EXITING FROM QE

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Abstract

We develop a regime-switching SVAR (structural vector autoregression) in which the monetary policy regime, chosen by the central bank responding to economic conditions, is endogenous and observable. There are two regimes, one of which is QE (quantitative easing). The model can incorporate the exit condition for terminating QE. We then apply the model to Japan, a country that has accumulated, by our count, 130 months of QE as of December 2012. Our impulse response and counter-factual analyses yield two findings about QE. First, an increase in reserves raises inflation and output. Second, terminating QE can be expansionary.

Keywords: quantitative easing, structural VAR, observable regimes, Taylor rule, impulse responses, Bank of Japan.

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1 Introduction and Summary

Since the recent global financial crisis, central banks of major market economies have adopted quantitative easing, or QE, which is to allow reserves held by depository institutions far above the required level while keeping the policy rate very close to zero. This paper uses an SVAR (structural vector autoregression) to evaluate macroeconomic effects of QE. Reliably estimating such a time-series model is difficult because only several years have passed since the crisis. We are thus led to examine Japan, a country that has already accumulated a history of, by our count, 130 months of QE as of December 2012. Those 130 QE months come in three installments, which allows us to evaluate the effect of exiting from QE as well.

Our SVAR has two monetary policy regimes: the zero-rate regime in which the policy rate is very close to zero, and the normal regime. In Section 2, we document for Japan that bank reserves are greater than required reserves (and often several times greater) when the policy rate is below 0.05% (5 basis points) per year. We say that the zero-rate regime is in place if and only if the policy rate is below this critical rate. Therefore, the regime is observable and, since reserves are substantially higher than the required level for all months under the zero-rate regime in data, the zero-rate regime and QE are synonymous. There are three spells of the zero-rate/QE regime: March 1999 - July 2000, March 2001 - June 2006, and December 2008 to date. (They are indicated by the shades in the time-series plot of the policy rate in Figure 1.) They account for the 130 months. Also documented in Section 2 is that for most of those months the BOJ (Bank of Japan) made a stated commitment of not exiting from the zero-rate regime unless inflation is above a certain threshold. That is, the exit condition in Japan is about inflation. Our SVAR model incorporates this exit condition.

The model is a natural extension of the standard recursive SVAR model developed by Christiano, Eichenbaum, and Evans (1999).[1] There are four variables: inflation, output (measured by the output gap), the policy rate, and excess reserves, in that order. We do not

[1] Their SVAR orders variables by placing non-financial variables (such as inflation and output) first, followed by monetary policy instruments (such as the policy rate and measures of money), and financial variables (such as stock prices and long-term interest rates).

impose any structure on inflation and output dynamics, so the first two equations of the system are reduced-form equations. The third equation is the Taylor rule providing a shadow policy rate, while the fourth equation specifies the central bank's supply of excess reserves under QE. We incorporate the exit condition by assuming that the central bank ends the zero-rate regime only if the shadow rate is positive (i.e., if the zero lower bound is not binding) *and* the inflation rate is above a certain threshold. The regime is endogenous because the regime evolution depends on inflation and output through the zero lower bound and the exit condition. In compliance with the Lucas critique, we allow the reduced-form coefficients for inflation and output to depend on the monetary policy regime. The model parameters are estimated by ML (maximum likelihood) that properly takes into account regime endogeneity.

We utilize the IRs (impulse responses) and other counter-factual analyses to describe the macroeconomic effects of various monetary policies, including those of a change in the monetary policy regime. The IRs we emply are a generalization, to non-linear systems such as ours, of the standard IRs for linear systems. To describe the effect of, for example, a cut in the policy rate in the base period t, we compare the path of inflation and output projected by the model given the baseline history up to t with the path given an alternative history that differs from the baseline history only with respect to the policy rate in t. We find:

- When the regime is the normal regime in both the baseline and alternative histories so that there is room for rate cuts, the IR of inflation to a policy rate cut is negative for many periods. Thus, consistent with the finding of the literature to be cited below, we observe the price puzzle for Japan.[2]

- Under the zero-rate/QE regime, the IR of inflation and output to an increase in excess reserves is positive. This, too, is consistent with the literature's finding.

- The IR analysis can be extended by allowing the two paths to differ in more than one respect in the base period t. As an example, we set t = July 2006, the month the zero-rate/QE regime was terminated, and consider an alternative and counter-factual history of not exiting from QE in t. The two histories differ at t not just in the regime but also in the policy rate and excess reserves. We find that output and (to a less extent) inflation are lower under the alternative of extending QE to July 2006. That is, exiting from QE in July 2006 was *expansionary*.

Turning to the relation of our paper to the literature, there is a rapidly expanding literature on the recent QE measures (called large-scale asset purchases (LSAPs)) by the U.S. Federal Reserve. Given the small sample sizes, researchers wishing to study macroeconomic effects of QE proceed in two steps, first documenting that QE lowered longer-term interest rates and then evaluating the effect of lower interest rates using macroeconomic models. In a recent review of the literature, Williams (2012) notes that there is a great deal of uncertainty surrounding the existing estimates. One reason he cites is that QE-induced interest rate declines may be atypical.

Were it not for the small-sample problem, time-series analysis of QE would complement nicely those model-based analyses. There are several SVAR studies about Japan's QE that exploit

[2] In a detailed examination of the price puzzle, Braun and Shioji (2006) show that the price puzzle is pervasive for both the U.S. and Japan in the recursive SVAR model of Christiano *et. al.* (1999) mentioned in footnote 1. For Japan, they use monthly data from 1981 to 1996 and find that a large and persistent price puzzle arises for a variety of choices for the financial variables including commodity prices, the Yen-Dollar exchange rate, oil prices, the wholesale price index, and the 10-year yield on government bonds. They also find that the puzzle arises when each of those financial variables are placed third after inflation and output. To corroborate their finding for the U.S., we estimated the 3-variable SVAR model of Stock and Watson (2001, to be presented in Section 3) on monthly U.S. data from 1960 to 2000 and found that the price puzzle lasts for several years (Stock and Watson (2001) estimated the model on quarterly U.S. data and found that the price puzzle lasts for only a couple of quarters). For a structural model for the U.S. that generates the price puzzle, see Christiano, Eichenbaum, and Evans (2005).

the many QE months noted above. They can be divided into three sets: (a) those assuming the regime is observable and exogenous, (b) those with exogenous but unobservable regimes, and (c) those (like our paper) with endogenous and observable regimes. All those studies assume the block-recursive structure of Christiano, *et. al.* (1999) mentioned in footnote 1. Honda *et. al.* (2007) and Kimura and Nakajima (2013) fall in category (a). Using Japanese monthly data covering only the zero-rate period of 2001 through 2006 and based on SVARs that exclude the policy rate (because it is zero), Honda *et. al.* (2007) find that the IR of inflation and output to an increase in reserves is positive. Kimura and Nakajima (2013) use quarterly data from 1981 and assume two spells of the QE regime (2001:Q1 - 2006:Q1 and 2010:Q1 on). They too find the expansionary effect of excess reserves under QE.[3] Falling in category (b) are Fujiwara (2006) and Inoue and Okimoto (2008).[4] Both papers apply the hidden-stage Markov switching SVAR model to Japanese monthly data. They find that the probability of the second state was very high in most of the months since the late 1990s. For those months, the IR of output to an increase in base money is positive and persistent. The regime in Iwata and Wu (2006) and Iwata (2010), in contrast, is necessarily endogenous because the policy rate in their VAR, being subject to the zero lower bound, is a censored variable. Thus these two papers fall in category (c). Like the other papers, they find that money is expansionary: the IR of inflation and output to base money is positive. They also find, as in some of the papers already cited, the price puzzle under the normal regime.

Because the regime is chosen by the central bank to honor the zero lower bound, or more generally, to respond to inflation and output, it seems clear that the regime must be treated as

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[3] Within each regime, they use the TVP-VAR (time-varying parameter VAR) model to allow coefficients and error variances to change stochastically. There are ohter studies on Japan's monetary policy that utilize TVP-VAR. They include Nakajima, Shiratsuka, and Teranishi (2010) and Nakajima and Watanabe (2011). They do not allow for discrete regime changes, though. For example, when the central bank enters the zero-rate/QE regime, the TV-VAR, ignorant of the regime change, does not shrink the coefficients in the policy rate equation immediately to zero. This sort of shrinking is enforced in Kimura and Nakajima (2013) cited in the text.

[4] A precurser to these two papers is the VAR study by Miyao (2002), which, using the conventional likelihood-ratio method, finds a structural break in 1995.

endogenous. And, as already argued above and will be argued more fully in the next section, a strong case can be made for the observability of the monetary policy regime. Our paper differs from Iwata and Wu (2006) and Iwata (2010), both of which treat the regime as observable and endogenous, in several respects. First, our SVAR incorporates the exit condition as well as the zero lower bound. Second, we consider IRs to regime changes. This allows us to examine the macroeconomic effect of exiting from QE. As already mentioned, our paper has a surprising result on this issue. Third, the interest rate equation in our SVAR is the Taylor rule rather than a reduced-form equation. Most existing estimates of the Taylor rule in Japan end the sample period at 1995 because there is little movements in the policy rate since then. Our estimation of the Taylor rule, with the sample including recent months of zero policy rates and allowing for regime endogeneity, should be of independent interest.

The rest of the paper is organized as follows. In Section 2, we present the case for the monetary policy regime observability. Section 3 describes our four-variable SVAR. Section 4 derives the ML estimator of the model, describes the monthly data, and reports our parameter estimates. Section 5 defines IRs for our regime-switching SVAR, displays estimated IRs, and then combines those IRs to calculate the effect of counter-factual policies. Section 6 considers several variations of the model to examine whether the major conclusions remain valid. Section 7 concludes.

2 Identifying the Zero-Rate Regime

Identification by the “L"

We identify the monetary policy regime on the basis of the relation between excess reserves and the policy rate. Figure 2a plots the policy rate measured by the overnight interbank rate (called the “Call rate" in Japan) against m, the *excess reserve rate* defined as the log of the ratio of the actual to required levels of reserves. The actual reserve level for the month is defined as the average of daily balances over the reserve maintenance period (between the 16th day of the month and the 15th day of the following month), not over the calendar month, because that is how the amount of required reserves is calculated. Accordingly, the policy rate for the month, to be denoted r, is the average of daily rates over the same reserve maintenance period. Because the BOJ (Bank of Japan) recently started paying interest on reserves, the vertical axis in the figure is not the policy rate r itself but the *net* policy rate $r - \bar{r}$ where $\bar{r}$ is the rate paid on reserves (0.1% since November 2008). It is the cost of holding reserves for commercial banks.

The plot in Figure 2a shows a distinct L shape. There are excess reserves (i.e., the excess reserve rate m is positive) for all months for which the net policy rate $r - \bar{r}$ is below some very low critical rate, and no excess reserves for most, but not all, months for which the net rate is above the critical rate.[5] We view those dots on or only slightly above the horizontal line below the critical rate as representing the *supply* of excess reserves chosen by the central bank, as banks would be indifferent between any two levels of excess reserves.

Turning to those dots above the critical rate with positive excess reserves, Figure 2b magnifies the plot near the origin for closer inspection. The dotted horizontal line is the critical rate of $r - \bar{r} = 0.05\%$ (5 basis points) below which excess reserves are supply-determined. Above the dashed line, those indicated by filled-in squares come from two periods (August 2000 - February 2001 and July 2006 - November 2008) between spells of very low net policy rates. The rest come from the late 1990s when the Japanese financial system was under stress. For example, $(m_t, r_t - \bar{r}_t) = (8.9\%, 0.22\%)$ in October 1998 when the Long-Term Credit Bank went bankrupt. We interpret those dots off the vertical axis from the late 1990s as representing the *demand* for excess reserves when the shock to reserve demand is very large due to precautionary reasons.

Regarding the filled-in squares, it appears that, until the Lehman crisis, precautionary demand was not the reason for banks to hold excess reserves. Industry sources indicate that, after months of near-zero interbank rate with large excess reserves, the response by smaller-scale banks when the policy rate turned positive from essentially zero was to delay re-entry to the

[5] The two months of significantly positive excess reserves when the policy rate is about 8% are February and March of 1991, when the Gulf war was about to end.

interbank market.[6] As more banks returned to the interbank market, however, the aggregate level of excess reserves steadily declined. This trend continued until the Lehman shock of September 2008, when smaller banks as well as large ones sharply increased excess reserves. In the empirical analysis below, we set the excess reserve value to zero for those months leading up to Lehman, as if banks either held the idle cash in the bank vault or converted it into some other form of short-term central bank liabilities. On the other hand, we view the positive excess reserves from September 2008 until the arrival of the next zero-rate period as representing demand and leave the excess reserve value as is.

Having argued that excess reserves are demand-determined when the net policy rate is above a critical rate and otherwise supply-determined, we are ready to state our definition of the zero-rate regime: we say that the *zero-rate regime* is in place if and only if the net policy rate $r - \bar{r}$ is below the critical rate of 0.05%. Since there are no incidents of near-zero excess reserves when the net rate is below the critical rate (see Figure 2b), the zero-rate regime is synonymous with QE (quantitative easing). For this reason we will use the term “the zero-rate regime" and “QE" interchangeably. Under our definition, there are three periods of the zero-rate/QE regime in Japan, indicated by the shades in Figure 1. They are:

QE1: March 1999 - July 2000,

QE2: March 2001 - June 2006,

QE3: December 2008 to date.

Our QE dating, which is based solely on the net policy rate, agrees with announced

[6] A breakdown of excess reserves by type of financial institutions since 2005, available from the BOJ’s homepage, shows that large banks quickly reduced their excess reserves after the zero-rate policy was terminated in July 2006 while other banks (regional banks, foreign banks, and trust banks) were slow to adjust. The average of excess reserves for July 2006 - August 2008 is only 0.1% of the average for January 2005 - June 2006 for large banks and 5.4% for other banks. In order to exploit the arbitrage opportunity presented by the positive interbank rates, banks need to train their employees afresh. The reason commonly cited for the slow adjustment (see, e.g., Kato (2010)) is that medium- to small-scale banks, after several years of near-zero overnight rates, didn’t find it profitable to incur this re-entry cost.

monetary policy changes. To substantiate this claim, we collected relevant announcements of the decisions made by the BOJ's Monetary Policy Meetings (Japanese equivalent of the U.S. FOMC, held every month and sometimes more often) in Table 1. For example, the end of our QE1 is followed by the 11 August 2000 BOJ announcement declaring the end of a zero-rate policy, and the 14 July 2006 BOJ announcement follows our QE2's end. The 19 March 2001 announcement marks the start of our QE2. The only discrepancy between our QE darting and the BOJ accouncements is the start of QE1. The 12 February 1999 BOJ announcement, which is to guide the policy rate as low as possible, is more than one month before the start of our QE1 (whose first month is the March 1999 reserve maintenance period). It took a while for the BOJ to lower the policy rate averaged over a reserve maintenance period below 0.05%.[7]

The Exit Condition

Several authors have noted that the BOJ's zero-interest rate policy is a combination of a zero policy rate and a stated commitment to a condition about inflation for exiting from the zero-rate regime.[8] Indeed, the BOJ statements collected in Table 1 indicate that during our three zero-rate/QE periods, the BOJ repeatedly expressed its commitment to the exit condition stated in terms of the year-on-year (i.e., 12-month) CPI (Consumer Price Index) inflation rate. For example, during QE1's very first reserve maintenance period (March 16, 1999 - April 15, 1999), the BOJ governor pledged to continue the zero rate "until the deflationary concern is dispelled" (see the 13 April 1999 announcement in the table). To be sure, the BOJ during the first twelve months of QE3 did not publicly mention the exit condition, until December 18, 2009. However, as Ueda (2012), a former BOJ board member, writes about this period: "At that time some observers thought that the BOJ was trying to target the lower end of the understanding of price stability, which was 0-2%." (Ueda (2012, p. 6)) We will assume in our SVAR analysis that the exit condition was in place during this episode as well.

[7] The net policy rate for February 1999 (which is the average over February 16 - March 15) was 0.075%. If we chose the critical rate to be this rate rather than 0.05%, we would have included February 1999 in the first zero-rate period, with a total zero-rate months increasing by one, from 130 to 131.

[8] See, e.g., Okina and Shiratsuka (2004), Ito (2009), and Ueda (2012).

The last several months of QE2 (ending in June 2006) requires some discussion. Table 2 has data for those and surrounding months. The 9 March 2006 announcement declared that the exit condition was now satisfied. However, the actual exit from the zero-rate regime did not take place until July 2006. To interpret this episode, we note that the year-on-year CPI inflation rate (excluding fresh food) for March 2006 was significantly above 0%, about 0.5%, if the CPI base year is 2000, but 0.1% (as shown in the table) if the base year is 2005. The 2005 CPI series was made public in August 2006. We assume that the BOJ postponed the exit until July because it became aware that inflation with the 2005 CPI series would be substantially below inflation with the 2000 CPI series.

3 The Regime-Switching SVAR

This section presents our four-variable SVAR (structural vector autoregression). A more formal exposition of the model is in Appendix 2.

The Standard Three-Variable SVAR

As a point of departure, consider the standard three-variable SVAR in the review paper by Stock and Watson (2001). The three variables are the monthly inflation rate from month $t-1$ to t (p_t), the output gap (x_t), and the policy rate (r_t).[9] The inflation and output equations are reduced-form equations where the regressors are (the constant and) lagged values of all three variables. The third equation is the Taylor rule that relates the policy rate to the contemporaneous values of the year-on-year inflation rate and the output gap. The error term in this policy rate equation is assumed to be uncorrelated with the errors in the reduced-form equations. This error covariance structure, standard in the structural VAR literature (see Christiano, Eichenbaum, and Evans (1999)), is a plausible restriction to make, given that our measure of the policy rate for the month is the average over the reserve maintenance period from the 16th of the month to the 15th of the

[9] In Stock and Watson (2001), the three variables are inflation, the unemployment rate, and the policy rate. We have replaced the unemployment rate by the output gap, because Okun's law does not seem to apply to Japan. The sampling frequency in Stock and Watson (2001) is a quarter.

next month.

As is standard in the literature (see, e.g., Clarida *etl. al.* (1998)), we consider the Taylor rule with interest rate smoothing. That is,

$$\text{(Taylor rule)} \qquad r_t = \rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}, \;\; r_t^* \equiv \alpha_r^* + \underset{(1\times 2)}{\boldsymbol{\beta}_r^{*\prime}} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix}, \;\; v_{rt} \sim \mathcal{N}(0, \sigma_r^2). \qquad (3.1)$$

Here, π_t, defined as $\pi_t \equiv \frac{1}{12}(p_t + \cdots + p_{t-11})$, is the year-on-year inflation rate over the past 12 months. If the adjustment speed parameter ρ_r equals unity, then this equation reduces to $r_t = r_t^* + v_{rt}$. We will call r_t^* the *desired Taylor rate*. In Taylor's (1993) original formulation, the vector of inflation and output coefficients $\boldsymbol{\beta}_r^*$ is $(1.5, 0.5)$, and the constant term α_r^* equals 1%, which is the difference between the constant equilibrium real interest rate of 2% and half times the target inflation rate of 2%.

Introducing Regimes

The three-variable SVAR just described does not take into account the zero lower bound on the policy rate. Given the interest rate $\bar{r}_t$ (≥ 0) paid on reserves, the lower bound is not zero but $\bar{r}_t$. The Taylor rule with the lower bound, which we call the *censored Taylor rule*, is

$$\text{(censored Taylor rule)} \quad r_t = \begin{cases} \underbrace{\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt},}_{\text{shadow Taylor rate}} \;\; v_{rt} \sim \mathcal{N}(0, \sigma_r^2) & \text{if } \rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt} > \bar{r}_t, \\ \bar{r}_t & \text{otherwise.} \end{cases} \qquad (3.2)$$

(That is, $r_t = \max[\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}, \bar{r}_t]$.) Now $\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}$ is a *shadow rate*, not necessarily equal to the actual policy rate.

It will turn out useful to rewrite this in the following equivalent way. Define the monetary policy regime indicator s_t by

$$s_t = \begin{cases} \text{P} & \text{if } \underbrace{\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}} > \bar{r}_t, \\ \text{Z} & \text{otherwise.} \end{cases} \qquad (3.3)$$

Then the censored Taylor rule can be written as

$$
\text{(censored Taylor rule)} \qquad r_t = \begin{cases} \underbrace{\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}}, \ \ v_{rt} \sim \mathcal{N}(0, \sigma_r^2) & \text{if } s_t = \text{P}, \\ \bar{r}_t & \text{if } s_t = \text{Z}. \end{cases} \tag{3.4}
$$

Note that $r_t - \bar{r}_t = 0$ if and only if $s_t = \text{Z}$. Thus, consistent with how we identified the regime in the previous section, we have $s_t = \text{P}$ (call it the *normal regime*) if the net policy rate $r_t - \bar{r}_t$ is positive and $s_t = \text{Z}$ (the *zero-rate regime*) if the rate is zero. An outside observer can tell, without observing the shadow Taylor rate, whether the regime is P or Z.

