Synchronization and Bias in a Simple Macroeconomic Model

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Strategic Uncertainty and Belief Synchronization

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- I study an environment where agents must learn to use the *correct* sunspot out of infinitely many options.
- Added ingredient: agents have some innate bias in predicting output (some are inherently optimistic, while others are pessimistic). There is no aggregate bias.

cont.

- This gives rise to complicated dynamics that can lead to:
  - Full Synchronization (all agents converge on playing a particular equilibrium).
  - Incoherence (the agents do not converge on an equilibrium)
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- This gives rise to complicated dynamics that can lead to:
  - Full Synchronization (all agents converge on playing a particular equilibrium).
  - Incoherence (the agents do not converge on an equilibrium)
  - Partial Synchronization (most agents converge on playing a particular equilibrium while others drift incoherently)
- Additionally, the system can fluctuate between synchronization and incoherence, spending long periods of time in one and then quickly switching to another.
  - Metronomes: http://youtu.be/Aaxw4zbULMs

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- It describes N oscillators whose phases ψ_t^i $(i = 1, \dots, n; \psi_t^j \in [-\pi, \pi])$, are coupled as described by the equation:

$$\frac{d}{dt}\psi_t^i = \omega^i - \frac{K}{N}\sum_{j=1}^{N} \sin(\psi_t^i - \psi_t^j), \quad i = 1, \cdots, N.$$

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- By defining $R_t e^{i\psi_t} = \frac{1}{N}\sum_{i=1}^{N} e^{i\psi_t^i}$, the equations take the more convenient form:

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- $R_t \in [0, 1]$ is a measure of the synchronization of the system (the order parameter). $R_t = 0$ is incoherence and $R_t = 1$ is full synchronization.

The Kuramoto Model - continuum limit

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- There is also a fully synchronized solution: $\psi^i_t = \bar{\omega} t + \phi^i$,

$$\bar{\omega} = \frac{1}{N}\sum_{i=1}^{N} \omega^i, \sin\phi^i = \frac{\omega^i - \bar{\omega}}{\bar{R}K}, \qquad \bar{R} = \frac{1}{N}\left|\sum_{i=1}^{N} e^{i\phi^i}\right|.$$

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- This solution requires that the natural frequencies not be too dispersed ($|\omega^i - \bar{\omega}| \ll K$). Otherwise, there is a partially synchronized solution.

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- For $K > K_c$ (strong coupling) with $N = \infty$, the incoherent solution is not stable, and the system tends (at $t \to \infty$) toward one of the synchronized solutions.
- In the finite N case, the system oscillates between the synchronized and the incoherent solution.
- Some common modifications of this model include:
  - Adding a stochastic term.
  - Allowing the coupling $K_{i,j}$ to depend on $|i - j|$.

Related Literature

- The model that I use today is based on Behanbib, Wang, Wen (2013);
- The approach to learning follows Marcet and Sargent (1989), see also Evans and Honkapohja (2012)
- Learning with multiple equilibria/sunspots Woodford (1990); Guesnerie and Woodford (1990); Evans et al. (1994); Evans and Honkapohja (2003*2); Honkapohja and Mitra (2004)...
- Synchronization phenomena: Kuramoto (1975), Strogatz (1994,2000), Acebrn et al. (2005).

Households

- Households maximize

$$\max E_0 \sum_{t=0}^{\infty} \beta^t [\log(C_t) - \psi N_t]$$

subject to:

$$C_t \leq \frac{W_t}{P_t} N_t + \frac{\Pi_t}{P_t}$$

- The first order conditions are:

$$C_t = \frac{1}{\psi} \cdot \frac{W_t}{P_t}$$

Final Good Producers

- Competitive final goods producers:

$$Y_t = \left[\int_0^1 \epsilon_{jt}^{\theta} Y_{jt}^{1-\theta} dj \right]^{\frac{1}{1-\theta}}$$

  where ϵ_{jt} are iid.

