An Estimate of the Risk Premiums in U.S. Treasury and Corporate Bond Yields

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Abstract

This paper describes an arbitrage-free dynamic term structure model of U.S. Treasury and corporate bond yields that is used to estimate the credit risk premiums in credit spreads of U.S. industrial corporate bonds.

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1 Introduction

This paper describes a reduced-form latent-factor arbitrage-free dynamic term structure model of U.S. Treasury and corporate bond yields that can be used to estimate the credit risk premiums in credit spreads of U.S. industrial corporate bonds. Specifically, we define the credit risk premium as the additional premium corporate bond investors demand above and beyond the expected excess return a risk-neutral investor would demand to assume the same exposure.

As for its structure, the model relies on the widely used yield curve function introduced by Nelson and Siegel (1987). However, the model is made theoretically consistent by exploiting the adjustment derived in Christensen et al. (2011) to ensure absence of arbitrage.

The rest of the paper is structured as follows. Section 2 details the properties of the yield data and motivates the model structure introduced in Section 3. Section 4 describes the model estimation and results, while Section 5 contains the analysis of the estimated bond risk premiums. Finally, Section 6 concludes.

2 Data Description and Model Motivation

In this section, the mechanics of the Nelson-Siegel yield curve model are detailed. Second, the U.S. Treasury yield data are described and it is explained why the Nelson-Siegel model is appropriate for that panel of data. Finally, the properties of the corporate bond yield data for U.S. industrial firms are explored.

2.1 The Nelson-Siegel Model

The model structure used throughout the analysis is closely related to the Nelson and Siegel (1987) model. In that model, the yield curve at a given point in time is assumed to take the following simple functional form[1]

$$y(\tau) = \beta_0 + \beta_1\Big(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\Big) + \beta_2\Big(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\Big),$$

where $y(\tau)$ is the zero-coupon yield with τ denoting the time to maturity, and β_0, β_1, β_2, and λ are model parameters. The three β parameters can be interpreted as factors and their corresponding factor loadings in the Nelson-Siegel yield curve function are illustrated in Figure 1.

Due to its flexibility this model is able to provide a good fit to cross sections of yields, which is the primary reason for its popularity among financial market practitioners. Although for some purposes such a static representation appears useful, a dynamic version is required to understand the evolution of the bond market over time. Diebold and Li (2006) achieve

[1] This is equation (2) in Nelson and Siegel (1987).

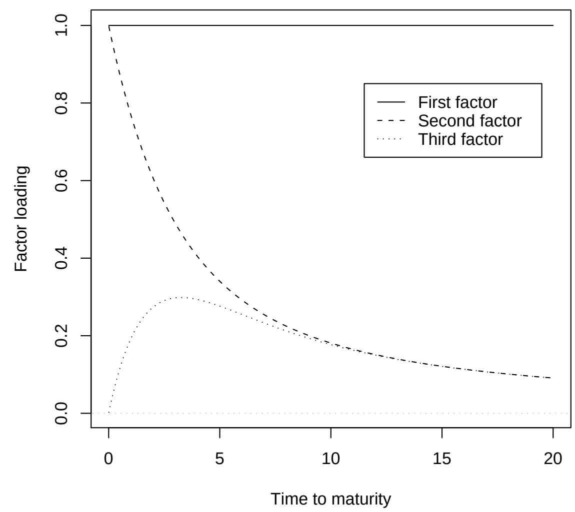

Figure 1: **Factor Loadings in the Nelson-Siegel Yield Function.**
Illustration of the factor loadings on the three state variables in the Nelson-Siegel model. The value for λ is 0.55 and maturity is measured in years.

this by introducing time-varying parameters

$$y_t(\tau) = L_t + S_t\Big(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\Big) + C_t\Big(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\Big),$$

where L_t, S_t, and C_t can be interpreted as level, slope, and curvature factors (given their associated Nelson-Siegel factor loadings). Furthermore, once the model is viewed as a factor model, a dynamic structure can be postulated for the three factors, which yields a fully dynamic version of the Nelson-Siegel model. In the following, it is demonstrated that the features of this model are relevant for modeling both Treasury bond yields as well as corporate bond credit spreads.

2.2 The Treasury Bond Yield Data

In the analysis, Treasury yields serve as a proxy for the risk-free rate used to extract the corporate bond credit spreads analyzed subsequently. Furthermore, this assumption establishes an empirical connection between Treasury and corporate bond markets.

The specific U.S. Treasury bond yields used are zero-coupon yields constructed by the method described in Gürkaynak at al. (2007).[2] The Treasury zero-coupon bond yields considered have the following 8 maturities: three-month, six-month, one-year, two-year, three-year, five-year, seven-year, and ten-year, and the sample is limited to end-of-month observations

[2]The Board of Governors in Washington D.C. updates the factors and parameters of this method daily, see also the related website http://www.federalreserve.gov/pubs/feds/2006/index.html

Maturity in months	Mean in %	Std. dev. in %	Skewness	Kurtosis
3	2.52	2.16	0.36	1.47
6	2.55	2.19	0.36	1.49
12	2.63	2.20	0.35	1.53
24	2.84	2.13	0.32	1.60
36	3.05	2.02	0.29	1.67
60	3.45	1.81	0.24	1.80
84	3.79	1.67	0.16	1.87
120	4.18	1.55	0.05	1.91

Table 1: **Summary Statistics for the U.S. Treasury Bond Yields**
Summary statistics for the sample of monthly U.S. Treasury zero-coupon bond yields covering the period from January 31, 1995, to December 31, 2018, a total of 288 observations.

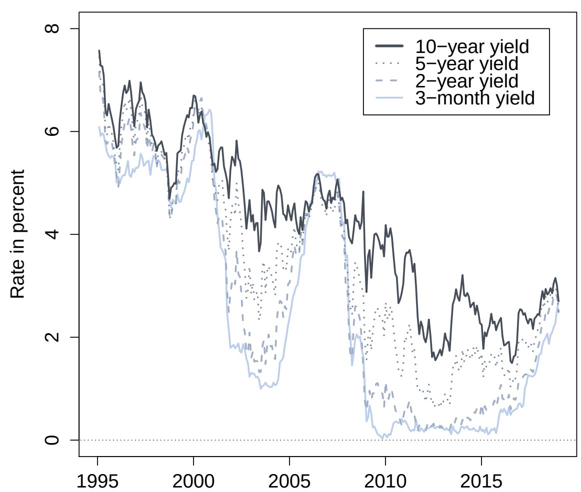

Figure 2: **Time Series of U.S. Treasury Bond Yields**
Illustration of the monthly U.S. Treasury zero-coupon bond yields covering the period from January 31, 1995, to December 31, 2018. The yields shown have maturities: three-month, two-year, five-year, and ten-year.

over the period from January 31, 1995, to December 31, 2018. The summary statistics are provided in Table 1, while Figure 2 illustrates the constructed time series of the three-month, two-year, five-year, and ten-year Treasury zero-coupon yields.