The Exit Condition

We have thus obtained a simple regime-switching three-variable SVAR by replacing the Taylor rule by its censored version. We expand this model to capture the two aspects of the zero-rate regime discussed in the previous section. One is the exit condition, the additional condition needed to end QE when the shadow rate $\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}$ has turned positive. As was documented in the previous section, that condition set by the BOJ is that the year-on-year inflation rate be above some threshold. We allow the threshold to be time-varying. More formally, we retain the censored Taylor rule (3.4) but modify (3.3) as follows.

$$
\text{If } \ s_{t-1} = \text{Z}, \ \ s_t = \begin{cases} \text{P} \quad \text{if} \ \ \underbrace{\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}} > \bar{r}_t \ \text{ and } \ \pi_t > \underbrace{\overline{\pi} + v_{\overline{\pi}t}}_{\text{period } t \text{ threshold}}, \ \ v_{\overline{\pi}t} \sim \mathcal{N}(0, \sigma_{\overline{\pi}}^2), \\ \text{Z} \quad \text{otherwise.} \end{cases} \tag{3.5}
$$

If $s_{t-1} = \text{P}$, the inflation exit condition is mute and the central bank picks the current regime s_t by (3.3). We assume that the stochastic component of the threshold ($v_{\overline{\pi}t}$) is i.i.d. over time.[10] It is still the case that $r_t - \bar{r}_t = 0$ if and only if $s_t = \text{Z}$, regardless of whether $s_{t-1} = \text{P}$ or Z. Thus an outside observer can tell the current monetary policy regime just by looking at the net policy rate: $s_t = \text{P}$ if $r_t - \bar{r}_t > 0$ and $s_t = \text{Z}$ if $r_t - \bar{r}_t = 0$.

[10] If we introduced serial correlation by allowing $v_{\overline{\pi}t}$ to be the AR(1) (the first-order autoregressive process) for example, we would have to deal with an unobservable state variable (which is $v_{\overline{\pi},t-1}$ for the AR(1) case) appearing only in an inequality. The usual filtering technique would not be applicable.

Adding m to the System

The second extension of the model is to add the excess reserve rate m_t (defined, recall, as the log of actual to required reserve ratio) to the system. This variable, while demand-determined in the normal regime P, becomes a monetary policy instrument in the zero-rate/QE regime Z. In either regime, it is a censored variable because excess reserves cannot be negative. If m_{dt} and m_{st} are (underlying) demand and supply of excess reserves, the actual m_t is determined as

$$m_t = \begin{cases} \max\left[m_{dt}, 0\right], & \text{if } s_t = \text{P}, \\ \max\left[m_{st}, 0\right], & \text{if } s_t = \text{Z}. \end{cases} \tag{3.6}$$

Our specification of m_{st} is analogous to the policy-rate Taylor rule and in the spirit of the McCallum rule (McCallum (1988)). That is, it is allowed to depend on the current value of inflation and output with partial adjustment:

$$\text{(excess reserve supply)} \qquad m_{st} \equiv \alpha_s + \underset{(1\times 2)}{\boldsymbol{\beta}_s{}'} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} + \gamma_s m_{t-1} + v_{st}, \;\; v_{st} \sim \mathcal{N}\left(0, \sigma_s^2\right). \tag{3.7}$$

The speed of adjustment is $1 - \gamma_s$. We expect the inflation (π_t) and output (x_t) coefficients to be negative, i.e., $\boldsymbol{\beta}_s < \mathbf{0}$, since the central bank would increase excess reserves when deflation worsens or output declines.

Regarding the excess reserve demand m_{dt}, we can leave it unspecified for now because zero excess reserves under P will be assumed in the IR (impulse response) and counter-factual analyses of Section 5. It will be shown in Section 6 that results are little affected when the demand for excess reserves is turned on.

Taking Lucas Critique into Account

Thus, the central bank sets the policy rate under the normal regime and the excess reserve level under the zero-rate/QE regime. Since the policy rule is different between the two regimes, the Lucas critique implies that the reduced-form equations describing inflation and output dynamics can shift with the regime. If the private sector in period t sets (p_t, x_t) in full anticipation of the period's regime to be chosen by the central bank, the period t reduced form should depend on the date t regime. Since we view this to be a very remote possibility, we assume that the reduced-form coefficients and error variance and covariances in period t depend, if at all, on the *lagged* regime s_{t-1}.

To Recapitulate

This completes our exposition of the regime-switching SVAR on four variables, p_t (monthly inflation), x_t (the output gap), r_t (policy rate), and m_t (the excess reserve rate). The underlying sequence of events leading up to the determination of the two policy instruments (r_t, m_t) can be described as follows. At the beginning of period t and given the previous period's regime s_{t-1}, nature draws two reduced-form errors, one for inflation and the other for output, from a bivariate distribution. The error variance and covariance and the reduced-form coefficients may depend on s_{t-1}. This determines (p_t, x_t) and hence the 12-month inflation rate $\pi_t \equiv \frac{1}{12}(p_t + \cdots + p_{t-11})$. The central bank then draws three policy shocks $(v_{rt}, v_{\overline{\pi}t}, v_{st})$ from $\mathcal{N}(\underset{(3\times1)}{\mathbf{0}}, \begin{bmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_{\overline{\pi}}^2 & 0 \\ 0 & 0 & \sigma_s^2 \end{bmatrix})$. It can now calculate: $\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}$ (the shadow Taylor rate given in (3.1)), $\overline{\pi} + v_{\overline{\pi}t}$ (the inflation threshold shown in (3.5)), and m_{st} (excess reserve supply, given in (3.7)). Suppose the previous regime was the normal regime (so $s_{t-1} = \text{P}$). Then the bank picks $s_t = \text{P}$ if $\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt} > \bar{r}_t$, and $s_t = \text{Z}$ otherwise. Suppose, on the other hand, that $s_{t-1} = \text{Z}$. Then the bank terminates the zero-rate/QE regime and picks $s_t = \text{P}$ only if $\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt} > \bar{r}_t \textit{ and } \pi_t > \overline{\pi} + v_{\overline{\pi}t}$. If $s_t = \text{P}$, the bank sets r_t to the shadow rate and the market sets m_t to 0; if $s_t = \text{Z}$, the bank sets r_t at $\bar{r}_t$ and m_t at $\max[m_{st}, 0]$.

The model's variables are $(s_t, \mathbf{y}_t)$ with $\mathbf{y}_t \equiv (p_t, x_t, r_t, m_t)$. The model provides a mapping from $(s_t, \mathbf{y}_t, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10})$ and date $t+1$ shocks (consisting of the reduced-form shocks and the policy shocks $(v_{r,t+1}, v_{\overline{\pi},t+1}, v_{s,t+1})$) to $(s_{t+1}, \mathbf{y}_{t+1})$. Ten lags are needed (even if the inflation and output reduced form does not involve that many lags) because the Taylor rule and the reserve supply in period $t+1$ involve the 12-month inflation rate $\pi_{t+1} = \frac{1}{12}(p_{t+1} + \cdots + p_{t-10})$. We note, for later reference, that the model can be expressed as a conditional density of $(s_{t+1}, \mathbf{y}_{t+1})$ given $(s_t, \mathbf{y}_t, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10})$.

4 Estimating the Model

This section has three parts. It summarizes the derivation in Appendix 2 of the model's likelihood function, and the data description of Appendix 1, followed by a discussion of the estimation results.

The Likelihood Function (Summary of Appendix 2)

Were it not for regime switching, it would be quite straightforward to estimate the model because of its block-recursive structure. As is well known, the regressors in each equation are predetermined, so the ML (maximum likelihood) estimator is OLS (ordinary least squares). With regime switching, the regressors are still predetermined, but regime endogeneity needs to be taken into account as described below.

Thanks to the block-recursive structure, the model's likelihood function has the convenient property of additive separability in a partition of the parameter vector, so the ML estimator of each subset of parameters can be obtained by maximizing the corresponding part of the log likelihood function. More specifically, the log likelihood can be written as

$$\text{log likelihood} = L_A(\boldsymbol{\theta}_A) + L_B(\boldsymbol{\theta}_B) + L_C(\boldsymbol{\theta}_C), \tag{4.1}$$

where $(\boldsymbol{\theta}_A, \boldsymbol{\theta}_B, \boldsymbol{\theta}_C)$ form the model's parameter vector.[11] The first subset of parameters, $\boldsymbol{\theta}_A$, is the reduced-form parameters for inflation and output. Because we allow the reduced form to depend on the (lagged) regime, the parameter vector $\boldsymbol{\theta}_A$ consists of two sets of parameters, one for P (the normal regime) and the other for Z (the zero-rate/QE regime). The second subset, $\boldsymbol{\theta}_B$, is the parameters of the Taylor rule with the exit condition appearing in (3.1) and (3.5). The third subset, $\boldsymbol{\theta}_C$, describe the excess reserve supply functions (3.7). More precisely,

$$\boldsymbol{\theta}_B = \left(\alpha_r^*, \underset{(2\times1)}{\boldsymbol{\beta}_r^*}, \rho_r, \sigma_r, \overline{\pi}, \sigma_{\overline{\pi}} \right) \text{ (7 parameters)}, \quad \boldsymbol{\theta}_C = \left(\alpha_s, \underset{(2\times1)}{\boldsymbol{\beta}_s}, \gamma_s, \sigma_s \right) \text{ (5 parameters)}.$$

The first term, $L_A(\boldsymbol{\theta}_A)$, being the log likelihood for the reduced-form for inflation and output, is entirely standard, with the ML estimator of $\boldsymbol{\theta}_A$ given by OLS. That is, the

[11] If the money demand shock is taken into account, there is an additional term, $L_D(\boldsymbol{\theta}_D)$, that depends only on the parameter vector $\boldsymbol{\theta}_D$ describing the demand for excess reserves. See Appendix 2.

reduced-form parameters for regime P can be obtained by OLS on the subsample for which the lagged regime s_{t-1} is P, and the same for Z. There is no need to correct for regime endogeneity because the reduced form errors for period t is independent of the *lagged* regime. Regarding the reserve supply parameters $\boldsymbol{\theta}_C$, which are estimated on subsample Z (i.e., those observations with $s_t = \text{Z}$, consisting of QE1, QE2, and QE3), the censoring implicit in the "max" operator in (3.6) calls for Tobit with m_t as the limited dependent variable. However, since there are no observations for which m_t is zero on subsample Z (which makes the zero-rate regime synonymous with QE as noted in Section 2), Tobit reduces to OLS. There is no need to correct for regime endogeneity because the current regime s_t is independent of the error term of the excess reserve supply equation.

Regime endogeneity *is* an issue for the second part $L_B(\boldsymbol{\theta}_B)$, because the shocks in the Taylor rule and the exit condition, $(v_{rt}, v_{\overline{\pi}t})$, affect regime evolution. If the exit condition were absent so that the censored Taylor rule (3.2) were applicable, then the ML estimator of $\boldsymbol{\theta}_B$ that controls for regime endogeneity would be Tobit on the whole sample composed of P and Z; subsample P, on which $r_t > \overline{r}_t$, provides "non-limit observations" while subsample Z, on which $r_t = \overline{r}_t$, is "limit observations". With the exit condition, the ML estimation is slightly more complicated because whether a given observation t is a limit observation or not is affected by the exit condition as well as the lower bound.

The Data (Summary of Appendix 1)

The model's variables are p_t (monthly inflation), x_t (output gap), r_t (the policy rate), and m_t (the excess reserve rate).

For the output measure underlying x_t, we desire a monthly series whose quarterly averages are quarterly GDP from the national accounts. The coincidental monthly series we use for monthly interpolation, which is available only since 1988, is a monthly index of all-industry production (which covers a much wider range of industries than the Index of Industrial Production) compiled by the METI (Ministry of Economy, Trade, and Industry of the Japanese government). For potential GDP, we use the official estimate by the Cabinet Office of the Japanese government (the Japanese equivalent of the U.S. Bureau of Economic Analysis). It is based on the Cobb-Douglas production function with the HP (Hodrick-Prescott) filtered Solow residual. The output gap is defined as 100 times the log difference between actual and potential GDP. Actual GDP and the official estimate of potential GDP are in Figure 3a. It shows the well-documented decline in the trend growth rate that occurred in the early 1990s, often described as the (ongoing) “lost decade(s)". It also shows that the output gap has rarely been above zero during the lost decades.[12]

The excess reserve rate m_t is defined as 100 times the log of the ratio of actual to required reserves. Data on actual and required reserves over monthly reserve maintenance periods are available from the BOJ’s website, way back to as early as 1960. We have argued in Section 2 that the positive excess reserves between QE spells (except September - November 2008) do not represent precautionary demand. For those months we set $m_t = 0$. Figure 3b has m_t since 1988. There is a spike during QE1 (March 1999-July 2000) in December 1999 when the BOJ provided ample liquidity to deal with the Y2K problem.

The policy rate r_t for month t is the average of daily values, over the reserve maintenance period from the 16th day of month t to the 15th day of month $t + 1$, of the overnight “Call" (i.e., interbank) rate. We ignore the variations of $r_t - \bar{r}_t$ within the 5 basis point band (shown in Figure 2) by setting $r_t - \bar{r}_t$ to zero for all observations in subsample Z.

The inflation rate is constructed from the CPI (consumer price index). The relevant CPI component is the so-called “core" CPI (the CPI excluding fresh food), which, as documented in Table 1, is the price index most often mentioned in BOJ announcements. (Confusingly, the core CPI in the U.S. sense, which excludes food and energy, is called the “core-core" CPI.) We made some adjustments to remove the effect of the increase in the consumption tax rate in 1989 and 1997 before performing a seasonal adjustment. We also adjusted for large movements in the

[12] We will show in Section 6 that most of the results, to be shown in Section 5 for the current choice of the output gap measure, remains valid if the HP-filtered log GDP is used as potential GDP.

energy component of the CPI between November 2007 and May 2009.[13] The monthly inflation rate p_t is at annual rates, 1200 times the log difference between month t and month $t-1$ values of the adjusted CPI. The year-on-year (i.e., 12-month) inflation rate π_t is calculated as 100 times the log difference between month t and $t-12$ values of the CPI, so $\pi_t = \frac{1}{12}(p_t + \cdots + p_{t-11})$. Figure 3c has π_t since 1970 along with the policy rate r_t.

Simple statistics of the relevant variables are in Table 3.

Parameter Estimates

Having described the estimation method and the data, we are ready to report parameter estimates. We start with $\boldsymbol{\theta}_B$.

Taylor rule with exit condition (θ_B).

Most existing estimates of the Taylor rule for Japan end the sample at 1995 because the policy rate shows very little movements near the lower bound since then.[14] In our ML estimation, which can incorporate the lower bound on the policy rate, the sample period can include all the many recent months of very low policy rates. On the other hand, the starting month is January 1988 at the earliest because that is when our monthly output series starts.

Before reporting our estimates, we mention two issues that turned out to affect the Taylor rule estimates.

- (Choice of starting month) If the sample starts at January 1988, the estimated speed of adjustment (ρ_r in (3.1)) is negative. This is probably because the equilibrium real interest rate,

[13] The "core" CPI (the CPI excluding fresh food) monthly inflation rate is set equal to that given by the "core-core" CPI (the CPI excluding food and energy) for those months. This is the only period during which the two CPI measures give substantially different inflation rates, see Appendix Figure 1. It appears that the large movement in the "core" CPI was discounted by the BOJ. The monetary policy announcement of August 19, 2008 (http://www.boj.or.jp/en/announcements/release_2008/k080819.pdf), which stated that the policy rate would remain at around 50 basis points, has the following passage: "The CPI inflation rate (excluding fresh food) is currently around 2 percent, highest since the first half of 1990s, due to increased prices of petroleum products and food."

[14] See Miyazawa (2010) for a survey.

which is assumed constant in our Taylor rule, declined during the transition period to the lost decades of low growth.[15] For this reason we decided to take the sample period to be the lost decades starting in January 1992.

- (The banking crisis dummy) Between September 1995 and July 1998, the policy rate remained low despite improvements in inflation and output. We surmise that the BOJ refrained from raising the policy rate to help alleviate the Japanese banking crisis of the late 1990s.[16] We view this as a temporary deviation from the Taylor rule and include a dummy for the period in the equation. Accordingly, the parameter vector $\boldsymbol{\theta}_B$ has now 8 parameters with the banking crisis dummy coefficient added.

Table 4 reports the ML estimate of the Taylor rule for the sample period of 1992-2012. The estimated speed of adjustment per month is 7.8%. The inflation and output coefficients in the desired Taylor rate ($\boldsymbol{\beta}_r^*$ in (3.1)) are estimated to be $(1.01, 0.04)$. The mean of the time-varying threshold inflation rate affecting the exit condition is mere 0.38% per year. As expected, the banking crisis dummy has a negative sign — the policy rate would have been higher on average by 28 basis points were it not for the banking crisis. The desired Taylor rate r_t^* implied by the ML estimate is shown in the red line in Figure 4. The portion indicated by the dotted line in the figure is the desired Taylor rate extrapolated back to 1988. The persistent and growing gap between the desired Taylor rate and the policy rate before 1992, which is responsible for the negative speed of adjustment when the sample period includes 1988-91, is probably due to higher real rates before the growth slowdown

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[15] For example, Hayashi and Prescott (2002) document that both the TFP (total factor productivity) and the rate of return on capital declined in the early 1990s. The Taylor rule in Braun and Waki (2006) allows the equilibrium real rate to vary with the TFP growth.

[16] The Bank of Japan started releasing minutes of the monetary policy meetings only since March 1998 (the 3 March 1998 release is about the meeting on January 16, 1998), so it is not possible for outside observers to substantiate the claim. However, those released minutes of the early part of 1998 do include frequent mentions of the financial system. For example, the minutes of the 16 January 1998 meeting has the following passage: "... a majority of the members commented that the sufficient provision of liquidity would contribute to stabilizing the financial system and to improving household and depositor sentiment."

It is instructive to compare the ML estimate, which incorporates the exit condition, to the Tobit estimate, which doesn't. Focus, for example, on QE2 (March 2001 - June 2006). The ML desired Taylor rate (which is proportional to the shadow Taylor rate because the lagged policy rate is zero) turned positive in the middle of the period. Yet the QE was not terminated. This is of course due to the exit condition, but Tobit, not being informed of the condition, takes it to be interest rate smoothing. Hence the Tobit estimate of the speed of adjustment is lower, at $\rho_r = 3.8\%$ (not shown in the table).

Excess reserve supply equation (θ_C).

We have already noted that the ML estimator can be obtained by regressing m_t on the constant, π_t, x_t, and m_{t-1} on subsample Z consisting of QE1, QE2, and QE3. As might have been clear from Figure 3b, however, m_t is much less persistent during QE1, with the estimated lagged m coefficient (not reported) of -0.20 (with the December 1999 Y2K spike in m dummied out). We thus estimate the equation on the pooled sample composed of QE2 and QE3 only. The results are in Table 5. Both the inflation and output coefficients pick up the expected sign.

Inflation and output reduced-form equations (θ_A).

As mentioned above, the ML estimate of the reduced form can be obtained by OLS on two separate subsamples, “lagged" subsample P (i.e., those months with $s_{t-1} = \text{P}$) and lagged subsample Z (with $s_{t-1} = \text{Z}$). The BIC (Baysian information criterion) instructs us to set the lag length to one in both the inflation and output equations and on both subsamples.[17]

Table 6 shows the estimates. First consider lagged subsample P. We take January 1992 as the first month (as in the Taylor rule estimation). This is because, for the output equation but not for the inflation equation, if the sample period includes the earlier months from 1988 and if the break date is January 1992, the Chow test detects a structural change (p-value is 0.0%). We include the banking crisis dummy in the set of regressors because the Lucas critique implies that the deviation from the Taylor rule during the bank crisis period could have shifted the reduced form equations. We exclude lagged m because it is essentially zero during regime P until the Lehman shock of September 2008. Lagged subsample P extends to December 2008 (the last t for

[17] In Section 6, we will set the lag length according to the AIC (Akaike information criterion).

which $s_{t-1} = \text{P}$ — recall that QE3 starts in that month), so there is some movement in m_{t-1} during the last four months of the subsample. We view this movement as proxying the Lehman shock component of the error term. Indeed, m_{t-1} when included picks up a negative and significant coefficient in the output equation, with the coefficients of the other regressors being affected very little.

There are two notable features about the inflation equation on lagged subsample P. First, inflation persistence is very low as indicated by the small lagged p coefficient of 0.10. Second, the lagged r coefficient is positive, large, and highly significant. A 1 percentage point cut in the policy rate *lowers* inflation by about 0.4 percentage points in the next period.[18] This will be seen as the primal source of the price puzzle in the next section's estimated IR (impulse response) of p to r.

Turn now to lagged subsample Z. Since, as noted above, the coefficients of the reserve supply equation differ between QE1 and QE2&QE3, the Lucas critique implies the reduced-form coefficients during QE1 could be different. For this reason the sample excludes QE1 and combines QE2 and QE3. The regressors include r_{t-1} because, although it is constant in each QE spell, it differs across spells ($r_{t-1} = 0$ during QE1 and QE2, $r_{t-1} = \bar{r} = 0.1\%$ during QE3 — recall that $\bar{r}$ (the rate paid on reserves) was raised from 0% to 0.1% in November 2008). The positive lagged m coefficients imply that inflation and output rise as excess reserves are increased. The effect of inflation is small and insignificant, though. The coefficient of 0.0052 in the output equation implies that a 100 percentage point increase in m raises the output gap by 0.52 ($= 0.0052 \times 100$) percentage points in the next period. We note for later reference that the intercept in the inflation equation is not well determined, with a t-value of only 0.3 on lagged subsample Z and −0.9 on subsample P.

