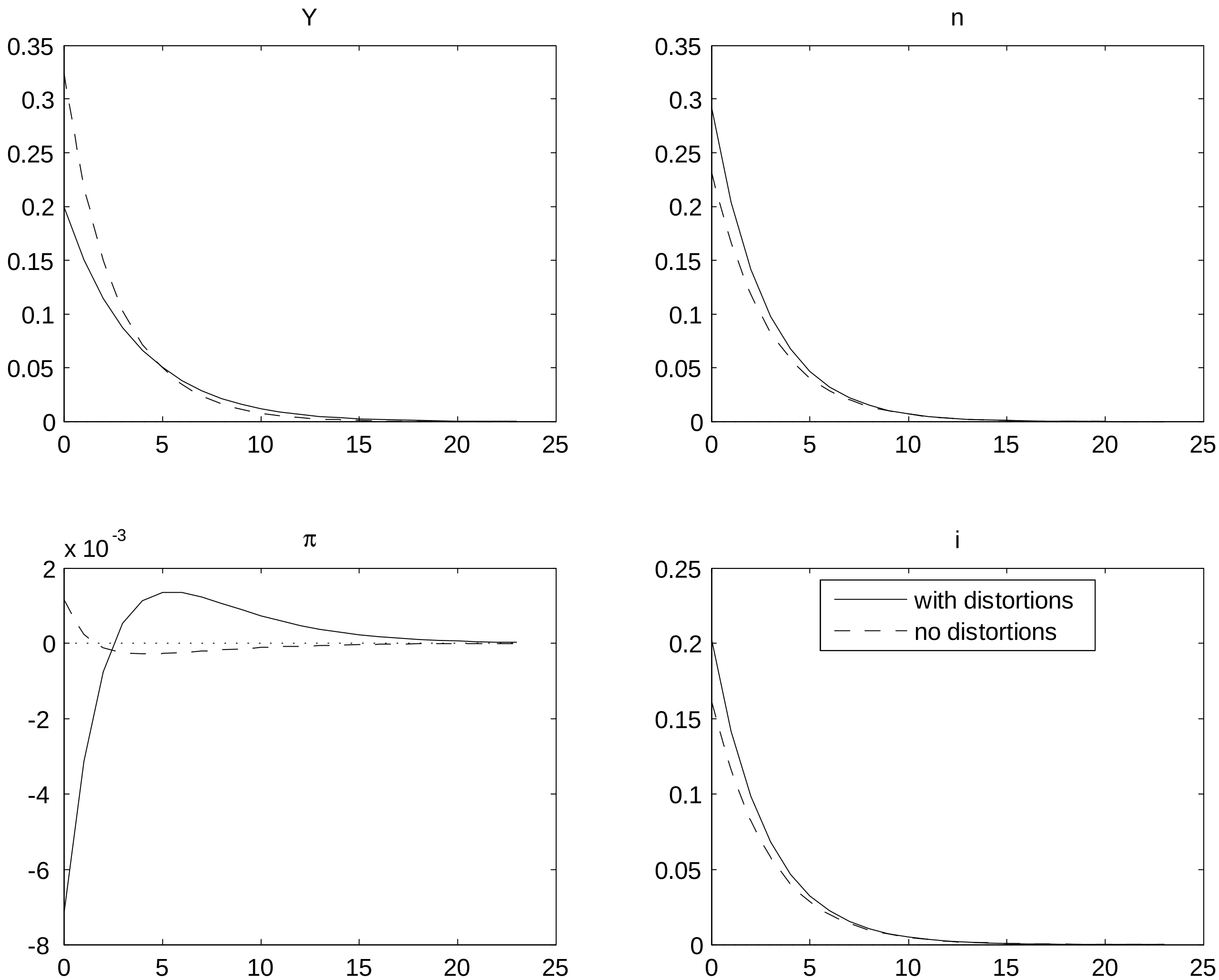


Figure 10: Responses to an increase in government expenditures (G) equivalent to one percent of the output, under the case with steady state distortions in compared to the case of no steady state distortions

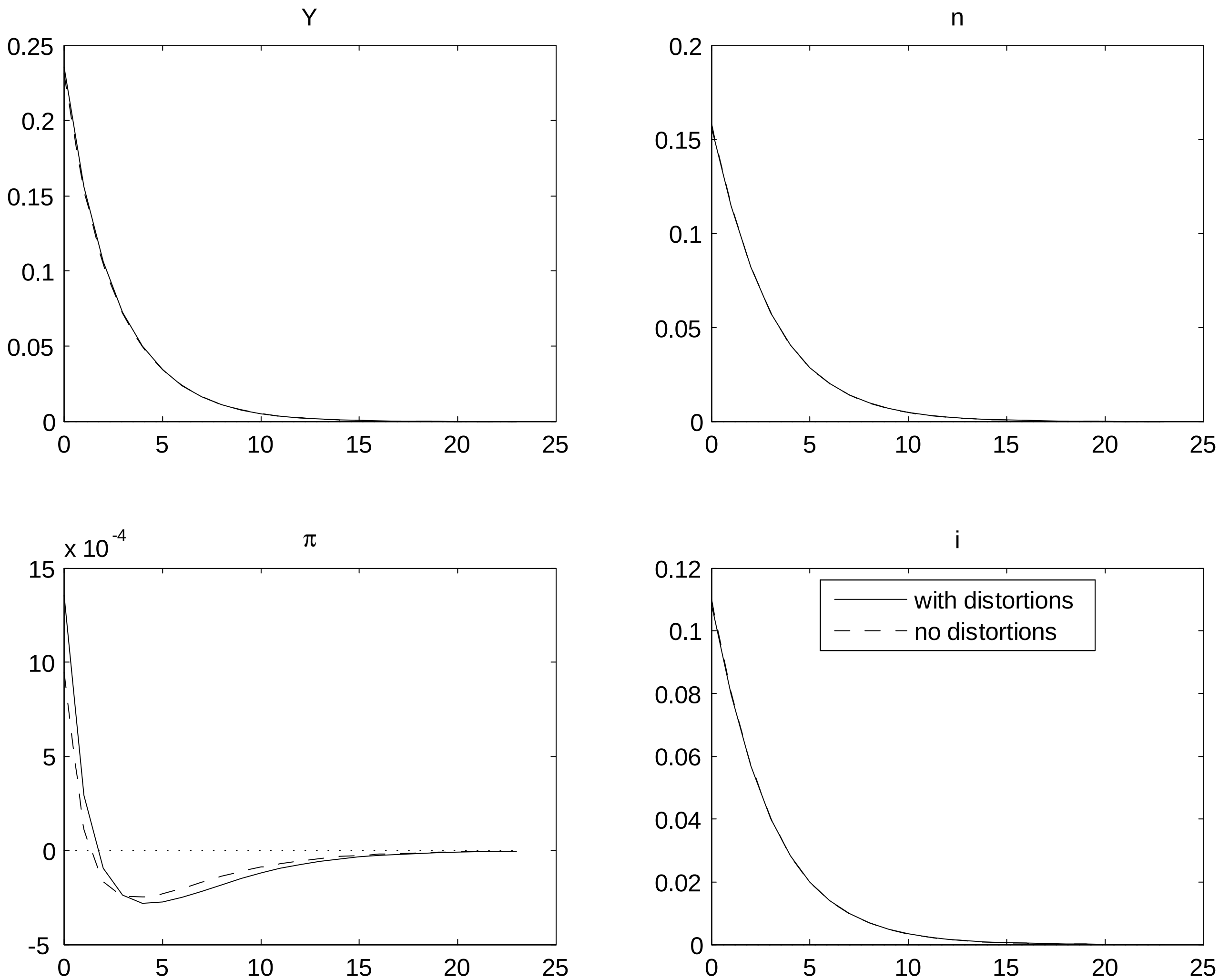


Figure 11: Responses to one percent increase in the shock to the marginal utility of consumption ($\bar{C}$), under the case with steady state distortions in compared to the case of no steady state distortions