In the literature on U.S. Treasury bond yields, researchers have found that three factors are sufficient to model the time-variation in cross sections of such yields, for an early example see Litterman and Scheinkman (1991). This applies to the current Treasury bond yield data as well. However, to get some insights about the characteristics of these three factors, focus on the eigenvectors that correspond to the first three principal components. They are reported

Maturity in months	Loading on		
	PC1	PC2	PC3
3	0.39	0.38	-0.49
6	0.39	0.36	-0.25
12	0.40	0.26	0.10
24	0.39	0.07	0.42
36	0.37	-0.09	0.46
60	0.32	-0.31	0.23
84	0.29	-0.46	-0.10
120	0.26	-0.58	-0.49
% explained	96.56	3.12	0.28

Table 2: **Loadings on the First Three Principal Components of U.S. Treasury Bond Yields**
Reported are the loadings of each maturity on the first (PC1), second (PC2), and third (PC3) principal components for zero-coupon Treasury yields. The final row shows the proportion of all bond yield variability accounted for by each principal component. The data consist of monthly U.S. Treasury zero-coupon bond yields from January 31, 1995, to December 31, 2018.

in Table 2. The first principal component explains 96.6% of the variation in the Treasury bond yields and its loading across maturities is uniformly positive. Thus, when there is a shock to the first principal component, it changes all yields in the same direction independent of maturity. This is referred to as a level factor. Likewise, it follows from the table that the second principal component explains about 3.1% of the variation in this data set. This factor has large negative loadings for the shorter maturities, while it has large positive loadings for the long maturities. Thus, a positive shock to this factor causes short-term yields to move higher while long-term yields go down, effectively creating a flattening of the yield curve. In case of a negative shock to the second principal component we get the reverse movements, short-term yields go down while long-term yields move up leading to a steepening of the yield curve. This is referred to as a slope factor as it determines the slope of the yield curve. Finally, the third principal component explains an additional 0.3% of the variation in the data. Its factor loading is an inverted U-shaped function of maturity with large negative loadings for the short and long maturities, while its loading is large positive for medium-term maturities. Thus, when there is a positive shock to the third principal component, the short and long end of the yield curve move down while the medium-term yields move up, effectively creating a hump shaped yield curve. Similarly, a negative shock to this factor will lead to an inverted hump shaped yield curve. For these reasons this factor is naturally interpreted as a curvature factor.

In summary, for this sample of Treasury bond yields three factors can easily explain 99.9% of the total variation. Focusing on the eigenvectors corresponding to the first three principal components, there is a clear pattern of level, slope, and curvature that is well approximated by the Nelson-Siegel yield function as illustrated in Figure 1. This explains our focus on that

Mat.	BBB		A	
	Mean	St. dev.	Mean	St. dev.
3	93.55	69.17	49.97	48.17
6	99.27	69.70	56.64	48.21
12	99.66	68.88	55.85	46.50
24	106.84	65.96	60.54	45.05
36	118.00	70.30	70.02	50.80
60	129.34	66.92	79.49	47.03
84	139.02	62.20	88.45	43.12
120	140.25	55.86	89.15	38.12

Table 3: **Summary Statistics for U.S. Industrial Corporate Bond Credit Spreads**
Summary statistics for the zero-coupon credit spreads of U.S. industrial corporate bonds across 8 maturities ranging from three months to ten years. The data are monthly and cover the period from January 31, 1995, to December 31, 2018. All numbers are measured in basis points.

model as a parsimonious and robust representation of the Treasury yield data.

2.3 The Corporate Bond Yield Data

The corporate bond data used in model estimation consist of representative zero-coupon yields on corporate bonds issued by U.S. industrial firms. The data are downloaded from Bloomberg and start in January 1995. To match the Treasury data, the sample contains the following 8 maturities: three-month, six-month, one-year, two-year, three-year, five-year, seven-year, and ten-year.

Since the Bloomberg data are annual discrete interest rates, the corporate bond yields are first converted into continuously compounded yields (see Appendix A). In order to convert the continuously compounded corporate zero-coupon bond yields into continuously compounded credit spreads, we deduct the corresponding observed Treasury zero-coupon yields. Thus, the credit spreads for each credit rating category c are given by

$$s_t^c(\tau) = y_t^c(\tau) - y_t^T(\tau), \quad c \in \{BBB, A\},$$

where $y_t^T(\tau)$ denotes the corresponding zero-coupon Treasury yield.

Summary statistics for the U.S. industrial credit spreads are provided in Table 3. Clearly, both the average credit spread and the credit spread volatility at a given maturity increase as credit quality deteriorates. Unreported results show that this pattern holds for credit spreads in other sectors as well.

To provide some preliminary, non-parametric evidence of the existence and characteristics of common factors in the credit spread data, the data for the two credit ratings (BBB, A) are pooled, a total of $2 \times 8 = 16$ time series. In a second step, the covariance matrix for these 16 time series is calculated and used in a principal component analysis. The results

Maturity in months	BBB		A	
	PC1	PC2	PC1	PC2
3	0.30	0.24	0.19	0.40
6	0.31	0.22	0.20	0.36
12	0.31	0.08	0.20	0.25
24	0.30	-0.08	0.20	0.12
36	0.32	-0.14	0.23	0.09
60	0.30	-0.25	0.21	-0.02
84	0.27	-0.35	0.19	-0.12
120	0.22	-0.47	0.15	-0.26

Table 4: **Loadings on the First Two Principal Components of Credit Spreads**
Reported are the loadings of each maturity on the first (PC1) and second (PC2) principal components for the zero-coupon credit spreads for U.S. industrial firms rated BBB and A covering the period from January 31, 1995, to December 31, 2018. The analysis is based on 16 time series, each with 288 monthly observations.

are reported in Table 4. The analysis reveals that the first two principal components explain 91.1% and 5.2%, respectively, of the total variation in these credit spreads. By implication, a two-factor model may explain as much as 96.3% of the observed variation in these 16 time series. Beyond these two factors each additional factor contributes very little.