- Profit maximization implies

$$Y_{jt} = (P_t / P_{jt})^{1/\theta} \epsilon_{jt} Y_t$$

  and

$$P_t^{1-1/\theta} = \int \epsilon_{jt} P_{jt}^{1-1/\theta} dj.$$

Intermediate Good Producers

- Intermediate good producers use labor only: $Y_{jt} = AN_{jt}$.
- They must make decisions before observing ϵ_{jt}, based on a signal generated from market research s_{jt}.
- After the intermediate firms produce, prices of their goods are set to clear the market (as in a Cournot competition).
- The intermediate firm's problem is

$$\max_{Y_{jt}} E_{jt} \left[\left(P_{jt} - \frac{W_t}{A} \right) Y_{jt} \middle| s_{jt} \right]$$

- Solved by:

$$Y_{jt} = \left\{ (1-\theta) \frac{A}{\psi} E_t \left[(\epsilon_{jt})^{\theta} Y_t^{\theta-1} \middle| s_{jt} \right] \right\}^{1/\theta}$$

Intermediate Good Producers - cont.

- Without loss of generality, choose A, such that

$$Y_{jt}^{\theta} = E_t \left[\epsilon_{jt}^{\theta} Y_t^{\theta - 1} \middle| s_{jt} \right] = E_t[\exp(\theta \varepsilon_{jt} - (1-\theta) y_t | s_{jt}]$$

- where ε_{jt} and y_t are the logs of ϵ_{jt} and Y_t respectively.
- Notice that firms are targeting:

$$\hat{y}_{jt} = \theta \varepsilon_{jt} - (1-\theta) y_t.$$

Forecasters

- There is a large number of forecasters that get to observe two random variables $z_t^i, i = 1, 2$; $z_t^i \sim \mathrm{N}(0, 1)$ iid.
- Forecaster i believes that output is related to these variables:

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- Basically, we limit the belief space to $\left(\phi^i, \xi^i\right) \in \mathbb{R}^3$.
- The firms get a signal that is a linear combination of their specific shock and the average forecast:

$$s_{jt} = \lambda \epsilon_{jt} + (1 - \lambda)\left(\langle \phi_t^i \rangle + \langle \xi_t^i \rangle \cdot z_t\right), \quad \lambda \in (0, 1).$$

- Also, each firm believes $y_t = \phi^j + \xi^j \cdot z_t$.

Learning

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- Both update their beliefs using an OLS estimator that can be written recursively:

$$\begin{pmatrix} \phi^j_{t+1} \\ \xi^j_{t+1} \end{pmatrix} = \begin{pmatrix} \phi^j_t \\ \xi^j_t \end{pmatrix} + g_t \begin{pmatrix} 1 \\ z_t \end{pmatrix} (y_t + \Delta\phi^j - \phi^j_t - \xi^j_t \cdot z_t).$$

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- $\Delta\phi^j$ is the persistent bias term, that is assumed to average to zero across agents.

The firm's decision

- Recall that firms want to know $x_{jt} = \theta\varepsilon_{jt} - (1-\theta)(y_t - \phi_0)$.
- Assume $\varepsilon_{jt} \sim \mathrm{N}(0, \sigma_\epsilon^2)$ iid across firms and time.
- Then, $x_{jt}|s_{jt} \sim \mathrm{N}(m(\|\xi^j\|^2)(s_{jt} - (1-\lambda)\phi^j), \hat{\Sigma}(\|\xi^j\|^2))$, where

$$m(\xi^2) = \frac{\theta\lambda\sigma_\epsilon^2 - (1-\theta)(1-\lambda)\xi^2}{\lambda^2\sigma_\epsilon^2 + (1-\lambda)^2\xi^2},$$

$$\hat{\Sigma}(\xi^2) = \frac{(\theta+\lambda-2\theta\lambda)^2\xi^2\sigma_\epsilon^2}{\lambda^2\sigma_\epsilon^2 + (1-\lambda)^2\xi^2}.$$

Output

- Firm's decision is

$$y_{jt} = (1-\theta^{-1})\phi^j + \theta^{-1}\left[m(\|\xi^j\|^2)(s_{jt} - (1-\lambda)\phi^j) + \frac{1}{2}\hat{\Sigma}(\|\xi^j\|^2)\right]$$