[18] The positive r_{t-1} coefficient may be due to the fact that r_{t-1} is the average over the period of the 16th of month $t-1$ and the 15th of month t. If the central bank can respond to price increases of the month by raising the policy rate in the first 15 days of the month, there will be a positive correlation between p_t and r_{t-1}. To check this, we replaced r_{t-1} by r_{t-2} and found a very similar coefficient estimate (the estimate is 0.38, $t = 3.8$).

5 Impulse Response (IR) and Other Counter-Factual Analyses

With the estimates of our model parameters in hand, we turn to the IR (impulse responses) and other counter-factual analyses. For linear models, the IR analysis is well known since Sims (1980). Our model, however, is nonlinear because the dynamics depends on the regime and also because of the nonnegativity constraint on excess reserves. In this section, we state the definition of IRs for our model and calculate responses of inflation and output to changes in monetary policy variables including the regime.

IRs for Nonlinear Processes in General

Consider for a moment a general strictly stationary process $\underset{(n\times 1)}{\mathbf{y}_t} \equiv (y_{1t}, y_{2t}, ..., y_{nt})$. Gallant, Rossi, and Tauchen (1993, particularly pp. 876-877) proposed to define an IR as the difference in conditional expectations under two alternative possible histories with one history being a perturbation of the other. The IR of the i-th variable to the j-th variable k-period ahead is defined as

$$\begin{aligned} &\mathrm{E}\Big(y_{i,t+k} \mid \underbrace{(y_{1t}, ..., y_{j-1,t}, y_{jt} + \delta, y^{(a)}_{j+1,t}, ..., y^{(a)}_{n-1,t}, y^{(a)}_{nt})}_{\mathbf{y}_t \text{ in the alternative history}}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, ... \Big) \\ &\quad - \mathrm{E}\Big(y_{i,t+k} \mid \underbrace{(y_{1t}, ..., y_{j-1,t}, y_{jt}, y^{(b)}_{j+1,t}, ..., y^{(b)}_{n-1,t}, y^{(b)}_{nt})}_{\mathbf{y}_t \text{ in the baseline history}}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, ... \Big), \;\; k = 1, 2, ..., \end{aligned} \tag{5.1}$$

where δ is the size of perturbation, $y^{(a)}_{\ell,t}$ $(\ell = j+1, ..., n)$ is the conditional expectation of $y_{\ell,t}$ conditional on the alternative history up to and including $y_{jt} + \delta$, and $y^{(b)}_{\ell,t}$ similarly is the expectation conditional on the baseline history up to and including y_{jt}. These expected values are “filled in" for the remaining elements $(\ell = j+1, ..., n)$ of $\mathbf{y}_t$ to trace out the effects of the shock to the j-th variable through the contemporaneous correlation among the variables.[19] This definition, when applied to linear processes, reduces to the orthogonalized IR of variable i to variable j, which for (block) recursive linear VARs is the standard IR.[20]

Adaptation to Our Model

In the model of Section 3, the model variables are $(s_t, \mathbf{y}_t)$ where $\mathbf{y}_t \equiv (p_t, x_t, r_t, m_t)$. As we noted at the end of Section 3, the model provides a conditional distribution of $(s_{t+1}, \mathbf{y}_{t+1})$ given $(s_t, \mathbf{y}_t, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10})$ (ten lags are needed because the 12-month inflation in $t+1$ depends on $(p_{t+1}, p_t, ..., p_{t-10})$ where p_t is the monthly inflation rate from month $t-1$ to t). So, what needs to be included in the conditioning set is, for $\mathbf{y}$, only its current value and ten lags, and for s, only its current value. The adaptation of the IR defined above to our model is easy to see for the last variable of the system, m_t.

m-IR (IRs to Changes in m)

Since the central bank has control over m only under the zero-rate regime, we assume $s_t = \mathrm{Z}$ and

[19] It may apear that a more natural definition is to do away with the filling-in. That is, we could alternatively define an IR as

$$\mathrm{E}\Big(y_{i,t+k} \mid (y_{1t}, ..., y_{j-1,t}, y_{jt} + \delta), \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, ... \Big) - \mathrm{E}\Big(y_{i,t+k} \mid (y_{1t}, ..., y_{jt}), \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, ... \Big).$$

The two definitions are equivalent if the process $\{\mathbf{y}_t\}$ is linear, but not necessarily so with nonlinear processes. We chose the definition (5.1) for two reasons (if you are interested). First, the difference is very minor for our model. Second, there is a subtlety in the above alternative definition when applied to Markov processes. To illustrate, consider a bivariate process with the conditional distribution of $\mathbf{y}_{t+1}$ that depends at most on two lags $(\mathbf{y}_t, \mathbf{y}_{t-1})$. In the IR of variable i to variable 1, look at the conditional expectation under the baseline history for example. In definition (5.1), it is: $\mathrm{E}\Big(y_{i,t+k} | (y_{1t}, y_{2t}^{(b)}), \mathbf{y}_{t-1}\Big)$. In the alternative definition, the conditioning information must be $(y_{1t}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2})$, not $(y_{1t}, \mathbf{y}_{t-1})$. Otherwise the alternative definition is not equivalent to definition (5.1) for linear processes. This is because in (5.1) the expected value $y_{2t}^{(b)}$ depends on $(\mathbf{y}_{t-1}, \mathbf{y}_{t-2})$.

[20] For a proof, see Hamilton (1994, Section 11.4 (particularly equation [11.4.19]) and Section 11.6).

define the IR to a change in m (denoted as m-IR) as:[21]

$$(m\text{-IR}) \qquad \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, \underbrace{(p_t, x_t, \bar{r}_t, m_t + \delta_m),}_{\mathbf{y}_t \equiv (p_t, x_t, r_t, m_t) \text{ in the alternative history}} \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big)$$
$$- \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, \underbrace{(p_t, x_t, \bar{r}_t, m_t),}_{\mathbf{y}_t \equiv (p_t, x_t, r_t, m_t) \text{ in the baseline history}} \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big), \quad y = p, x, r, m. \tag{5.2}$$

In both the baseline and alternative histories, we set $r_t = \bar{r}_t$ because that is what is implied by the regime $s_t = \mathrm{Z}$. We calculate the conditional expectations in the definition by utilizing the model-implied conditional distribution. Two aspects of the calculation need to be mentioned:

- (Monte Carlo integration) We compute numerically the conditional expectations by drawing a large number of sample paths from the (estimated) conditional distribution and then taking the average of those simulated paths. In the estimated IRs and counter-factual simulations to be reported below, 2000 simulations are generated.

[21] Stating the definition of m-IR equivalently in terms of the shocks dated t is more complicated because, thanks to the exit condition, there are multiple prior histories with the same information indicated in the conditioning set. Nevertheless it can be done. The shocks to the system are: $(\boldsymbol{\varepsilon}_t, v_{rt}, v_{\bar{\pi}t}, v_{st})$ where $\boldsymbol{\varepsilon}_t$ is the bivariate shock to the reduced-form equations for (p_t, x_t), v_{rt} is the Taylor-rule shock in (3.4), $v_{\bar{\pi}t}$ is the stochastic component of the threshold inflation rate in (3.5), and v_{st} is the excess reserve supply shock in (3.7). Consider the information $(s_t = \mathrm{Z}, \mathbf{y}_t, ..., \mathbf{y}_{t-10})$ that conditions the conditional expectation for the baseline history (the argument below can be adapted easily to the alternative history by replacing m_t in $\mathbf{y}_t$ by $m_t + \delta_m$). There are two sets of histories up to $t-1$, $(s_{t-1}, \mathbf{y}_{t-1}, s_{t-2}, \mathbf{y}_{t-2}, ...)$, that are consistent with the same information when combined with the date t shocks. One set of histories, call history set P here, has $s_{t-1} = \mathrm{P}$ and the other set has $s_{t-1} = \mathrm{Z}$. Take history set P first. $\boldsymbol{\varepsilon}_t$ is such that the value of (p_t, x_t) in the information is implied by the reduced-form equations. Given the history up to $t-1$ and given (p_t, x_t), define $r_t^e \equiv \rho_r r_t^* + (1-\rho_r) r_{t-1}$. v_{rt} is any value that satisfies $r_t^e + v_{rt} \le \bar{r}_t$ (so that $s_t = \mathrm{Z}$). Because the exit condition is irrelevant when $s_{t-1} = \mathrm{P}$, $v_{\bar{\pi}t}$ can be any real number. v_{st} is such that the value of m_t in the information is implied by the excess reserve supply equation. Next, consider history set Z. $\boldsymbol{\varepsilon}_t$ and v_{st} are defined in the same way as in the case of history set P. A difference arises for $(v_{rt}, v_{\bar{\pi}t})$ due to the exit condition: $(v_{rt}, v_{\bar{\pi}t})$ is such that $r_t^e + v_{rt} \le \bar{r}_t$ or $\pi_t \le \bar{\pi} + v_{\bar{\pi}t}$. The conditional expectations for the baseline history in the definition of m-IR in the text do not depend on which history set is to be used. It equals the conditional expecation given history set P (with $s_{t-1} = \mathrm{P}$) and the associated $(\boldsymbol{\varepsilon}_t, v_{rt}, v_{\bar{\pi}t}, v_{st})$, which in turn equals the conditional expectation given history set Z (with $s_{t-1} = \mathrm{Z}$) and the associated $(\boldsymbol{\varepsilon}_t, v_{rt}, v_{\bar{\pi}t}, v_{st})$.

- (projected sequence of exogenous variables) There are two exogenous variables in the system: $\bar{r}$ (the rate paid on reserves) and the banking crisis dummy. Each sample path of $(s, \mathbf{y})$ from the base period t depends on the projected path from t on of those exogenous variables.[22] We assume static point expectation about the path of exogenous variables. Therefore, the projected path of $\bar{r}$ is assumed to be constant at 0% if the base period t is before November 2008 and constant at 0.1% if t is November 2008 or later. Likewise, if t is before or after the crisis period of September 1995-July 1998, then the projected path of the crisis dummy is constant at 0. This assumption would be problematic if t were during the crisis (because the crisis would not be expected to last forever). In the IR and counter-factual analyses below, we will not take the base period during the crisis, so the crisis dummy can be ignored because their value is zero.

r-IR (IRs to Changes in r)

A change in the policy rate is possible only under regime P. The IR to a policy rate change, denoted r-IR, then, is

$$(r\text{-IR}) \qquad \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{P}, \underbrace{(p_t, x_t, r_t + \delta_r, 0),}_{\mathbf{y}_t \equiv (p_t, x_t, r_t, m_t) \text{ in the alternative history}} \mathbf{y}_{t-1}, \ldots, \mathbf{y}_{t-10}\Big) - \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{P}, \underbrace{(p_t, x_t, r_t, 0),}_{\mathbf{y}_t \equiv (p_t, x_t, r_t, m_t) \text{ in the baseline history}} \mathbf{y}_{t-1}, \ldots, \mathbf{y}_{t-10}\Big), \quad y = p, x, r, m. \tag{5.3}$$

Under the assumption (to be relaxed in Section 6) of zero excess reserve demand, the excess reserve rate m is zero under P. So m_t is set to 0 in both the baseline and alternative histories.

PZ-IR (IRs to a Change in Regime from P to Z)

To define IRs to changes in the regime s_t, we would require that the regime be the only difference between the two possible histories. So set r_t to $\bar{r}_t$ in both histories because that is the rate set by

[22] Therefore, the expectations operator should have a subscript t (E_t rather than E). We won't carry this sub t for notational simplicity.

the central bank under $s_t = \text{Z}$ and set m_t to 0, the value of m under P. Thus,[23]

$$\text{(PZ-IR)} \qquad \text{E}\Big(y_{t+k}\,|\,s_t = \text{Z}, \underbrace{(p_t, x_t, \bar{r}_t, 0),}_{\mathbf{y}_t \,\equiv\, (p_t, x_t, r_t, m_t) \text{ in the alternative history}} \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big)$$
$$- \text{E}\Big(y_{t+k}\,|\,s_t = \text{P}, \underbrace{(p_t, x_t, \bar{r}_t, 0),}_{\mathbf{y}_t \,\equiv\, (p_t, x_t, r_t, m_t) \text{ in the baseline history}} \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big), \;\; y = p, x, r, m. \tag{5.4}$$

Some Analytics on the Impact Effect

Since our model is nonlinear, neither history independence nor the proportionality to the perturbation size holds for the IRs thus defined. The exception is the impact effect on (p, x), namely the IR at $k = 1$ (one period ahead). This is because (p_{t+1}, x_{t+1}) depends linearly on $\mathbf{y}_t$ ($\equiv (p_t, x_t, r_t, m_t)$) and the relevant state is the lagged state s_t. To provide the analytical expression for the impact effect, write the reduced-form equations for period $t + 1$ as[24]

$$\begin{bmatrix} p_{t+1} \\ x_{t+1} \end{bmatrix} = \underset{(2\times1)}{\mathbf{c}(s_t)} + \underset{(2\times1)}{\boldsymbol{\phi}_p(s_t)} p_t + \underset{(2\times1)}{\boldsymbol{\phi}_x(s_t)} x_t + \underset{(2\times1)}{\boldsymbol{\phi}_r(s_t)} r_t + \underset{(2\times1)}{\boldsymbol{\phi}_m(s_t)} m_t + \underset{(2\times1)}{\boldsymbol{\varepsilon}_{t+1}}. \tag{5.5}$$

Our estimates of the coefficients can be read off from Table 6. For example,

$$\mathbf{c}(\text{P}) = \begin{bmatrix} -0.22 \\ -0.20 \end{bmatrix}, \;\; \mathbf{c}(\text{Z}) = \begin{bmatrix} 0.16 \\ -1.22 \end{bmatrix}, \;\; \boldsymbol{\phi}_m(\text{Z}) = \begin{bmatrix} 0.0002 \\ 0.0052 \end{bmatrix}, \;\; \boldsymbol{\phi}_r(\text{P}) = \begin{bmatrix} 0.39 \\ 0.02 \end{bmatrix}, \;\; \boldsymbol{\phi}_m(\text{P}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Clearly, for the m-IRs of (5.2) and r-IRs of (5.3), the impact effect is given by

$$\begin{bmatrix} m\text{-IR of } p_{t+1} \\ m\text{-IR of } x_{t+1} \end{bmatrix} = \boldsymbol{\phi}_m(\text{Z})\,\delta_m, \;\; \begin{bmatrix} r\text{-IR of } p_{t+1} \\ r\text{-IR of } x_{t+1} \end{bmatrix} = \boldsymbol{\phi}_r(\text{P})\,\delta_r. \tag{5.6}$$

---

[23] It is true that, in our model, the policy rate r_t is greater than the rate paid on reserves $\bar{r}_t$ under P, so the baseline history in the second conditional expectation in the definition (5.4) is not possible. We can, however, make this conditional expectation well-defined as the limit as the policy rate falls arbitrarily close to $\bar{r}_t$:

$$\text{the second conditional expectation in (5.4)} \equiv \lim_{r \downarrow \bar{r}_t} \text{E}\Big(y_{t+k}\,|\,s_t = \text{P}, (p_t, x_t, r, 0), \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big).$$

[24] There is no need to include the banking crisis dummy in the reduced form because its value is zero in all the relevant simulations.

Regarding PZ-IR, since the only difference is in s_t and since $m_t = 0$, the impact effect comes from the shifts in the reduced-form coefficients of (p, x, r):

$$\begin{bmatrix} \text{PZ-IR of } p_{t+1} \\ \text{PZ-IR of } x_{t+1} \end{bmatrix} = [\mathbf{c}(\mathrm{Z}) - \mathbf{c}(\mathrm{P})] + \left[\boldsymbol{\phi}_p(\mathrm{Z}) - \boldsymbol{\phi}_p(\mathrm{P})\right] p_t + \left[\boldsymbol{\phi}_x(\mathrm{Z}) - \boldsymbol{\phi}_x(\mathrm{P})\right] x_t + \left[\boldsymbol{\phi}_r(\mathrm{Z}) - \boldsymbol{\phi}_r(\mathrm{P})\right] \bar{r}_t. \tag{5.7}$$

Estimated IRs

The IR profiles revert to the horizontal axis because the two conditional expectations, one under the baseline scenario and the other under the alternative scenario, converge to the same long-run value for each y $(= p, x, r, m)$ as the horizon k goes to infinity. The long-run expected value of (p, x, r, m) is $(-0.4, -3, 0.1, 70)$ and the long-run frequency of the zero-rate regime is about three quarters. Thus, under the observed monetary policy rule, the economy has the tendency to slip into chronic deflation.[25]

In the next several figures, we display estimated IRs with error bands computed by a Monte Carlo method. The error bands are obtained as follows. Draw a parameter vector from the estimated asymptotic distribution and do the Monte Carlo integration described above for the parameter vector.[26] Continue this until we accumulate 300 “valid" IRs.[27] Finally, pick the 84 and 16 percentiles for each horizon (so the coverage rate is 68%, corresponding to one-standard error bands).

---

[25] That the output gap remains far below 0, at −3%, is partly due to our choice of the potential GDP. If the HP filtered GDP is used for potential GDP, the long-run value of the output gap is −0.6%.

[26] Let $\mathrm{Avar}(\widehat{\boldsymbol{\theta}}_T)$ be the asymptotic variance of the estimator and let $\widehat{\mathrm{Avar}(\widehat{\boldsymbol{\theta}}_T)}$ its consistent estimator. Each draw is done by generating a random vector from $\mathcal{N}\left(\mathbf{0}, \frac{1}{T}\widehat{\mathrm{Avar}(\widehat{\boldsymbol{\theta}}_T)}\right)$ and adding the vector to $\widehat{\boldsymbol{\theta}}_T$. An alternative method, described in Gallant, Rossi, and Tauchen (1993), is to obtain a set of the parameter vector by bootstrapping. That is, use the estimated model to draw sample paths (100 in number, say) of $(s_t, \mathbf{y}_t)$ for t =January 1992 - December 2012, and then for each sample path use it as data to estimate the model parameters as described in the previous section. We did not employ this procedure because of its possible computational burden.

[27] Let $IR(i,k)$ be the k-period ahead IR of variable i and let n be the IR horizon. For each i, define $v_{1i} \equiv \sum_{k=1}^{\ell}(IR(i,k))^2$ and $v_{2i} \equiv \sum_{k=\ell+1}^{n}(IR(i,k))^2$ where ℓ is the largest integer not exceeding $0.8n$. We declare the IR “valid" if $\min_i v_{2i}/v_{1i} \le 0.1$. We set n (the IR horizon) to 120.

m-IRs

The general shape of the m-IR does not depend very much on the choice of the base period t. Figure 5a shows the m-IR for the base period of February 2004 (the peak QE month) when $m_t = 185\%$, about 6.4 (= exp(185/100)) times required reserves. The south-west panel shows the response of m, so its intercept at horizon $k = 0$ (the base period) equals the perturbation δ_m. Its size is chosen so that its ratio to the estimated standard deviation of the reserve supply shock v_{st} (which is 13.1 from Table 5) roughly equals the ratio of $-\delta_r$ (the perturbation in r-IR) to the estimated standard deviation of the policy rate shock v_{rt} (0.11 from Table 4). We will set $\delta_r = -1\%$ in the r-IRs below and $\delta_m = 100\%$.

The estimated response of the output gap (x) is shown in the north-east panel of the figure. Its impact effect (the IR at $k = 1$), by the formula given in (5.6), is about 0.52% (= 0.0052×100). Because of the persistence in the output dynamics reported in Table 6, the IR builds on the impact effect and goes up above 1% in several months. The response of monthly inflation (p) is very modest, only about 0.02% (= 0.0002×100) on impact at $k = 1$, with a very modest peak after several months. Because both output and inflation rise, regime P is more likely to occur under the alternative scenario. This is why the response of the policy rate (r) gradually rises from zero with the response of m turning negative. This also explains why the average duration from the base period of the initial regime (which is Z in both the base and alternative scenarios) is shorter under the alternative scenario with 13 months than under the base scenario with 19 months.

r-IRs

For r-IR, we wish to examine, as we did with m-IRs, expansionary monetary policies. So we take the policy rate perturbation δ_r to be negative 1 percentage point ($\delta_r = -1\%$). In order to calculate the response of a 1 percentage point cut in the policy rate, however, the base period has to be May 1995 or before, when the policy rate is above 1 percent. On the other hand, we argued in the previous section that the excess reserve supply rule during QE1 (March 1999 - July 2000) was different from the one during QE2 and QE3. Therefore, for our model, which does not allow for multiple zero-rate regimes, to be applicable, the base period cannot be before QE1.[28]

With that caveat in mind, we proceed as follows. We take the base period t to be the earliest month in the sample period, January 1992, when the policy rate, at $r_t = 5.6\%$, was comfortably above zero. Figure 5b has the profiles of r-IRs. The price puzzle emerges: the IR of p to the rate cut is *negative* and remains so for the entire horizon of 5 years. The impact effect is -0.4% by (5.6). It remains significantly negative (the error band does not include 0) for 2 to 3 years. The output effect is essentially absent. Because of the high initial policy rate of 5.6%, the system rarely switches to QE in the simulations (the average duration of the initial regime of P is about 5 to 6 years under either scenario, baseline or alternative), which explains the almost no response of m as shown in the south-east panel of the figure. Therefore, the IR would have looked similar if we had used different parameter estimates for the excess reserve supply equation and the reduced form under Z.