The first principal component has a clear pattern of a level factor across both rating categories, while the second principal component can be characterized as a slope factor. For these reasons the credit spreads are modeled with a level and a slope factor in the style of the Nelson-Siegel model as in Christensen and Lopez (2012). As described below, each rating category is allowed to load more or less intensely on the two common credit risk factors.

3 The Model of U.S. Treasury and Corporate Bond Yields

In this section, the arbitrage-free model of U.S. Treasury and corporate bond yields is described in detail.

To begin the analysis, a model framework is needed. To that end, the model is set within the reduced-form credit risk model framework[3] using the assumption of Recovery of Market Value (see Duffie and Singleton 1999). Denote the risk-free short rate by r_t, the default intensity by $\lambda_t^{\mathbb{Q}}$, and the recovery rate by $\pi_t^{\mathbb{Q}}$.[4] Under these assumptions the price of a representative zero-coupon bond is given by

$$V_t(\tau) = E^{\mathbb{Q}}\big[e^{-\int_t^{t+\tau}(r_t+(1-\pi_s^{\mathbb{Q}})\lambda_s^{\mathbb{Q}})ds}\big].$$

[3]See Lando (1998) for technical details on the reduced-form approach to modeling credit risk.

[4]If a jump risk premium exists, the default intensity under the P-measure may deviate by a factor from the default intensity under the $\mathbb{Q}$-measure. Since we only observe bond yields, the model-implied default intensities and recovery rates are only meaningful when interpreted under the $\mathbb{Q}$-measure as indicated by the notation.

Since the loss rate $1 - \pi_t^{\mathbb{Q}}$ and the default intensity $\lambda_t^{\mathbb{Q}}$ only appear as a product under the RMV modeling assumption, their individual components are not econometrically identifiable. Instead, their product is replaced with the instantaneous credit spread s_t, which is without any loss of generality. It is this credit spread process that needs to be specified along with the risk-free rate r_t.

One important thing to note is that the corporate bonds in each credit rating category are priced in isolation without regard for possible rating transitions. Although this is a theoretical inconsistency, it is unlikely to prevent the model from extracting any common risk factors across rating categories, which is the main goal of the empirical model implementation. Taking the rating transitions into consideration is a second-order effect and refinement that will not materially change any of the results.

With the general modeling framework settled, the next step is to decide on the assumed dynamics of the risk-free rate r_t and the credit spread process s_t. The details are provided in the following subsections.

3.1 The Risk-Free Rate Model

The risk-free rate is modeled using the affine arbitrage-free approximation of the Nelson-Siegel term structure model presented in Christensen et al. (2011). This is a three-factor model where the latent state variables, $X_t^T = (L_t^T, S_t^T, C_t^T)$,[5] can be given the interpretation of level, slope, and curvature by imposing a fixed set of restrictions on the $\mathbb{Q}$-dynamics of a canonical affine three-factor Gaussian term structure model

$$\begin{aligned} r_t &= \delta_0 + \delta_1' X_t^T, \\ dX_t^T &= K^{T,\mathbb{Q}}(\theta^{T,\mathbb{Q}} - X_t^T)dt + \Sigma^T dW_t^{T,\mathbb{Q}}. \end{aligned}$$

The first key assumption is to define the instantaneous risk-free rate as the sum of the level and the slope factor

$$r_t = L_t^T + S_t^T.$$

The second key assumption is that the mean-reversion matrix under the $\mathbb{Q}$-measure must have the following simple form

$$K^{T,\mathbb{Q}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda^T & -\lambda^T \\ 0 & 0 & \lambda^T \end{pmatrix},$$

where λ^T will be identical to λ in the standard Nelson-Siegel model.

[5] Superscript T is used to indicate that the primary role of the risk-free rate model is to fit the Treasury yields in the sample.

Finally, following Christensen et al. (2011), the mean vector under the $\mathbb{Q}$-measure can be fixed at zero, $\theta^{T,\mathbb{Q}} = 0$, which is without loss of generality.

Imposing the above structure on the general affine model, default risk-free zero-coupon yields will be given by

$$y_t^T(\tau) = L_t^T + \frac{1 - e^{-\lambda^T \tau}}{\lambda^T \tau} S_t^T + \Big[\frac{1 - e^{-\lambda^T \tau}}{\lambda^T \tau} - e^{-\lambda^T \tau}\Big] C_t^T + \frac{A^T(\tau)}{\tau}.$$

Note that the Nelson-Siegel factor loadings for the level, slope, and curvature are preserved. In addition, the yield function contains a maturity-dependent term, $\frac{A^T(\tau)}{\tau}$, which arises from imposing absence of arbitrage on the dynamic Nelson-Siegel model.[6] Furthermore, building on the empirical findings reported in Christensen et al. (2011), the volatility matrix is restricted to a diagonal specification.

3.2 The Credit Spread Model

Krishnan, Ritchken, and Thomson (2007) include the risk-free level, slope, and curvature factor from their Treasury bond yield analysis in their model of firm-specific credit spreads and report performance improvements.

Inspired by these results the factors of the risk-free rate are included directly in the formulation of the instantaneous credit spread process. As the Treasury curvature factor is absent in the instantaneous short-rate process r_t^T, it will also be absent from the instantaneous credit spread process. Thus, the instantaneous credit spread for credit rating category c is assumed to be a function of the level and the slope factor from the risk-free yield term structure in addition to two common credit risk factors

$$s_t^c = \alpha_0^c + \alpha_{L^T}^c L_t^T + \alpha_{S^T}^c S_t^T + \alpha_L^c L_t^S + \alpha_S^c S_t^S.$$

This structure implies that each credit rating category c can load separately on each of these four factors and do so independently of the remaining credit rating categories.