- Integrating over all firms, we get

$$(1-\theta)y_t = \log \int_0^1 e^{\frac{\sigma_\epsilon^2}{2}[\theta+(\theta^{-1}-1)\lambda m(\|\xi^j\|^2)]^2} \times$$
$$\times e^{+(1-\theta)\{(1-\theta^{-1})\phi^j+\theta^{-1}[(1-\lambda)m(\|\xi^j\|^2)(\langle\phi^i\rangle-\phi^j+\langle\xi^i\rangle\cdot z_t)+\frac{1}{2}\hat{\Sigma}(\|\xi^j\|^2)]\}} dj$$

REEE without bias

- Set $\Delta\phi^i = 0$, and let all agents have common beliefs.
- The last equation defines a mapping from perceived to actual law of motion

$$\phi \to -\frac{(1-\theta)}{\theta}\phi + \frac{1}{2\theta}\hat{\Sigma}(\|\xi\|^2) + \frac{[\theta + (\theta^{-1} - 1)m(\|\xi\|^2)\lambda]^2\sigma_\epsilon^2}{2(1-\theta)},$$

$$\xi \to \frac{1}{\theta}m(\|\xi\|^2)(1-\lambda)\xi.$$

REEE without bias

- The mapping has two types of fixed points:
  1. A deterministic equilibrium:

$$\phi^C = \frac{\theta\sigma_\epsilon^2}{2(1-\theta)}, \quad \xi^C = 0$$

  2. A circle of stochastic equilibria, only when $\lambda < 1/2$

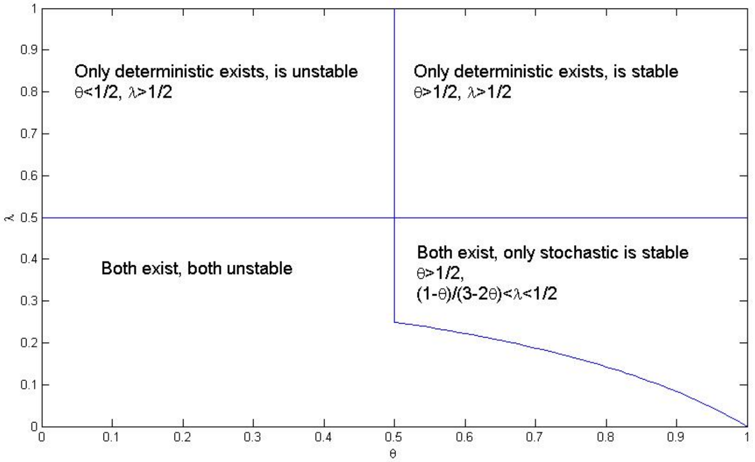
$$\phi^S = \phi^C\left(1 - \frac{(1-\theta)(1-2\lambda)}{1-\lambda}\right), \quad \|\xi^S\|^2 = \frac{\theta\lambda(1-2\lambda)}{(1-\lambda)^2}\sigma_\varepsilon^2.$$

- Note that the stochastic equilibrium is Pareto inferior.