Counter-factual Analysis

More interesting counter-factual analyses are possible if we combine the three IRs. To illustrate, we examine the episode of the winding-down of QE2. The data on $(s_t, m_t, r_t, p_t, \pi_t, x_t)$ during the episode are in Table 2.

The last month of QE2 is June 2006 and the normal regime P resumed in July 2006. If QE2 were allowed to continue until July 2006, what difference would it have made? We can answer the question by setting t = July 2006 (when the regime was P) and taking Z as the counter-factual alternative regime. The difference we calculate, then, is

$$\mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, m_t^e), \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big) - \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{P}, (p_t, x_t, r_t, 0), \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big). \quad (5.8)$$

Thus, the perturbation occurs to not just one but three variables: r_t, m_t, and s_t. Here, m_t^e, which is the perturbation to m, is the level of excess reserves that can be expected given the history leading

[28] One way to accommodate multiple zero-rate regimes is to assume that the central bank, conditional on having chosen Z, flips a coin to choose between a “strong" zero-rate regime and a “weak" regime.

up to (p_t, x_t) and given the excess reserve supply equation:[29]

$$m_t^e \equiv \mathrm{E}\Big[\max[m_{st}, 0] \,|\, p_t, x_t, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big] = \mathrm{E}_{v_{st}}\Big[\max[m_{st}, 0] \,|\, \pi_t, x_t, m_{t-1}\Big] \quad \text{with } m_{st} \text{ given by (3.7).} \tag{5.9}$$

The estimate of this m_t^e for t = July 2006 is 43%, which is about 1.5 (= $\exp(43/100)$) times required reserves, about a quarter of the ratio (of 6.4) observed at the peak QE month of February 2002.

The estimated profile of the difference (5.8) for $y = p, x, r, m$ is in Figure 5c. The perturbations to m of $\delta_m = 43\%$ and to r of $\delta_r = -0.26\%$ ($r_t = 0.26\%$ and $\bar{r}_t = 0\%$ in July 2006) can be read off from the profiles as the value of m and r at horizon $k = 0$. Surprisingly, despite the increase in m, both inflation and output *decline* (the inflation rate rises to 0.4% on impact, but it is quickly followed by deflation). The output gap declines by 0.7% on impact (at $k = 1$) and in several months reaches a trough of about −1.5%.

To see why continuing QE2 would have been contractionary (namely, terminating QE2 was expansionary), decompose the (overall) difference (5.8) as the sum of m-IR, PZ-IR, and r-IR:

$$\begin{aligned}
(5.8) = {} & \underbrace{\mathrm{E}\Big(y_{t+k} \,|\, s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, m_t^e), ... \Big) - \mathrm{E}\Big(y_{t+k} \,|\, s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, 0), ... \Big)}_{m\text{-IR}} \\
& + \underbrace{\mathrm{E}\Big(y_{t+k} \,|\, s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, 0), ... \Big) - \mathrm{E}\Big(y_{t+k} \,|\, s_t = \mathrm{P}, (p_t, x_t, \bar{r}_t, 0), ... \Big)}_{\text{PZ-IR}} \\
& + \underbrace{\mathrm{E}\Big(y_{t+k} \,|\, s_t = \mathrm{P}, (p_t, x_t, \bar{r}_t, 0), ... \Big) - \mathrm{E}\Big(y_{t+k} \,|\, s_t = \mathrm{P}, (p_t, x_t, r_t, 0), ... \Big)}_{r\text{-IR}}.
\end{aligned} \tag{5.10}$$

By the formulas (5.6) and (5.7), the overall impact effect (at $k = 1$) on (p, x) of (0.4%, −0.7%)

[29] This conditional expectation can be computed analytically by one of the standard Tobit formulas. Consider the Tobit model $y = \max[\mathbf{x}'\boldsymbol{\beta} + u, c]$ where $u \sim \mathcal{N}(0, \sigma)$. We have: $\mathrm{E}(y|\mathbf{x}) = [1 - \Phi(v)] \times [\mathbf{x}'\boldsymbol{\beta} + \sigma\lambda(v)] + \Phi(v)c$, where $v \equiv (c - \mathbf{x}'\boldsymbol{\beta})/\sigma$ and $\lambda(v) \equiv \phi(v)/[1 - \Phi(v)]$. Here, ϕ and Φ are the pdf and cdf of the standard normal distribution.

can be decomposed as (noting $p_t = 1.3\%$ and $x_t = -0.7\%$ for July 2006 from Table 2):

$$m\text{-IR:} \quad \underbrace{\begin{bmatrix} 0.0002 \\ 0.0052 \end{bmatrix}}_{\phi_m} \times \underbrace{43}_{\delta_m = m_t^e} = \begin{bmatrix} 0.01 \\ 0.22 \end{bmatrix}, \quad r\text{-IR:} \quad \underbrace{\begin{bmatrix} 0.39 \\ 0.02 \end{bmatrix}}_{\phi_r} \times \underbrace{(-0.26)}_{\delta_r = -(r_t - \bar{r})} = \begin{bmatrix} -0.10 \\ 0.00 \end{bmatrix},$$

$$\text{PZ-IR:} \quad \underbrace{\begin{bmatrix} 0.15 - (-0.23) \\ -1.21 - (-0.20) \end{bmatrix}}_{\mathbf{c}(\text{Z}) - \mathbf{c}(\text{P})} + \underbrace{\begin{bmatrix} 0.22 - 0.10 \\ -0.02 - (-0.0) \end{bmatrix}}_{\phi_p(\text{Z}) - \phi_p(\text{P})} \times \underbrace{1.3}_{p_t} + \underbrace{\begin{bmatrix} 0.16 - 0.14 \\ 0.77 - 0.93 \end{bmatrix}}_{\phi_x(\text{Z}) - \phi_x(\text{P})} \times \underbrace{(-0.7)}_{x_t} = \begin{bmatrix} 0.52 \\ -0.92 \end{bmatrix}. \tag{5.11}$$

This makes clear that the overall impact effect of continuing QE2 (of $(0.4\%, -0.7\%)$) is heavily influenced by the impact effect of PZ-IR (of $(0.52\%, -0.92\%)$), which in turn is largely determined by the difference in the intercept between regimes, $\mathbf{c}(\text{Z}) - \mathbf{c}(\text{P})$. This, and the fact that the intercept term in the inflation equation was not well-determined for either P and Z, are responsible for the positive but insignificant overall impact effect on p of 0.4% in Figure 5c. For the output gap, the negative overall impact of -0.7% is significantly different from 0.

The whole profile of PZ-IR for t = July 2006 is in Figure 5d. The output gap reaches a trough of about -1.2%, which is smaller in absolute value than the trough in the overall effect shown in Figure 5c. Because the response of output in m-IR is positive, this means that the output effect of cutting the policy rate from 0.26% to zero would have been substantially negative in July 2006, in contrast to the r-IR shown in Figure 5b for the base period of January 1992. That this is indeed the case is shown in Figure 5e. When the policy rate is very low, lowering the rate further makes it more likely that the regime switches from P to Z in the following period with all the contractionary effect of PZ-IR.

A question then arises: if ending QE by switching to P in July 2006 was expansionary, would it have been better to end it earlier? We can answer this question by considering the opposite of (5.8) for the base period t before July 2006. That is, take Z as the baseline regime and take P as the counter-factual alternative regime. So the difference we calculate is

$$
\begin{aligned}
&\mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{P},(p_t, x_t, \bar{r}_t, 0), \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big) - \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, m_t), \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big) \\
&\qquad = -\Big[\underbrace{\mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, 0), ...\Big) - \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{P}, (p_t, x_t, \bar{r}_t, 0), ...\Big)}_{\text{PZ-IR}}\Big] \\
&\qquad\quad - \Big[\underbrace{\mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, m_t), ...\Big) - \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, 0), ...\Big)}_{m\text{-IR}}\Big]
\end{aligned}
\tag{5.12}
$$

for any of the Z months preceding July 2006. There is no r-IR component because the policy rate is set at $\bar{r}$ in both the baseline and alternative scenarios. The PZ-IR component of this difference is, as just seen, negative for $y = p, x$. Whether the overall difference (5.12) is positive or not (namely, whether ending QE would have been expansionary or not) depends on the strength of the m-IR component which, in turn, depends on the size of m_t. If m_t is not large enough, the PZ-IR component dominates and the profiles of the overall difference for (p, x) would be the opposite of those in Figure 5c. This is indeed the case for t = June 2006 (with $m_t = 46$ as shown in Table 2), May 2006 (with $m_t = 55$ or the actual-to-required reserve ratio of 1.7), but not for April 2006 with $m_t = 100\%$ or the actual-to-required ratio of 2.7. Figure 5f has the profiles for t = April 2006. It shows that exiting from QE in April 2006 would have been contractionary.

6 Robustness to Variations

In this section, we examine whether the results about the IR and counter-factual analyses of Section 5 are robust to changes to the model and to the simulation specifications. We consider one variation at a time.

Turning Excess Reserve Demand On

In all the simulations underlying the Monte Carlo integration, we turned the demand for excess reserves off by setting $m_{t+k} = 0$ when $s_{t+k} = \mathrm{P}$ for $k = 1, 2, ...$ We now allow for positive excess reserves under regime P. Recall from Section 3 that the observed excess reserve rate m_t equals

$\max[m_{dt}, 0]$ under P where m_{dt} is the demand for excess reserves. The specification of m_{dt} we consider here relates the excess reserve demand to the current values of π (the 12-month inflation rate), x (the output gap), r (the policy rate) and the lagged value of m. The sample is those months under P between January 1992 and November 2008 (the last month under P). As was true in the estimation of the excess reserve supply, there is no need to correct for regime endogeneity because the excess reserve demand shock is independent of the regime. The estimation method is Tobit because of the censoring in $\max[m_{dt}, 0]$. We define the limit observations as the months for which $m < 0.5\%$. There are 103 such months.[30] The estimated equation is (t-values in brackets)[31]

$$\begin{gathered} m_{dt} = \underset{[-1.7]}{-4.7} + \underset{[1.8]}{6.7}\,\pi_t - \underset{[-2.7]}{2.4}\,x_t - \underset{[-1.97]}{10.5}\,r_t + \underset{[2.8]}{0.72}\,m_{d,t-1}, \\ \text{estimated standard deviation of the error} = 5.8 \text{ (s.e.} = 1.0), \\ \text{sample size} = 122, \text{number of limit observations} = 103. \end{gathered} \tag{6.1}$$

The output coefficient is negative, perhaps because commercial banks desire excess reserves in recessions. The estimated error size (measured by its standard deviation) of 5.8 percentage points is large, but the fitted value of m_{dt} is about -13 percentage points on average. So m_t under P, which is $\max[m_{dt}, 0]$, is positive only very rarely. The IR and counter-factual analyses with the excess reserve demand turned on look almost identical to those without. So we won't show any graphs here.[32]

[30] Recall that we have set $m_t = 0$ between QE spells (except the Lehman crisis months of September to November 2008), on the ground that banks kept excess reserves to postpone costs of re-entering the interbank market. So the value of m under P reflects precautionary reserve demand only.

[31] For two months, August 2000 and July 2006, the previous month is the last month of a QE and the lagged value of m is far above 0. We assume that the precautionary reserve demand in that previous month is zero. This amounts to setting $m_{d,t-1} = 0$ for t = August 2000 and July 2006.

[32] With the excess reserve demand on, the definition of r-IR needs to be modified slightly. The zero for m_t in the alternative history must be replaced by $m_t^{(a)}$, the value of the excess reserve rate that would be expected given the history up to $r_t + \delta_r$. Likewise, the zero for m_t in the baseline history must be replaced by $m_t^{(b)}$, the value of the excess reserve rate that would be expected given the history up to r_t. Similarly, the zero for m_t in the baseline history in the definition of PZ-IR must be replaced by $m_t^{(b)}$, the value of the excess reserve rate that would be expected given the history up to $\bar{r}_t$.

Lag Length

For the inflation and output reduced-form, we have set the lag length to 1 because that is what is instructed by the BIC (Schwartz information criterion). We now select the lag length by the AIC (Akaike information criterion). To preserve the degrees of freedom, we allow the lag length to differ across equations, with n_p for the inflation (p) equation and n_x for the output (x) equation. If K is the total number of coefficients (including the intercepts) of the bivariate system, we have $K = 2 + 4(n_p + n_x)$ for lagged subsample Z (with $s_{t-1} = \text{Z}$). For lagged subsample P (with $s_{t-1} = \text{P}$), we have $K = 4 + 3(n_p + n_x)$ because lagged m is absent but the banking crisis dummy is present. Let T be the sample size and $\widehat{\varepsilon_t}$ be the 2×1 matrix of estimated reduced-form residuals with the lag length configuration of (n_p, n_x). The information criterion to be minimized is

$$\log\left(\left|\frac{1}{T}\sum_{t=1}^{T}\widehat{\varepsilon_t}\widehat{\varepsilon_t}'\right|\right) + K \cdot C(T)/T, \tag{6.2}$$

where $C(T) = 2$ for the AIC and $\log(T)$.[33] Given the choice of the maximum lag length n_{max}, we search over all possible combinations of $n_p, n_x = 1, 2, ..., n_{max}$ to minimize this objective function.

The AIC picks $(n_p, n_x) = (3, 1)$ for lagged subsample P and and $(n_p, n_x) = (3, 3)$ for Z when n_{max}=6.[34] Figure 6 shows the result about the QE2 extension, comparable to Figure 5c, when the lag length is as given by the AIC. The only notable difference from Figure 5c is the response of p over the initial months. The positive impact effect on p of 0.4% shown in Figure 5c is a fragile result.

HP Filtered GDP as Potential GDP

Next we consider an alternative measure of potential GDP that underlies the output gap. Figure 7 plots log monthly GDP and its HP (Hodrick-Prescott) filtered series from 1988.[35] Compared to Figure 3a, where potential GDP is the official measure constructed by the Cabinet Office, the

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[33] See, e.g., Hayashi (2000, p. 398).

[34] The AIC picks $(n_p, n_x) = (10, 1)$ for P and $(n_p, n_x) = (3, 3)$ for Z if $n_{max} = 12$. Given the moderate sample sizes, we decided not to include as many as 10 lags.

[35] The smoothness parameter for the HP filter is 1600×3^4, which is the value recommended by Ravin and Uhlig (2002) for monthly series.

HP-filtered GDP tracks actual GDP more closely. For example, the output gap has been positive since September 2011.

Rather surprisingly, results change very little with the different measure of the output gap. The reason is that the two output gap measures differ primarily in the mean, not in the serial correlation properties. To illustrate, Table 7 shows the inflation and output reduced-form estimates with the HP-filtered GDP. Compared to Table 6, which are based on the Cabinet Office potential GDP, the difference occurs at the intercepts ($\mathbf{c}$(P) and $\mathbf{c}$(Z) in (5.5)) and also at the r_{t-1} coefficient on lagged subsample Z. Since r_{t-1} is constant during QE3, the difference in the r_{t-1} coefficient mainly reflects the level difference in the output gap measure after the Lehman crisis. The different value of the r_{t-1} coefficient, however, does not affect the IR and counter-factual simulations because we chose the base period t to be before the Lehman crisis with the projected path of $\bar{r}_{t+k}$ (which equals r_{t+k} under Z) to be constant at 0%. Regarding the difference in the intercepts $\mathbf{c}$(P) and $\mathbf{c}$(Z), recall that the impact effect in the PZ-IR depends on the intercepts only through the differential $\mathbf{c}(\mathrm{Z}) - \mathbf{c}(\mathrm{P})$. The estimated value of this differential is similar across the two tables.

Figure 8 shows the result about the QE2 extension. Again, the qualitative and quantitative features are about the same as those in Figure 5c. The long-run expected value of (p, x, r, m) and the long-run frequency of the zero-rate regime, which were $(-0.4, -3, 0.1, 70)$ and three-quarters under the Cabinet Office potential GDP, are very similar under the HP trend, except that the long-run value of x is now -0.5. This is to be expected because, as just observed, the two output gap measures differ primarily in the mean.

7 Conclusions

We have constructed a regime-dependent SVAR model in which the regime is determined by the central bank responding to economic conditions. The model was used to study the dynamic effect of not only changes in the policy rate and the reserve supply but also changes in the regime chosen by the central bank. Several conclusions emerge.

- The Taylor rule, estimated on the sample period including the many recent period of zero policy rates, indicates that the policy rate responded strongly to inflation and less so to output.
- Our IR (impulse response) analysis indicates that a cut in the policy rate lowers inflation. Thus, consistent with the existing Japanese literature, the price puzzle is observed for Japan as well.
- An increase in the reserve supply under QE raises both inflation and output. The significance of this result relative to the existing Japanese literature is that this conclusion is obtained while regime endogeneity is taken into account.
- Surprisingly, exiting from QE is expansionary if the actual-to-required reserve ratio is not too large. In the episode of exiting from QE in 2006, the critical value of the reserve ratio under which ending QE is expansionary is somewhere between 1.7 and 2.7. Bringing the ratio down to this range during QE, however, is contractionary.

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Table 1: Policy Announcements by the Bank of Japan, 1999-2012

date	quotes and URLs
1999.2.12	“The Bank of Japan will provide more ample funds and encourage the uncollateralized overnight call rate to move as low as possible." http://www.boj.or.jp/en/announcements/release_1999/k990212c.htm/
1999.4.13	“(The Bank of Japan will) continue to supply ample funds until the deflationary concern is dispelled." (A remark by governor Hayami in a Q & A session with the press. Translation by authors.) http://www.boj.or.jp/announcements/press/kaiken_1999/kk9904a.htm/
1999.9.21	“The Bank of Japan has been pursuing an unprecedented accommodative monetary policy and is explicitly committed to continue this policy until deflationary concerns subside." http://www.boj.or.jp/en/announcements/release_1999/k990921a.htm/
2000.8.11	“... the downward pressure on prices ... has markedly receded. ... deflationary concern has been dispelled, the condition for lifting the zero interest rate policy." http://www.boj.or.jp/en/announcements/release_2000/k000811.htm/
2001.3.19	“The main operating target for money market operations be changed from the current uncollateralized overnight call rate to the outstanding balance of the current accounts at the Bank of Japan. Under the new procedures, the Bank provides ample liquidity, and the uncollateralized overnight call rate will be determined in the market ... The new procedures for money market operations continue to be in place until the consumer price index (excluding perishables, on a nationwide statistics) registers stably a zero percent or an increase year on year." http://www.boj.or.jp/en/announcements/release_2001/k010319a.htm/
2003.10.10	“The Bank of Japan is currently committed to maintaining the quantitative easing policy until the consumer price index (excluding fresh food, on a nationwide basis) registers stably a zero percent or an increase year on year." http://www.boj.or.jp/en/announcements/release_2003/k031010.htm/
2006.3.9	“... the Bank of Japan decided to change the operating target of money market operations from the outstanding balance of current accounts at the Bank to the uncollateralized overnight call rate... The Bank of Japan will encourage the uncollateralized overnight call rate to remain at effectively zero percent. ... The outstanding balance of current accounts at the Bank of Japan will be reduced towards a level in line with required reserves. ... the reduction in current account balance is expected to be carried out over a period of a few months.... Concerning prices, year-on-year changes in the consumer price index turned positive. Meanwhile, the output gap is gradually narrowing. ... In this environment, year-on-year changes in the consumer price index are expected to remain positive. The Bank, therefore, judged that the conditions laid out in the commitment are fulfilled." http://www.boj.or.jp/en/announcements/release_2006/k060309.htm/
2006.7.14	“... the Bank of Japan decided ... to change the guideline for money market operations... The Bank of Japan will encourage the uncollateralized overnight call rate to remain at around 0.25 percent." http://www.boj.or.jp/en/announcements/release_2006/k060714.pdf/
2008.12.19	“... it (author note: meaning the policy rate) will be encouraged to remain at around 0.1 percent (author note: which is the rate paid on reserves)..." http://www.boj.or.jp/en/announcements/release_2008/k081219.pdf
2009.12.18	“The Policy Board does not tolerate a year-on-year rate of change in the CPI equal to or below 0 percent." http://www.boj.or.jp/en/announcements/release_2009/un0912c.pdf
2010.10.5	“The Bank will maintain the virtually zero interest rate policy until it judges, on the basis of the "understanding of medium- to long-term price stability" that price stability is in sight..." http://www.boj.or.jp/en/announcements/release_2010/k101005.pdf
2012.2.14	“The Bank will continue pursuing the powerful easing until it judges that the 1 percent goal is in sight..." http://www.boj.or.jp/en/announcements/release_2012/k120214a.pdf

Table 2: Winding-down of QE2, March to August 2006

	March	April	May	June	July	August
regime (P for normal, Z for zero-rate/QE)	Z	Z	Z	Z	P	P
ratio of actual to required reserves	4.5	2.7	1.7	1.6	1.0	1.0
m, log of the above ratio (%)	151	100	55	46	0	0
r, the policy rate (% per year)	0.0	0.0	0.0	0.0	0.26	0.25
p, monthly inflation rate (% per year)	1.1	−1.7	0.1	1.2	1.3	2.3
π, year-on-year inflation rate (% per year)	0.1	−0.1	0.0	0.2	0.2	0.3
x, output gap (%)	−0.7	−0.3	−0.6	−0.4	−0.7	−0.4

Note: The ratio of actual to required reserves for July and August 2006, which was 1.2 (July) and 1.1 (August), is set to 1.0. The policy rate under the zero-rate regime is set equal to $\bar{r}$ (the rate paid on reserves) which before November 2008 is 0%.