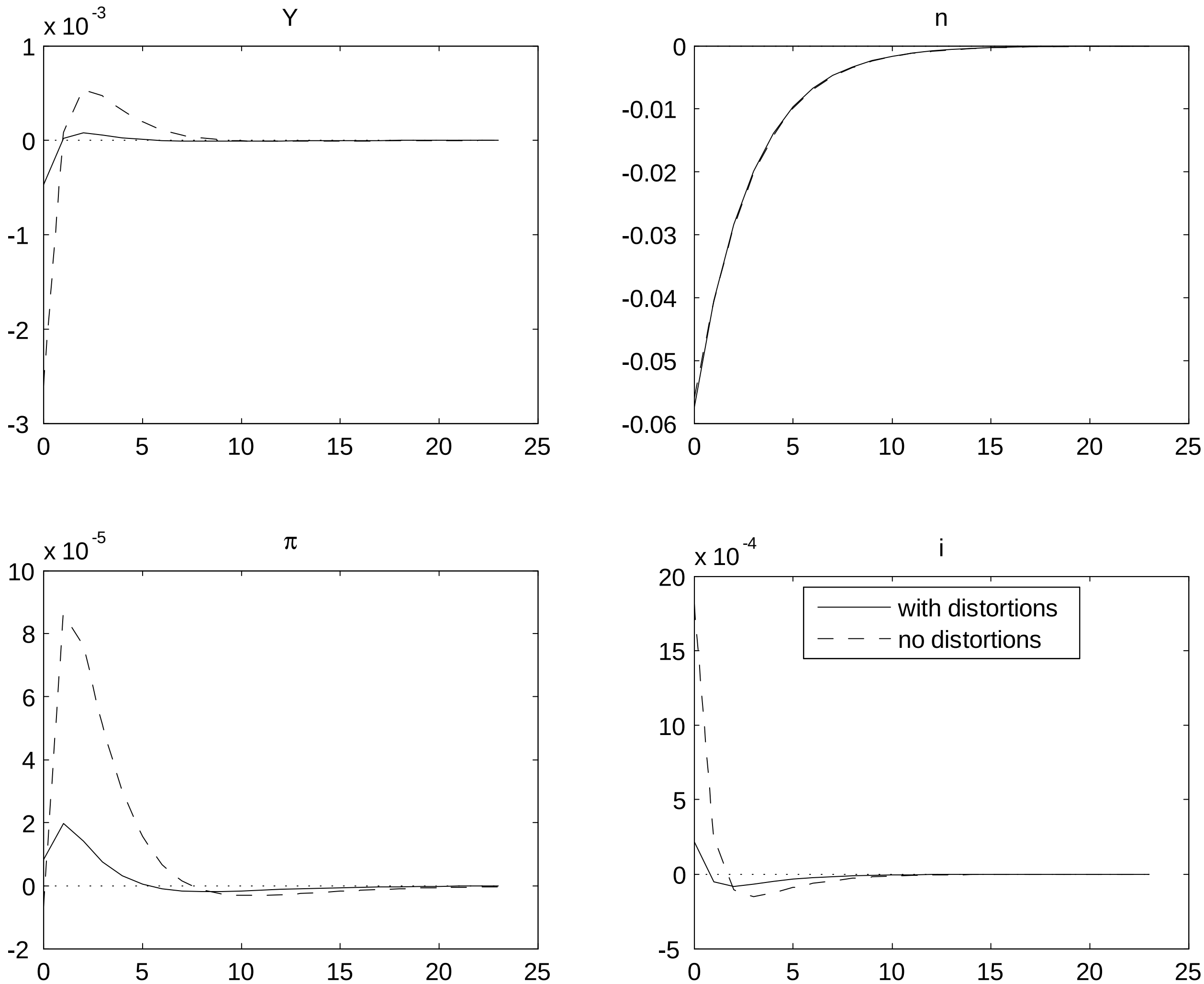


Figure 12: Responses to one percent increase in the shock to the transaction costs (χ), under the case with steady state distortions in compared to the case of no steady state distortions

Alternative Policy Rule

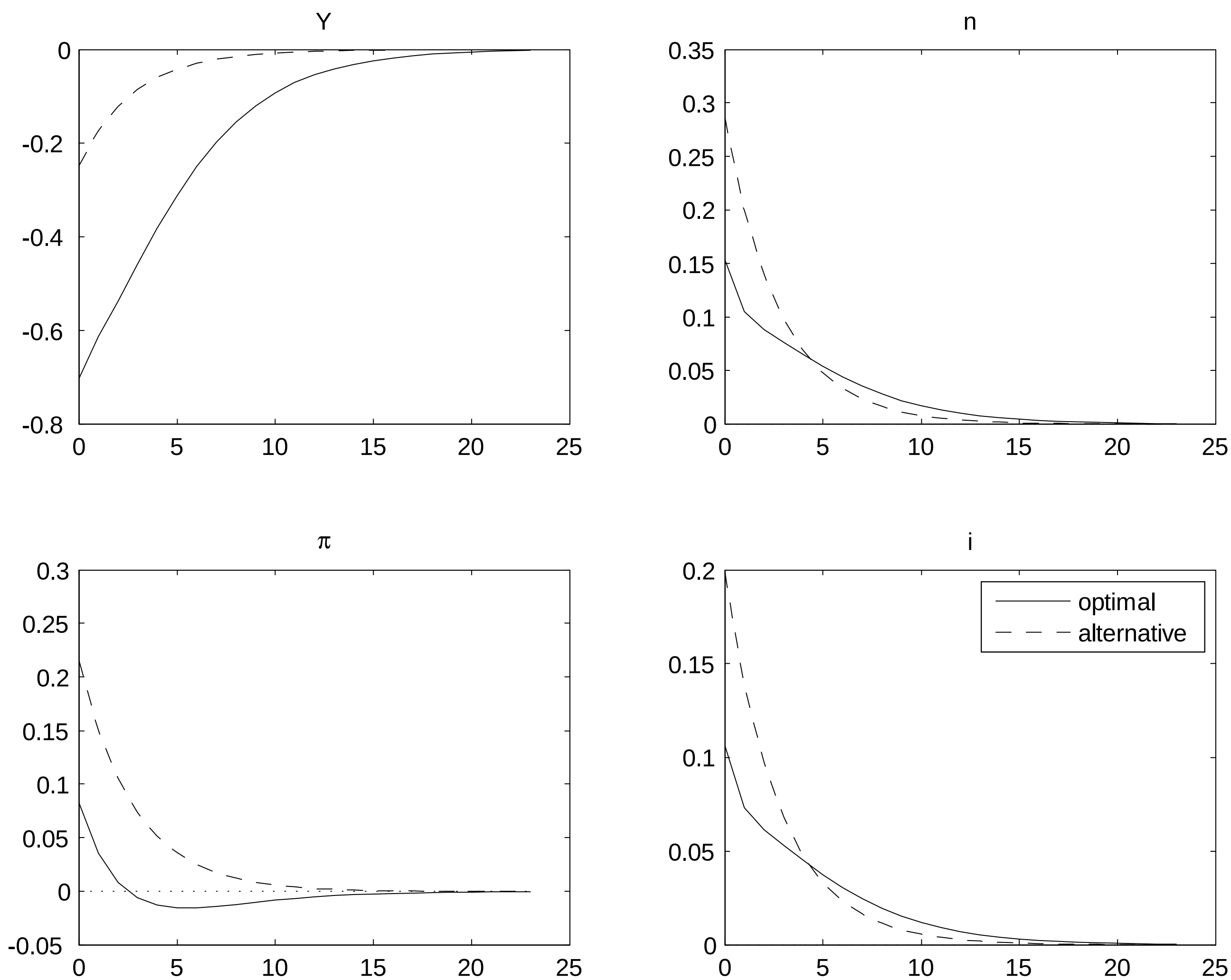


Figure 13: Responses to one percentage point increase in the tax rate (τ), under an alternative policy rule compared to the optimal policy

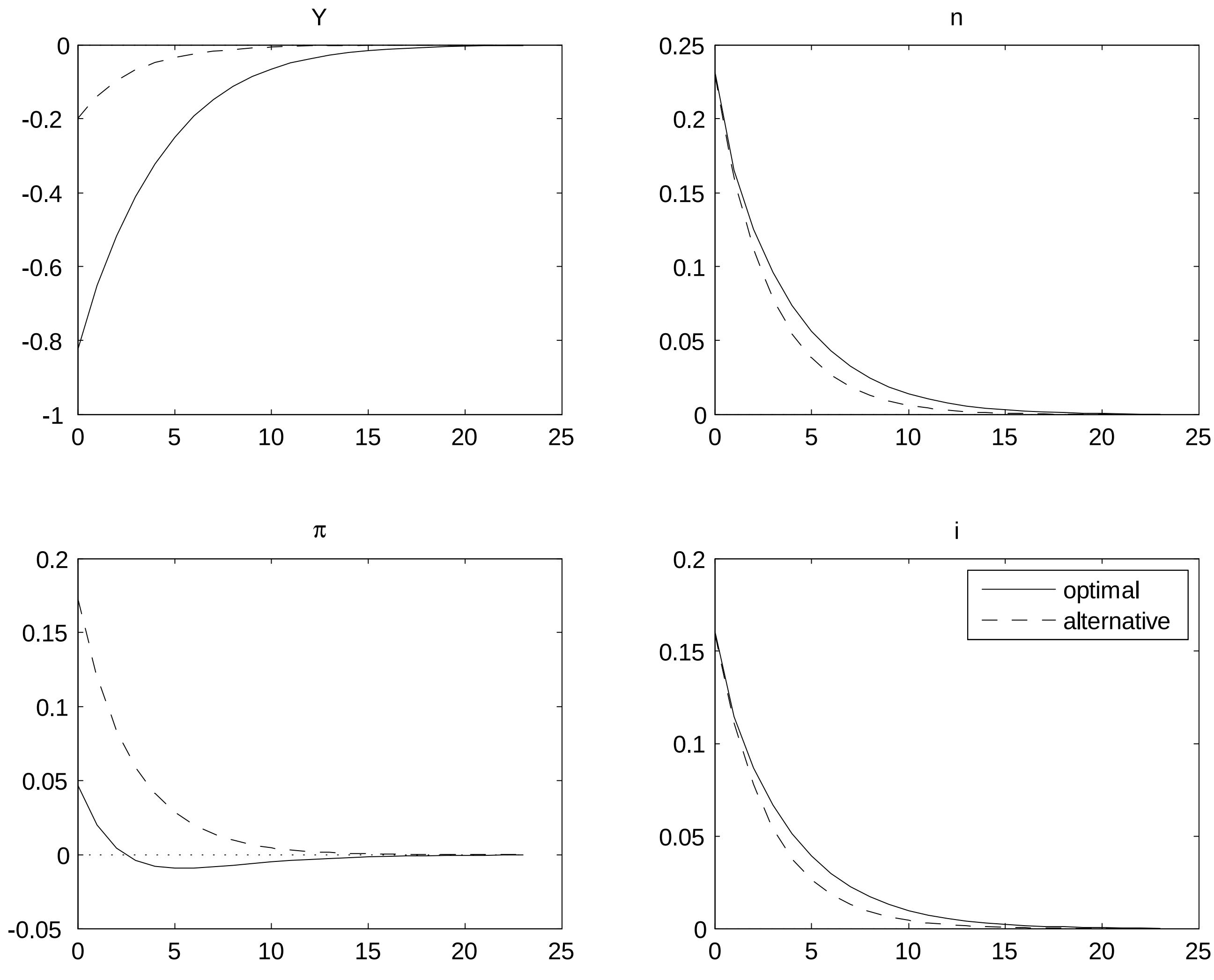


Figure 14: Responses to one percent increase in the makup (μ), under an alternative policy rule compared to the optimal policy

