Bond Risk Premia in Consumption-based Models *

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Abstract

The literature on recursive preference attributes all the time variation in bond risk premia to stochastic volatility. We introduce another source: time-varying prices of risk that co-move with inflation and consumption growth through a preference shock. We find that a time-varying price of risk driven by inflation dominates stochastic volatility in contributing to time variation in term premia. Once preference shocks are present, term premia are economically the same with or without stochastic volatility.

Keywords: consumption-based model; term structure of interest rates; recursive preferences; term premia; stochastic volatility.

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1 Introduction

The risk premium in the bond market, the term premium, is a key object of interest for central banks. It helps determine the effectiveness of a central bank's monetary policy because it influences how manipulations of short term interest rates are transmitted into the real economy through borrowing at longer maturities. For example, Greenspan labeled the period from 2004 through 2005 a "conundrum," when the Federal Reserve raised short term interest rates in response to a better economic environment yet long term interest rates remained low. Many researchers attribute this behavior of the yield curve to a decline in the term premium; see, e.g. Rudebusch, Swanson, and Wu(2006). Our paper develops a new consumption based asset pricing model with recursive preferences that realistically captures the dynamics of term premia and provides an economic interpretation for its source.

Our model introduces two potential channels of time variation in term premia: time-varying prices of risk and quantities of risk. The primary channel for time-varying prices of risk are through preferences shocks, while time-varying quantities of risk are driven by stochastic volatility. In our model, correlation between shocks to preferences and shocks to the state vector driving consumption growth and inflation introduce time-varying risk premia, even when shocks are homoskedastic. Comparing a range of models within our framework, we find the key driving force behind fluctuations in term premia is a time-varying price of risk that co-moves with expected inflation.

The two potential sources for time-varying term premia have been studied in separate but related literatures. Gaussian affine term structure models (ATSMs), the primary modeling tool for interest rates in central banks worldwide, attribute all the variation in term premia to a time-varying price of risk while the quantity of risk is constant; see, Wright(2011) and Bauer, Rudebusch, and Wu(2012). Although Gaussian ATSMs provide a good description of the yield curve, the lack of micro-foundations makes it hard to interpret what economic mechanism ultimately drives the term premium. Conversely, structural models with recursive preferences generate time-varying risk premia through stochastic volatility; see Bansal and

Yaron(2004) and Bansal and Shaliastovich(2013). When a model with recursive preferences has stochastic volatility, there is both a time-varying price and quantity of risk. However, these models strongly restrict the price of risk to be completely driven by stochastic volatility alone. Estimating several models for the dynamics of consumption growth and inflation, we find that models with recursive preferences and stochastic volatility produce implausible term premia. Compared to reduced form evidence from Gaussian ATSMs, they can have a combination of the wrong sign, are economically insignificant, and have infeasible dynamics. We also find that estimated term premia from stochastic volatility models (without preference shocks) are highly sensitive to the dynamics of consumption growth and inflation.

Our new model with preference shocks allows for greater flexibility in how the prices of risk can vary through time. This allows us to disentangle the driving force of term premia and pin down which is a more plausible explanation for its variability: a time-varying price of risk or quantity of risk. Adding preference shocks enables us to produce realistic term premia. The general pattern is the term premia rise for the first half of the sample from 1959 through the late 1970s, drop sharply in the early 80s, then keep steady, for the five year maturity between 1% and 2% afterwards. They also resemble the cyclical pattern discussed in Bauer, Rudebusch, and Wu(2012). Term premia increase before recessions and drop post recessions. During the Great Recession, term premia drop sharply due to the flight to quality argument. The basic dynamics of term premia are not sensitive to the time series process of inflation and consumption, nor to whether we include stochastic volatility. This evidence points to preference shocks as the main driving force for the time variation in term premia through price of risk.

We also investigate whether these models are able to match other key moments from the data. Most empirical work on the term structure in either endowment economies or DSGE models has focused on matching the unconditional slope of the term structure. We show that Gaussian models with recursive preferences can match the unconditional slope of the yield curve with or without preference shocks, consistent with the literature, see Piazzesi and Schneider(2007).

Conversely, adding stochastic volatility without preference shocks inhibits the models' ability to fit the unconditional yield curve. The yield curve slopes downward in models with stochastic volatility. Although stochastic volatility is introduced with the intention of making models more flexible, it can have the opposite effect. In structural models with recursive preferences, there are only three parameters in the representative agent's utility function (the time discount factor, risk aversion, and the intertemporal elasticity of substitution). These three parameters need to fit the original bond loadings on inflation, consumption, and the intercept across different maturities that are present in Gaussian models plus the bond loadings that are introduced when stochastic volatility factors are added. This consequently prevents the model from fitting a fundamental moment of the yield curve. Preference shocks relax this tension and enable us to fit the unconditional yield curve as well as a Gaussian model does.

Empirical examination of the asset pricing implications of recursive preferences requires solving for the stochastic discount factor.[1] However, a solution does not always exist. We provide conditions on the model's parameters guaranteeing the existence of a solution when the state vector driving consumption follows a general affine process. The general rule is that agents cannot be too patient, i.e. their rate of time preference cannot be too high. In special cases with no preference shocks, the upper bound depends on how risk averse the agents are. When the intertemporal elasticity of substitution is lower than 1, the more risk averse the agent is the more impatient he needs to be. In the opposite case, the less risk averse the agent is the more patient he is allowed to be. A direct implication of the general rule is that an extremely patient agent faces more restrictions in their risk aversion and intertemporal elasticity of substitution. These conditions partition the parameter space and make it cumbersome for econometricians implementing either an optimization-based

[1] Specifically, we implement a solution method developed by Bansal and Yaron(2004) that is widely used in the macroeconomics and finance literatures; see, e.g. Bollerslev, Tauchen, and Zhou(2009), Bansal, Kiku, and Yaron(2012) and Schorfheide, Song, and Yaron(2014). Rudebusch and Swanson(2012) and Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao(2012) describe other solution methods.

estimator or Bayesian Markov chain Monte Carlo algorithm.

This paper continues as follows. In Subsection 1.1, we discuss the literature. In Section 2, we introduce preference shocks into models with recursive preferences when consumption and inflation follow a homoskedastic, Gaussian process. In Section 3, we generalize the dynamics of consumption and inflation to include stochastic volatility. In Section 4, we discuss sufficient conditions for the existence of a solution to the (approximate) Euler equation and the restrictions this places on the model's structural parameters. In Section 5, we evaluate the ability of the models to fit key moments of the data including the unconditional slope of the term structure. Section 6 examines the model's ability to generate realistic term premia and trace for the underlying economic source. The paper concludes in Section Section 7 with a discussion of possible extensions.

1.1 Relationship to the literature

A large literature in macroeconomics and finance uses recursive preferences as developed by Kreps and Porteus(1978), Epstein and Zin(1989), and Weil(1989). These preferences separate intertemporal substitution from risk aversion, which are directly linked under power utility. The combination of recursive preferences and affine dynamics of the state variables is particularly attractive because it generates (approximate) closed-form solutions for bond and equity prices. A prime example is the long-run risk model of Bansal and Yaron(2004).

Our specification for preference shocks is motivated by Campbell and Cochrane(1999). In their model, preference shocks induced by habit formation are driven by shocks to consumption growth and are scaled by a risk sensitivity function. The interaction between the consumption shock and the risk sensitivity function introduces a time-varying price of risk into the stochastic discount factor, which is a highly autocorrelated function of consumption growth. Consequently, it exhibits business cycle variation. We introduce risk sensitivity functions as in Campbell and Cochrane(1999) that allow the representative agent's utility and their marginal rate of substitution to depend on the state of the economy. State dependent preferences are intuitive. During bad times (times when consumption growth is low), agents may be more risk averse and more sensitive to shocks. They consequently require a larger risk premium in order to hold risky assets. Albuquerque, Eichenbaum, and Rebelo(2014) and Schorfheide, Song, and Yaron(2014) also analyze models with recursive preferences and preference shocks. Our model nests their models as a special case but we also include the risk sensitivity function which is critical for generating flexible time-varying prices of risk into the model.

Several authors have developed consumption based models with endowment economies to study the yield curve. Wachter(2006) studies a model of habit formation with Campbell and Cochrane(1999) style preferences. Piazzesi and Schneider(2007) build a model of recursive preferences with unitary intertemporal elasticity of substitution and homoskedastic, Gaussian shocks to the state vector. Bansal and Shaliastovich(2013) evaluate a model with recursive preferences where shocks to consumption growth and inflation have stochastic volatility. Le and Singleton(2010) build a model of recursive preferences without stochastic volatility that has a time-varying price of risk. The main focus of our paper is to build a model that generates realistic term premia and decomposes the sources of risk premia. We achieve this by (i) introducing a preference shock that generates flexible time-varying prices of risk, and (ii) allowing shocks to be homoskedastic or heteroskedastic.

Hansen and Scheinkman(2012) discuss theoretical conditions that guarantee a solution to the representative agent's problem under recursive preferences. Campbell, Giglio, Polk, and Turley(2014) discuss existence conditions for their ICAPM model with stochastic volatility.

2 Time-varying risk premia in Gaussian models

In this section, we introduce a model with recursive preferences that generates a time-varying market price of risk when consumption and inflation follow a homoskedastic, Gaussian process. The source of the variation is a preference shock. We use Gaussian models to demonstrate the intuition, which we extend to stochastic volatility models in Section 3.

2.1 Basic framework

Preferences We consider a discrete-time endowment economy. The representative agent optimizes over his lifetime utility

$$V_t = \left[(1-\beta)\, \Upsilon_t C_t^{1-\eta} + \beta \left\{ \mathrm{E}_t \left[V_{t+1}^{1-\gamma} \right] \right\}^{\frac{1-\eta}{1-\gamma}} \right]^{\frac{1}{1-\eta}}, \tag{1}$$

where β is the time discount factor, γ measures risk aversion, and $\psi = \frac{1}{\eta}$ is the elasticity of intertemporal substitution when there is no uncertainty. For convenience, let $\vartheta \equiv \frac{1-\gamma}{1-\eta}$. Our formulation of recursive preferences (1) includes a stochastic rate of time preference with Υ_t as in Albuquerque, Eichenbaum, and Rebelo(2014). Like the habit formation models of Abel(1990) and Campbell and Cochrane(1999), we assume that the representative agent treats it as external to the model when making decisions about consumption.