Table 3: Simple Statistics

sample period is January 1992 - Dec. 2012					
	p (monthly inflation rate, % per year)	π (12-month inflation rate, %)	x (output gap, %)	$r-\bar{r}$ (net policy rate, % per year)	m (excess reserve rate, %)
subsample P (sample size=122)					
mean	0.4	0.5	−1.0	1.32	0.7
std. dev.	1.6	0.8	1.7	1.43	2.4
max	6.3	2.6	2.3	5.64	20.6
min	−3.5	−0.9	−4.2	0.08	0.0
subsample Z (sample size=130)					
mean	−0.4	−0.4	−3.0	0.0	105.3
std. dev.	1.5	0.4	1.8	0.0	61.7
max	4.1	0.3	−0.3	0.0	184.9
min	−4.7	−1.3	−10.4	0.0	4.1

Table 4: Taylor Rule, January 1992 - December 2012 (sample size = 252)

banking crisis dummy coefficient (% per year)	inflation coefficient	output coefficient	speed of adjutment (ρ_t, % per month)	std. dev. of error (σ_r, % per year)	mean of threshold ($\overline{\pi}$, % per year)	std. dev. of threshold ($\sigma_{\overline{\pi}}$, % per year)
−0.28 [−0.9]	1.01 [4.0]	0.04 [0.5]	7.8 [4.2]	0.11 (0.0073)	0.38 (0.25)	0.24 (0.16)

Note: Estimation by the ML (maximum likelihood) method described briefly in the text and more fully in Appendix 2. t-values in brackets and standard errors in parentheses. The Taylor rule controls the shadow rate in the censored Taylor rule:

$$\text{(censored Taylor rule)} \qquad r_t = \begin{cases} \underbrace{\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}}, \ v_{rt} \sim \mathcal{N}(0, \sigma_r^2) & \text{if } s_t = \text{P}, \\ \overline{r}_t & \text{if } s_t = \text{Z}, \end{cases}$$

where the desired Taylor rate r_t^* and the regime s_t is defined by

$$r_t^* \equiv \alpha_r^* + \underset{(1\times 2)}{\boldsymbol{\beta}_r^{*\prime}} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix}, \ s_t = \begin{cases} \text{P} \quad \text{if} \ \underbrace{\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}} > \overline{r}_t, \\ \text{Z} \quad \text{otherwise.} \end{cases}$$

The banking crisis dummy (1 for September 1995-July 1998, 0 otherwise) is added to the constant term α_r^*. The inflation and output coefficients are the first and second element of $\boldsymbol{\beta}_r^*$. The speed of adjustment is ρ_r in the shadow rate. The mean of threshold $\overline{\pi}$ appears in the exit condition:

$$\text{If } s_{t-1} = \text{Z, then } s_t = \begin{cases} \text{P} \quad \text{if} \ \underbrace{\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}} > \overline{r}_t \ \text{and} \ \pi_t > \underbrace{\overline{\pi} + v_{\overline{\pi}t}}_{\text{period } t \text{ threshold}}, \ v_{\overline{\pi}t} \sim \mathcal{N}(0, \sigma_{\overline{\pi}}^2), \\ \text{Z} \quad \text{otherwise.} \end{cases}$$

Table 5: Excess Reserve Supply Equation

t is in	coefficient of				σ_s (%)	R^2
	const	π_t	x_t	m_{t-1}		
QE2 & QE3 (113 obs.)	−3.3 [−0.5]	−1.7 [−0.4]	−2.0 [−2.6]	0.99 (0.033)	13.1 (0.87)	0.94

Note: Estimation by OLS. t-values in brackets and standard errors in parentheses. The equation estimated here is

$$m_t = \alpha_s + \boldsymbol{\beta}_s{}' \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} + \gamma_s m_{t-1} + v_{st}, \quad v_{st} \sim \mathcal{N}(0, \sigma_s^2).$$

Here, m_t is the exces reserve rate in percents. This is what the reserve supply equation $m_t = \max[m_{st}, 0]$ (where m_{st} is given in (3.7)) reduces to when $m_t > 0$ for all t. σ_s (standard deviation of the error) is estimated as $\widehat{\sigma_s} = \sqrt{SSR/n}$ where n is the sample size. The standard error of $\widehat{\sigma_s}$ is calculated as $\frac{\widehat{\sigma_s}}{\sqrt{2n}}$.

Table 6: Inflation and Output Reduced Form, January 1992 - December 2012

lagged subsample P (set of t's such that s_{t-1} = P)								
		coefficient of						
$t-1$ is in	dependent variable	const.	p_{t-1}	x_{t-1}	r_{t-1}	m_{t-1}	bank crisis dummy	R^2
P (123 obs.)	inflation (p_t)	−0.23 [−0.9]	0.10 [1.1]	0.14 [1.7]	0.39 [3.6]		0.39 [1.2]	0.19
	output (x_t)	−0.20 [−1.4]	−0.00 [−0.1]	0.93 [21]	0.02 [0.3]		0.08 [0.5]	0.80

lagged subsample Z (set of t's such that s_{t-1} = QE1, QE2 or QE3)							
		coefficient of					
$t-1$ is in	dependent variable	const.	p_{t-1}	x_{t-1}	r_{t-1}	m_{t-1}	R^2
QE2 & QE3 (112 obs.)	inflation (p_t)	0.15 [0.3]	0.22 [2.4]	0.16 [1.8]	0.05 [0.0]	0.0002 [0.1]	0.11
	output (x_t)	−1.21 [−3.3]	−0.02 [−0.3]	0.77 [14]	−0.98 [−0.5]	0.0052 [2.6]	0.75

Note: Estimation by OLS. t-values in brackets. p is the monthly inflation rate stated at annual rates, x is the output gap in percents, r is the policy rate in percents per year, and m is the excess reserve rate in percents. The bank crisis dummy takes the value of 1 if September 1995 ≤ t ≤ July 1998 and 0 otherwise. The value of r_{t-1} is 0% for (QE1 and) QE2, and 0.1% for QE3. The reduced form equations on lagged subsamples Z does not include the bank crisis dummy because the crisis period is when s_{t-1} = P.

Table 7: Inflation and Output Reduced Form with HP-Filtered Trend, Jan. 1992 - Dec. 2012

lagged subsample P (set of t's such that s_{t-1} = P)

$t-1$ is in	dependent variable	coefficient of const.	p_{t-1}	x_{t-1}	r_{t-1}	m_{t-1}	bank crisis dummy	R^2
P (123 obs.)	inflation (p_t)	−0.50	0.11	0.13	0.45		0.40	0.19
		[−2.1]	[1.2]	[1.4]	[4.1]		[1.2]	
	output (x_t)	−0.01	0.00	0.90	−0.02		0.07	0.76
		[0.0]	[0.0]	[18]	[−0.3]		[0.4]	

lagged subsample Z (set of t's such that s_{t-1} = QE1, QE2 or QE3)

$t-1$ is in	dependent variable	coefficient of const.	p_{t-1}	x_{t-1}	r_{t-1}	m_{t-1}	R^2
QE2 & QE3 (112 obs.)	inflation (p_t)	−0.26	0.23	0.15	−1.4	0.0008	0.10
		[−0.5]	[2.4]	[1.7]	[−0.4]	[0.3]	
	output (x_t)	−0.66	−0.02	0.78	1.3	0.0043	0.70
		[−2.2]	[−0.4]	[14]	[0.6]	[2.2]	

Note: Estimation by OLS. t-values in brackets. p is the monthly inflation rate stated at annual rates, x is the output gap in percents, r is the policy rate in percents per year, and m is the excess reserve rate in percents. The trend output underlying the output gap is the HP-filtered log GDP. The bank crisis dummy takes the value of 1 if September 1995 $\leq t \leq$ July 1998 and 0 otherwise. The value of r_{t-1} is 0% for (QE1 and) QE2, and 0.1% for QE3. The reduced form equations on lagged subsamples Z does not include the bank crisis dummy because the crisis period is when s_{t-1} = P.

Appendix 1 Data Description

This appendix describes how the variables used in the paper — p (monthly inflation), π (12-month inflation), x (output gap), r (the policy rate), $\bar{r}$ (the interest rate paid on reserves), and m (the excess reserve rate), — are derived from various data sources.

Monthly and Twelve-Month Inflation Rates (p and π)

The monthly series on the monthly inflation rate (appearing in the inflation and output reduced-form) and the 12-month inflation rate (in the Taylor rule and the excess reserve supply equation) are constructed from the CPI (consumer price index). The Japanese CPI is compiled by the Ministry of Internal Affairs and Communications of the Japanese government. The overall CPI and its various subindexes can be downloaded from the portal site of official statistics of Japan called"e-Stat". The URL for the CPI is
`http://www.e-stat.go.jp/SG1/estat/List.do?bid=000001033702&cycode=0`.
This page lists a number of links to CSV files. One of them,
`http://www.e-stat.go.jp/SG1/estat/Csvdl.do?sinfid=000011288575`
has the "core" CPI (CPI excluding fresh food), the "core-core" CPI (CPI excluding food and energy), and other components from January 1970. They are seasonally *un*adjusted series and combine different base years from January 1970. For how the Ministry combines different base years, see Section III-6 of the document (in Japanese) downloadable from
`http://www.stat.go.jp/data/cpi/2010/kaisetsu/index.htm#p3`
Briefly, to combine base years of 2005 and 2010, say, the Ministry multiplies one of the series by a factor called the "link factor" whose value is such that the two series agree on the average of monthly values for the year 2005.

Twelve-month inflation rates constructed from the (seasonally unadjusted) "core" CPI and the "core-core" CPI are shown in Appendix Figure 1. The two humps for 1989 and 1997 are due to the increases in the consumption tax. The two inflation rates behave similarly, except for the period November 2007 - May 2009.

The above URL has another CSV file, whose link is
`http://www.e-stat.go.jp/SG1/estat/Csvdl.do?sinfid=000011288581`,
has *seasonally adjusted* series for various subindexes (including the "core-core" CPI), but only from January 2005. As explained below, we use the "core-core" CPI between November 2007 and May 2009 that is seasonally adjusted, along with the seasonally unadjusted "core" CPI, in order to construct p (monthly inflation) and π (12-month inflation). The construction involves three steps.

Adjustment for Consumption Tax Hikes. The consumption tax rate rose from 0% to 3% in April 1989 and to 5% in April 1997. We compute the 12-month inflation rate from the seasonally unadjusted index (as the log difference between the current value of the index and the value 12 months ago) and subtract 1.2% for t = April 1989,..., March 1990 (to remove the effect of the April 1989 tax hike) and 1.5% for t = April 1997,..., March 1998 (to remove the effect of the April 1997 tax hike). These two numbers (1.2% and 1.5%) are taken from *Price Report* (various years) by the Economic Planning Agency of the Japanese government (which became a part of the Cabinet Office). We then calculate the index so that its implied 12-month inflation agrees with the tax-adjusted 12-month inflation.

Seasonal Adjustment. We apply Kitagawa and Gersch's (1984) seasonal adjustment method. It uses the following state-space model, known as "Decomp". For the time series y_t in question, the observation equation is

$$y_t = T_t + S_t + A_t + u_{0t}$$

and the state equations are

$$\Delta^d T_t = u_{1t}, \quad S_t = -S_{t-1} - \cdots - S_{t-p} + u_{2t}, \quad A_t = a_1 A_{t-1} + \cdots + a_q A_{t-q} + u_{3t}.$$

Here, $(u_{0t}, u_{1t}, u_{2t}, u_{3t})$ are mutually and serially independent normal errors with mean 0, T is the trend component, S is the seasonal component, A is the stationary component, and Δ is the difference operator. The seasonal adjusted series is $T_t + A_t + u_{0t}$. The parameter values we chose are: $d = 2$, $p = 11$ $(= 12 - 1)$, $q = 4$. So the trend component is allowed to be quadratic. The seasonally adjustment can be performed on-line at `http://ssnt.ism.ac.jp/inets2/title.html`. We apply this method on the seasonally unadjusted (but tax-adjusted) "core" index from January 1970 through December 2012 (43 years).

Adjustment for the 2007-2008 Energy Price Swing. Let CPI_{1t} be the seasonally adjusted "core" CPI obtained from this operation for t = January 1970,..., December 2012. Let CPI_{2t} be the seasonally adjusted "core-core" CPI for t = January 2005,..., December 2012 that is directly available from the above CSV file. Our CPI measure (call it CPI) is CPI_1, except that we switch from CPI_1 to CPI_2 between November 2007 and May 2009 to remove the large movement in the energy component of the "core" CPI. More precisely,

$$CPI_t = \begin{cases} CPI_{1t} & \text{for } t = \text{January 1970}, ..., \text{October 2007}, \\ CPI_{t-1} \times \dfrac{CPI_{2t}}{CPI_{2,t-1}} & \text{for } t = \text{November 2007}, ..., \text{May 2009}, \\ CPI_{t-1} \times \dfrac{CPI_{1t}}{CPI_{1,t-1}} & \text{for } t = \text{June 2009}, ..., \text{December 2012}. \end{cases} \tag{A1.1}$$

The monthly inflation rate for month t, p_t, is calculated as

$$p_t \equiv 1200 \times [\log(CPI_t) - \log(CPI_{t-1})]. \tag{A1.2}$$

The 12-month inflation rate for month t, π_t, is

$$\pi_t \equiv 100 \times [\log(CPI_t) - \log(CPI_{t-12})]. \tag{A1.3}$$

Excess Reserve Rate (m)

Monthly series on actual and required reserves are available from September 1959. The source is the BOJ's portal site `http://www.stat-search.boj.or.jp/index_en.html/`. The value for month t is defined as the average of daily balances over the reserve maintenance period of the 16th day of month t to the 15th day of month $t+1$. We define the excess reserve rate for month t (m_t) as

$$m_t \equiv 100 \times [\log(\text{actual reserve balance for month } t) - \log(\text{required reserve balance for month } t)]. \tag{A1.4}$$

The Policy Rate (r)

The monthly time series on the policy rate from January 1970 is a concatenation of three series.

August 1985 - December 2012. We obtained daily data on the uncollateralized overnight "Call" rate (the Japanese equivalent of the U.S. Federal Funds rate) since immediately after the inception of the market (which is July 1985) from *Nikkei* (a data vendor maintained by a subsidiary of *Nihon Keizai Shinbun* (the Japan Economic Daily)). The policy rate for month t, r_t, for t = August 1985,...,December 2012 is the average of the daily values over the reserve maintenance period of the 16th of month t to the 15th of month $t+1$.

October 1978 - July 1985. Daily data on the collateralized overnight "Call" rate from October 1978 are available from *Nikkei*. The policy rate for month t, r_t, for t = October 1978,..., July 1985 is the average of the daily values over the reserve maintenance period of the 16th of month t to the 15th of month $t+1$ plus a risk premium of 7.5 basis points. The risk premium estimate of 7.5 basis points is the difference in the August 1985 reserve maintenance period average between the uncollateralized call rate (6.305%) and the collateralized call rate (6.230%).

January 1970 - September 1978. Monthly averages (over calendar months, not over reserve maintenance periods) of the collateralized rate are available from the above BOJ portal from January 1960. The policy rate for month t in this period of January 1970 - September 1978 is this monthly average for month t plus the risk premium of 7.5 basis points.

Interest Rate paid on Reserves ($\bar{r}$)

$\bar{r}_t$ is 0% until October 2008 and 0.1% since November 2008.

Output Gap (x), with GDP as the Output Measure

Three series go into our output gap construction: (i) quarterly seasonally adjusted real GDP (from the National Income Accounts (NIA), compiled by the Cabinet Office of the Japanese government), (ii) the "all-industry activity index" (compiled by the Ministry of Economy, Trade, and Industry of the Japanese government (METI) available from January 1988), and (iii) the GDP gap estimate by the Cabinet Office.

Quarterly NIA GDP. The Japanese national accounts adopted the chain-linking method in 2004. We obtained the chain-linked quarterly seasonally-adjusted real GDP series from two SCV files, available from

- `http://www.esri.cao.go.jp/en/sna/data/sokuhou/files/2010/qe104_2/__icsFiles/afieldfile/2012/02/29/gaku-jk1042.csv`,
- `http://www.esri.cao.go.jp/jp/sna/data/data_list/sokuhou/files/2013/qe134/__icsFiles/afieldfile/2014/02/13/gaku-jk1341.csv`.

The first SCV file has the real quarterly GDP series (call it GDP_2000 here) for a benchmark year of 2000 from 1980:Q1 to 2010:Q4. The second (call it GDP_2005 here) is for a benchmark year of 2005 from 1994:Q1 from 2013:Q4. The two series are linked together at 1994:Q1. That is, let GDP be the linked series. It is constructed as

$$GDP_t = \begin{cases} GDP\_2000_t \times \lambda \ \text{ for } t = \text{1980:Q1 - 1993:Q4}, \\ GDP\_2005_t \ \text{ for } t = \text{1994:Q1 - 2013:Q4}, \end{cases} \tag{A1.5}$$

where λ is the ratio of GDP_2005_t for t = 1994:Q1 to GDP_2000_t for t = 1994:Q1.

METI's All-Industry Activity Index. This index is a Laspeyres index combining four subindexes: a construction industry index, the IP (the Index of Industrial Producion), a services industry index, and a government services index. It therefore excludes agriculture. The latest base year is 2005, with a weight of 18.3% for the IP. METI has released two series, one whose base year is 2005 and the other (what the "link index") that combines various past series with different base years, and the latter series is adjusted so that the two series can be concatenated to form a consistent series. The two seasonally adjusted series, along with a very brief documentation, can be downloaded from `http://www.meti.go.jp/statistics/tyo/zenkatu/index.html`.

Monthly Interpolation. Given the METI all-industry activity index, the allocation of quarterly GDP between the three months constituting the quarter is done as follows. Let Y here be the quarterly GDP at annual rates, and (v_1, v_2, v_3) be the value of the all-industry index. Our estimate of monthly GDP at annual rates for the three months, (Y_1^e, Y_2^e, Y_3^e), is chosen so that the average equals the value of quarterly GDP and the growth rates within the quarter are the same as those of the given output measure. That is, (Y_1^e, Y_2^e, Y_3^e) solves the following system of three equations:

$$(Y_1^e + Y_2^e + Y_3^e)/3 = Y, , \tag{A1.6}$$

$$\log(Y_2^e) - \log(Y_1^e) = \log(v_2) - \log(v_1), \tag{A1.7}$$

$$\log(Y_3^e) - \log(Y_2^e) = \log(v_3) - \log(v_2). \tag{A1.8}$$

Construction of Potential Monthly GDP and Output Gap. In constructing potential quarterly GDP, the Cabinet Office uses a production function approach . A documentation (in Japanese) can be found in: `http://www5.cao.go.jp/j-j/wp/wp-je07/07f61020.html`. To summarize the document, the production function is Cobb-Douglas with 0.33 as capital's share. Capital input is defined as an estimate of the capital stock (available from the National Income Accounts) times capacity utilization. Labor input is the number of persons employed times hours worked per person. The TFP (total factor productivity) level implied by this production function and actual quarterly, real, seasonally adjusted GDP is smoothed by the HP (Hodrick-Prescott) filter. Potential GDP is defined as the value implied by the production function with the smoothed TFP level. The capital and labor in this potential GDP calculation is also HP smoothed. The (quarterly) GDP gap is defined as: 100×(actual GDP - potential GDP)/potential GDP.

The Cabinet Office's GDP gap series for 1980:Q1 - 2013:Q3 is as follows:
0.3 -1.2 -0.1 1.0 0.6 0.9 -0.4 -0.6 -0.1 -0.4 -0.9 -0.5 -1.4 -1.6 -1.1 -1.3 -1.2 -0.7 -0.5 -1.4 0.0 0.9 1.1 1.3 0.6 -0.8 -1.4 -1.7 -2.9 -2.1 -1.3 0.0 1.2 -0.1 0.9 0.8 2.3 -0.2 0.4 2.4 0.9 3.0 3.7 2.6 2.5 2.9 1.9 1.8 1.2 0.9 0.4 -0.8 -0.2 -1.3 -2.5 -2.3 -1.8 -3.2 -1.7 -3.1 -2.9 -1.8 -1.5 -1.8 -1.3 -0.6 -0.8 0.4 0.9 -0.3 -0.1 -0.4 -2.4 -3.1 -3.0 -2.7 -3.7 -3.5 -3.8 -3.5 -2.1 -2.1 -2.6 -2.1 -1.7 -2.1 -3.4 -3.7 -4.2 -3.4 -3.1 -2.9 -3.7 -2.8 -2.6 -1.9 -1.1 -1.4 -1.5 -2.0 -2.1 -1.1 -0.9 -1.0 -0.8 -0.6 -0.9 0.2 1.0 1.0 0.5 1.2 1.7 0.4 -0.7 -4.1 -8.0 -6.6 -6.5 -5.0 -3.8 -2.9 -1.6 -2.1 -3.4 -3.9 -2.7 -2.7 -1.7 -2.1 -3.1 -3.2 -2.3 -1.6 -1.3.