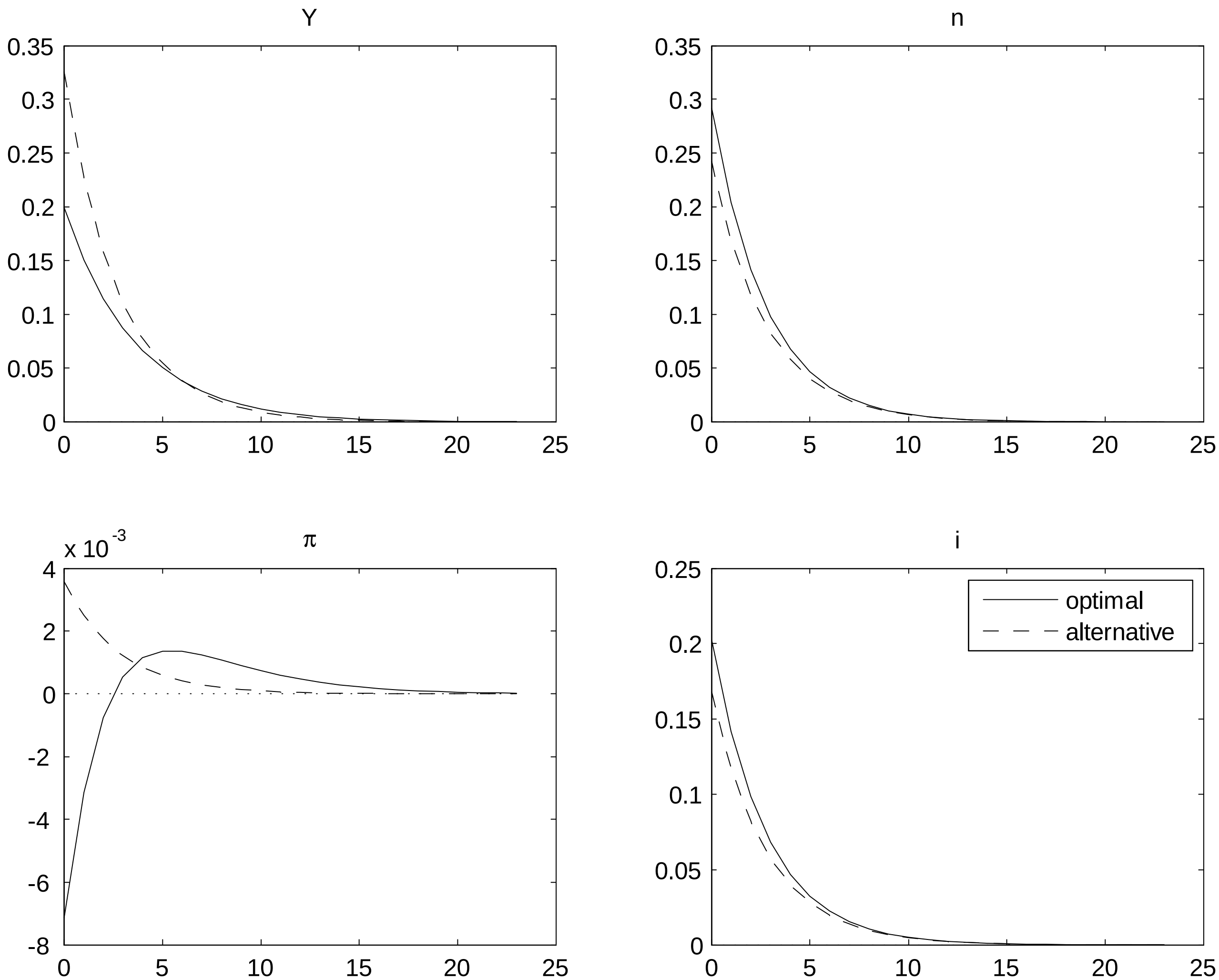


Figure 15: Responses to an increase in government expenditures (G) equivalent to one percent of the output, under an alternative policy rule compared to the optimal policy

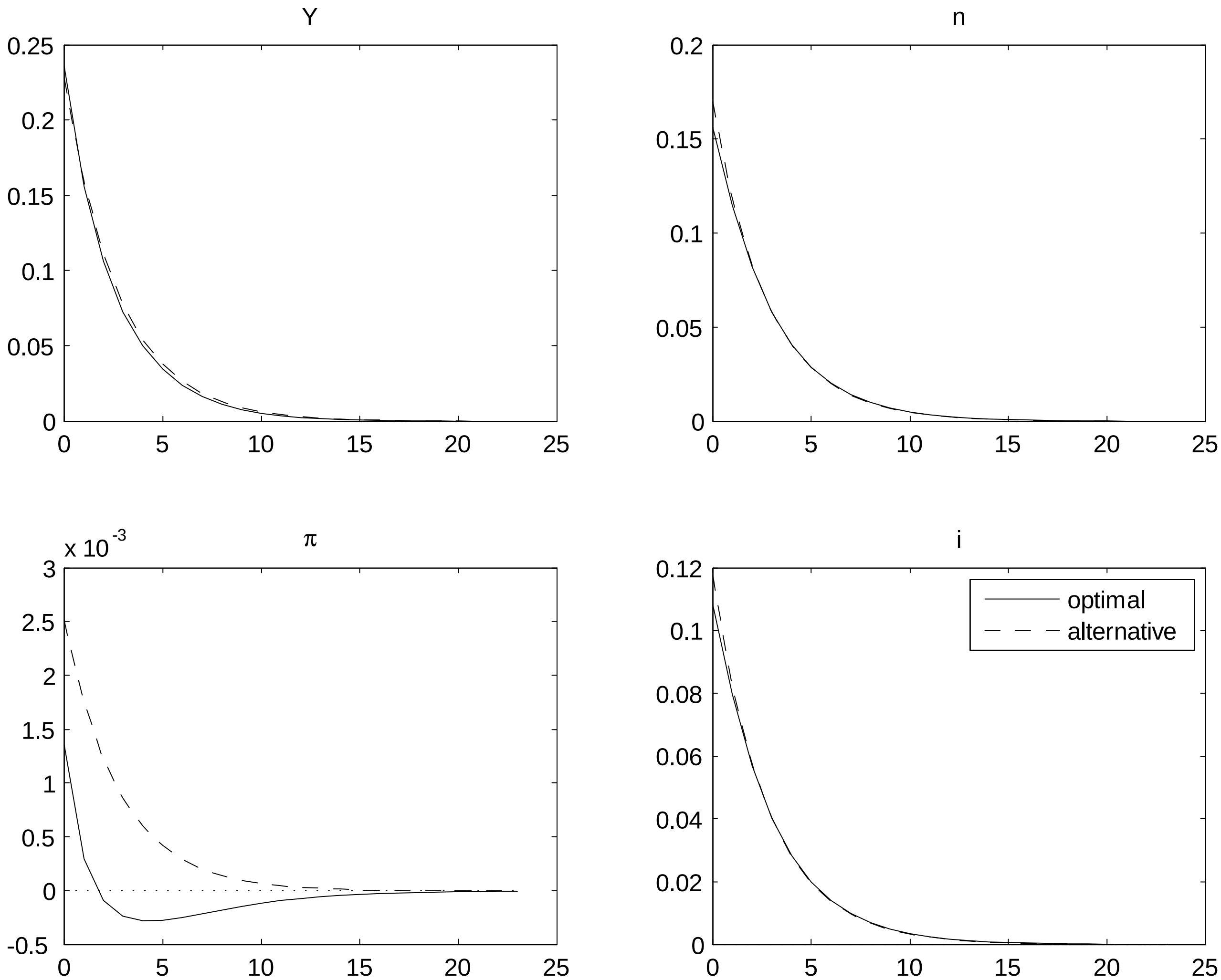


Figure 16: Responses to one percent increase in the shock to the marginal utility of consumption ($\bar{C}$), under an alternative policy rule compared to the optimal policy