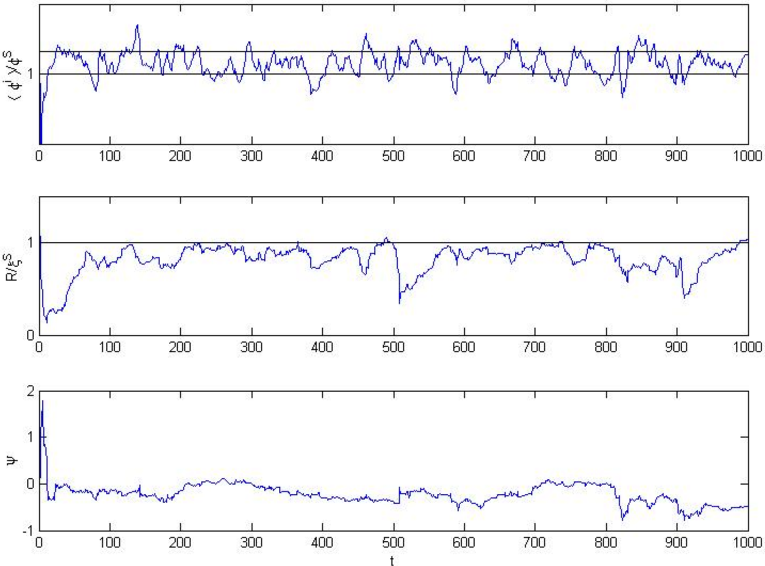
Stability

- Locally, stability has to do with the eigenvalues of the Jacobian matrix of the mapping PLM$\rightarrow$ALM.
- Stability under RLS with $g_t = 1/t$, is equivalent to the eigenvalues having real parts smaller than 1.
  - Theorem: For $\lambda > 1/2$ only the deterministic equilibrium exists and it is stable under OLS learning. For $\lambda < 1/2$, both equilibria exist but only the stochastic ones are stable.
- With constant gains the situation is more complicated. The eigenvalues also need to be larger than -1, for there to be stability with any gain value. This results are depicted in the following graph.

Stability with const. gains



Full Simulation



Simulations - Results

- For large enough $V(\Delta\phi^i)$, the system does not converge, but does not diverge either.
- Coordination builds up slowly and falls abruptly.
- With small bias, $|\Delta\phi^j| \ll \phi^S$, the system quickly converges and stays near $R_t = \xi^S$. Output in the latter case is symmetric and mesokurtic.
- With high bias the system stays near $R_t = 0$ and the resulting time series for output, y_t, is right-skewed and heavy tailed.
- Non-intuitive: the economy is more volatile when beliefs are better synchronized!

Learning about phases only

- To better understand the results, consider a version of the model where all agents share the beliefs: $\phi = \phi^S$ and $\|\xi\| = \xi^S$, and are only trying to learn about the phases, i.e.:

$$\phi_t^j = \phi^S, \quad \xi_t^j = \xi^S(\cos\psi_t^j, \sin\psi_t^j).$$

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- Also define $z_t = r_t(\cos\zeta_t, \sin\zeta_t)$.

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- The actual law of motion is

$$y_t = \phi^S + \frac{1}{1-\theta}\log\int_0^1 e^{(1-\theta)\xi^S r_t \cos(\psi^j - \zeta_t)} dj,$$

cont.

- The evolution is

$$\psi^j_{t+1} = \psi^j_t - \frac{g_t r_t}{\xi^S} \sin(\psi^j_t - \zeta_t) \times$$
$$\times \left(\xi^S r_t \left\{ \left\langle \cos(\psi^k_t - \zeta_t) \right\rangle^* - \cos(\psi^j_t - \zeta_t) \right\} + \Delta\phi^j \right)$$
$$\left\langle \cos(\psi^k_t - \zeta_t) \right\rangle^* = \frac{1}{(1-\theta)(\xi^S r_t)} \log \int_0^1 e^{(1-\theta)\xi^S r_t \cos(\psi^k_t - \zeta_t)} dk$$

cont.

- When the ψ^j's are not too dispersed, we can further approximate and get

$$\psi_{t+1}^j = \psi_t^j - g_t r_t \sin(\psi_t^j - \zeta_t)\left[\sin(\psi_t^j - \zeta_t)\int_0^1 \sin(\psi_t^j - \psi_t^k)dk + \frac{\Delta\phi^j}{\xi^S}\right]$$

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- Compare:

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- The second: $\propto \sin(\psi^j_t - \zeta_t)\Delta\phi^j$ creates dispersion, since empirically $\rho(\psi^j_t, \Delta\phi^j)$ is almost always near ± 1.

Summary

- A simple macro model where volatility changes dynamically as agents' beliefs synchronize and de-synchronize.
- Heavy-tailed growth series.
- Volatility is inversely related to belief-dispersion.
- A connection to the Kuramoto model.
- Demonstration: http://youtu.be/tlR1Ksv6cuI

Future research

- Understanding the stochastic coupling.
- Adding persistence (tricky).
- Exploring alternative couplings $K_{i,j}$.