The Cabinet Office releases this quarterly output gap but *not* their underlying estimate of potential quarterly GDP. When we back out potential GDP from the output gap and actual GDP, the backed-out quarterly potential GDP has very erratic movements in the growth rate between 1980 and 1993. We therefore decided to smooth the backed-out potential GDP from 1980 to 2012 by the HP filter with the usual quarterly smoothness parameter of 1600. This

HP-smoothed quarterly potential GDP is converted to a monthly series by the same interpolation procedure described above, with both $\log(v_2) - \log(v_1)$ in (A1.7) and $\log(v_3) - \log(v_2)$ in (A1.8) set equal to $1/3$ times the quarterly growth rate in potential GDP from the previous to the current quarter (for the first quarter, 1980:Q1, the monthly values are assumed to be the same). Finally, using this smoothed monthly potential GDP (call it GDP_t^*) and the monthly actual GDP (call it GDP_t) obtained above, we define the monthly output gap for month t, x_t, as

$$x_t \equiv 100 \times [\log(GDP_t) - \log(GDP_t^*)]. \tag{A1.9}$$

Appendix 2 The Model and Derivation of the Likelihood Function

The Model

The state vector of the model consists of a vector of continuous state variables $\mathbf{y}_t$ and a discrete state variable s_t $(= \mathrm{P}, \mathrm{Z})$. The continuous state $\mathbf{y}_t$ has the following elements:

$$\underset{(4\times1)}{\mathbf{y}_t} \equiv \begin{bmatrix} \underset{(2\times1)}{\mathbf{y}_{1t}} \\ r_t \\ m_t \end{bmatrix}, \quad \underset{(2\times1)}{\mathbf{y}_{1t}} \equiv \begin{bmatrix} p_t \\ x_t \end{bmatrix}, \tag{A2.1}$$

where p = monthly inflation rate, x = output gap, r = policy rate, and m = excess reserve rate.

The model is a mapping from $(s_{t-1}, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-11})$ to $(s_t, \mathbf{y}_t)$. (We need to include 11 lags of $\mathbf{y}$ because of the appearance of the 12-month inflation rate in the model, see (A2.3) below.) The mapping depends on: (i) the sequence of two exogenous variables $\bar{r}_t$ (the interest rate paid on reserves) and d_t (the banking crisis dummy, 1 if September 1995 $\leq t \leq$ July 1998), (ii) the model parameters listed in (A2.7) below, and (iii) a shock vector $(\boldsymbol{\varepsilon}_t, v_{rt}, v_{\overline{\pi}t}, v_{st}, v_{dt})$ (to be defined below) that are mutually and serially independent. The mapping itself can be described recursively as follows.

(a) ($\mathbf{y}_{1t}$ determined) $\underset{(2\times1)}{\boldsymbol{\varepsilon}_t}$ is drawn from $\mathcal{N}(\mathbf{0}, \boldsymbol{\Omega}(s_{t-1}))$ and $\mathbf{y}_{1t}$ (the first two elements of $\mathbf{y}_t$) is given by

$$\underset{(2\times1)}{\mathbf{y}_{1t}} = \underset{(2\times1)}{\mathbf{c}(s_{t-1})} + \underset{(2\times1)}{\mathbf{a}(s_{t-1})} \underset{(1\times1)}{d_t} + \underset{(2\times4)}{\boldsymbol{\Phi}(s_{t-1})} \underset{(4\times1)}{\mathbf{y}_{t-1}} + \underset{(2\times1)}{\boldsymbol{\varepsilon}_t}. \tag{A2.2}$$

Here, only one lag is allowed, strictly for expositional purposes; more lags can be included without any technical difficulties.

(b) (s_t determined) Given $\mathbf{y}_{1t}$ and $(\mathbf{y}_{t-1}, ..., \mathbf{y}_{t-11})$, the central bank calculates (through $(p_t, ..., p_{t-11}, x_t, r_{t-1})$)

$$\pi_t \equiv \frac{1}{12}\left(p_t + \cdots + p_{t-11}\right), \quad r_t^e \equiv \alpha_r + \delta_r d_t + \boldsymbol{\beta}_r' \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} + \gamma_r r_{t-1}. \tag{A2.3}$$

The central bank draws $(v_{rt}, v_{\overline{\pi}t})$ from $\mathcal{N}(\mathbf{0}, \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_{\overline{\pi}}^2 \end{bmatrix})$, and determines s_t as

$$\text{If } s_{t-1} = \mathrm{P}, \quad s_t = \begin{cases} \mathrm{P} & \text{if } r_t^e + v_{rt} > \bar{r}_t, \\ \mathrm{Z} & \text{otherwise.} \end{cases} \tag{A2.4a}$$

$$\text{If } s_{t-1} = \text{Z}, \quad s_t = \begin{cases} \text{P} & \text{if } r_t^e + v_{rt} > \bar{r}_t \text{ and } \pi_t > \overline{\pi} + v_{\overline{\pi}t}, \\ \text{Z} & \text{otherwise.} \end{cases} \tag{A2.4b}$$

(c) (r_t determined) Given s_t, r_t is determined as

$$\text{If } s_t = \text{P}, \text{ then } r_t = r_t^e + v_{rt}. \tag{A2.5a}$$

$$\text{If } s_t = \text{Z}, \text{ then } r_t = \bar{r}_t. \tag{A2.5b}$$

Note that r_t in (A2.5a) is guaranteed to be $> \bar{r}_t$ under P because by (A2.4a) and (A2.4b) $r_t^e + v_{rt} > \bar{r}_t$ if $s_t = \text{P}$.

(d) (m_t determined) Finally, the central bank draws v_{st} from $\mathcal{N}(0, \sigma_s^2)$ and the market draws v_{dt} from $\mathcal{N}(0, \sigma_d^2)$. The excess reserve rate m_t is determined as

$$\text{If } s_t = \text{P}, \text{ then } m_t = \max\left[m_{dt}^e + v_{dt}, 0\right]. \tag{A2.6a}$$

$$\text{If } s_t = \text{Z}, \text{ then } m_t = \max\left[m_{st}^e + v_{st}, 0\right]. \tag{A2.6b}$$

Here, m_{dt}^e and m_{st}^e are functions of $(\mathbf{y}_{1t}, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-11})$ and r_t. For example, the specification of the demand for excess reserves in the text has

$$m_{st}^e \equiv \alpha_s + \boldsymbol{\beta}_s' \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} + \gamma_s m_{t-1}.$$

Let $\boldsymbol{\theta}$ be the model's parameter vector. It will turn out useful to divide it into 4 sets:

$$\begin{cases} \boldsymbol{\theta}_A = \Big(\underset{(2\times1)}{\mathbf{c}(s)}, \underset{(2\times1)}{\mathbf{a}(s)}, \underset{(2\times4)}{\boldsymbol{\Phi}(s)}, \underset{(2\times2)}{\boldsymbol{\Omega}(s)}, s = \text{P, Z} \Big), \\ \boldsymbol{\theta}_B = \Big(\alpha_r, \delta_r, \underset{(2\times1)}{\boldsymbol{\beta}_r}, \gamma_r, \sigma_r, \overline{\pi}, \sigma_{\overline{\pi}} \Big) \quad \text{(8 parameters)}, \\ \boldsymbol{\theta}_C = \Big(\alpha_s, \underset{(2\times1)}{\boldsymbol{\beta}_s}, \gamma_s, \sigma_s \Big) \quad \text{(5 parameters)}, \end{cases} \tag{A2.7}$$

and $\boldsymbol{\theta}_D$ that is composed of σ_d and the coefficients in m_{dt}^e.

There is a one-to-one mapping between the Taylor rule parameters in the text (see equation (3.1)) and the $\boldsymbol{\theta}_B$ here. The mapping is given by

$$\rho_r = 1 - \gamma_r, \ \alpha_r^* = \alpha_r / \rho_r, \ \boldsymbol{\beta}_r^* = \boldsymbol{\beta}_r / \rho_r. \tag{A2.8}$$

Derivation of the Likelihood Function

With the mapping from $(s_{t-1}, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-11})$ to $(s_t, \mathbf{y}_t)$ in hand, we proceed to derive the likelihood function. The likelihood of the data is

$$\mathcal{L} \equiv p\left(s_1, ..., \ s_T, \mathbf{y}_1, ..., \mathbf{y}_T \,|\, s_0, \mathbf{y}_0, \mathbf{y}_{-1}, ..., \mathbf{y}_{-11}\right). \tag{A2.9}$$

Here, $p(.)$ is the joint density-distribution function of $(s_1, ..., s_T)$ and $(\mathbf{y}_1, ..., \mathbf{y}_T)$ conditional on $(s_0, \mathbf{y}_0, \mathbf{y}_{-1}, ..., \mathbf{y}_{-10})$. It is also conditional on the path of the two exogenous variables, $(\overline{r}_1, ..., \overline{r}_T, d_1, ..., d_T)$, but this fact is not made explicit here for notational simplicity. Since the distribution of $\{s_t, \mathbf{y}_t\}$ depends on the history up to $t-1$ only through $(s_{t-1}, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-11})$, the usual sequential factorization yields

$$\mathcal{L} = \prod_{t=1}^{T} p\left(s_t, \mathbf{y}_t \,|\, s_{t-1}, \mathbf{x}_{t-1}\right), \quad \text{where } \mathbf{x}_{t-1} \equiv \left(\mathbf{y}_{t-1}, ..., \mathbf{y}_{t-11}\right). \tag{A2.10}$$

The likelihood for period t, $p\left(s_t, \mathbf{y}_t \,|\, s_{t-1}, \mathbf{x}_{t-1}\right)$, can be rewritten as (recall: $\mathbf{y}_t = (\mathbf{y}_{1t}, r_t, m_t)$)

$$\begin{aligned} p\left(s_t, \mathbf{y}_t \,|\, s_{t-1}, \mathbf{x}_{t-1}\right) = {} & p\left(m_t \,|\, r_t, s_t, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}\right) \\ & \times p\left(r_t \,|\, s_t, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}\right) \\ & \times \text{Prob}\left(s_t \,|\, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}\right) \\ & \times p\left(\mathbf{y}_{1t} \,|\, s_{t-1}, \mathbf{x}_{t-1}\right). \end{aligned} \tag{A2.11}$$

In what follows, we rewrite each of the four terms on the right hand side of this equation in terms of the model parameters.

The Fourth Term, $p\left(\mathbf{y}_{1t} \,|\, s_{t-1}, \mathbf{x}_{t-1}\right)$

This term is entirely standard:

$$p\left(\mathbf{y}_{1t} \,|\, s_{t-1}, \mathbf{x}_{t-1}\right) = b\Big(\mathbf{y}_{1t} - \big(\mathbf{c}(s_{t-1}) + \mathbf{a}(s_{t-1})d_t + \mathbf{\Phi}(s_{t-1})\mathbf{y}_{t-1}\big); \mathbf{\Omega}(s_{t-1})\Big), \tag{A2.12}$$

where $b(.; \mathbf{\Omega})$ is the density of the bivariate normal with mean $\underset{(2\times1)}{\mathbf{0}}$ and variance-covariance matrix $\underset{(2\times2)}{\mathbf{\Omega}}$.

The Third Term, $\text{Prob}\left(s_t \,|\, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}\right)$

This is the transition probability matrix for $\{s_t\}$. The probabilities depend on (r_t^e, π_t) (which in term can be calculated from $(\mathbf{y}_{1t}, \mathbf{x}_{t-1})$, see (A2.3)). They are easy to derive:

s_{t-1} \ s_t	P	Z
P	P_{rt}	$1-P_{rt}$
Z	$P_{rt}P_{\pi t}$	$1-P_{rt}P_{\pi t}$

Here,

$$P_{rt} \equiv \text{Prob}\left(r_t^e + v_{rt} > \bar{r}_t \,|\, r_t^e\right) = \Phi\left(\frac{r_t^e - \bar{r}_t}{\sigma_r}\right), \tag{A2.13}$$

$$P_{\pi t} \equiv \text{Prob}\left(\pi_t > \overline{\pi} + v_{\overline{\pi} t} \,|\, \pi_t\right) = \Phi\left(\frac{\pi_t - \overline{\pi}}{\sigma_{\overline{\pi}}}\right), \tag{A2.14}$$

where $\Phi(.)$ is the cdf of $\mathcal{N}(0,1)$.

The First Term, $p\left(m_t \,|\, r_t, s_t, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}\right)$

m_t is given by (A2.6a) and (A2.6b). So this term is the Tobit distribution-density function given by

$$h_{jt} \equiv \left[\frac{1}{\sigma_j}\phi\left(\frac{m_t - m_{jt}^e}{\sigma_j}\right)\right]^{1(m_t>0)} \times \left[1 - \Phi\left(\frac{m_{jt}^e}{\sigma_j}\right)\right]^{1(m_t=0)}, \tag{A2.15}$$
$$j = d \text{ if } s_t = \text{P and } j = s \text{ if } s_t = \text{Z},$$

where $1(.)$ is the indicator function, $\phi(.)$ and $\Phi(.)$ are the density and the cdf of $\mathcal{N}(0,1)$.

The Second Term, $p\left(r_t \,|\, s_t, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}\right)$

If s_t = Z, then $r_t = \bar{r}_t$ with probability 1, so this term can be set to 1. If s_t = P, there are two cases to consider.

- For s_{t-1} = P,

$$\begin{aligned} &p(r_t \,|\, s_t = \text{P}, \mathbf{y}_{1t}, s_{t-1} = \text{P}, \mathbf{x}_{t-1}) \\ &\quad = p\left(r_t^e + v_{rt} \,|\, r_t^e + v_{rt} > \bar{r}_t, r_t^e\right) \quad \text{(by (A2.4a) and (A2.5a))} \\ &\quad = \frac{p\left(r_t^e + v_{rt} \,|\, r_t^e\right)}{\text{Prob}\left(r_t^e + v_{rt} > \bar{r}_t \,|\, r_t^e\right)} \quad \text{(see, e.g., Hayashi, p. 512)} \\ &\quad = \frac{\frac{1}{\sigma_r}\phi\left(\frac{v_{rt}}{\sigma_r}\right)}{\text{Prob}\left(r_t^e + v_{rt} > \bar{r}_t \,|\, r_t^e\right)} \quad \text{(b/c } r_t^e + v_{rt} \sim \mathcal{N}\left(r_t^e, \sigma_r^2\right)) \\ &\quad = \frac{\frac{1}{\sigma_r}\phi\left(\frac{r_t - r_t^e}{\sigma_r}\right)}{P_{rt}} \quad \text{(b/c } P_{rt} = \text{Prob}\left(r_t^e + v_{rt} > \bar{r}_t | r_t^e\right)) \end{aligned} \tag{A2.16}$$

- For $s_{t-1} = \mathrm{Z}$,

$$\begin{aligned} &p(r_t \,|\, s_t = \mathrm{P}, \mathbf{y}_{1t}, s_{t-1} = \mathrm{Z}, \mathbf{x}_{t-1}) \\ &\quad = p\left(r_t^e + v_{rt} \,|\, r_t^e + v_{rt} > \bar{r}_t, \pi_t > \overline{\pi} + v_{\overline{\pi}t}, r_t^e, \pi_t\right) \quad \text{(by (A2.4b) and (A2.5a))} \\ &\quad = p\left(r_t^e + v_{rt} \,|\, r_t^e + v_{rt} > \bar{r}_t, r_t^e\right) \quad \text{(b/c } v_{rt} \text{ and } v_{\overline{\pi}t} \text{ are independent)} \\ &\quad = \frac{\frac{1}{\sigma_r}\phi\left(\frac{r_t - r_t^e}{\sigma_r}\right)}{P_{rt}} \qquad \text{(as above).} \end{aligned} \tag{A2.17}$$

Putting All Pieces Together

Putting all those pieces together, the likelihood for date t, (A2.11), can be written as

$s_t \| s_{t-1}$	$p\left(m_t \,\|\, r_t, s_t, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}\right)$	$p\left(r_t \| s_t, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}\right)$	$\mathrm{Prob}\left(s_t \| \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}\right)$	$f\left(\mathbf{y}_{1t} \| s_{t-1}, \mathbf{x}_{t-1}\right)$
P|P	h_{dt}	$\frac{g_t}{P_{rt}}$	P_{rt}	$f_{\mathrm{P}t}$
P|Z	h_{dt}	$\frac{g_t}{P_{rt}}$	$P_{rt}P_{\pi t}$	$f_{\mathrm{Z}t}$
Z|P	h_{st}	1	$1 - P_{rt}$	$f_{\mathrm{P}t}$
Z|Z	h_{st}	1	$1 - P_{rt}P_{\pi t}$	$f_{\mathrm{Z}t}$

Here,

$$f_{\mathrm{P}t} \equiv b\Big(\mathbf{y}_{1t} - \mathbf{c}(\mathrm{P}) - \mathbf{a}(\mathrm{P})d_t - \mathbf{\Phi}(\mathrm{P})\mathbf{y}_{t-1}; \mathbf{\Omega}(\mathrm{P})\Big),$$
$$f_{\mathrm{Z}t} \equiv b\Big(\mathbf{y}_{1t} - \mathbf{c}(\mathrm{Z}) - \mathbf{a}(\mathrm{Z})d_t - \mathbf{\Phi}(\mathrm{Z})\mathbf{y}_{t-1}; \mathbf{\Omega}(\mathrm{Z})\Big),$$
$$g_t \equiv \frac{1}{\sigma_r}\phi\left(\frac{r_t - r_t^e}{\sigma_r}\right), \;\; P_{rt} \equiv \Phi\left(\frac{r_t^e - \bar{r}_t}{\sigma_r}\right), \;\; P_{\pi t} \equiv \Phi\left(\frac{\pi_t - \overline{\pi}}{\sigma_{\overline{\pi}}}\right),$$

h_{jt} is defined in (A2.15) and $b(.; \mathbf{\Omega})$ is the density function of the bivariate normal distribution with mean $\underset{(2\times1)}{\mathbf{0}}$ and variance-covariance matrix $\underset{(2\times2)}{\mathbf{\Omega}}$.

Dividing it into Pieces

Taking the log of both sides of (A2.10) with (A2.11) and substituting the entries in the table, we obtain the log likelihood of the sample:

$$L \equiv \log(\mathcal{L}) = \sum_{t=1}^{T} \log\left[p\left(s_t, \mathbf{y}_t \,|\, s_{t-1}, \mathbf{x}_{t-1}\right)\right] = L_A + L_1 + L_2 + L_D,$$

where

$$L_A = \sum_{s_{t-1}=\mathrm{P}} \log[f_{\mathrm{P}t}] + \sum_{s_{t-1}=\mathrm{Z}} \log[f_{\mathrm{Z}t}], \tag{A2.18}$$

$$L_1 = \sum_{s_t=\mathrm{P}} \log[P_{rt}] + \sum_{s_t\,|\,s_{t-1}=\mathrm{P}\,|\,\mathrm{Z}} \log[P_{\pi t}] + \sum_{s_t\,|\,s_{t-1}=\mathrm{Z}\,|\,\mathrm{P}} \log[1-P_{rt}] + \sum_{s_t\,|\,s_{t-1}=\mathrm{Z}\,|\,\mathrm{Z}} \log[1-P_{rt}P_{\pi t}], \tag{A2.19}$$

$$L_2 = \sum_{s_t=\mathrm{P}} [\log(g_t) - \log(P_{rt})] + \sum_{s_t=\mathrm{Z}} \log[h_{st}], \tag{A2.20}$$

$$L_D = \sum_{s_t=\mathrm{P}} \log[h_{dt}]. \tag{A2.21}$$

The terms in $L_1 + L_2$ can be regrouped into L_B and L_C, as in

$$L = L_A + \underbrace{L_B + L_C}_{=L_1+L_2} + L_D, \tag{A2.22}$$

where

$$L_B = \sum_{s_t=\mathrm{P}} \log[g_t] + \sum_{s_t\,|\,s_{t-1}=\mathrm{P}\,|\,\mathrm{Z}} \log[P_{\pi t}] + \sum_{s_t\,|\,s_{t-1}=\mathrm{Z}\,|\,\mathrm{P}} \log[1-P_{rt}] + \sum_{s_t\,|\,s_{t-1}=\mathrm{Z}\,|\,\mathrm{Z}} \log[1-P_{rt}P_{\pi t}], \tag{A2.23}$$

$$L_C = \sum_{s_t=\mathrm{Z}} \log[h_{st}]. \tag{A2.24}$$

L_A, L_B, L_C and L_D can be maximized separately, because L_j depends only on $\boldsymbol{\theta}_j$ ($j = A, B, C, D$) (($\boldsymbol{\theta}_A, \boldsymbol{\theta}_B, \boldsymbol{\theta}_C$) was defined in (A2.7) above).