Now, the two common credit risk factors (L_t^S, S_t^S) are assumed to have a Nelson-Siegel factor loading structure. To achieve this, the dynamics of the common credit risk factors under the $\mathbb{Q}$-measure are assumed to be given by

$$\begin{pmatrix} dL_t^S \\ dS_t^S \end{pmatrix} = - \begin{pmatrix} 0 & 0 \\ 0 & \lambda^S \end{pmatrix} \begin{pmatrix} L_t^S \\ S_t^S \end{pmatrix} dt + \begin{pmatrix} \sigma_{L^S} & 0 \\ 0 & \sigma_{S^S} \end{pmatrix} \begin{pmatrix} dW_t^{L^S,\mathbb{Q}} \\ dW_t^{S^S,\mathbb{Q}} \end{pmatrix}.$$

Thus, the dynamics under the $\mathbb{Q}$-measure of the five factors that affect the corporate bond

[6]The analytical formula for $\frac{A^T(\tau)}{\tau}$ is provided in Christensen et al. (2011).

yields are given by the following system of stochastic differential equations

$$\begin{pmatrix} dL_t^T \\ dS_t^T \\ dC_t^T \\ dL_t^S \\ dS_t^S \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^T & -\lambda^T & 0 & 0 \\ 0 & 0 & \lambda^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^S \end{pmatrix} \begin{pmatrix} L_t^T \\ S_t^T \\ C_t^T \\ L_t^S \\ S_t^S \end{pmatrix} dt + \begin{pmatrix} \sigma_{L^T} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{S^T} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{C^T} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{L^S} & 0 \\ 0 & 0 & 0 & 0 & \sigma_{S^S} \end{pmatrix} \begin{pmatrix} dW_t^{L^T,\mathbb{Q}} \\ dW_t^{S^T,\mathbb{Q}} \\ dW_t^{C^T,\mathbb{Q}} \\ dW_t^{L^S,\mathbb{Q}} \\ dW_t^{S^S,\mathbb{Q}} \end{pmatrix}.$$

Given this dynamic structure under the $\mathbb{Q}$-measure, the yield of a representative zero-coupon bond with credit rating c and maturity in τ years can be shown to be given by

$$\begin{aligned} y_t^c(\tau) &= L_t^T + \frac{1-e^{-\lambda^T\tau}}{\lambda^T\tau}S_t^T + \Big[\frac{1-e^{-\lambda^T\tau}}{\lambda^T\tau} - e^{-\lambda^T\tau}\Big]C_t^T \\ &\quad + \alpha_{L^T}^c L_t^T + \alpha_{S^T}^c \frac{1-e^{-\lambda^T\tau}}{\lambda^T\tau}S_t^T + \alpha_{S^T}^c\Big[\frac{1-e^{-\lambda^T\tau}}{\lambda^T\tau} - e^{-\lambda^T\tau}\Big]C_t^T \\ &\quad + \alpha_0^c + \alpha_L^c L_t^S + \alpha_S^c \frac{1-e^{-\lambda^S\tau}}{\lambda^S\tau}S_t^S + \frac{A^c(\tau)}{\tau}. \end{aligned}$$

By implication, the corresponding zero-coupon credit spread is given by

$$\begin{aligned} s_t^c(\tau) &= y_t^c(\tau) - y_t^T(\tau) \\ &= \alpha_{L^T}^c L_t^T + \alpha_{S^T}^c \frac{1-e^{-\lambda^T\tau}}{\lambda^T\tau}S_t^T + \alpha_{S^T}^c\Big[\frac{1-e^{-\lambda^T\tau}}{\lambda^T\tau} - e^{-\lambda^T\tau}\Big]C_t^T \\ &\quad + \alpha_0^c + \alpha_L^c L_t^S + \alpha_S^c \frac{1-e^{-\lambda^S\tau}}{\lambda^S\tau}S_t^S + \frac{A^c(\tau)}{\tau} - \frac{A^T(\tau)}{\tau}, \end{aligned}$$

where

$$\begin{aligned} \frac{A^c(\tau)}{\tau} &= -\frac{\sigma_{L^T}^2(1+\alpha_{L^T}^c)^2}{6}\tau^2 - \sigma_{S^T}^2(1+\alpha_{S^T}^c)^2\Big[\frac{1}{2(\lambda^T)^2} - \frac{1}{(\lambda^T)^3}\frac{1-e^{-\lambda^T\tau}}{\tau} + \frac{1}{4(\lambda^T)^3}\frac{1-e^{-2\lambda^T\tau}}{\tau}\Big] \\ &\quad -\sigma_{C^T}^2(1+\alpha_{S^T}^c)^2\Big[\frac{1}{2(\lambda^T)^2} + \frac{1}{(\lambda^T)^2}e^{-\lambda^T\tau} - \frac{1}{4\lambda^T}\tau e^{-2\lambda^T\tau} - \frac{3}{4(\lambda^T)^2}e^{-2\lambda^T\tau}\Big] \\ &\quad -\sigma_{C^T}^2(1+\alpha_{S^T}^c)^2\Big[\frac{5}{8(\lambda^T)^3}\frac{1-e^{-2\lambda^T\tau}}{\tau} - \frac{2}{(\lambda^T)^3}\frac{1-e^{-\lambda^T\tau}}{\tau}\Big] \\ &\quad -\frac{(\sigma_{L^S})^2(\alpha_L^c)^2}{6}\tau^2 \\ &\quad -(\sigma_{S^S})^2(\alpha_S^c)^2\Big[\frac{1}{2(\lambda^S)^2} - \frac{1}{(\lambda^S)^3}\frac{1-e^{-\lambda^S\tau}}{\tau} + \frac{1}{4(\lambda^S)^3}\frac{1-e^{-2\lambda^S\tau}}{\tau}\Big]. \end{aligned}$$

The description so far has detailed the dynamics under the pricing measure and, by implication, determined the functions that are fitted to the observed yield data. The above structure places no restrictions on the dynamic drift components under the real-world $\mathbb{P}$-measure beyond the requirement of constant volatility; therefore, to facilitate the empirical implementation, the essentially affine risk premium specification introduced in Duffee (2002) is used. In the Gaussian framework, this specification implies that the risk premiums, Γ_t, depend on the state variables:

$$\Gamma_t = \gamma^0 + \gamma^1 X_t,$$

where $\gamma^0 \in \mathbf{R}^5$ and $\gamma^1 \in \mathbf{R}^{5\times 5}$ contain unrestricted parameters. The relationship between the real-world yield curve dynamics under the $\mathbb{P}$-measure and the risk-neutral dynamics under the $\mathbb{Q}$-measure is given by

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \Gamma_t dt.$$

Thus, in general, the $\mathbb{P}$-dynamics of the state variables can be written as

$$dX_t = K^{\mathbb{P}}(\theta^{\mathbb{P}} - X_t)dt + \Sigma dW_t^{\mathbb{P}},$$

where $K^{\mathbb{P}}$ and $\theta^{\mathbb{P}}$ are both allowed to vary freely relative to their counterparts under the $\mathbb{Q}$-measure.