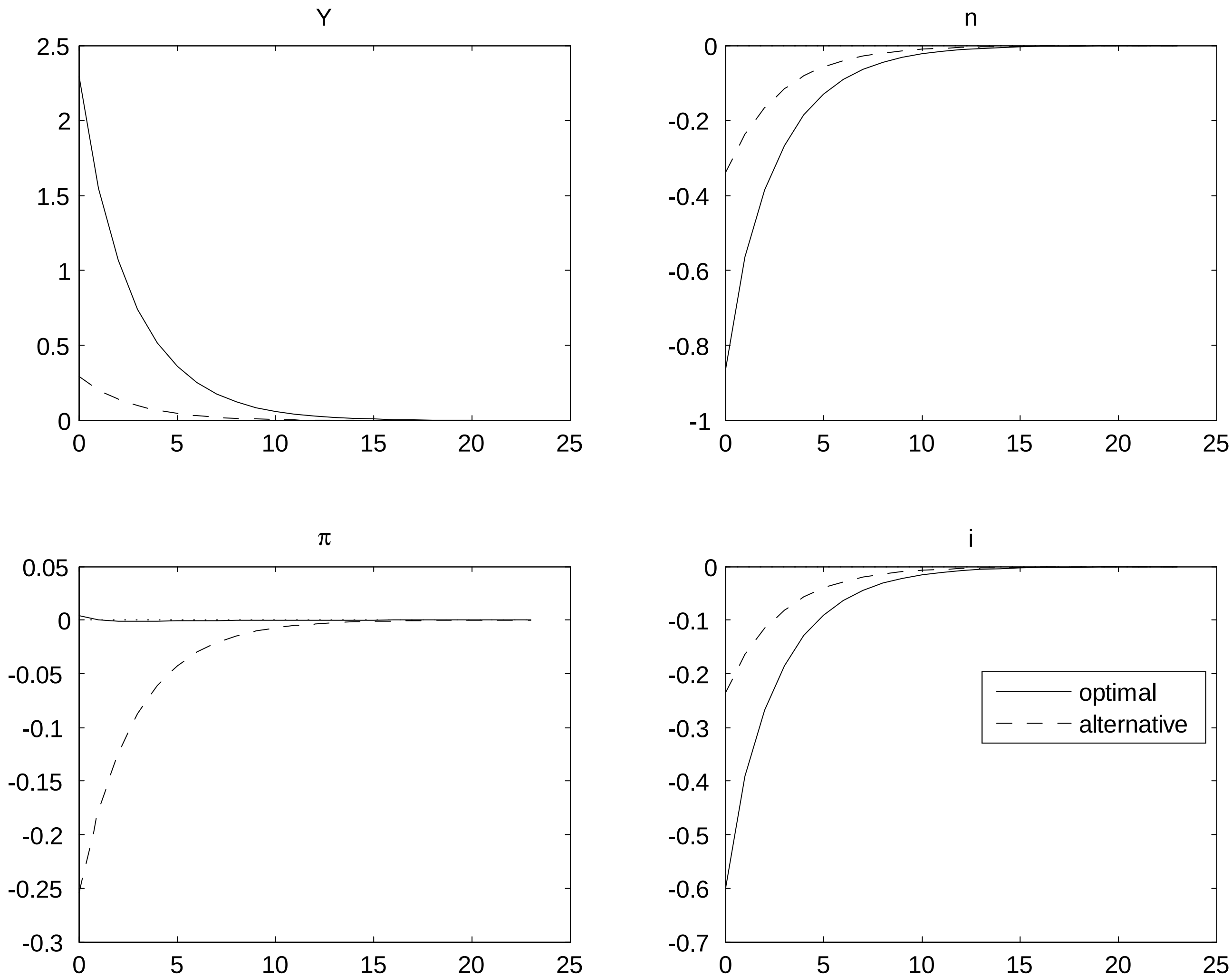


Figure 17: Responses to one percent increase in productivity (A), under an alternative policy rule compared to the optimal policy

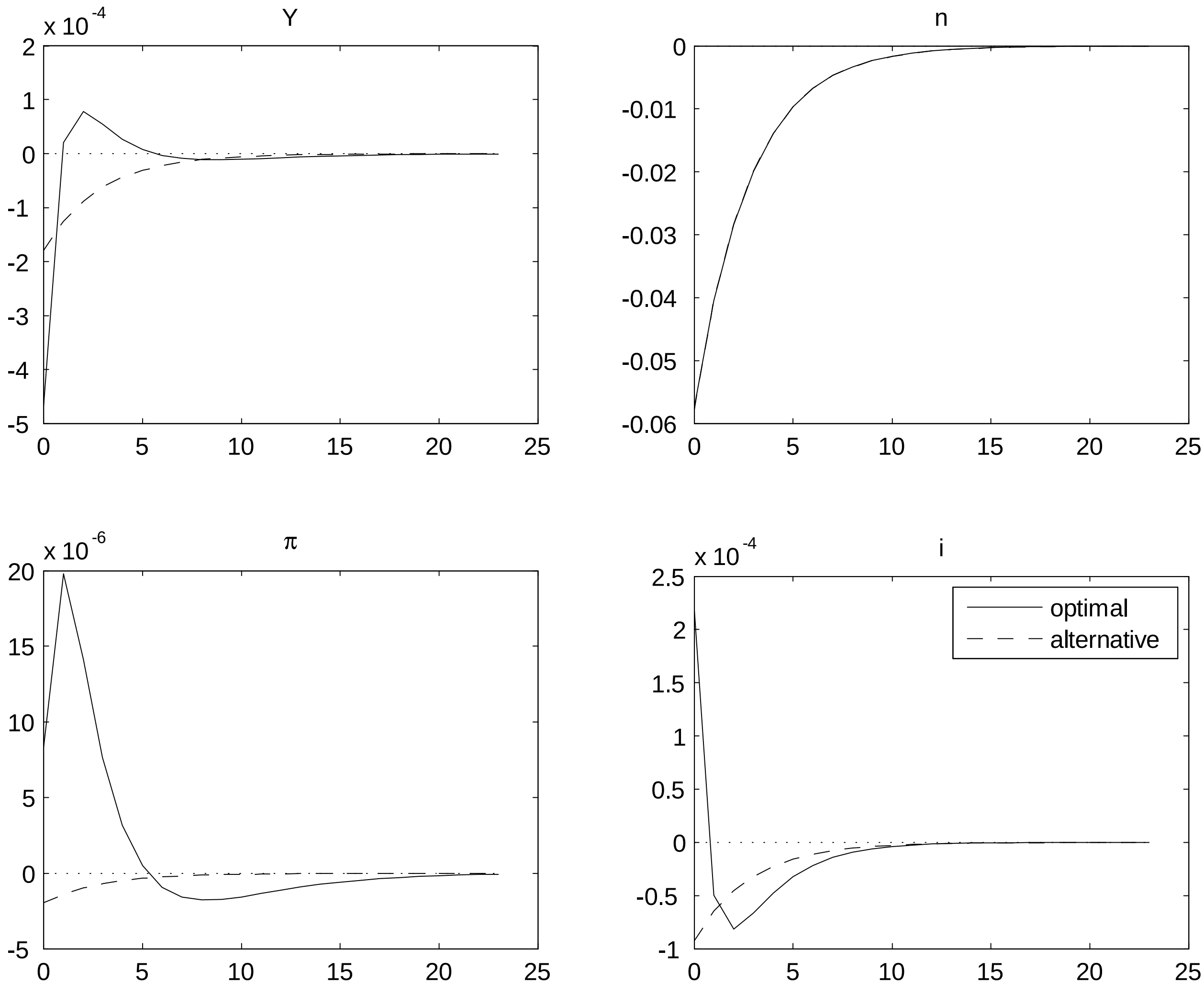


Figure 18: Responses to one percent increase in the shock to the transaction costs (χ), under an alternative policy rule compared to the optimal policy

B Smets Wouters model

B.1 Calibration

β	0.99	time discount factor
α_p	0.83	Calvo probability of not adjusting prices
θ	0.29	index of habit persistance in consumption
δ	0.025	rate of capital depreciation
α_w	0.79	Calvo probability of not adjusting wages
γ_p	0.3	fraction of households subject to indexation in price setting
γ_w	0.7	fraction of households subject to indexation in price setting
σ_C	2.19	Risk aversion coefficient in consumer preferences
σ_L	1.49	Inverse of the Frisch elasticity of labor supply
μ	1.2	steady state price mark up
μ_W	1.2	steady state wage mark up
$\overline{rc}$	4.02	Elasticity of capacity utilisation cost
ϕ_i	1.18	Sensitivity of investment adjustment cost
p_1	1.5	Inflation related term in Taylor rule
p_2	0.2	Output related term in Taylor rule
ρ_r	0.85	Nominal interest rate autocorrelation coeff. in inertial Taylor r.