Agents maximize utility (1) subject to the budget constraint

$$W_{t+1} = (W_t - C_t)\, R_{c,t+1},$$

where $R_{c,t+1}$ is the gross return on the consumption asset and W_t is wealth.

Stochastic discount factor The log stochastic discount factor (SDF) in this model is

$$m_{t+1} = \vartheta \ln(\beta) + \vartheta \Delta v_{t+1} - \eta \vartheta \Delta c_{t+1} + (\vartheta - 1)\, r_{c,t+1}, \tag{2}$$

where $c_t = \ln(C_t)$, $r_{c,t+1} = \ln(R_{c,t+1})$ is the continuously compounded return and $v_{t+1} = \ln\left(\frac{\Upsilon_{t+1}}{\Upsilon_t}\right)$ is the preference shock. Nominal assets are priced using the nominal pricing kernel

$$m_{t+1}^{\$} = m_{t+1} - \pi_{t+1}, \tag{3}$$

where inflation is $\pi_{t+1} = \ln(\Pi_{t+1}) - \ln(\Pi_t)$ and Π_t is the nominal price level.

Dynamics The state of the economy is summarized by a $G \times 1$ vector g_t, which includes consumption growth Δc_t and inflation π_t

$$\Delta c_t = Z_c' g_t, \tag{4}$$

$$\pi_t = Z_\pi' g_t, \tag{5}$$

where Z_c and Z_π are $G \times 1$ selection vectors containing only zeros and ones. The state vector follows a Gaussian process, summarized in companion form as

$$g_{t+1} = \mu_g + \Phi_g g_t + \Sigma_{0,g} \varepsilon_{g,t+1} \qquad \varepsilon_{g,t+1}, \sim \mathrm{N}(0, I). \tag{6}$$

Preference shock Our specification of the preference shocks is more general than in recent papers; see, e.g. Albuquerque, Eichenbaum, and Rebelo(2014) and Schorfheide, Song, and Yaron(2014). We model their stochastic process as

$$\upsilon_{t+1} = Z_\upsilon' g_{t+1} + \Lambda_1(g_t) + \Lambda_2(g_t)' \varepsilon_{g,t+1}, \tag{7}$$

where Z_υ is a $G \times 1$ selection vector and $\Lambda_1(g_t)$ and $\Lambda_2(g_t)$ are risk sensitivity functions. The key function $\Lambda_2(g_t)$ is defined as

$$\Lambda_2(g_t) = -\eta \Sigma_{0,g}^{-1} (\lambda_0 + \lambda_g g_t). \tag{8}$$

It introduces time-varying prices of risk into the Gaussian model. As we explain below, the parameter λ_g determines how important "state dependence" is to the representative agent's utility. The term $\Lambda_1(g_t) = -\frac{\vartheta \eta^2}{2} (\lambda_0 + \lambda_g g_t)' \left(\Sigma_{0,g} \Sigma_{0,g}'\right)^{-1} (\lambda_0 + \lambda_g g_t) + \bar{\Lambda}$ is introduced to keep the model inside the affine family, while the constant $\bar{\Lambda}$ ensures the shocks have mean zero, see Appendix B.

Solution method The SDF in (2) is a function of the return on the consumption asset $r_{c,t+1}$, which is generally regarded as unobserved in the data. We eliminate this variable from the SDF using a standard approach in the finance literature, see Bansal and Yaron(2004) and Bansal, Kiku, and Yaron(2012).[2] We apply the log-linearization technique of Campbell and Shiller(1989) and write it as a function of the price to consumption ratio $r_{c,t+1} = \kappa_0 + \kappa_1 pc_{t+1} - pc_t + \Delta c_{t+1}$, where κ_0 and κ_1 are log-linearization constants that depend on the average price to consumption ratio $\bar{pc}$. As the real pricing kernel in (2) must also price the consumption good with return of $r_{c,t+1}$, we can solve the coefficients in $pc_t = D_0 + D_g' g_t$ as functions of the underlying parameters, including κ_0, κ_1. This is a fixed point problem: pc_t depends on κ_0, κ_1 through D_0, D_g, which in turn depend on $\bar{pc} = \mathrm{E}\left[pc_t\right]$. We discuss this fixed point problem and its solutions in more detail in Appendix C.

2.2 Sources of risk premia

Using the solution method described above, the nominal log-SDF in deviation from the mean form becomes

$$m_{t+1}^{\$} - \mathrm{E}_t\left[m_{t+1}^{\$}\right] \;=\; -\lambda_{g,t}^{\$,\prime}\varepsilon_{g,t+1}, \tag{9}$$

Shocks to the SDF are heteroskedastic with a time-varying price of risk $\lambda_{g,t}^{\$}$ due to the risk sensitivity functions. The time-varying risk premium can be decomposed into the following terms

$$\begin{aligned}
\Sigma_{0,g}\lambda_{g,t}^{\$} \;&=\; \Sigma_{0,g}\Sigma_{0,g}'\left(\gamma Z_c + Z_\pi\right) && \leftarrow \text{power utility} \\
&-\; \kappa_1\frac{(\eta-\gamma)}{(1-\eta)}\Sigma_{0,g}\Sigma_{0,g}'D_g && \leftarrow \text{recursive preferences} \\
&-\; \vartheta\Sigma_{0,g}\Sigma_{0,g}'Z_\upsilon + \eta\vartheta\lambda_0 + \eta\vartheta\lambda_g g_t, && \leftarrow \boxed{\text{preference shocks}}
\end{aligned} \tag{10}$$

[2] The solution method used by Campbell, Giglio, Polk, and Turley(2014) for their ICAPM model is similar, only they substitute out consumption instead of the return on the consumption asset.

where the first term is inherited from power utility, the second line comes from recursive preferences, and the terms in the third line are due to preference shocks.

One contribution of our paper is to introduce time-varying prices of risk into a Gaussian model through the preference shocks. The key term in (10) is $\lambda_g g_t$. Only when λ_g is non-zero does the model have a time-varying price of risk and a time-varying term premia. The affine form of the market prices of risk is similar to the expressions found in Gaussian ATSMs; see, e.g. Duffee(2002). It is this feature that has enabled Gaussian ATSMs to become the benchmark model in the term structure literature. Conversely, term premia are constant in Gaussian models if $\lambda_g = 0$. This includes existing models without preference shocks, or with preference shocks through $Z_v \neq 0$ as in Albuquerque, Eichenbaum, and Rebelo(2014) and Schorfheide, Song, and Yaron(2014).

The remaining terms in (10) are intuitive. Consider the standard case when there are no preference shocks and $Z_v = \lambda_0 = \lambda_g = 0$. If in addition $\eta = \gamma$, the model reduces to power utility with only a constant risk premium term. The magnitude of the risk adjustment is small for any reasonable value of the risk aversion parameter γ. When $\gamma = 0$, investors are risk neutral.

The second term in (10) is due to recursive preferences and the separation of the inverse of the intertemporal elasticity of substitution from risk aversion $\eta \neq \gamma$. The sign of this term depends on whether η is greater or less than γ.[3] If $\gamma > \eta$, the representative agent prefers an earlier resolution of uncertainty, and this term adds a positive value to the risk premium. If on the other hand $\gamma < \eta$, then this term contributes negatively to the risk premium. The magnitude of this term is a function of how far apart γ and η are from one another. The larger their difference the greater the impact recursive preferences have on asset prices. In the other extreme, when $\eta = \gamma$, this term disappears and the model collapses to power utility.

We can alternatively characterize risk aversion as the difference between the parameters

[3]The demonimator $(1-\eta)$ gets canceled out with $D_g = (1-\eta)\left(I_G - \Phi_g'\right)^{-1} \Phi_g' Z_c$

of the nominal risk neutral measure $\mathbb{Q}^{\$}$ and the physical measure $\mathbb{P}$ as

$$\Sigma_{0,g}\lambda_{g,t}^{\$} = \left(\mu_g - \mu_g^{\mathbb{Q},\$}\right) + \left(\Phi_g - \Phi_g^{\mathbb{Q},\$}\right) g_t.$$

The implied dynamics of g_t under $\mathbb{Q}^{\$}$ are also a Gaussian vector autoregression

$$g_{t+1} = \mu_g^{\mathbb{Q},\$} + \Phi_g^{\mathbb{Q},\$} g_t + \Sigma_{0,g}\varepsilon_{g,t+1}^{\mathbb{Q},\$}, \qquad \varepsilon_{g,t+1}^{\mathbb{Q},\$} \sim \mathrm{N}\left(0, I\right). \tag{11}$$

and the relationship between the $\mathbb{P}$ and nominal $\mathbb{Q}^{\$}$ parameters is

$$\mu_g^{\mathbb{Q},\$} = \mu_g - \Sigma_{0,g}\Sigma_{0,g}'\left(\gamma Z_c + Z_\pi\right) + \kappa_1 \frac{(\eta-\gamma)}{(1-\eta)}\Sigma_{0,g}\Sigma_{0,g}' D_g + \vartheta\Sigma_{0,g}\Sigma_{0,g}' Z_\upsilon - \eta\vartheta\lambda_0, \tag{12}$$

$$\Phi_g^{\mathbb{Q},\$} = \Phi_g - \eta\vartheta\lambda_g, \tag{13}$$

If $\lambda_g = 0$, then $\Phi_g = \Phi_g^{\mathbb{Q},\$}$. The parameters of the real risk neutral measure $\mathbb{Q}$ can be found by setting $Z_\pi = 0$.

2.3 Bond prices

The price of a zero-coupon real bond with maturity n at time t is the expected price of the same asset at time $t+1$ discounted by the stochastic discount factor

$$P_t^{(n)} = \mathrm{E}_t\left[\exp\left(m_{t+1}\right) P_{t+1}^{(n-1)}\right]. \tag{14}$$

Using standard techniques for bond pricing in Gaussian models (see Ang and Piazzesi(2003), Creal and Wu(2015a)), the real yield can be expressed as a linear function of the state vector

$$y_t^{(n)} \equiv -\frac{1}{n}\ln\left(P_t^{(n)}\right) = a_n + b_{n,g}' g_t. \tag{15}$$

Similarly, we can derive nominal yields as

$$y_t^{\$,(n)} \equiv -\frac{1}{n}\ln\left(P_t^{\$,(n)}\right) = a_n^{\$} + b_{n,g}^{\$,\prime} g_t, \tag{16}$$

where the real and nominal bond-loadings (a_n, b_n) and $(a_n^{\$}, b_n^{\$})$ follow difference equations given in Appendix D.