4 Model Estimation and Results

In this section, we first describe the model estimation based on the standard Kalman filter before we proceed to a discussion of the estimation results.

Thanks to the affine structure of our model, its measurement equations used in the model estimation take the form:

$$y_t = \begin{pmatrix} y_t^T \\ y_t^{Indu} \end{pmatrix} = \begin{pmatrix} A^T \\ A^{Indu} \end{pmatrix} + \begin{pmatrix} B^T \\ B^{Indu} \end{pmatrix} X_t + \varepsilon_t.$$

The data vector y_t is a (24×1) vector consisting of y_t^T with 8 Treasury yields and y_t^{Indu} with 16 industrial corporate bond yields. Correspondingly, the constant term consists of an (8×1) vector A^T and a (16×1) vector A^{Indu}. The loading matrix for the five factors consists of an (8×5) matrix B^T and a (16×5) matrix B^{Indu}.

For identification, the A-rated corporate bond yields are chosen to be the benchmark credit rating category, that is, its constant α_0^A is set equal to zero, and the factor loadings on the two spread factors have unit sensitivity for this rating category, i.e., $\alpha_L^A = 1$ and $\alpha_S^A = 1$. This choice is not restrictive and simply implies that the sensitivities to changes in the two spread factors are measured relative to those of A-rated firms and that the estimated values of those factors represent the absolute sensitivity of the benchmark A-rated corporate credit spreads.

For continuous-time Gaussian models, the conditional mean vector and the conditional covariance matrix are given by

$$\begin{aligned} E^{\mathbb{P}}[X_T|\mathcal{F}_t] &= (I-\exp(-K^{\mathbb{P}}\Delta t))\theta^{\mathbb{P}} + \exp(-K^{\mathbb{P}}\Delta t)X_t, \\ V^{\mathbb{P}}[X_T|\mathcal{F}_t] &= \int_0^{\Delta t} e^{-K^{\mathbb{P}}s}\Sigma\Sigma' e^{-(K^{\mathbb{P}})'s}ds, \end{aligned}$$

where $\Delta t = T - t$ and $\exp(-K^{\mathbb{P}}\Delta t)$ is a matrix exponential. Stationarity of the system under the $\mathbb{P}$-measure is ensured provided the real components of all the eigenvalues of $K^{\mathbb{P}}$ are positive. This condition is imposed in all estimations, so we can start the Kalman filter at the unconditional mean and covariance matrix[7]

$$\widehat{X}_0 = \theta^{\mathbb{P}} \quad \text{and} \quad \widehat{\Sigma}_0 = \int_0^{\infty} e^{-K^{\mathbb{P}}s}\Sigma\Sigma' e^{-(K^{\mathbb{P}})'s}ds.$$

The transition state equation for the Kalman filter is given by

$$X_{t_i} = \Phi^0_{\Delta t_i} + \Phi^1_{\Delta t_i} X_{t_{i-1}} + \eta_{t_i},$$

where $\Delta t_i = t_i - t_{i-1}$ and

$$\Phi^0_{\Delta t_i} = (I - \exp(-K^{\mathbb{P}}\Delta t_i))\theta^{\mathbb{P}}, \quad \Phi^1_{\Delta t_i} = \exp(-K^{\mathbb{P}}\Delta t_i), \quad \text{and} \quad \eta_{t_i} \sim N\Big(0, \int_0^{\Delta t_i} e^{-K^{\mathbb{P}}s}\Sigma\Sigma' e^{-(K^{\mathbb{P}})'s}ds\Big).$$

All measurement errors are assumed to be independently and identically distributed white noise with an error structure given by

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix}\right].$$

For parsimony, all Treasury yields are assumed to have a common error standard deviation, and a similar assumption is imposed on all corporate bond yields. The associated four standard error parameters are denoted σ^T_ε and σ^S_ε.

The linear least-squares optimality of the Kalman filter requires that the white noise transition and measurement errors be orthogonal to the initial state, i.e.

$$E[f_0\eta'_t] = 0, \quad E[f_0\varepsilon'_t] = 0.$$

[7] Throughout conditional and unconditional covariance matrices are calculated using the analytical solutions provided in Fisher and Gilles (1996).

$K^{\mathbb{P}}$	$K^{\mathbb{P}}_{\cdot,1}$	$K^{\mathbb{P}}_{\cdot,2}$	$K^{\mathbb{P}}_{\cdot,3}$	$K^{\mathbb{P}}_{\cdot,4}$	$K^{\mathbb{P}}_{\cdot,5}$	$\theta^{\mathbb{P}}$		Σ
$K^{\mathbb{P}}_{1,\cdot}$	0.0340	-0.0927	-0.0054	-0.1012	-0.0035	0.0602	σ_{11}	0.0067
	(0.2441)	(0.1978)	(0.1567)	(0.6311)	(0.5784)	(0.0240)		(0.0003)
$K^{\mathbb{P}}_{2,\cdot}$	0.6825	0.8843	-0.6593	1.3743	0.4083	-0.0272	σ_{22}	0.0104
	(0.4615)	(0.3245)	(0.2364)	(1.3679)	(1.1334)	(0.0155)		(0.0007)
$K^{\mathbb{P}}_{3,\cdot}$	0.9699	0.1996	1.0280	4.1037	-1.6530	-0.0243	σ_{33}	0.0253
	(0.8716)	(0.5669)	(0.3739)	(2.3114)	(1.9409)	(0.0173)		(0.0010)
$K^{\mathbb{P}}_{4,\cdot}$	0.0699	-0.0424	0.0132	0.0552	-0.2019	0.0126	σ_{44}	0.0027
	(0.1012)	(0.0799)	(0.0674)	(0.2645)	(0.1929)	(0.0055)		(0.0002)
$K^{\mathbb{P}}_{5,\cdot}$	-0.0676	-0.0837	0.0866	0.7310	0.5517	-0.0100	σ_{55}	0.0031
	(0.1111)	(0.0793)	(0.0593)	(0.3874)	(0.3244)	(0.0074)		(0.0004)

Table 5: **Estimated Dynamic Parameters**
This table shows the estimated parameters of the $K^{\mathbb{P}}$ matrix, $\theta^{\mathbb{P}}$ vector, and diagonal Σ matrix for the joint five-factor model of U.S. Treasury and corporate bond yields with unrestricted $K^{\mathbb{P}}$ matrix. λ^T is estimated at 0.5007 (0.0041), while λ^S is estimated at 0.1895 (0.0096). The maximum log likelihood value is 36,493.98. The data used are monthly covering the period from January 31, 1995, to December 31, 2018.