Table 3: Baseline parameters

B.2 Model's wage equations

Households set their wage on a staggered basis. Each period, any household faces a constant probability $1-\alpha_W$ of changing its wage. In such a case, the wage is set to $\widetilde{w}_t$, taking into account that it will not be re-optimized in the near future. Otherwise, wages are adjusted following an indexation rule on CPI inflation and central bank objective:

$$W_t^h = \left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_W} \overline{\pi}^{1-\gamma_W} W_{t-1}^h$$

and, in cumulated term,

$$W_{t+j}^h = \Pi_{h=0}^{j-1} \pi_{t+h}^{\gamma_W} \overline{\pi}^{1-\gamma_W} W_t^h.$$

The consumer in the choice of the wages such that he maximizes the welfare and the FOC is:

$$\mathbb{E}_t \sum_{j=0}^{\infty} (\beta\alpha_W)^s \left[(1-\tau_{W,t}) \left(\Pi_{h=0}^{j-1} \pi_{t+h}^{\gamma_W} \overline{\pi}^{1-\gamma_W}\right) L_{t+j}^h \frac{\Lambda_{t+j}}{P_{t+j}} - \varepsilon_W \left(1-\tau_{W,t}\right) W_{t+j}^h \frac{L_{t+j}^h}{W_t^h} \Lambda_{t+j} \right.$$
$$\left. + \varepsilon_W \overline{L} \left(L_{t+j}^h\right)^{\sigma_L} E_t^L E_t^B \frac{L_{t+j}^h}{W_t^h} \right] = 0$$

and

$$\mathbb{E}_t\left[\sum_{j=0}^{\infty}(\beta\alpha_W)^s L^h_{t+j}\left\{\frac{(1-\tau_{W,t})(\varepsilon_W-1)}{\varepsilon_W}\frac{\widetilde{w}_t}{P_t}(\Pi_{h=0}^{j-1}\frac{\pi_{t+h}^{\gamma_W}\overline{\pi}^{1-\gamma_W}}{\pi_{t+h+1}})\Lambda_{t+j}-\overline{L}\left(L^h_{t+j}\right)^{\sigma_L}E^L_t E^B_t\right\}\right]=0.$$

When wages are perfectly flexible, this relation collapses to

$$\frac{\varepsilon_W}{(\varepsilon_W-1)(1-\tau_{W,t})}U^h_{L,t}=\Lambda_{t+j}\frac{W^h_t}{P_t}.$$

The real wage is equal to a constant markup $\frac{\mu_W}{1-\tau_W}$ (with $\mu_W=\frac{\varepsilon_W}{\varepsilon_W-1}$) over the marginal rate of substitution between consumption and labour.

Finally, the dynamics of the aggregate wage index given that a share $(1-\alpha_W)$ of workers adjust threir wage (assuming symmetry) and a fraction α_W follow an indexation rule is given by:

$$W_t^{1-\varepsilon_W}=\alpha_W\left[\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_W}\overline{\pi}^{1-\gamma_W}W_{t-1}\right]^{1-\varepsilon_W}+(1-\alpha_W)\left(\widetilde{w}_t\right)^{1-\varepsilon_W}.$$

The first order condition can be written as, using $L^h_t=\left(\frac{W^h_t}{W_t}\right)^{-\epsilon_W}L_t$:

$$\begin{aligned}
&\frac{1}{\mu_W}\frac{\widetilde{w}_t}{P_t}\mathbb{E}_t\left\{\sum_{j=0}^{\infty}(\beta\alpha_W)^j(1-\tau_{W,t})(\Pi_{h=0}^{j-1}\frac{\pi_{t+h}^{\gamma_W}\overline{\pi}^{1-\gamma_W}}{\pi_{t+h+1}})\left(\frac{\Pi_{h=0}^{j-1}\pi_{t+h}^{\gamma_W}\overline{\pi}^{1-\gamma_W}W^h_t}{W_{t+j}}\right)^{-\epsilon_W}L_{t+j}\Lambda_{t+j}\right\}\\
=\;&\mathbb{E}_t\left\{\sum_{j=0}^{\infty}(\beta\alpha_W)^j\overline{L}\left(\left(\frac{\Pi_{h=0}^{j-1}\pi_{t+h}^{\gamma_W}\overline{\pi}^{1-\gamma_W}W^h_t}{W_{t+j}}\right)^{-\epsilon_W}L_{t+j}\right)^{1+\sigma_L}E^L_{t+j}\right\}
\end{aligned}$$

which leads to define

$$\begin{aligned}
\frac{1}{\mu_W}\frac{\widetilde{w}_t}{P_t}\mathbb{H}_t\left(W^h_t\right)^{-\epsilon_W} &\equiv \mathbb{G}_t\left(W^h_t\right)^{-\epsilon_W(1+\sigma_L)}\\
\frac{1}{\mu_W}\frac{(\widetilde{w}_t)^{1-\epsilon_W+\epsilon_W(1+\sigma_L)}}{P_t^{1-\epsilon_W+\epsilon_W(1+\sigma_L)+\epsilon_W-\epsilon_W(1+\sigma_L)}}\mathbb{H}_t &\equiv \mathbb{G}_t\\
\frac{1}{\mu_W}\frac{(\widetilde{w}_t)^{1-\epsilon_W+\epsilon_W(1+\sigma_L)}}{P_t^{1-\epsilon_W+\epsilon_W(1+\sigma_L)}}\frac{\mathbb{H}_t}{P_t^{\epsilon_W}} &\equiv \frac{\mathbb{G}_t}{P_t^{\epsilon_W(1+\sigma_L)}}\\
\frac{1}{\mu_W}\left(\frac{\widetilde{w}_t}{P_t}\right)^{1-\epsilon_W+\epsilon_W(1+\sigma_L)} &\equiv \frac{\mathbb{G}_t}{P_t^{\epsilon_W(1+\sigma_L)}}\times\frac{P_t^{\epsilon_W}}{\mathbb{H}_t}\\
\frac{1}{\mu_W}\left(\frac{\widetilde{w}_t}{P_t}\right)^{\frac{\mu_W(1+\sigma_L)-1}{\mu_W-1}} &\equiv \frac{\mathbb{G}_t}{P_t^{\epsilon_W(1+\sigma_L)}}\times\frac{P_t^{\epsilon_W}}{\mathbb{H}_t}
\end{aligned}$$

From this:

$$\mathbb{G}_t=\mathbb{E}_t\left\{\sum_{j=0}^{\infty}(\beta\alpha_W)^j\overline{L}\left(\left(\frac{\Pi_{h=0}^{j-1}\pi_{t+h}^{\gamma_W}\overline{\pi}^{1-\gamma_W}W^h_t}{W_{t+j}}\right)^{-\epsilon_W}L_{t+j}\right)^{1+\sigma_L}\exp(E^L_{t+j})\right\}$$

and so

$$
\begin{aligned}
Z_{W1,t} &= \frac{\mathbb{G}_t}{P_t^{\epsilon_W(1+\sigma_L)}} \\
&= \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} (\beta\alpha_W)^j \overline{L} \frac{P_{t+j}^{\epsilon_W(1+\sigma_L)}}{P_t^{\epsilon_W(1+\sigma_L)}} \frac{W_{t+j}^{\epsilon_W(1+\sigma_L)}}{P_{t+j}^{\epsilon_W(1+\sigma_L)}} \left(\Pi_{h=0}^{j-1} \pi_{t+h}^{\gamma_W} \overline{\pi}^{1-\gamma_W}\right)^{-\epsilon_W(1+\sigma_L)} (L_{t+j})^{1+\sigma_L} \exp(E_{t+j}^L) \right\} \\
&= \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} (\beta\alpha_W)^j \overline{L} \frac{W_{t+j}^{\epsilon_W(1+\sigma_L)}}{P_{t+j}^{\epsilon_W(1+\sigma_L)}} \left(\frac{\Pi_{h=1}^{j} \pi_{t+h}}{\Pi_{h=0}^{j-1} \pi_{t+h}^{\gamma_W} \overline{\pi}^{1-\gamma_W}} \right)^{\epsilon_W(1+\sigma_L)} (L_{t+j})^{1+\sigma_L} \exp(E_{t+j}^L) \right\} \\
&= \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} (\beta\alpha_W)^j \overline{L} W_R^{\epsilon_W(1+\sigma_L)} \left(\frac{\Pi_{h=1}^{j} \pi_{t+h}}{\Pi_{h=0}^{j-1} \pi_{t+h}^{\gamma_W} \overline{\pi}^{1-\gamma_W}} \right)^{\epsilon_W(1+\sigma_L)} (L_{t+j})^{1+\sigma_L} \exp(E_{t+j}^L) \right\} \\
&= \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} (\beta\alpha_W)^j \overline{L} W_{Rt}^{\frac{(1+\sigma_L)\mu_W}{\mu_W-1}} \left(\frac{\Pi_{h=1}^{j} \pi_{t+h}}{\Pi_{h=0}^{j-1} \pi_{t+h}^{\gamma_W} \overline{\pi}^{1-\gamma_W}} \right)^{\frac{(1+\sigma_L)\mu_W}{\mu_W-1}} (L_{t+j})^{1+\sigma_L} \exp(E_{t+j}^L) \right\}
\end{aligned}
$$

where W_R is $\frac{W}{P}$ and $\epsilon_W(1+\sigma_L) = \frac{(1+\sigma_L)\mu_W}{\mu_W-1}$. Therefore in a recursive form:

$$
Z_{W1,t} = \overline{L} W_{Rt}^{\frac{(1+\sigma_L)\mu_W}{\mu_W-1}} L_t^{1+\sigma_L} E_t^L + \beta\alpha_W \mathbb{E}_t \left(Z_{W1,t+1} \left[\frac{\pi_{t+1}}{\pi_t^{\gamma_W} \overline{\pi}^{1-\gamma_W}} \right]^{\frac{(1+\sigma_L)\mu_W}{\mu_W-1}} \right)
$$