Real yields can be expressed as the expected average future path of consumption between t and $t+n$ scaled by the inverse of the elasticity of intertemporal substitution plus a risk premium and Jensen's inequality terms

$$y_t^{(n)} = -\ln(\beta) + \frac{1}{\psi}\frac{1}{n}\sum_{j=1}^{n}\mathrm{E}_t\left[\Delta c_{t+j}\right] + \frac{1}{\psi}\frac{1}{n}\sum_{j=1}^{n}\left(\mathrm{E}_t^{\mathbb{Q}}\left[\Delta c_{t+j}\right] - \mathrm{E}_t\left[\Delta c_{t+j}\right]\right) + \text{Jensen's Ineq.}$$

where we have set $Z_v = 0$. The risk premium is the difference between the expected average path of consumption under the $\mathbb{P}$ and $\mathbb{Q}$ measures. It varies over time if $\lambda_g \neq 0$ otherwise the difference between expected consumption is a constant.

A nominal yield has a similar expression only it takes into account the expected average future path of inflation as well as consumption

$$\begin{aligned} y_t^{\$,(n)} &= -\ln(\beta) + \frac{1}{\psi}\frac{1}{n}\sum_{j=1}^{n}\mathrm{E}_t\left[\Delta c_{t+j}\right] + \frac{1}{\psi}\frac{1}{n}\sum_{j=1}^{n}\left(\mathrm{E}_t^{\mathbb{Q},\$}\left[\Delta c_{t+j}\right] - \mathrm{E}_t\left[\Delta c_{t+j}\right]\right) \\ &\quad + \frac{1}{n}\sum_{j=1}^{n}\mathrm{E}_t\left[\pi_{t+j}\right] + \frac{1}{n}\sum_{j=1}^{n}\left(\mathrm{E}_t^{\mathbb{Q},\$}\left[\pi_{t+j}\right] - \mathrm{E}_t\left[\pi_{t+j}\right]\right) + \text{Jensen's Ineq.} \end{aligned}$$

where we have set $Z_v = 0$. If the agent is sensitive to inflation shocks, he requires additional compensation for uncertainty about future inflation. In standard macroeconomic models, the difference between nominal and real yields of a given maturity $y_t^{\$,(n)} - y_t^{(n)}$ reflects average expected inflation over this period. Our expressions suggest that this difference can include a risk premium component that may vary through time.

The nominal term premium is defined as the difference between the model implied yield

$y_t^{\$,(n)}$ in (16) and the average of expected future short rates over the same period

$$tp_t^{\$,(n)} = y_t^{\$,(n)} - \frac{1}{n}\mathrm{E}_t\left[r_t^{\$} + r_{t+1}^{\$} + \ldots, + r_{t+n-1}^{\$}\right]. \tag{17}$$

The term premium has a simple portfolio interpretation. An investor can buy an n-period bond and hold it until maturity or he can purchase a sequence of 1 period bonds, repeatedly rolling them over for n periods. The term premium measures the additional compensation a risk averse agent needs to choose one option over another. Under the expectations hypothesis, term premia are constant. For Gaussian models with homoskedastic shocks, this coincides with setting $\lambda_g = 0$ and eliminating time-variation in the risk sensitivity functions.

2.4 Discussion on preference shocks

Preference shocks have been used to add flexibility to the SDF in (2) in order to better explain variation in stock prices. We focus on their implications for bond prices, especially bond risk premia. Time variation of risk premia has important implications for market participants as well as central bankers, but realistic dynamics are difficult to capture in a consumption-based model. We use the preference shock to achieve both.

The risk sensitivity functions (8) are motivated by the habit formation model of Campbell and Cochrane(1999). In their model, preference shocks are induced by habit formation with their influence on the SDF determined by past shocks to aggregate consumption. Importantly, past shocks to consumption are scaled by a risk sensitivity function like $\Lambda_2(g_t)$ that generates a time-varying market price of risk. The sensitivity function (8) is different than Campbell and Cochrane(1999) for two reasons. First, it may depend on all shocks $\varepsilon_{g,t}$ that hit the state vector g_t in (6) instead of only shocks to consumption growth. This introduces flexibility into the price of risk. Specifically, the non-zero columns of the matrix λ_g determine which elements of the state vector g_t have an impact on agent's preferences. If columns of λ_g are set to zero, it shuts down state dependence in the corresponding element of the state vector. Second, the risk sensitivity functions $\Lambda_1(g_t)$ and $\Lambda_2(g_t)$ result in closed-form real and nominal bond prices as shown in Section 2.3, whereas the traditional risk sensitivity functions in models of habit formation do not; see, e.g. Wachter(2006).[4]

The process for υ_{t+1} in (7) allows two sources of variation in the preference shocks. If there are no risk sensitivity functions $\Lambda_1(g_t) = \Lambda_2(g_t) = 0$ and $Z_\upsilon = 0$, preference shocks υ_{t+1} can be present as a latent variable in the state vector g_t. This is consistent with most of the macroeconomics literature, see also Albuquerque, Eichenbaum, and Rebelo(2014) and Schorfheide, Song, and Yaron(2014). Latent factors can help explain variation in asset prices and improve the fit of a model but they do not introduce a time-varying market price of risk without the risk sensitivity functions. Alternatively, if we shut down latent factors by setting $Z_\upsilon = 0$, preference shocks do not have their own source of variation and are driven entirely by shocks to consumption growth and inflation but scaled by the risk sensitivity functions. In this case, the key term is $\Lambda_2(g_t)$. It introduces time-varying market prices of risk into agent's preferences. Finally, our approach allows for both latent factors and risk sensitivity functions, making the model a Gaussian ATSM with macro factors.

Piazzesi and Schneider(2007) study an economy with recursive preferences when the inverse of the elasticity of intertemporal substitution equals one, $\eta = 1$. In this case, recursive preferences can be solved analytically; see, e.g. Tallarini(2000), Hansen, Heaton, and Li(2008), and Appendix C.3. The difference between our work and theirs for Gaussian models are the preference shocks, which are critical for introducing time-variation into the term premium.

3 Interaction between price and quantity of risk

In this section, we introduce preference shocks into models where shocks to consumption growth and inflation have stochastic volatility. In models with recursive preferences, stochas-

[4]When $\gamma = \eta$, the model reduces to power utility. The risk sensitivity functions here can be chosen to make the SDF in (2) observationally equivalent to the SDF of the habit formation model in Wachter(2006) by setting $\Lambda_1 = (1 - \phi_\upsilon)\bar{\upsilon} + \phi_\upsilon \upsilon_t$ and $\Lambda_2 = \frac{1}{\Upsilon}\sqrt{1 - 2(\upsilon_t - \bar{\upsilon})} - 1$.

tic volatility is typically the sole source of time variation in term premia, e.g. Bansal and Yaron(2004). We demonstrate how the preference shock, introduced in Section 2, induces another source of variation in addition to stochastic volatility. Then, in Section 5, we decompose the contribution each channel has on the term premium as we see in the data.

Dynamics We generalize the dynamics of the state vector in (6) to

$$
\begin{aligned}
g_{t+1} &= \mu_g + \Phi_g g_t + \Phi_{gh} h_t + \Sigma_{gh}\varepsilon_{h,t+1} + \Sigma_{g,t}\varepsilon_{g,t+1} \qquad \varepsilon_{g,t+1} \sim \mathrm{N}(0, I) && (18)\\
\Sigma_{g,t}\Sigma'_{g,t} &= \Sigma_{0,g}\Sigma'_{0,g} + \sum_{i=1}^{H} \Sigma_{i,g}\Sigma'_{i,g} h_{it} \\
h_{t+1} &\sim \mathrm{NCG}(\nu_h, \Phi_h, \Sigma_h) && (19)\\
\varepsilon_{h,t+1} &= h_{t+1} - \mathrm{E}_t[h_{t+1}|h_t]
\end{aligned}
$$

where h_t is a $H \times 1$ vector following a non-central gamma (NCG) process as in Creal and Wu(2015a). This is an affine process that is the exact discrete time equivalent of a multivariate Cox, Ingersoll, and Ross(1985) process. The conditional mean is $\mathrm{E}_t[h_{t+1}|h_t] = \Sigma_h \nu_h + \Phi_h h_t$ meaning that Φ_h controls the autocovariances and $\Sigma_h \nu_h$ is the intercept. Σ_h is a matrix of scale parameters and ν_h are a vector of shape parameters. The vector $\varepsilon_{h,t+1}$ are shocks to volatility with conditional heteroskedasticity $\mathrm{V}_t[h_{t+1}|h_t] = \Sigma_{h,t}\Sigma'_{h,t}$, and Σ_{gh} measures the covariance between Gaussian and non-Gaussian shocks, i.e. the leverage effect. Further details on properties of the model can be found in Appendix A.[5]

[5]The timing of how volatility scales the shocks in discrete-time models such as (18) is an outstanding issue in financial econometrics. Both timings $\Sigma_{g,t}\varepsilon_{g,t+1}$ and $\Sigma_{g,t+1}\varepsilon_{g,t+1}$ lead to log-SDF's that are linear in the state variables and produce affine bond prices. Our analysis focuses on the timing in (18), which is common in finance. However, the alternative timing $\Sigma_{g,t+1}\varepsilon_{g,t+1}$ means the representative agent faces greater short term uncertainty. Volatility has an immediate and potentially more significant impact on the short rate (and consequently all yields) even when there is no leverage effect.

Preference shocks We generalize the preference shocks in (7) so that they can potentially depend on shocks to volatility.

$$\upsilon_{t+1} = Z_{\upsilon}' g_{t+1} + \Lambda_1 (g_t, h_t) + \Lambda_2 (g_t, h_t)' \varepsilon_{g,t+1} + \Lambda_3 (h_t)' \varepsilon_{h,t+1}$$

Our specification for the risk sensitivity functions is a straightforward generalization of the Gaussian model

$$\begin{aligned}
\Lambda_1 (g_t, h_t) &= \bar{\Lambda} - \frac{\vartheta \eta^2}{2} (\lambda_0 + \lambda_g g_t + \lambda_{gh} h_t)' \left(\Sigma_{g,t} \Sigma_{g,t}' \right)^{-1} (\lambda_0 + \lambda_g g_t + \lambda_{gh} h_t), \\
\Lambda_2 (g_t, h_t) &= -\eta \Sigma_{g,t}^{-1} (\lambda_0 + \lambda_g g_t + \lambda_{gh} h_t), \\
\Lambda_3 (h_t) &= -\lambda_h.
\end{aligned}$$

In theory, we can make $\Lambda_3 (h_t)$ time-varying as well. But to keep the model simple and see how much mileage we gain from the single source of variation in $\Lambda_2 (g_t, h_t)$, we only allow the sensitivity function for the Gaussian shock $\Lambda_2 (g_t, h_t)$ to vary over time. Under these assumptions, the key term is again $\Lambda_2 (g_t, h_t)$. It introduces flexibility into the cross-section of asset prices. The constant $\bar{\Lambda}$ can be found in Appendix B.