Finally, the standard deviations of the estimated parameters are calculated as

$$\Sigma(\widehat{\psi}) = \frac{1}{T}\Big[\frac{1}{T}\sum_{t=1}^{T}\frac{\partial \log l_t(\widehat{\psi})}{\partial \psi}\frac{\partial \log l_t(\widehat{\psi})}{\partial \psi}'\Big]^{-1},$$

where $\widehat{\psi}$ denotes the optimal parameter set.

Please note that the Gaussian distributional assumption is used here as in most of the dynamic term structure literature. This assumption presents modeling issues for the Treasury yields in particular, which have been near the zero lower bound in recent years. However, it should be noted that it would be straightforward to cast the risk-free rate as a shadow-rate process that respects a lower bound using the formulas provided in Christensen and Rudebusch (2015). Thus, such a refinement can easily be achieved with a minimum of modifications to the model framework as presented.

4.1 Estimation Results

In this section, the results of the model estimation with an unrestricted $K^{\mathbb{P}}$ matrix are briefly summarized.

The model's estimated dynamic mean-reversion parameters are reported in Table 5. The table also reports the estimated mean and volatility parameters. For the three Treasury factors the estimated means and volatility parameters are similar to the results reported by Christensen et al. (2014b).[8] Thus, overall, the estimated factor dynamics are consistent with those reported in previous studies that have used AFNS models.

[8]These estimated model parameters are also similar to those reported by Christensen et al. (2011), even

Rating	U.S. industrial credit spreads				
	α_0^c	α_{LT}^c	α_{ST}^c	α_{LS}^c	α_{SS}^c
BBB	-0.0007	-0.0256	-0.1730	1.4568	1.28226
	(0.0002)	(0.0389)	(0.0119)	(0.0079)	(0.0088)
A	0	-0.0091	-0.0761	1	1
		(0.0268)	(0.0093)		

Table 6: **Estimated Factor Loadings in the Corporate Credit Spread Functions**
The estimated factor loadings for each of the credit rating categories in the joint five-factor model of U.S. Treasury and corporate bond yields with unrestricted $K^{\mathbb{P}}$ matrix. The data used are monthly covering the period from January 31, 1995, to December 31, 2018. The numbers in parentheses are the estimated standard deviations of the parameter estimates.

Table 6 reports the estimated factor loadings of the state variables in the corporate bond credit spread function for each credit rating category. Note that lower credit quality implies higher sensitivities to the common credit risk factors. Generally speaking this result indicates that bonds issued by firms with lower credit quality tend to have higher and steeper credit spread curves. Furthermore, in terms of the credit spread sensitivity to the Treasury slope factor, there is another systematic pattern with lower credit rating implying greater credit spread sensitivity to this factor.[9] When the Treasury curve is steep like in the 2009-2015 period, the Treasury slope factor is negative and below its historical mean. In that case, the credit spread term structures are higher and less steep than what the credit risk factors in isolation would imply, and this effect is stronger the lower the credit rating.

Table 7 provides statistics for the model's ability to fit the Treasury and corporate bond yields. First, the fit of the Treasury yields is good and on par with models of only Treasury yields, see Christensen et al. (2011). Second, for the corporate bond yields, the reported mean errors indicate a small overall bias in their fit. Focusing on the root mean squared fitted errors (RMSEs), we note that the corporate bond yields are fitted with somewhat less accuracy than the Treasury bond yields as could be expected given the greater heterogeneity in the corporate bond market.

5 Bond Yield Decomposition

In this section, we describe how we decompose Treasury and corporate bond yields into their respective expectations and risk premium components.

The term premium of the Treasury yield with maturity in τ years is conventionally defined as

$$TP_t^T(\tau) = y_t^T(\tau) - \frac{1}{\tau}\int_t^{t+\tau} E_t^{\mathbb{P}}[r_s]ds. \qquad (1)$$

though that study used unsmoothed Fama-Bliss Treasury yields.

[9]Due to the assumed AFNS model structure, the credit spread sensitivities to the Treasury curvature factor are identical to those of the Treasury slope factor, see the credit spread equations in Section 3.2.

Mat. in months	Treasury yields	Corporate yields BBB	Corporate yields A
		Mean	
3	-1.61	-2.87	-0.41
6	-0.60	0.85	4.80
12	1.40	-1.97	1.56
24	2.32	-1.93	-0.30
36	0.56	1.36	1.25
60	-2.98	0.00	-2.66
84	-2.64	2.96	-1.05
120	3.29	1.94	-3.22
		RMSE	
3	7.35	13.45	13.30
6	2.92	9.25	10.57
12	6.58	9.68	10.02
24	5.07	8.39	9.54
36	3.48	8.25	11.18
60	5.84	7.29	11.02
84	4.51	10.32	10.11
120	6.41	11.64	12.37

Table 7: **Summary Statistics of Fitted Errors**
This table provides the mean and RMSE of the model fitted yield errors in basis points.