In a similar manner for the second term:

$$
\begin{aligned}
Z_{W2,t} &= \frac{\mathbb{H}_t}{P_t^{\epsilon_W}} \\
&= \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} (\beta\alpha_W)^j (1-\tau_{W,t}) \frac{P_{t+j}^{\epsilon_W}}{P_t^{\epsilon_W}} \frac{W_{t+j}^{\epsilon_W}}{P_{t+j}^{\epsilon_W}} (\Pi_{h=0}^{j-1} \frac{\pi_{t+h}^{\gamma_W} \overline{\pi}^{1-\gamma_W}}{\pi_{t+h+1}}) \left(\Pi_{h=0}^{j-1} \pi_{t+h}^{\gamma_W} \overline{\pi}^{1-\gamma_W}\right)^{-\epsilon_W} L_{t+j} \Lambda_{t+j} \right\} \\
&= \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} (\beta\alpha_W)^j (1-\tau_{W,t}) W_R^{\epsilon_W} (\Pi_{h=0}^{j-1} \pi_{t+h+1})^{-1+\epsilon_W} \left(\Pi_{h=0}^{j-1} \pi_{t+h}^{\gamma_W} \overline{\pi}^{1-\gamma_W}\right)^{1-\epsilon_W} L_{t+j} \Lambda_{t+j} \right\} \\
&= \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} (\beta\alpha_W)^j (1-\tau_{W,t}) W_R^{\epsilon_W} \left(\frac{\Pi_{h=1}^{j} \pi_{t+h}}{\Pi_{h=0}^{j-1} \pi_{t+h}^{\gamma_W} \overline{\pi}^{1-\gamma_W}} \right)^{\frac{1}{\mu_W-1}} L_{t+j} \Lambda_{t+j} \right\}
\end{aligned}
$$

where $1-\epsilon_W = 1 - \frac{\mu_W}{\mu_W-1} = \frac{1}{-\mu_W+1}$. Therefore:

$$
Z_{W2,t} = (1-\tau_{W,t}) W_R^{\frac{\mu_W}{\mu_W-1}} L_t \Lambda_t + \beta\alpha_W \mathbb{E}_t \left(Z_{W2,t+1} \left[\frac{\pi_{t+1}}{\pi_t^{\gamma_W} \overline{\pi}^{1-\gamma_W}} \right]^{\frac{1}{\mu_W-1}} \right)
$$

Given that

$$\left(\frac{\widetilde{w}_t}{P_t}\right)^{\frac{\mu_W(1+\sigma_L)-1}{\mu_W-1}} \equiv \mu_W \frac{Z_{W1,t}}{Z_{W2,t}}$$

then

$$\left(\frac{\widetilde{w}_t}{P_t}\right)^{\frac{1}{1-\mu_W}} \equiv \left(\mu_W \frac{Z_{W1,t}}{Z_{W2,t}}\right)^{\frac{-1}{\mu_W(1+\sigma_L)-1}}.$$

The real wage setting equations can be rewritten in the following recursive form:

$$(1-\alpha_W)\left(\frac{\widetilde{w}_t}{P_t}\right)^{\frac{1}{1-\mu_W}} = (1-\alpha_W)\left(\mu_w \frac{Z_{W1,t}}{Z_{W2,t}}\right)^{-\frac{1}{\mu_W(1+\sigma_L)-1}} = W_{R,t}^{\frac{1}{1-\mu_W}} - \alpha_W W_{R,t-1}^{\frac{1}{1-\mu_W}} \left(\frac{\pi_t}{\pi_{t-1}^{\gamma_W}\overline{\pi}_t^{1-\gamma_W}}\right)^{\frac{-1}{1-\mu_W}} \tag{B.1}$$

$$Z_{W1,t} = \overline{L} W_{R,t}^{\frac{(1+\sigma_L)\mu_W}{\mu_W-1}} L_t^{1+\sigma_L} E_t^L E_t^B + \beta\alpha_W \mathbb{E}_t\left(Z_{W1,t+1}\left[\frac{\pi_{t+1}}{\pi_t^{\gamma_W}\overline{\pi}^{1-\gamma_W}}\right]^{\frac{(1+\sigma_L)\mu_W}{\mu_W-1}}\right) \tag{B.2}$$

$$Z_{W2,t} = (1-\tau_{W,t}) W_{R,t}^{\frac{\mu_W}{\mu_W-1}} L_t \Lambda_t + \beta\alpha_W \mathbb{E}_t\left(Z_{W2,t+1}\left[\frac{\pi_{t+1}}{\pi_t^{\gamma_W}\overline{\pi}^{1-\gamma_W}}\right]^{\frac{1}{\mu_W-1}}\right) \tag{B.3}$$