As a special case, consider simplifying step (b) of the mapping above by replacing (A2.4a) and (A2.4b) by

$$s_t = \begin{cases} \mathrm{P} & \text{if } r_t^e + v_{rt} > \bar{r}_t, \\ \mathrm{Z} & \text{otherwise.} \end{cases} \tag{A2.25}$$

Namely, drop the exit condition. This is equivalent to constraining $P_{\pi t}$ to be 1, so L_B becomes

$$L_B = \sum_{s_t=\mathrm{P}} \log[g_t] + \sum_{s_t=\mathrm{Z}} \log[1-P_{rt}], \tag{A2.26}$$

which is the Tobit log likelihood function.

Figure 1: Policy Rate in Japan, 1988-2012

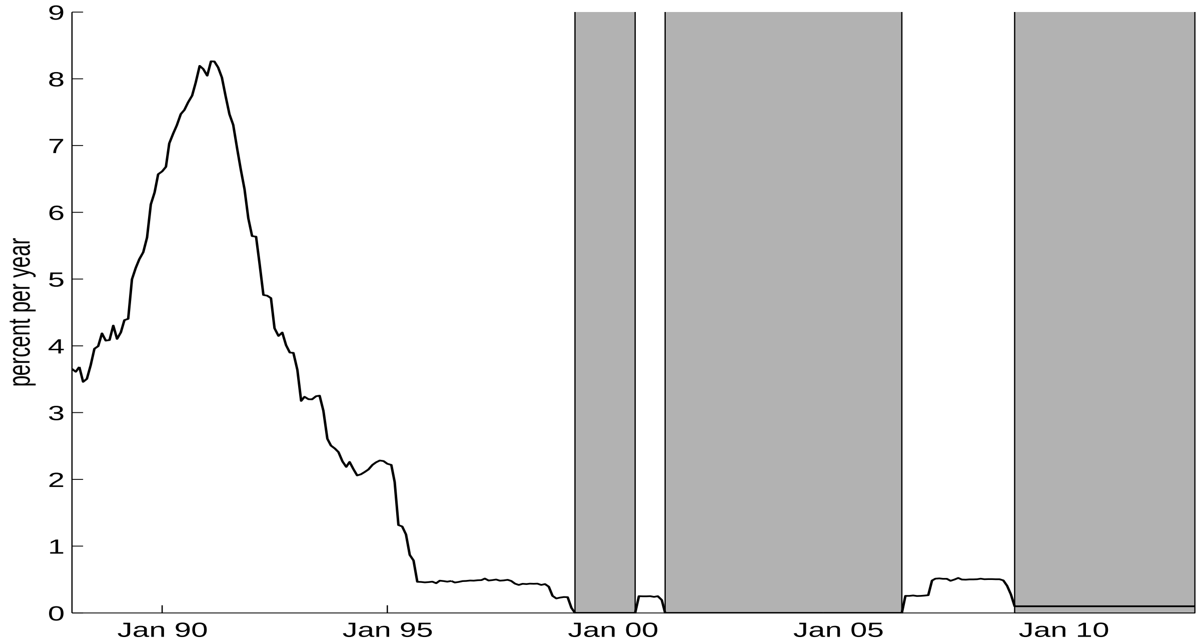


Figure 2a: Plot of Net Policy Rate against Excess Reserve Rate, 1988-2012

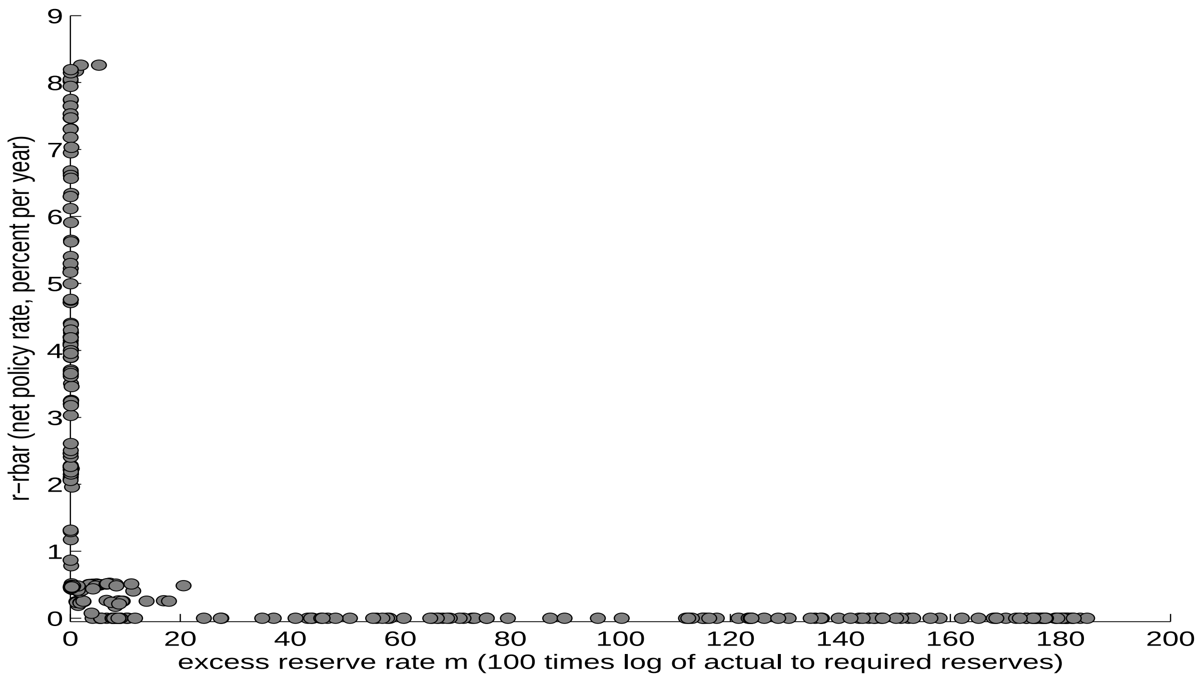


Figure 2b: Plot of Net Policy Rate against Excess Reserve Rate, Near Origin

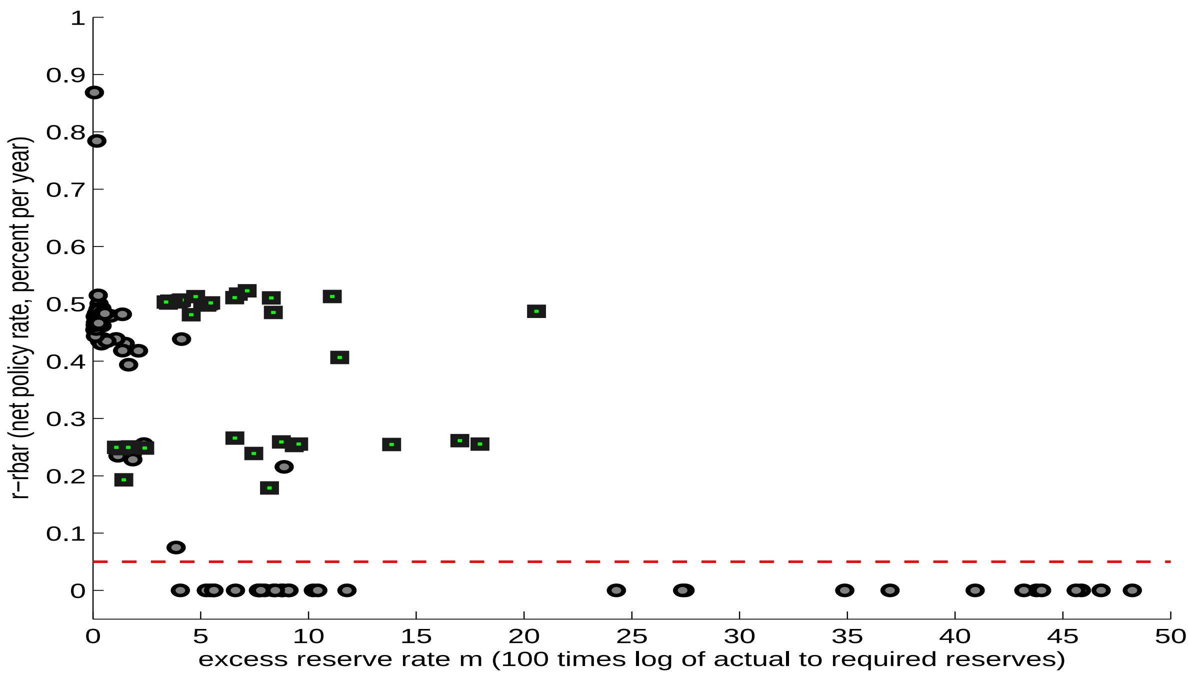


Figure 3a: Actual and Potential Monthly GDP, 1988 - 2012

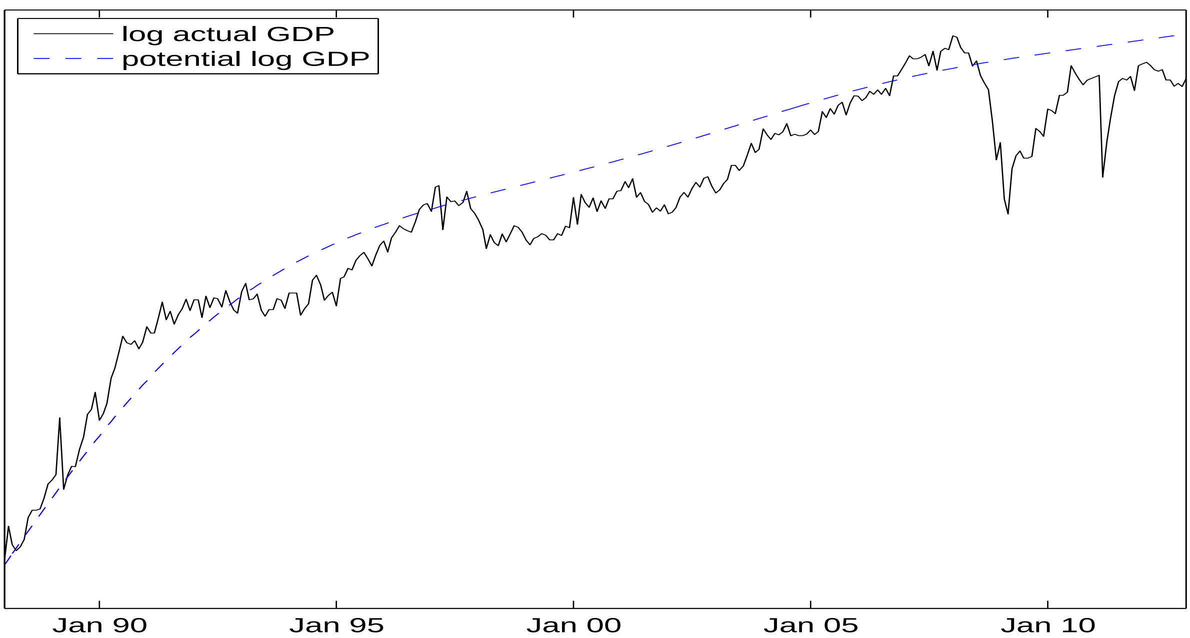


Figure 3b: Excess Reserve Rate, 1988 - 2012

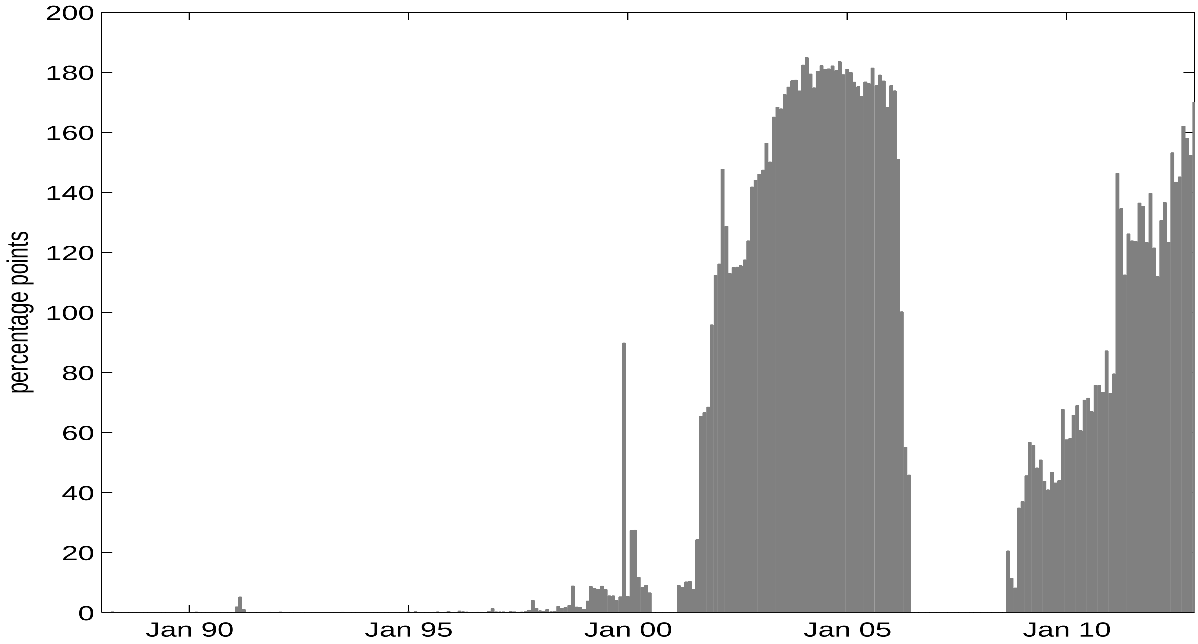


Figure 3c: Policy Rate and 12-Month Inflation Rate, January 1970 - December 2012

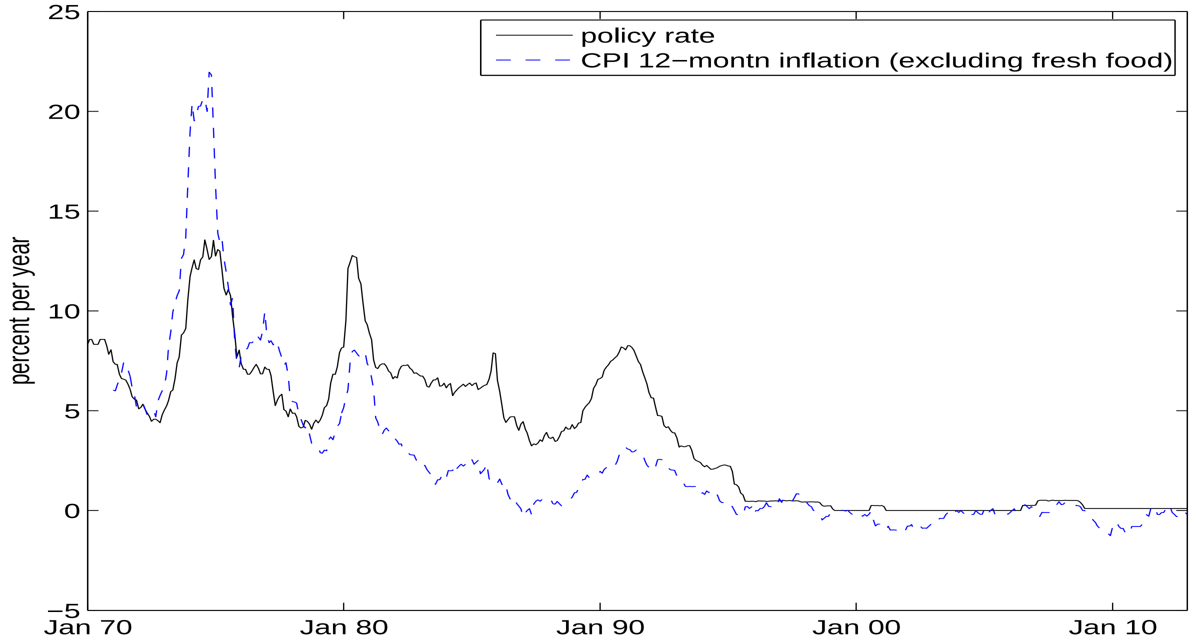