SDF To obtain an expression for the SDF, we follow the same procedure as the Gaussian model. The only difference is the price to consumption ratio is also a function of the volatility $pc_t = D_0 + D_g' g_t + D_h' h_t$. In Section 4, we describe the solution method in detail, and discuss how the existence of a solution to the model restricts the structural parameters (β, γ, ψ).

The nominal log-SDF can be written as

$$m_{t+1}^{\$} - \mathrm{E}_t \left[m_{t+1}^{\$} \right] = -\lambda_{g,t}^{\$,\prime} \varepsilon_{g,t+1} - \lambda_{h,t}^{\$,\prime} \tilde{\varepsilon}_{h,t+1} \tag{20}$$

where the shock $\tilde{\varepsilon}_{h,t+1} = \Sigma_{h,t}^{-1} \varepsilon_{h,t+1}$ is the standardized shock to volatility with unit variance.

Sources of risk premia First, we decompose the risk premium for Gaussian shocks into their sources

$$
\begin{aligned}
\Sigma_{g,t}\lambda_{g,t}^{\$} \quad = \quad & \Sigma_{0,g}\Sigma_{0,g}'\left(\gamma Z_c + Z_\pi\right) - \kappa_1\frac{(\eta-\gamma)}{(1-\eta)}\Sigma_{0,g}\Sigma_{0,g}'D_g - \vartheta\Sigma_{0,g}\Sigma_{0,g}'Z_v + \eta\vartheta\lambda_0 + \eta\vartheta\lambda_g g_t \\
& +(\gamma Z_c + Z_\pi \otimes I_G)'\,\tilde{S}_g h_t \qquad\qquad \leftarrow \text{power utility} \\
& -\kappa_1\frac{(\eta-\gamma)}{(1-\eta)}\left(D_g \otimes I_G\right)'\tilde{S}_g h_t \qquad\qquad \leftarrow \text{recursive preferences} \\
& +\eta\vartheta\lambda_{gh}h_t - \vartheta\left(Z_v \otimes I_G\right)'\tilde{S}_g h_t \qquad\qquad \leftarrow \text{preference shock} \\
& + \text{leverage effect} \qquad\qquad (21)
\end{aligned}
$$

where $\tilde{S}_g$ is a $G^2 \times H$ matrix with columns vec $\left(\Sigma_{g,i}\Sigma_{g,i}'\right)$ for $i = 1, \ldots, H$.

The terms in the first line of (21) are the same as the Gaussian model in (10). Importantly, Gaussian factors g_t only impact the price of risk through the preference shock when $\lambda_g \neq 0$; otherwise, the price of risk is not a function g_t. Economically, if $\lambda_g = 0$, time variation in the risk premium cannot be driven by any elements of g_t such as (expected) inflation and/or consumption growth.

Stochastic volatility introduces the remaining terms. The second line is due to the interaction between power utility and stochastic volatility. It has a similar functional form as the power utility term in the Gaussian model except that the new term varies through time with h_t. The third line comes from recursive preferences. Similar to the Gaussian model in (10), the sign and magnitude of this term depends on the relative sizes of η and γ. If $\gamma > \eta$, the representative agent prefers an earlier resolution of uncertainty, and this term adds a positive value to the risk premium. The two terms in line 4 come from preference shocks. The main difference between the terms in line 2-4 and their counterparts in line 1 is that they all vary across time as a function of the volatility h_t. The last line comes from the leverage effect, which is 0 when $\Sigma_{gh} = 0$. A detailed expression for this term can be found in Appendix A.2.

Next, we decompose the prices of risk for the non-Gaussian shocks $\lambda_{h,t}^{\$}$ as

$$\begin{aligned} \Sigma_{h,t}^{\prime,-1}\lambda_{h,t}^{\$} &= \Sigma_{gh}^{\prime}\left(\gamma Z_c + Z_\pi\right) && \leftarrow \text{power utility} \\ &-\kappa_1 \frac{(\eta-\gamma)}{(1-\eta)}\left(\Sigma_{gh}^{\prime} D_g + D_h\right) && \leftarrow \text{recursive preference} \\ &-\vartheta\left(\Sigma_{gh}^{\prime} Z_v - \lambda_h\right) && \leftarrow \text{preference shock} \end{aligned} \tag{22}$$

The three terms have similar features and functional forms as those in (10) and (21). Power utility only has an impact on the price of volatility risk through the leverage effect when $\Sigma_{gh} \neq 0$, while recursive preferences will generate a price of risk even when $\Sigma_{gh} = 0$.

Under the $\mathbb{Q}^{\$}$ measure, the factors have the same dynamics as under the $\mathbb{P}$ measure and are a Gaussian vector autoregression with stochastic volatility as in (18)-(19) but with updated parameters for $(\mu_g^{\mathbb{Q},\$}, \Phi_g^{\mathbb{Q},\$}, \Phi_{gh}^{\mathbb{Q},\$}, \Phi_h^{\mathbb{Q},\$}, \Sigma_h^{\mathbb{Q},\$})$, see Appendix A.2 for detailed expressions. We can write the risk premium for Gaussian shocks as a relationship between the $\mathbb{P}$ and $\mathbb{Q}^{\$}$ parameters as

$$\Sigma_{g,t}\lambda_{g,t}^{\$} = \left(\mu_g - \mu_g^{\mathbb{Q},\$}\right) + \left(\Phi_g - \Phi_g^{\mathbb{Q},\$}\right) g_t + \left(\Phi_{gh} - \Phi_{gh}^{\mathbb{Q},\$}\right) h_t, \tag{23}$$

and, when Σ_h is diagonal, $\lambda_{h,t}^{\$}$ can be written as

$$\lambda_{h,t}^{\$} = \Sigma_h^{-1}\left(\Sigma_h^{\mathbb{Q},\$}\right)^{-1}\Sigma_h - \Sigma_h^{-1}. \tag{24}$$

Like the Gaussian model in Section 2, the price of risk $\lambda_{g,t}^{\$}$ is not a function of g_t without preference shocks. This is because the physical and risk-neutral measures are equal $\Phi_g = \Phi_g^{\mathbb{Q},\$}$ when $\lambda_g = 0$. In standard models with recursive preferences and no preference shocks such as Bansal and Yaron(2004), time-variation in the price of risk $\lambda_{g,t}^{\$}$ is driven by stochastic volatility because in these models the risk neutral and physical measures for volatility are typically not equal $\Phi_{gh} \neq \Phi_{gh}^{\mathbb{Q},\$}$.

Bond prices Following Creal and Wu(2015a), we derive real and nominal yields for discrete-time models with stochastic volatility using the standard formula for asset prices (14). Yields are an affine function of both the Gaussian state vector and volatility

$$y_t^{(n)} = a_n + b_{n,g}' g_t + b_{n,h}' h_t, \tag{25}$$

$$y_t^{\$,(n)} = a_n^{\$} + b_{n,g}^{\$,\prime} g_t + b_{n,h}^{\$,\prime} h_t, \tag{26}$$

where the bond-loadings are in Appendix D.

4 Model properties

Empirical examination of the asset pricing implications of recursive preferences requires solving for the SDF. In the approach used here, we need to solve for the return on the consumption asset $r_{c,t+1}$ in (2) as a function of the underlying state of the economy. Whether there exists an economy that is consistent with recursive preferences, mathematically amounts to a fixed point problem. In this section, we provide the conditions that lead to a valid solution for the Euler equation and asset prices. We base our analysis on the approximation method used by Bansal, Kiku, and Yaron(2012) and Schorfheide, Song, and Yaron(2014), among many others.

We partition the vector of all parameters of the model $\theta = (\beta, \psi, \gamma, \theta^r)$ into the structural parameters (β, ψ, γ) and the rest of the parameters θ^r. We condition our analysis on θ^r and characterize the restrictions on the parameter space for the structural parameters.

4.1 Gaussian models

For models with Gaussian dynamics defined in Section 2, the following proposition provides a sufficient condition for the fixed point problem to have a solution.

Proposition 1 *There is a value $\bar{\beta}(\psi, \gamma, \theta^r)$ such that if $\beta < \bar{\beta}$, then there exists a solution for the fixed point problem.*

Proof: See Appendix E.1.

The proposition provides a general condition that guarantees a solution to the representative agent's problem for any Gaussian model and characterizes the joint restrictions that exist between the structural parameters (β, γ, ψ). Given the dynamics of the economy and the parameters driving the preference shock in θ^r, agents' risk appetite γ, and the intertemporal elasticity of substitution ψ, the representative agent needs to be sufficiently impatient (small β) in order for a solution to exist.

For some special cases with no preference shocks, we can simplify the general condition between (β, γ, ψ) to only restrict γ. The following corollary also characterizes the upper bound $\bar{\beta}$ as a monotonic function in γ.

Corollary 1 *If there is no preference shock, i.e., $\lambda_0 = 0, \lambda_g = 0, \bar{\Lambda} = 0, Z_v = 0$, then*

1. *If $Z_1^{\infty\prime}\mu_g \leq 0$, then $\vartheta = \frac{1-\gamma}{1-\eta} < 0$ guarantees the existence of a solution.*
2. *If $\beta = 1$, there is a value $\bar{\gamma}(\theta^r)$ such that $\frac{\bar{\gamma}-\gamma}{1-\eta} < 0$ guarantees a solution.*
3. *For any ψ, $\bar{\beta}$ is monotonic in γ: for $\psi > 1$, then $\frac{d\bar{\beta}}{d\gamma} > 0$; for $\psi < 1$, then $\frac{d\bar{\beta}}{d\gamma} < 0$.*

Under the condition specified in part 1 of Corollary 1, a solution exists if $(\gamma > 1,\ \psi > 1)$ or $(\gamma < 1,\ \psi < 1)$. This divides the parameter space for (γ, ψ) into four quadrants, and only two of these four have a solution. Part 2 of Corollary 1 says that if an agent is perfectly patient with $\beta = 1$, then $(\gamma > \bar{\gamma},\ \psi > 1)$ or $(\gamma < \bar{\gamma},\ \psi < 1)$ guarantees a solution. Again, two out of the four quadrants have a solution, similar to part 1. The intuition is also similar. Although the cutoff for ψ is always 1, the difference is the boundary on γ now depends on the parameters driving the dynamics of the model in θ^r.