By extension, it is possible to define a corporate bond yield term premium as

$$\begin{aligned} TP_t^{c,i}(\tau) &= y_t^{c,i}(\tau) - \frac{1}{\tau}\int_t^{t+\tau} E_t^{\mathbb{P}}[r_s^{c,i}]ds \\ &= y_t^T(\tau) + s_t^c(\tau) - \frac{1}{\tau}\int_t^{t+\tau} E_t^{\mathbb{P}}[r_s + s_s^{c,i}]ds \end{aligned}$$

where i continues to indicate the rating category and $s_t^{c,i}$ is the associated instantaneous credit spread rate. From this formula it follows that we can define a credit spread risk premium as

$$CRP_t^{c,i}(\tau) = s_t^c(\tau) - \frac{1}{\tau}\int_t^{t+\tau} E_t^{\mathbb{P}}[s_s^{c,i}]ds, \quad (2)$$

so that the corporate term premium can be written as

$$TP_t^{c,i}(\tau) = TP_t^T(\tau) + CRP_t^{c,i}(\tau).$$

In the definition of the credit risk premium, the term $\frac{1}{\tau}\int_t^{t+\tau} E_t^{\mathbb{P}}[s_s^{c,i}]ds$ represents the expected risk-neutral excess return, while the credit risk premium itself represents the residual compensation that investors demand above and beyond the expected excess return to assume long-term exposures in the corporate bond market.

For a start, Figure 3 shows the classic decomposition of the ten-year Treasury yield into

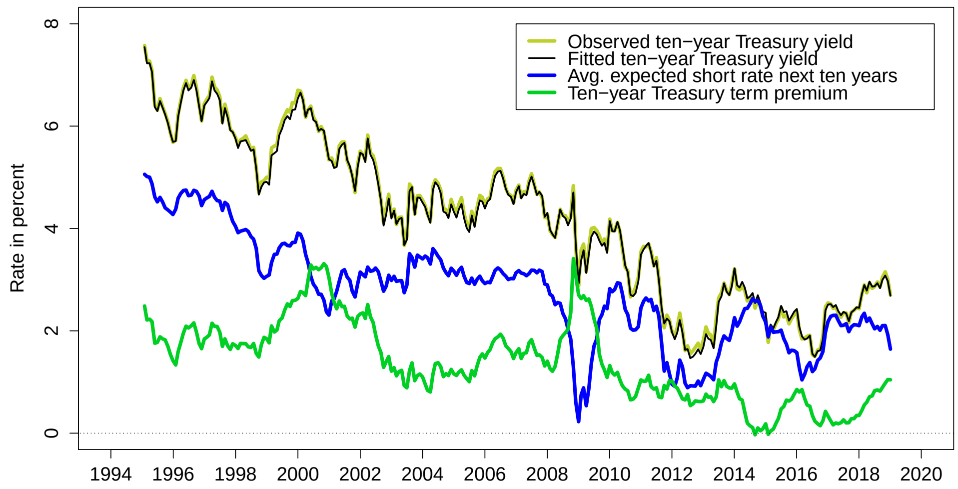

Figure 3: **Ten-Year Treasury Yield Decomposition**

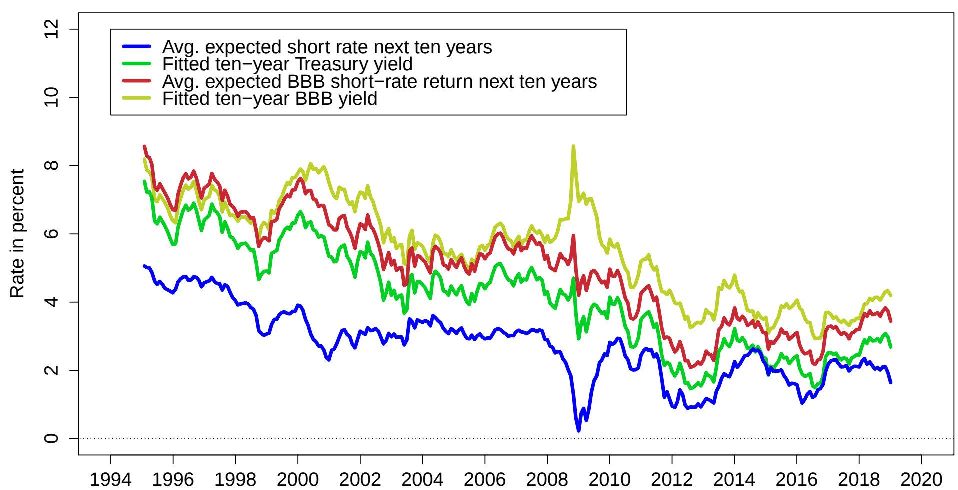

Figure 4: **Ten-Year BBB Yield Decomposition**

its expectations and residual term premium components.

Figure 4 shows a breakdown of the key components determining the level of the BBB-rated corporate bond yields in our data.

Finally, Figures 5 and 6 show the decomposition of the BBB- and A-rated credit spreads into their respective expectations and risk premium components.

Figure 7 shows the resulting two credit risk premium series, where we note that the BBB-rated credit risk premium is slightly more volatile than the A-rated credit risk premium, but