B.3 Responses to shocks

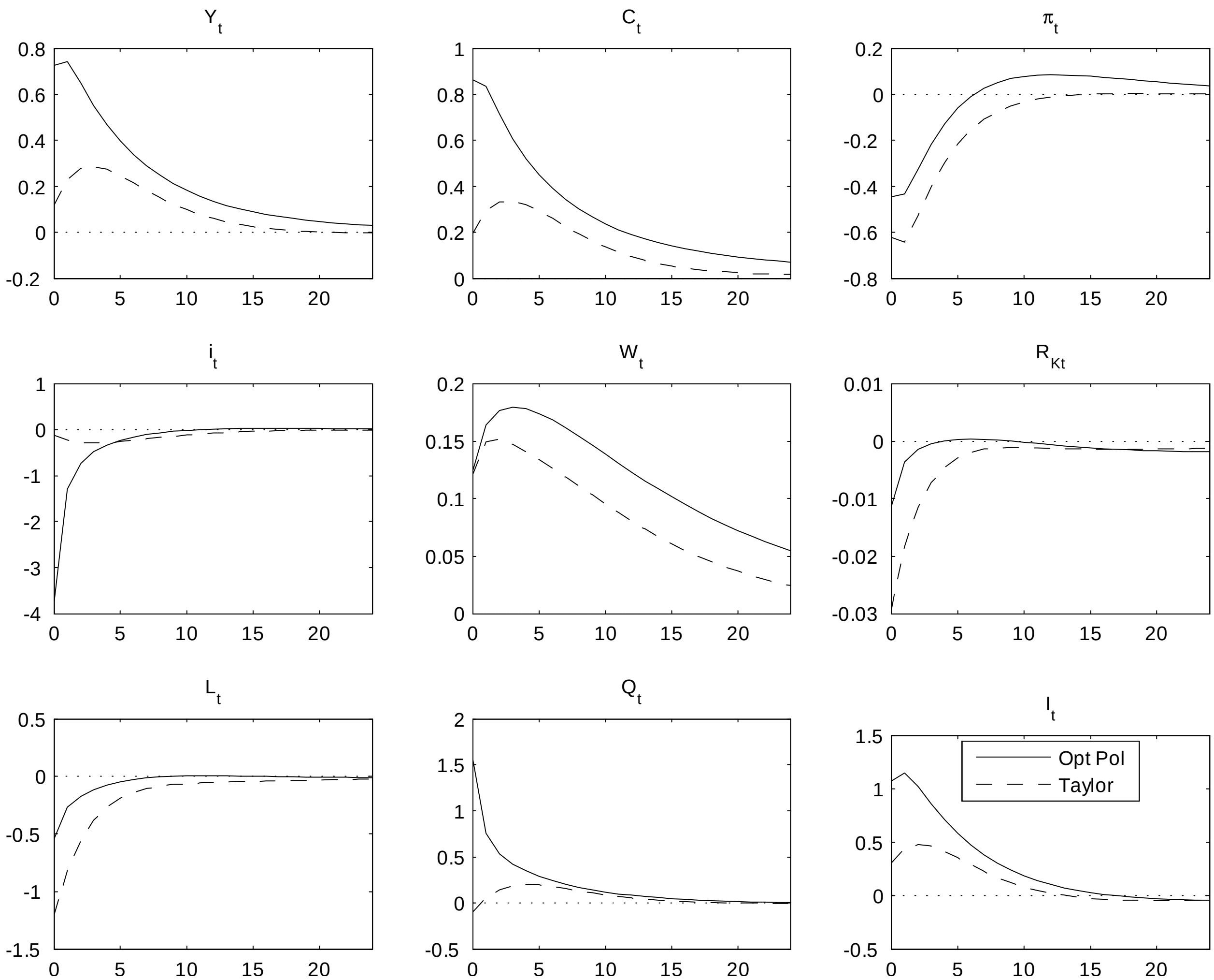


Figure 19: Productivity shock

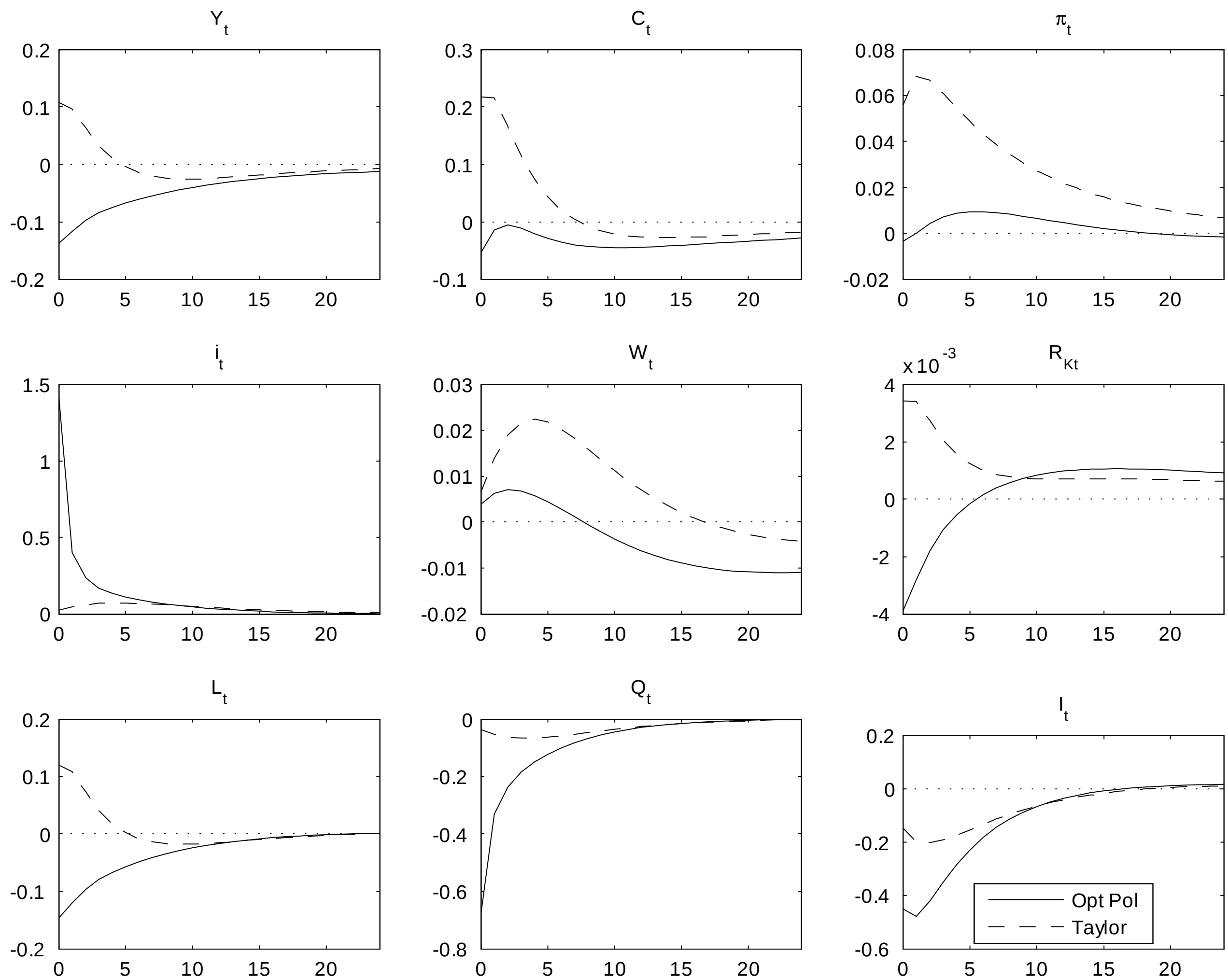


Figure 20: Preferences shock

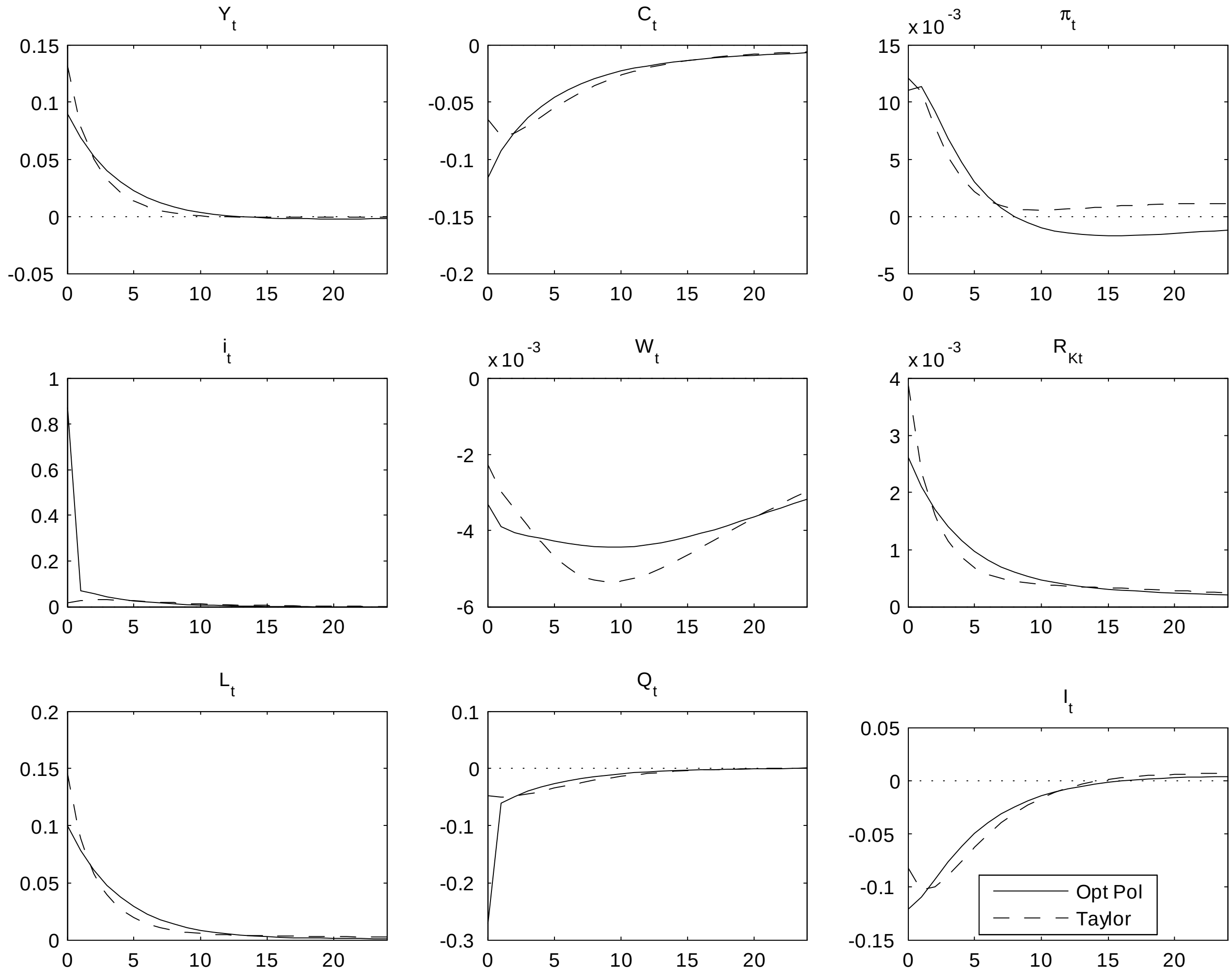


Figure 21: Government expenditure shock