The upper left panel of Figure 1 provides a visualization for parts 1-2 of Corollary 1. This plot is based on the Gaussian dynamics of Model #1 (long-run risk) in Section 5.1 with no preference shocks. We set $\beta = 1$, which is consistent with the assumption in part 2 of the corollary. The remaining parameters of the model are fixed at their posterior mean values from Table 1. The plot illustrates how the parameter space for (γ, ψ) is partitioned into 4 regions, and the areas that have a solution are the lower-left and upper-right quadrants. As stated in the corollary, the cutoff for ψ is always 1 while the boundary for γ is determined by the unconditional mean of g_t and the variance, see Appendix E.1.

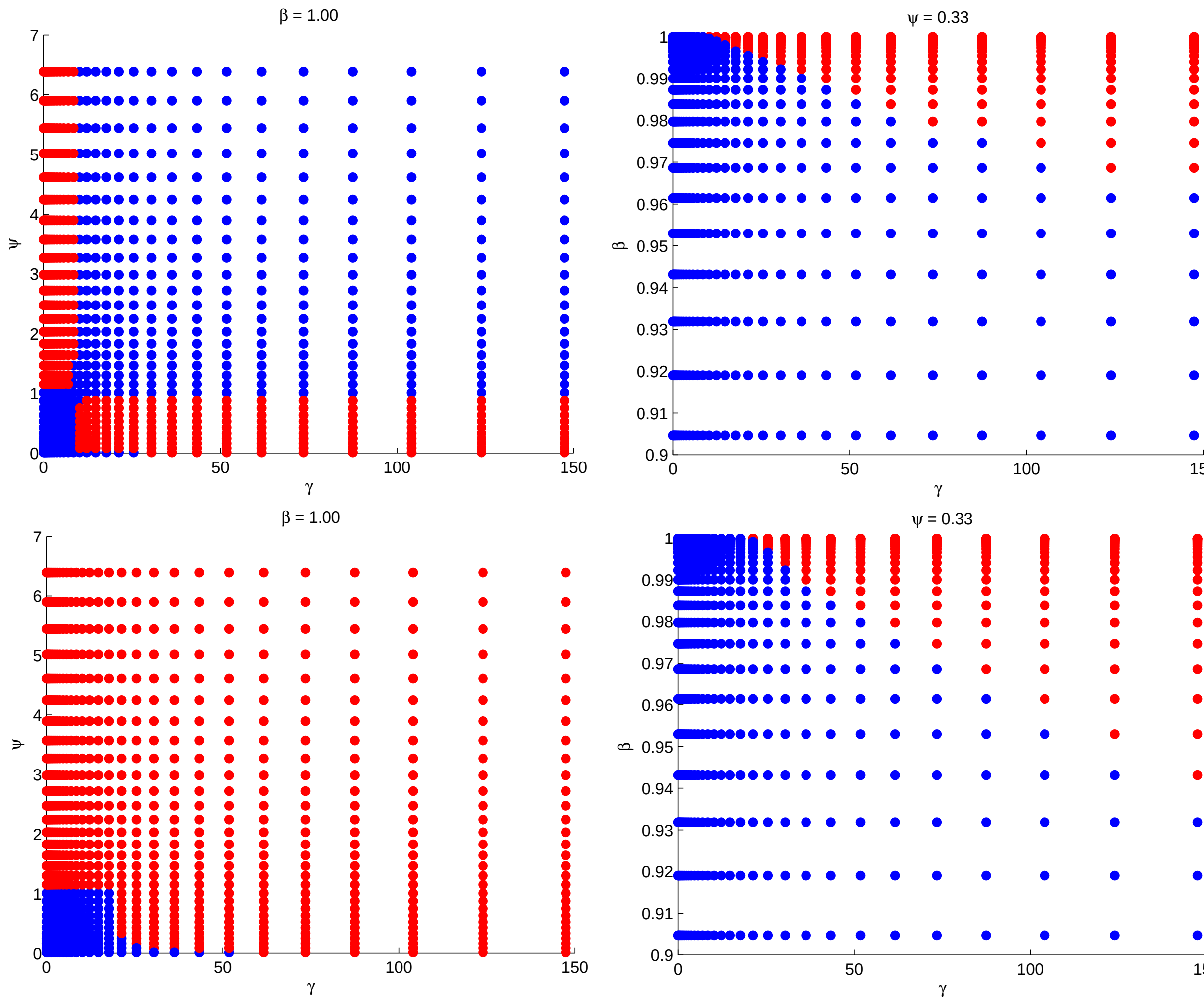
The separation of the parameter space into quadrants makes estimation more challenging. For example, if the optimum is within the upper-right region and we start from the lower left region, a numerical optimization algorithm or a Bayesian MCMC algorithm, can have a hard time getting through the tiny bottleneck and reaching the correct part of the parameter space. In practice, we observe these algorithms hitting the (red) regions where no solution exists and often stopping. Estimation gets more complicated when the structural parameters interact with the remaining parameters of the model as the boundaries can shift creating strong dependencies among the model's parameters.

Corollary 1 part 3 states the relationship between the upper bound for β and γ. If the substitution effect dominates the wealth effect, then the more risk averse the agent is the more patient he must be. If the wealth effect dominates the substitution effect, then a more risk averse agents need to be less patient and discount the future faster. The case $\psi < 1$ is shown in the right panel of Figure 1, where we use $\psi = 0.33$ for demonstration purposes. The graph shows a downward sloping line that separates the parameter space for (β, γ) into feasible (blue) and infeasible (red) regions. The larger the value of risk aversion γ gets, the smaller the discount rate β needs to be to remain in a region with a valid solution.

4.2 Non-Gaussian models

Stochastic volatility introduces several new restrictions on the parameter space. A solution to the fixed point problem requires a solution for the loadings D_h in the price to consumption ratio $pc_t = D_0 + D_g' g_t + D_h' h_t$. A real solution for D_h does not always exist nor is it unique.

Figure 1: Feasible and infeasible regions of the parameter space.



Feasible (blue) and infeasible (red) regions of the parameters space. Figures are based on Gaussian Model #1 (long-run risk) with no preference shocks. Top row has no stochastic volatility and bottom row has stochastic volatility. Top left: Parameter space for (γ, ψ) *with* $\beta = 1.0$. *Top right: Parameter space for* (γ, β) *with* $\psi = 0.33$. *Bottom left: Parameter space for* (γ, ψ) *with* $\beta = 1.0$. *Bottom right: Parameter space for* (γ, β) *with* $\psi = 0.33$. *Parameters* θ^r *are equal to the posterior mean values from Table 1.*

The solution can be calculated in closed-form when both Σ_h and Φ_h are lower-triangular or when Σ_h is diagonal. For simplicity, we derive the conditions when Σ_h and Φ_h are diagonal.

Assumption 1 *For any real* $\bar{pc}$*, the loadings* D_h *have a real solution.*

Assumption 2 *For any real* $\bar{pc}$ *and* D_h*, the conditional Laplace transform exists.*

Both Assumptions 1 and 2 are necessary conditions that impose restrictions on the parameter space. Assumption 1 amounts to the existence of a real solution for a series of H quadratic equations in H unknowns. The solution of each equation requires their respective discriminant to be positive.[6] Assumption 2 is a necessary condition for the existence of the conditional expectations when stochastic volatility follows a multivariate Cox, Ingersoll, and Ross(1985) process. Appendix C discusses these conditions in more detail.

Conditional on these being satisfied, we have the following proposition.

Proposition 2 *Given Assumption 1-2, there is a value $\bar{\bar{\beta}}(\psi, \gamma, \theta^r)$ such that if $\beta < \bar{\bar{\beta}}$, then there exists a real solution for the fixed point problem.*

Proof: See Appendix E.2.

Proposition 2 provides a sufficient condition for the existence of a real solution for the fixed point problem, that is a counterpart to Proposition 1 for the Gaussian model. The difference from the Gaussian model are the additional restrictions that Assumptions 1-2 impose on the parameter space and the function $\bar{\bar{\beta}}(\psi, \gamma, \theta^r)$ is more general. The nature of the fixed point problem requires that all three conditions be jointly satisfied for a solution to exist.

In Figure 1, we plot the structural parameters (β, γ, ψ) that illustrate the conditions above. First, we focus on the case when $\beta = 1$ and the representative agent is extremely patient, see the bottom left plot. The upper-left and lower-right regions remain infeasible as before with similar intuition as Gaussian models. The difference is now the upper-right region becomes infeasible in addition to the earlier regions in order to satisfy Assumptions 1-2. This emphasizes that in stochastic volatility models both the intertemporal elasticity of substitution and risk aversion need to be modest. The implications are two-fold. First, much of the economics literature evaluates a model's success according to whether or not it can produce a small value of γ. We need to interpret this result with caution. As we show,

[6] This condition is similar to an existence condition discussed by Campbell, Giglio, Polk, and Turley(2014) in their ICAPM model. They do not provide a condition guaranteeing a solution to the fixed point problem.

for stochastic volatility models with no preference shocks, a small value of γ is required to satisfy the constraints of the model and is not necessarily a feature of a specific consumption-based model. Second, stochastic volatility models have much smaller feasible regions of the parameter space, and they are more likely to encounter numerical problems and boundaries.

The bottom right panel in Figure 1 is similar to the upper right plot. Again the downward sloping line that divides the blue and red regions indicates that with a small intertemporal elasticity of substitution, an agent needs to be more patient as their risk aversion increases. This replicates the result from the Gaussian models.

5 Slope of the yield curve

In the next two sections, we investigate the ability of consumption based models to fit key moments of the term structure empirically. The moments that we focus on are the unconditional slope of the yield curve (this section) and the variation in the term premium (next section).