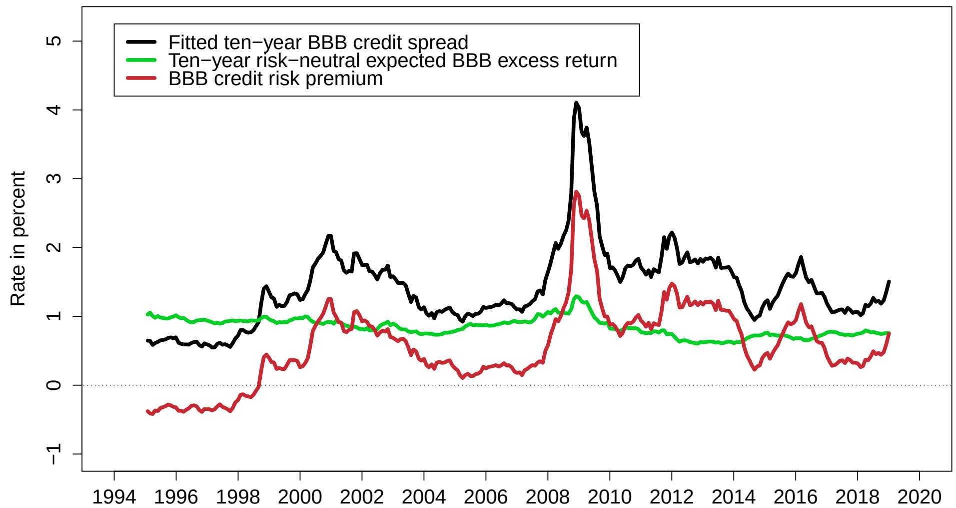

Figure 5: **Ten-Year BBB Credit Spread Decomposition**

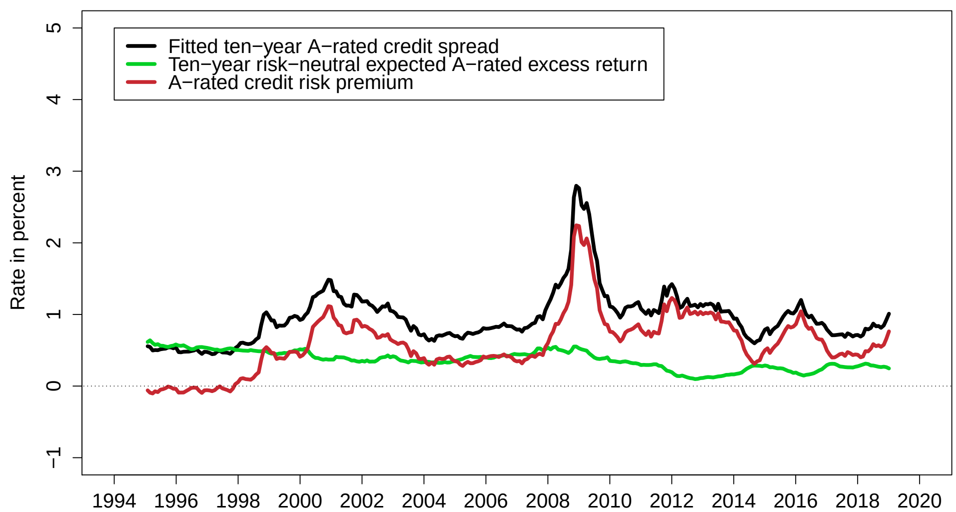

Figure 6: **Ten-Year A Credit Spread Decomposition**

overall the two series are very similar as one could expect. This implies that the level difference in credit spreads between A- and BBB-rated exposures are due primarily to differences in the expected excess returns and not driven by differences in risk premiums across rating categories.

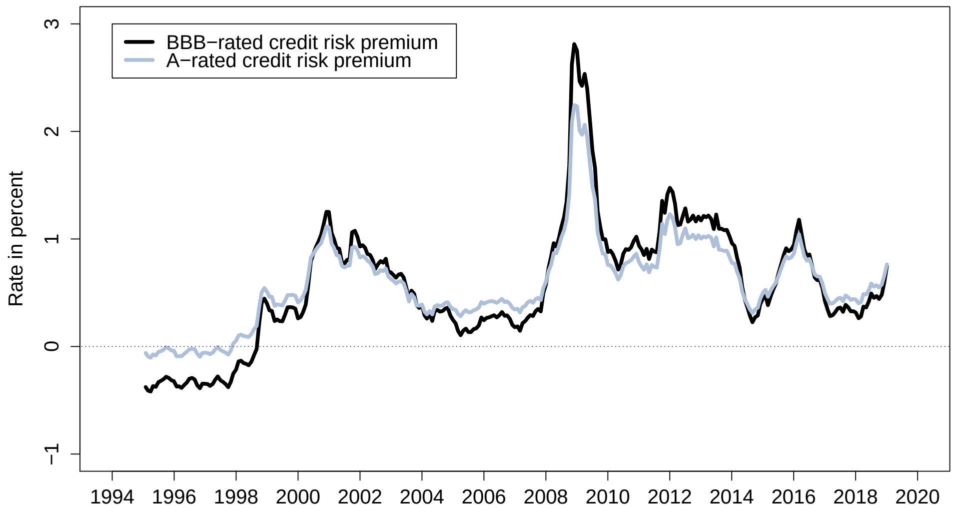

Figure 7: **Ten-Year Credit Risk Premium Comparison**

6 Conclusion

In this paper, we introduce a joint five-factor dynamic term structure model of U.S. Treasury and corporate bond yields that can be used to estimate the credit risk premiums embedded in corporate bond credit spreads.

The results reveal that the level difference in credit spreads between A- and BBB-rated exposures are due primarily to differences in the expected excess returns and not driven by differences in risk premiums across rating categories, which are very similar and highly positively correlated.

We note that the presented model framework can be extended in numerous ways. For greater accuracy it may be worthwhile to consider the generalized arbitrage-free Nelson-Siegel models derived in Christensen et al. (2009). Also, as already noted, issues related to the zero lower bound of Treasury yields can be handled by casting the risk-free rate as a shadow-rate using the formulas provided in Christensen and Rudebusch (2015). If stochastic yield volatility is a requirement, Christensen et al. (2014a) expand the arbitrage-free Nelson-Siegel model class to accommodate that. Furthermore, the data set can be expanded to encompass Treasury inflation-protected securities (TIPS) through a joint modeling of nominal and real yields as described in Christensen et al. (2010). Lastly, liquidity premiums in the bond price data can be accounted for using the augmented model structure developed in Andreasen et al. (2018). However, we leave all of these avenues for future research.

Appendix A: Conversion of Interest Rate Data

The Bloomberg fair-value, zero-coupon yield curves are generated for particular (sector,rating) segments of the corporate bond market using individual bond prices, both indicative and executable, as quoted by price contributors over a specified time window. Based on these bond datasets, option-adjusted spreads are generated, and these adjusted bond yields are converted into zero-coupon yield curves using piecewise linear functions.

We convert the Bloomberg data for financial corporate bond rates into continuously compounded yields. The n-year yield at time t, $r_t(n)$, the corresponding zero-coupon bond price, $P_t(n)$, and the continuously compounded yield, $y_t(n)$, are related by

$$P_t(n) = \frac{1}{(1+r_t(n))^n} = e^{-y_t(n)n} \iff y_t(n) = -\frac{1}{n}\ln\frac{1}{(1+r_t(n))^n} = \ln(1+r_t(n)).$$

For maturities shorter than one year, we assume the standard convention of linear interest rates. For example, the zero-coupon bond price corresponding to the six-month yield is calculated as

$$P_t(6m) = \frac{1}{1+0.5r_t(6m)} = e^{-0.5y_t(6m)},$$

and the corresponding continuously compounded yield as

$$y_t(6m) = -2\ln\frac{1}{1+0.5r_t(6m)} = 2\ln(1+0.5r_t(6m)).$$

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