Figure 4: Policy Rate and Desired Taylor Rates, 1988 - 2012

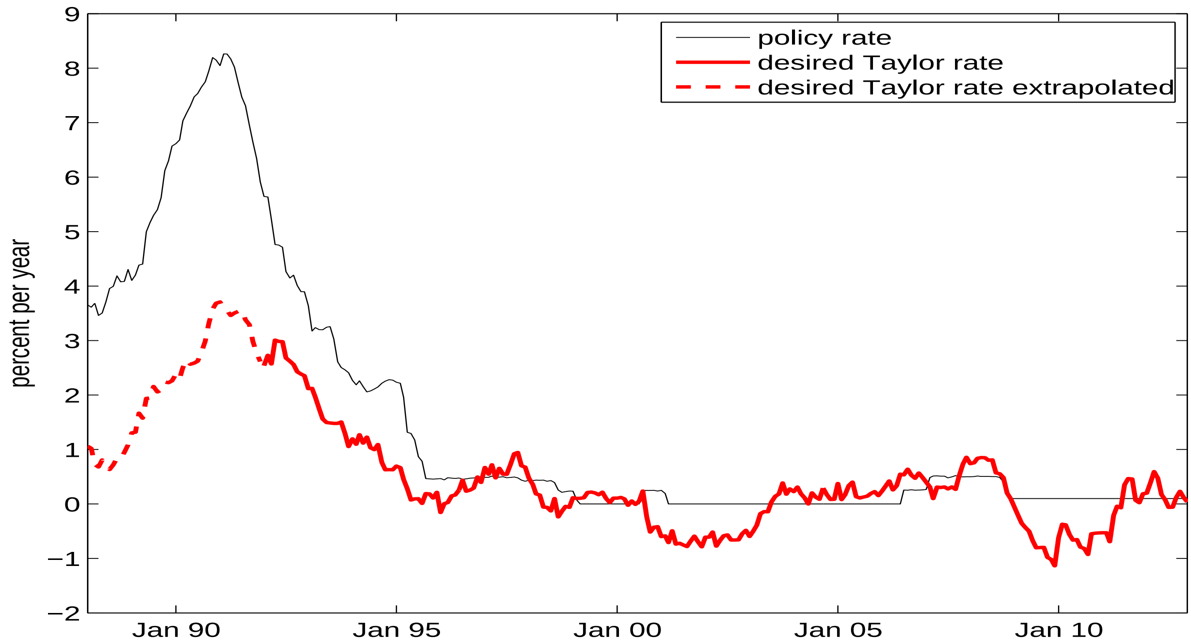


Figure 5a: m-IR (Impulse Response to m), the base period is February 2004

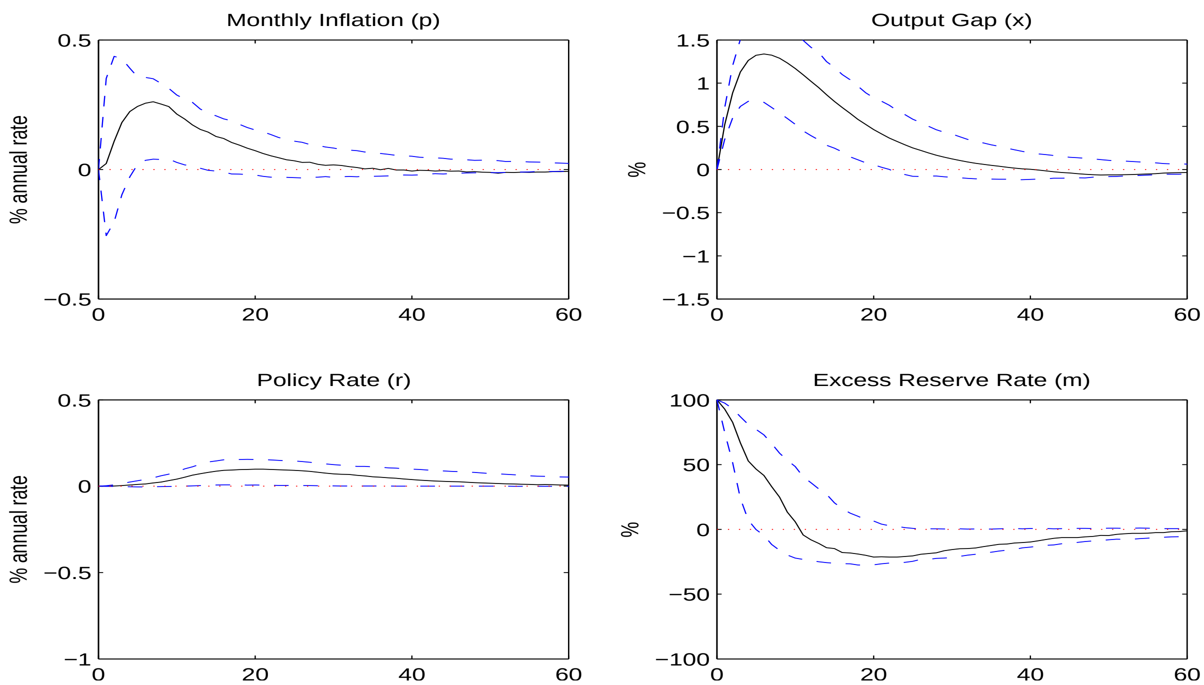


Figure 5b: r-IR (Impulse Response to r), the base period is January 1992

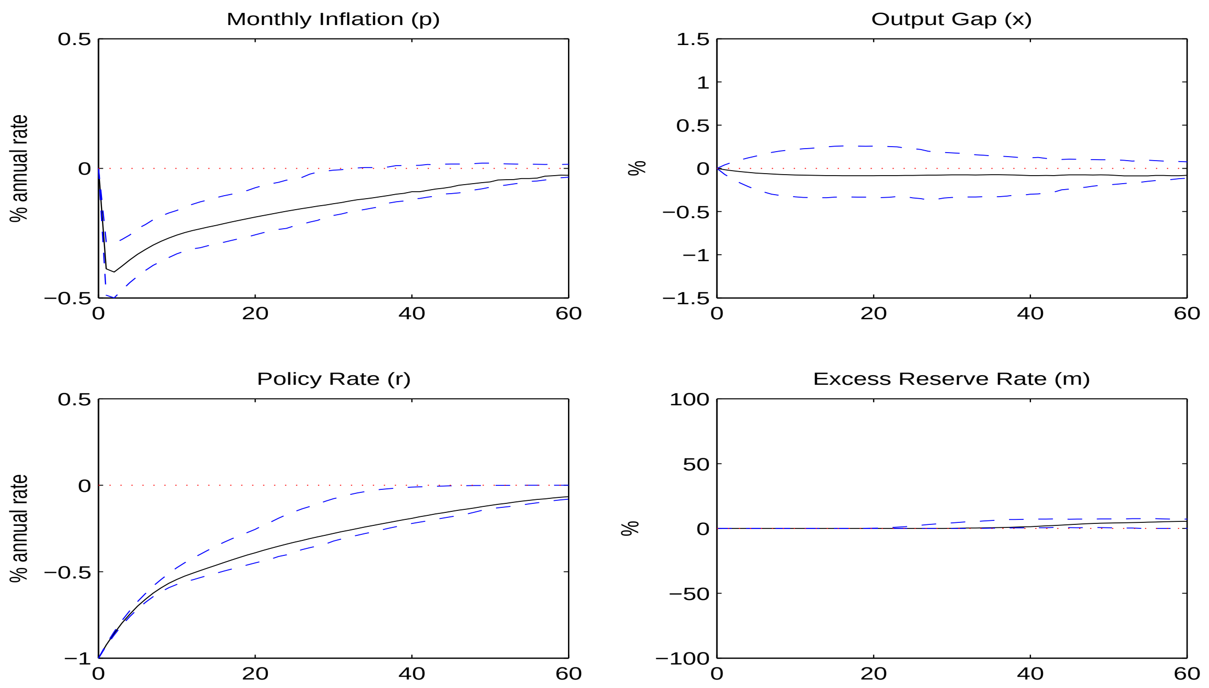


Figure 5c: Effect of Extending QE2 to July 2006

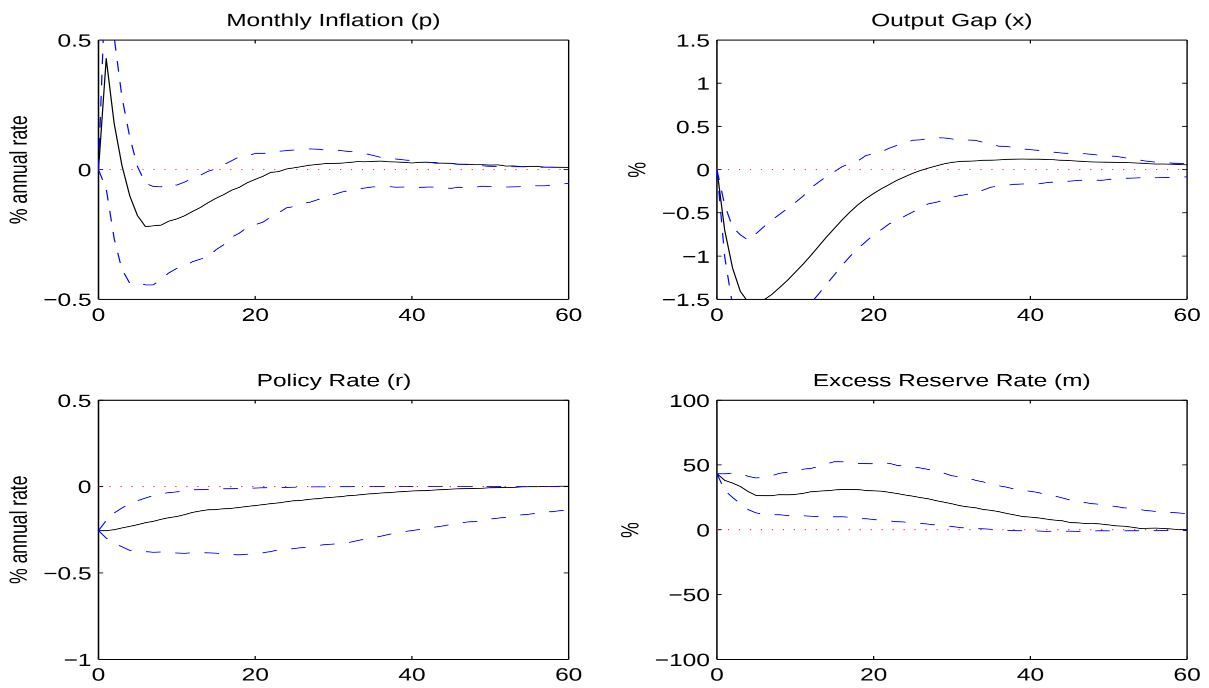


Figure 5d: PZ-IR, the base period is July 2006

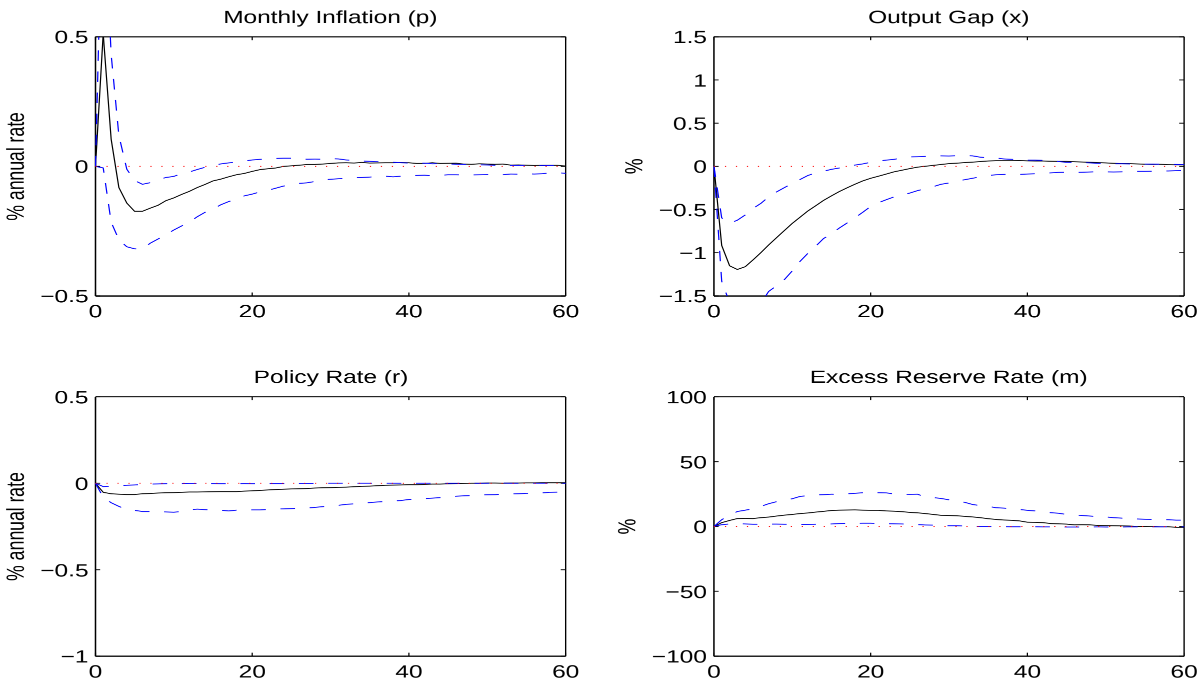


Figure 5e: r-IR, the base period is July 2006

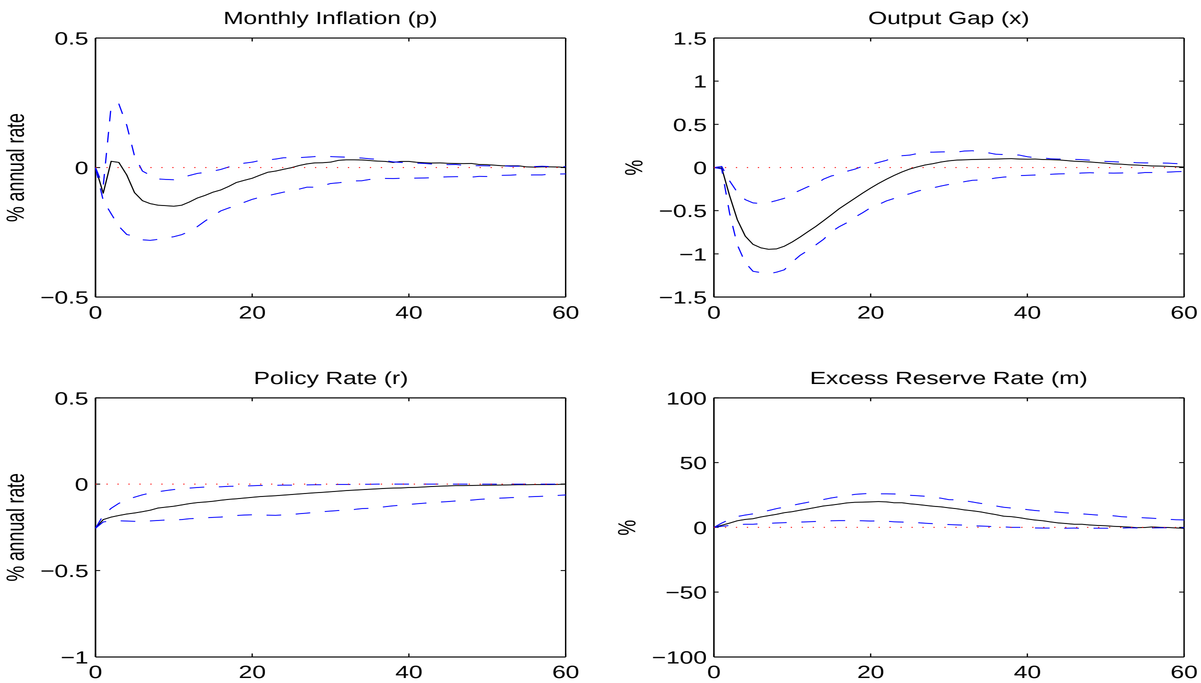


Figure 5f: Effect of Terminating QE2 in April 2006

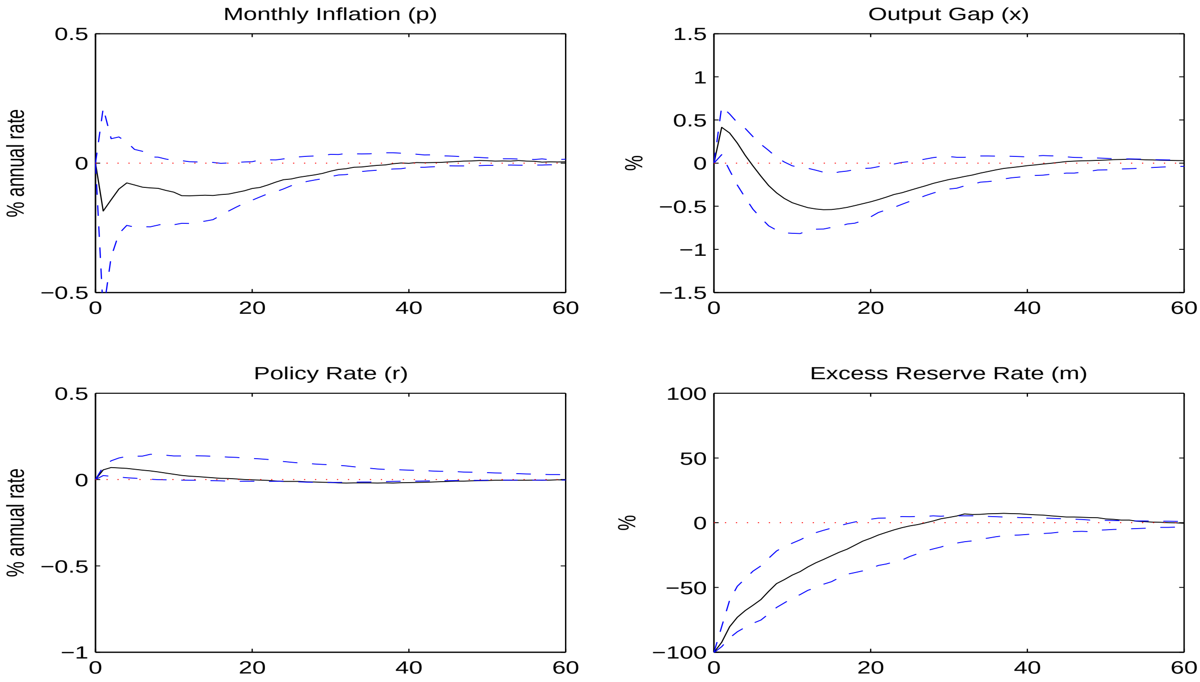


Figure 6: Effect of Extending QE2 to July 2006, with more lags

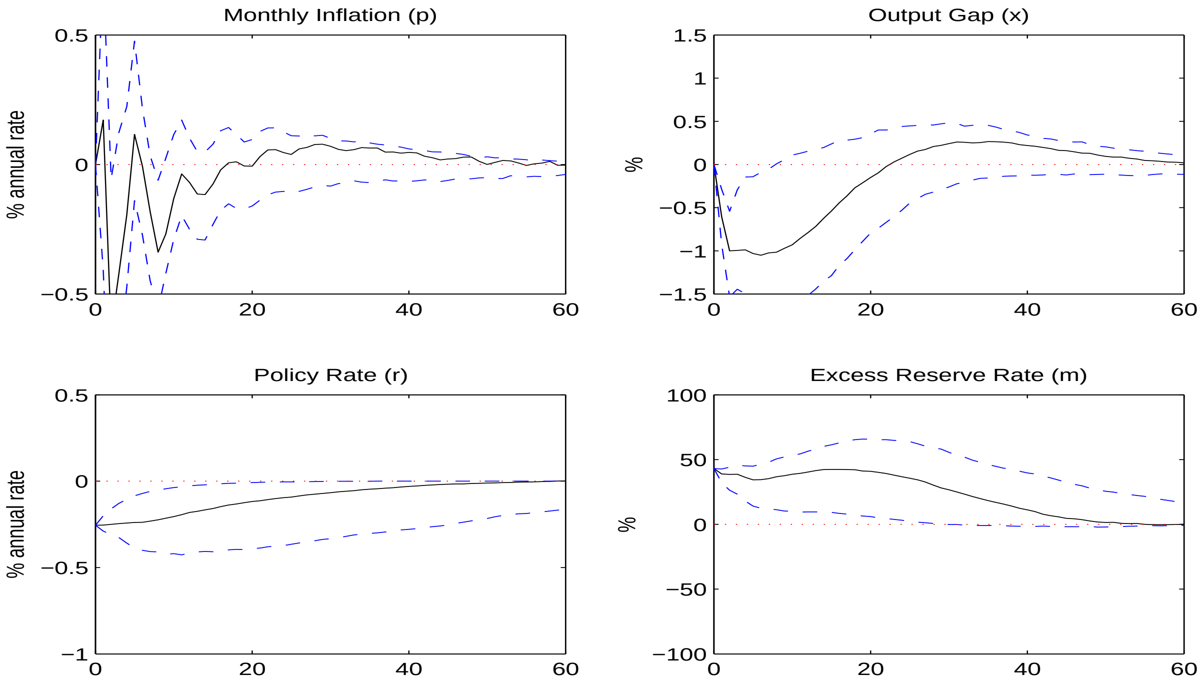


Figure 7: Actual and HP-filtered Monthly GDP, 1988 - 2012

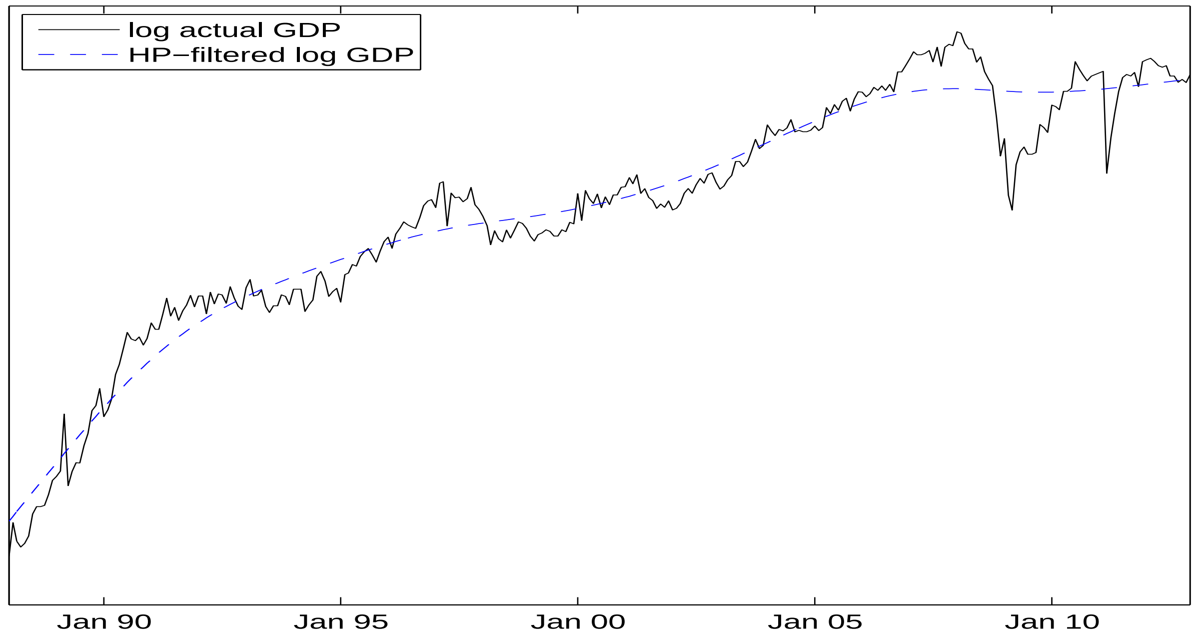
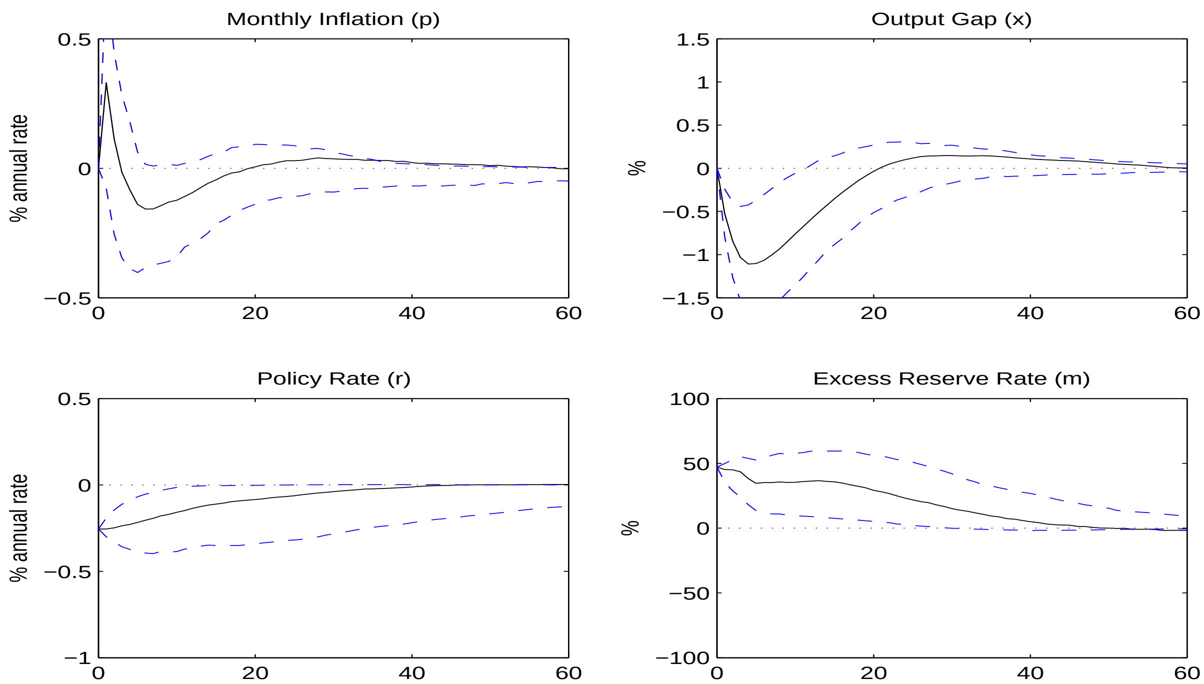


Figure 8: Effect of Extending QE2 to July 2006, Trend Output is HP-Filtered Output



Appendix Figure 1: Twelve-Month CPI Inflation Rate, 1988 - 2012