5.1 Dynamics for consumption and inflation

Model #1: Long-run risk in consumption and inflation Consider a model with long-run risks for consumption growth as in Bansal and Yaron(2004) and a long-run risk factor in inflation similar to Stock and Watson(2007). The model is

$$\pi_{t+1} = \bar{\pi}_t + \varepsilon_{\pi_1,t+1} \qquad \varepsilon_{\pi_1,t+1} \sim \mathrm{N}\left(0, h_{t,\pi_1}\right) \tag{27}$$

$$\Delta c_{t+1} = \bar{c}_t + \varepsilon_{c_1,t+1} \qquad \varepsilon_{c_1,t+1} \sim \mathrm{N}\left(0, h_{t,c_1}\right) \tag{28}$$

$$\bar{\pi}_{t+1} = \mu_\pi + \phi_\pi \bar{\pi}_t + \varepsilon_{\pi_2,t+1} \qquad \varepsilon_{\pi_2,t+1} \sim \mathrm{N}\left(0, h_{t,\pi_2}\right) \tag{29}$$

$$\bar{c}_{t+1} = \mu_c + \phi_c \bar{c}_t + \sigma_{c,\pi}\varepsilon_{\pi_2,t+1} + \varepsilon_{c_2,t+1} \qquad \varepsilon_{c_2,t+1} \sim \mathrm{N}\left(0, h_{t,c_2}\right) \tag{30}$$

where $\bar{c}_t$ is the long-run growth rate of consumption (long-run risk) and $\bar{\pi}_t$ is the trend in inflation. The shocks $\varepsilon_{c_1,t}$ and $\varepsilon_{\pi_1,t}$ are transitory, and determine the high-frequency movements in their respective series whereas $\varepsilon_{\pi_2,t}$ and $\varepsilon_{c_2,t}$ are shocks to their persistent components. In our model, shocks to long run inflation have a contemporaneous impact on the long run risk, and all shocks have stochastic volatility. The state vectors are $g_t = (\pi_t, \Delta c_t, \bar{\pi}_t, \bar{c}_t)'$ and $h_t = (h_{t,\pi_1}, h_{t,c_1}, h_{t,\pi_2}, h_{t,c_2})'$. See Appendix A.3 for more details.

Model #2: VARMA(1,1) Similar to Wachter(2006) and Piazzesi and Schneider(2007), we also consider a VARMA(1,1) model for consumption growth and inflation given by

$$\begin{aligned}
\pi_{t+1} &= \mu_\pi + \phi_\pi \pi_t + \phi_{\pi,c}\Delta c_t + \varepsilon_{\pi,t+1} + \psi_\pi \varepsilon_{\pi,t} & \varepsilon_{\pi,t+1} \sim \mathrm{N}\left(0, h_{t,\pi}\right) \quad (31)\\
\Delta c_{t+1} &= \mu_c + \phi_{c,\pi}\pi_t + \phi_c \Delta c_t + \sigma_{c,\pi}\varepsilon_{\pi,t+1} + \varepsilon_{c,t+1} + \psi_c \varepsilon_{c,t} & \varepsilon_{c,t+1} \sim \mathrm{N}\left(0, h_{t,c}\right) \quad (32)
\end{aligned}$$

Shocks to consumption growth and inflation are correlated and both have stochastic volatility. The state vectors are $g_t = (\pi_t, \Delta c_t, \varepsilon_{\pi,t}, \varepsilon_{c,t})'$ and $h_t = (h_{t,\pi}, h_{t,c})'$. See Appendix A.4 for more details.

5.2 Data and estimation

It is well established in the Gaussian ATSM literature that if we allow latent factors, any procedure to fit the yield curve will select the level, slope and curvature as factors. Some factors in (27)-(32) are inherently latent (e.g., the long run inflation $\bar{\pi}_t$ and consumption $\bar{c}_t$). If we simultaneously fit yields, inflation and consumption data, the estimated latent factors, such as $\bar{\pi}_t$ and $\bar{c}_t$, will look like level and slope, and lose their original purpose of having a structural economic underpinning. The reasons are simple. First, we have more yield data than consumption or inflation. Second, the conditional variance of yields is small compared to consumption growth and inflation, which have greater high frequency variation. Therefore, any sensible estimation procedure will emphasize the fit of yields first while fitting consumption growth and inflation poorly.

To maintain the economic interpretation of any latent factors, we constrain the factors to fit the macroeconomic variables through a two step estimation procedure.[7] First, we estimate time series models for consumption growth and inflation, which gives us estimates of the $\mathbb{P}$ parameters $\theta^{\mathbb{P}}$ and the pricing factors g_t and h_t. Conditional on these estimates, we run cross-sectional regressions of yields on these factors to estimate the structural parameters.

Data Our measure of monthly real per capita consumption growth is constructed from nominal non-durables and services data downloaded from the NIPA tables at the U.S. Bureau of Economic Analysis. We deflate each of these series by their respective price indices, add them together, and divide by the civilian population. The population series and monthly U.S. CPI inflation are downloaded from the Federal Reserve Bank of St. Louis. Yields are the Fama and Bliss(1987) zero coupon bond data available from the Center for Research in Securities Prices (CRSP) with maturities of (1, 3, 12, 24, 36, 48, 60) months. The data spans from February 1959 through June 2014 for a total of $T = 665$ observations.

State space form Stacking yields $y_t^{\$,(n)}$ and bond loadings in order for N different maturities $n_1, n_2, ..., n_N$ gives $y_t^{\$} = \left(y_t^{\$,(n_1)}, y_t^{\$,(n_2)}, \ldots, y_t^{\$,(n_N)}\right)$, $A^{\$} = (a_{n_1}^{\$}, \ldots, a_{n_N}^{\$})'$, $B_g^{\$} = (b_{g,n_1}^{\$,\prime}, ..., b_{g,n_N}^{\$,\prime})'$,and $B_h^{\$} = (b_{h,n_1}^{\$,\prime}, ..., b_{h,n_N}^{\$,\prime})'$. The observation equations for yields are measured with errors

$$y_t^{\$} = A + B_g^{\$} g_t + B_h^{\$} h_t + \eta_t$$

where η_t is a vector of pricing errors.

Estimation of the time series dynamics We estimate time series models for consumption growth and inflation by a particle Gibbs sampler which is an algorithm recently devel-

[7]Formally, our procedure can be interpreted as an indirect inference estimator where there are more moment conditions than parameters to estimate. The weighting matrix is implicitly chosen so that the $\mathbb{P}$ parameters are estimated entirely from the time series using no cross-sectional information.

Table 1: Estimates of Model #1 and #2 with and without stochastic volatility

	Pr. m	Pr. St. D.	Post. m	Post. St. D.		Pr. m	Pr. St. D.	Post. m	Post. St. D.
μ_π	0.001	0.010	1.12e-04	5.22e-05	μ_π	0.001	0.010	0.62e-04	2.85e-05
μ_c	0.001	0.010	0.50e-04	2.53e-05	μ_c	0.001	0.010	1.34e-03	5.06e-05
ϕ_π	0.950	0.030	0.966	0.014	ϕ_π	0.950	0.030	0.975	0.010
ϕ_c	0.950	0.030	0.962	0.016	ϕ_c	0.950	0.030	0.900	0.026
σ_{π_1}	0.9e-03	4.62e-04	0.0021	0.092e-03	σ_{h,π_1}	1.48e-04	5.26e-05	1.61e-04	3.31e-04
σ_{c_1}	0.9e-03	4.62e-04	0.0034	0.098e-03	σ_{h,c_1}	2.22e-04	5.99e-05	1.38e-04	3.79e-04
σ_{π_2}	0.47e-03	1.37e-04	6.64e-04	0.125e-03	σ_{h,π_2}	2.74e-06	2.76e-07	2.40e-07	3.75e-08
σ_{c_2}	0.47e-03	1.37e-04	2.64e-04	0.042e-03	σ_{h,c_2}	2.38e-06	3.64e-07	2.72e-06	2.78e-08
$\sigma_{c,\pi}$	0	1.70e-04	-1.28e-04	0.074e-03	$\sigma_{c,\pi}$	0	1	-0.1122	0.1484
					Φ_{h,π_1}	0.975	0.01	0.956	0.0113
					Φ_{h,c_1}	0.975	0.01	0.980	0.0066
					Φ_{h,π_2}	0.975	0.01	0.979	0.0064
					Φ_{h,c_2}	0.975	0.01	0.987	0.0049
					ν_{h,π_1}	2	0.20	1.771	0.164
					ν_{h,c_1}	2	0.20	2.091	0.162
					ν_{h,π_2}	2	0.20	1.932	0.178
					ν_{h,c_2}	2	0.20	2.024	0.168
μ_π	0.001	0.01	0.2641	0.1394	μ_π	0.001	0.01	1.451e-04	1.092e-04
μ_c	0.001	0.01	1.14e-03	0.2011	μ_c	0.001	0.01	1.28e-03	1.73e-04
ϕ_π	0.95	0.10	0.884	0.038	ϕ_π	0.95	0.10	0.911	0.032
$\phi_{c,\pi}$	0	0.10	-0.091	0.030	$\phi_{\pi,c}$	0	0.10	-0.091	0.040
$\phi_{\pi,c}$	0	0.10	0.079	0.026	$\phi_{\pi,c}$	0	0.10	0.068	0.025
ϕ_c	0.65	0.10	0.379	0.091	ϕ_c	0.65	0.10	0.142	0.053
ψ_π	-0.2	0.32	-0.474	0.076	ψ_π	-0.2	0.32	-0.552	0.074
ψ_c	0	0.32	-0.479	0.092	ψ_c	0	0.32	-0.242	0.032
σ_π	1.49e-03	5.40e-04	2.36e-03	6.81e-05	$\sigma_{h,\pi}$	1.48e-04	5.26e-05	3.84e-05	9.81e-05
σ_c	1.49e-03	4.35e-04	3.02e-05	6.54e-06	$\sigma_{h,c}$	2.22e-04	5.99e-05	1.22e-04	6.81e-06
$\sigma_{c,\pi}$	0	7.06e-04	-3.44e-04	9.59e-05	$\sigma_{c,\pi}$	0	1	-0.190	0.148
					$\Phi_{h,\pi}$	0.975	0.01	0.976	0.007
					$\Phi_{h,c}$	0.975	0.01	0.971	0.009
					$\nu_{h,\pi}$	2	0.20	1.66	0.331
					$\nu_{h,c}$	2	0.20	2.77	0.744

Prior and posterior mean and standard deviation of Model #1 (top) and #2 (bottom) with and without stochastic volatility.

oped in the Bayesian statistics literature; see, e.g. Andrieu, Doucet, and Holenstein(2010). The particle Gibbs sampler is an MCMC algorithm that uses a particle filter to draw from distributions that are intractable; see Creal(2012) for a survey on particle filtering.