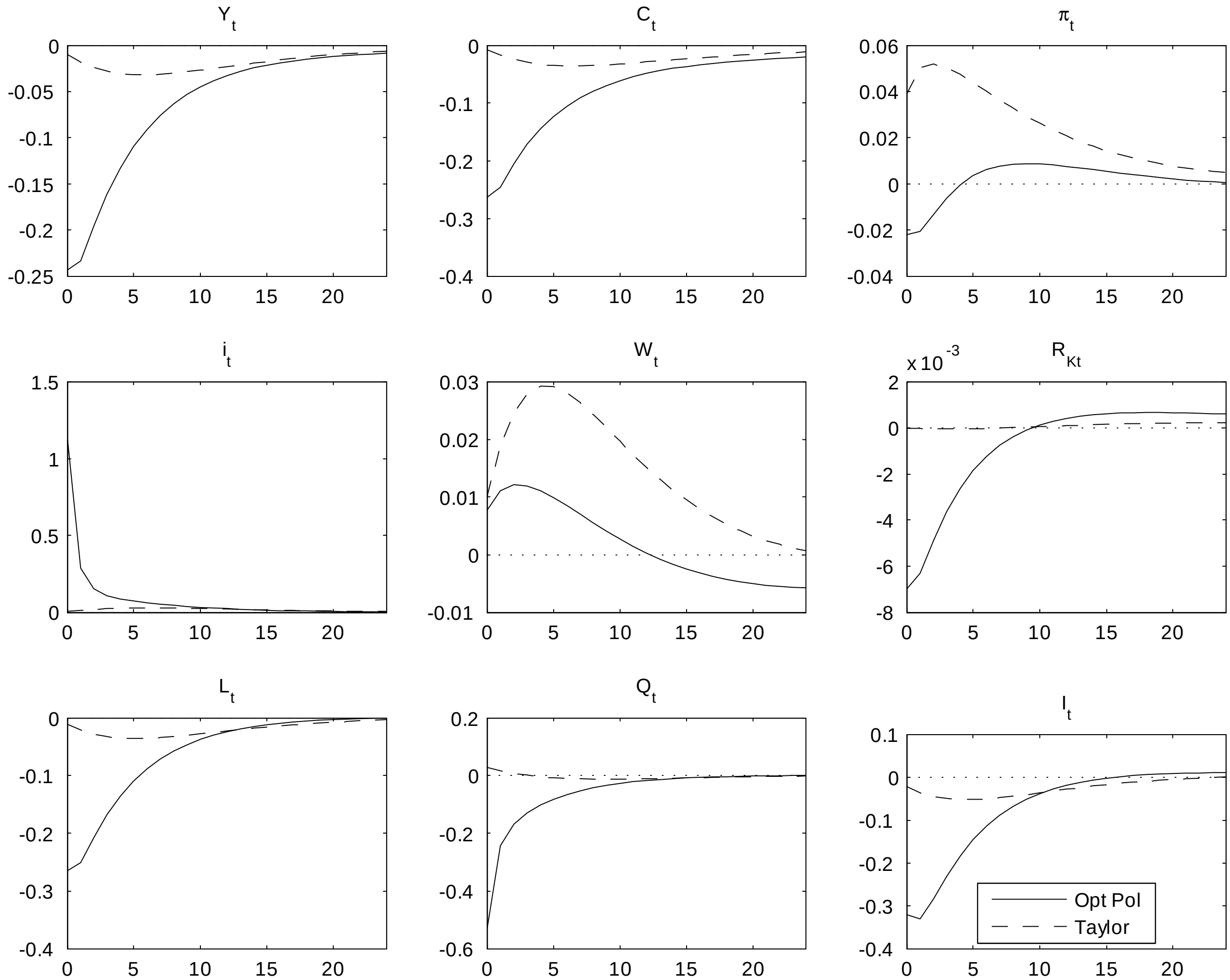


Figure 22: Labour supply shock (negative shock)

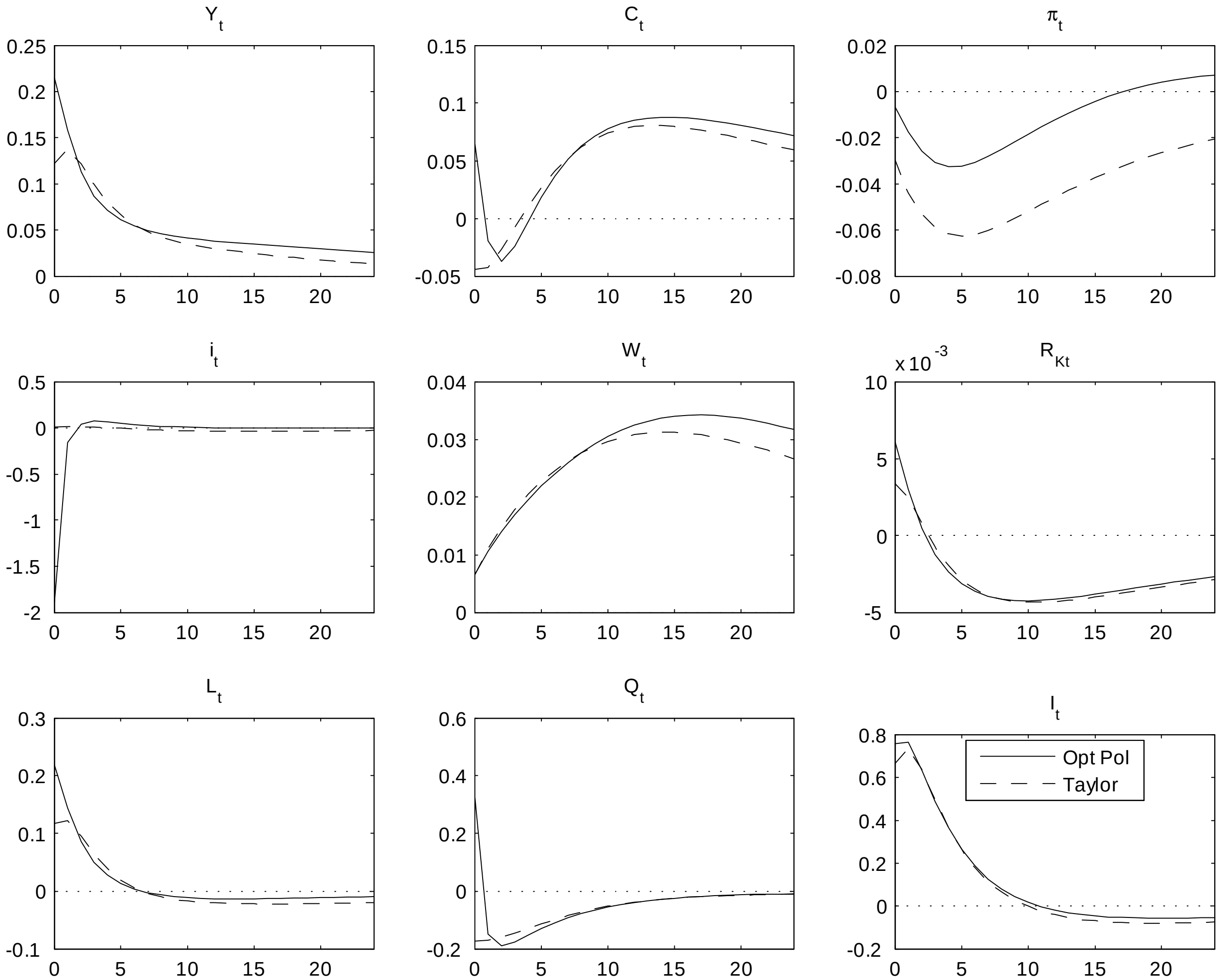


Figure 23: Investment shock

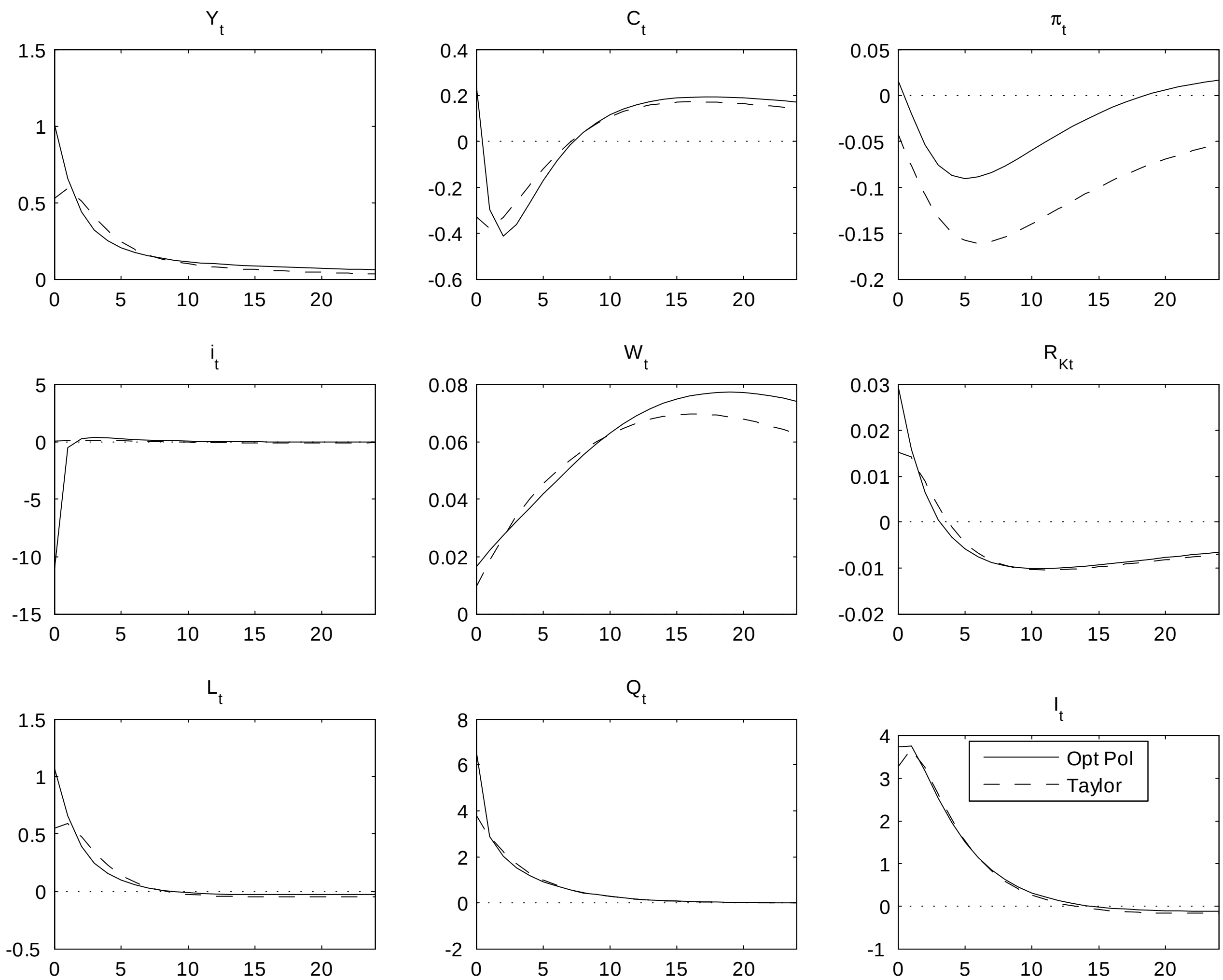


Figure 24: Equity premium shock