Given a prior distribution $p\left(\theta^{\mathbb{P}}\right)$ for the $\mathbb{P}$ parameters, we sample from the joint posterior distribution

$$p\left(\theta^{\mathbb{P}}, g_{1:T}, h_{0:T} | Y_{1:T}\right) \quad \propto \quad p\left(Y_{1:T} | g_{1:T}, h_{0:T}, \theta\right) p\left(g_{1:T} | h_{0:T}\right) p\left(h_{0:T} | \theta\right) p\left(\theta^{\mathbb{P}}\right) \tag{33}$$

where $Y_t = (\Delta c_t, \pi_t)$ and $x_{t:t+k} = (x_t, \ldots, x_{t+k})$. Starting with an initial value for the parameters $\theta^{\mathbb{P},(0)}$, the particle Gibbs sampler draws from this distribution by iterating for $j = 1, \ldots, M$ between the two full conditional distributions

$$(g_{1:T}, h_{0:T})^{(j)} \sim p\left(g_{1:T}, h_{0:T} | Y_{1:T}, \theta^{\mathbb{P},(j-1)}\right) \tag{34}$$

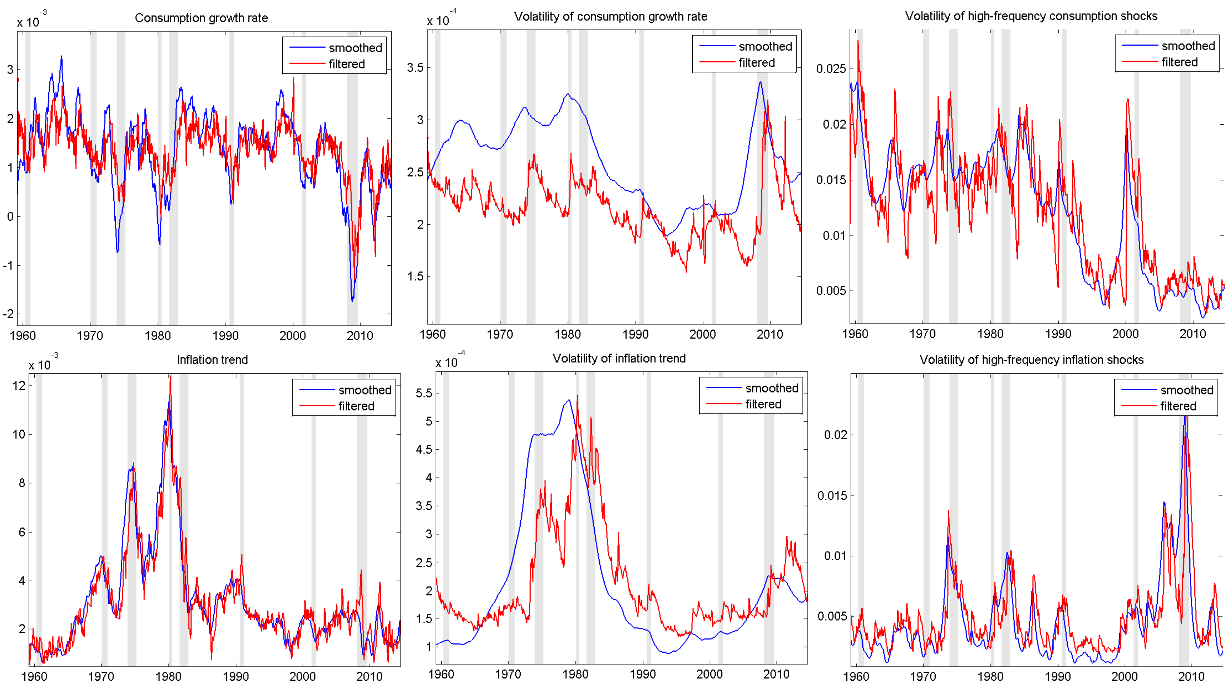
$$\theta^{\mathbb{P},(j)} \sim p\left(\theta^{\mathbb{P}} | Y_{1:T}, g_{1:T}^{(j)}, h_{0:T}^{(j)}\right) \tag{35}$$

This produces a Markov chain whose stationary distribution is the posterior (33). The models for consumption growth and inflation (27)-(32) are non-linear, non-Gaussian state space models. In these models, the full conditional distribution of the latent state variables given the data and model's parameters (34) is not easy to sample. The particle Gibbs sampler overcomes this limitation by using a particle filter to jointly sample paths of the state variables $(g_{1:T}, h_{0:T})$ in large blocks. Consequently, it improves the mixing of the MCMC algorithm and the efficiency with which the Markov chain explores the parameter space. Further details of the algorithm can be found in Appendix F.1, see also Creal and Tsay(2015) for a longer discussion.

Using the particle Gibbs sampler, we estimate Models #1 and #2 with and without stochastic volatility. Posterior means and standard deviations for the parameters of the model are in Table 1. Filtered and smoothed estimates of the latent state variables are plotted in Figures 2 and 3, respectively.

In Model #1, there is considerable variation in the long-run risk factor $\bar{c}_t$ of consumption growth (top left). It shows a noticeable decline during each recession, with the largest decline during the Great Recession. The pattern replicates the long run risk in the literature. While the volatility of the long run growth rate (top middle) is economically small and does not vary much, the stochastic volatility of the high frequency component (top right) is larger with more variation. The volatility of trend inflation increases during the mid-1970's, peaks during the early 1980's, and declines gradually until late 1990s, and keeps at a low level

Figure 2: Estimated factors from Model #1: long-run risk in consumption and inflation.



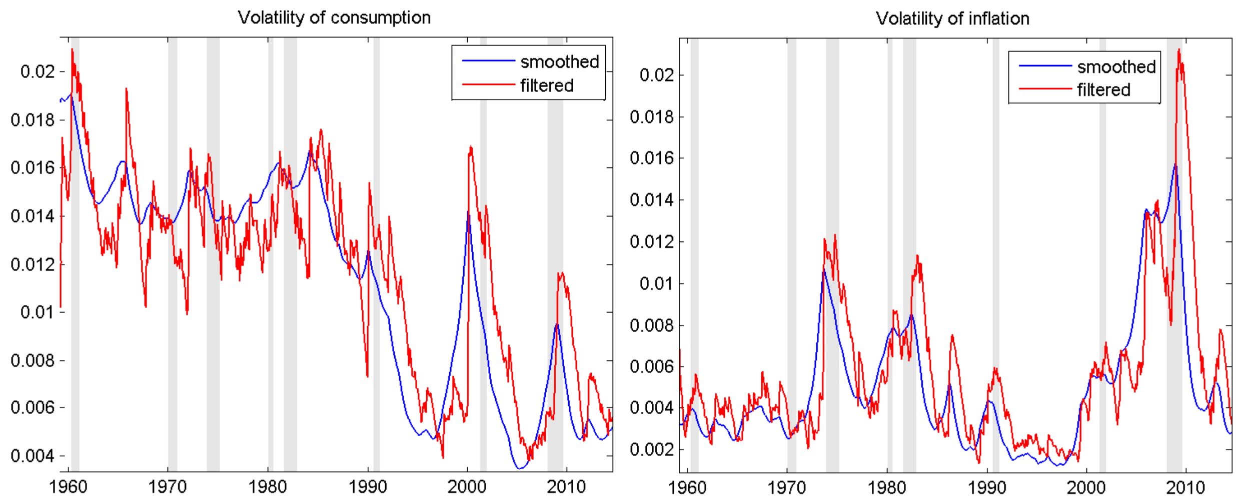
Filtered (red) and smoothed (blue) estimates of the factors from Model # 1. Top row is consumption growth and bottom row is inflation. Top left to right: long-run risk $\bar{c}_t$, long-run risk volatility h_{t,c_2}, high-frequency volatility h_{t,c_1}. Bottom left to right: inflation trend $\bar{\pi}_t$, trend volatility h_{t,π_2}, high-frequency volatility h_{t,π_1}.

afterwards. The estimates of stochastic volatility from this model of inflation are similar to those found by Stock and Watson(2007) and Creal(2012). The stochastic volatilities from Model #2 in Figure 3 resemble the high frequency components in Model #1.

Estimation of the structural parameters Given the estimated parameters $\hat{\theta}^{\mathbb{P}}$ and the filtered factors $\hat{g}_t$ and $\hat{h}_t$ from the time series dynamics, we estimate the structural parameters $\theta^u = (\beta, \gamma, \psi)$ and the risk sensitivity parameters θ^λ that enter the bond loadings. This is a cross-sectional regression of yields on the factors

$$y_t^{\$} \;=\; A^{\$}\left(\theta^u, \theta^\lambda\right) + B_g^{\$}\left(\theta^u, \theta^\lambda\right)\hat{g}_t + B_h^{\$}\left(\theta^u, \theta^\lambda\right)\hat{h}_t + \eta_t \qquad \eta_t \sim \mathrm{N}\left(0, \Omega\right) \qquad (36)$$

Figure 3: Estimated factors from Model #2.



Filtered (red) and smoothed (blue) estimates of volatility from Model # 2. Consumption volatility $h_{t,c}$ and inflation volatility $h_{t,\pi}$

The parameters have to satisfy the constraints required for the existence of a solution as discussed in Section 4. The bond loadings are also a function of the $\mathbb{P}$ parameters $\theta^{\mathbb{P}}$ which we have fixed at their posterior mean estimates from Table 1. For many models, the Gaussian state vector g_t can be rotated so that the bond loadings $B_g^{\$}$ are zero for some of the factors. Let G^* denote the minimal amount of Gaussian factors which have non-zero bond loadings.[8]

To estimate the parameters, we run a least squares regression of yields on the filtered estimates of the factors. This produces reduced form estimates of the loadings $\hat{A}^{\$,r}, \hat{B}_g^{\$,r}$ and $\hat{B}_h^{\$,r}$ that do not impose the restrictions from the structural model. Next, we estimate the structural parameters θ^u and θ^λ by minimizing the distance between the reduced form parameters $\hat{\zeta}^r = \left(\hat{A}^{\$,r}, \mathrm{vec}\left(\hat{B}_g^{\$,r}\right), \mathrm{vec}\left(\hat{B}_h^{\$,r}\right)\right)$ and the loadings implied by the structural model $\zeta\left(\theta^u, \theta^\lambda\right)$. The criterion function is

$$\underset{\theta^u,\theta^\lambda}{\operatorname{argmin}} \; T\left(\hat{\zeta}^r - \zeta\left(\theta^u, \theta^\lambda\right)\right)' W \left(\hat{\zeta}^r - \zeta\left(\theta^u, \theta^\lambda\right)\right)$$

[8]For example, in Model #1, the state vector has dimension $G = 4$ and is defined as $g_t = (\pi_t, \Delta c_t, \bar{\pi}_t, \bar{c}_t)'$. However, the bond loadings on inflation and consumption are zero and only expected inflation $\bar{\pi}_t$ and expected consumption growth $\bar{c}_t$ have non-zero bond loadings making $G^* = 2$. Similarly, in Model #2, the state vector can be rotated to include expected consumption growth and expected inflation making $G^* = 2$.

Table 2: Estimated structural and preference shock parameters

	Model 1	Model 1 PS	Model 1 SV	Model 1 SV-PS	Model 2	Model 2 PS	Model 2 SV	Model 2 SV-PS
β	0.9822	0.9999	0.9987	0.9994	0.9999	0.9951	0.9999	0.9999
ψ	19.01	1.995	2.741	1.887	0.478	0.439	0.622	0.999
γ	374.91	11.3127	0.001	9.03e-07	140.06	470.01	46.83	2.868
$\bar{pc}$	3.363	4.288	7.795	7.621	7.135	5.552	7.365	17.801
λ_π	–	0.0035	–	-0.0335 e-07	–	-0.0255	–	-0.1121 e-05
λ_c	–	0.0031	–	3.900 e-03	–	0.161 e-03	–	-0.0484 e-05

Estimated structural and preference shock parameters from eight models. These are Model #1 and #2 without and stochastic volatility (SV) and preference shocks (PS).

where W is a weighting matrix. In our work, the weighting matrix is $W = -\frac{1}{T}\mathrm{E}\left[\frac{\partial^2 \mathcal{L}^r}{\partial \varsigma^r \partial \varsigma^{r,\prime}}\right]$, where $\mathcal{L}^r$ is the reduced-form log-likelihood from (36). This procedure produces a minimum chi-square estimator, because the optimized objective function yields a chi-square statistic. A similar procedure has been used in Gaussian ATSMs by Hamilton and Wu(2012).

There are two advantages to this procedure as opposed to direct maximum likelihood estimation of (36). First, the weighting matrix W is well-conditioned for any value of the structural parameters and it only needs to be evaluated once as opposed to every iteration within an optimizer. Second, we obtain estimates of the reduced form parameters which measure the "best" fit we could hope to achieve for a given set of pricing factors. Then, the value of the objective function tells us how far away we are from the best fit, and how severe the restrictions imposed by the structural models are.

For the models with preference shocks, to keep the model minimal, we only allow the price of inflation risk to depend on inflation, and the price of consumption risk to depend on consumption. This means that we only introduce two new parameters into the matrix λ_g while the remaining preference shock parameters are set to zero $Z_v = \lambda_0 = \lambda_{gh} = \lambda_h = 0$, see Appendix A for details. If there are no preference shocks, there are three free parameters to fit $N \times (G^* + H + 1)$ reduced form moments in a stochastic volatility model. In models with preference shocks, there are a total of five free parameters.

The parameter estimates from these models are in Table 2. Across all models, the time discount factor β is estimated to be close to 1. The intertemporal elasticity of substitution ψ varies from 0.4 to a high of 19, although most estimates are close to the former rather than the latter. Estimates of the risk aversion parameter γ vary markedly from homoskedastic models to stochastic volatility models. For models without stochastic volatility, the estimates are high and range between 11 to 470. Estimates of γ in stochastic volatility models are considerably lower.

5.3 Slope of the yield curve

In standard models with no preference shocks, there are only 3 structural parameters (β, γ, ψ) that can be used to fit the cross section of the yield curve. The literature has thus far primarily focused on a model's ability to fit the unconditional slope of the yield curve with these parameters.

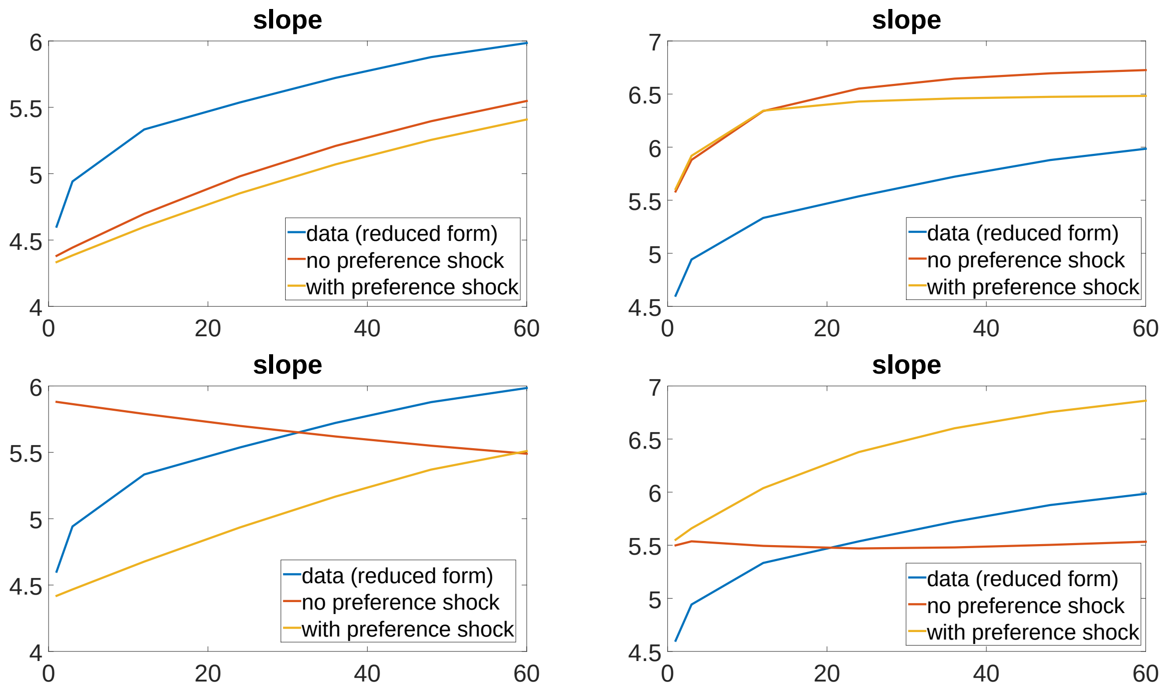
Figure 4 plots the unconditional yield curve from the data (in blue) along with the unconditional slope from Models' #1 (in the left panels) and #2 (in the right panels) with (yellow) and without (red) the preference shock parameter λ_g set to zero. The top row are homoskedastic, Gaussian models and the bottom row are models with stochastic volatility.

The four Gaussian models are all able to generate an upward sloping yield curve, replicating a key finding in the literature. Piazzesi and Schneider(2007) argue that a contributing factor of this result are the negatively correlated shocks to consumption growth and inflation. Our estimates of the covariance are also negative and significant (see Table 1).

In the bottom row, we can see that both stochastic volatility models without preference shocks have flat if not downward sloping yield curves (in red). Stochastic volatility models have a harder time fitting the unconditional yield curve. This is because the addition of stochastic volatility introduces more moment conditions that need to be fit without adding any free structural parameters. Adding stochastic volatility to generate time-varying risk premia actually makes it increasingly difficult to fit the unconditional yield curve. In contrast, adding preference shocks allows the SV models (yellow lines at the bottom) as well as the Gaussian models (yellow lines on the top).

In Figures 5 and 6, we plot the time series of yields with 1, 12, and 60 month maturities

Figure 4: Unconditional yield curves from Models #1 and #2



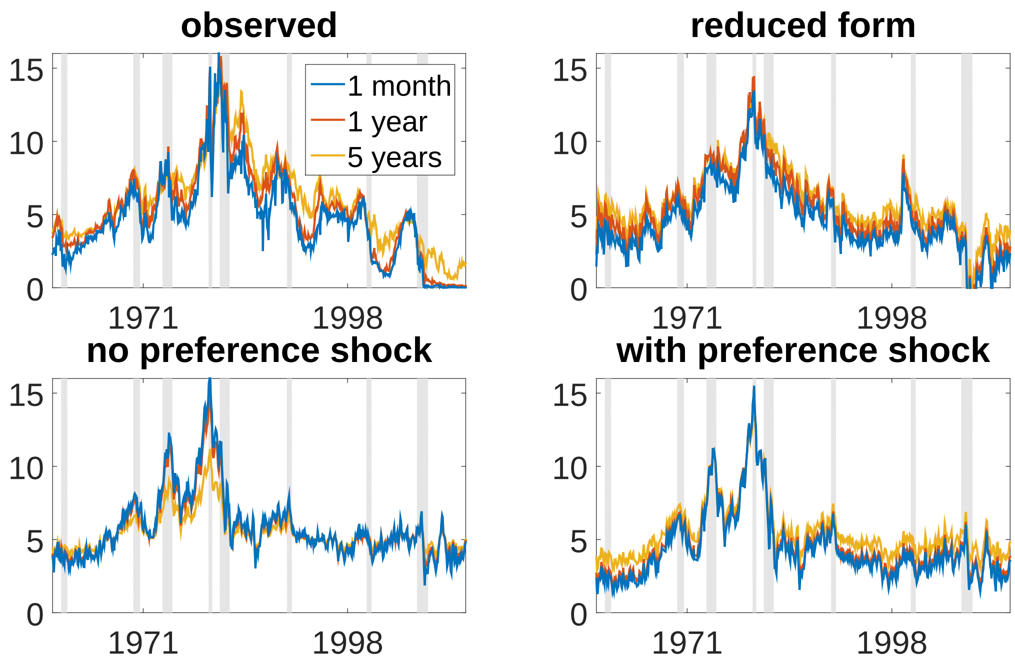
Unconditional yield curves as a function of maturity in blue. Red (yellow) are estimates from models without (with) preference shocks. Top left: Model #1 (long-run risk) without stochastic volatility. Top right: Model #2 (VARMA) without stochastic volatility. Bottom left: Model #1 (long-run risk) with stochastic volatility. Bottom right: Model #2 (VARMA) with stochastic volatility.

from the data (upper left), the unrestricted reduced form (upper right) and the implied yields with (bottom right) and without (bottom left) preference shocks. These are for the stochastic volatility Models #1 and #2, respectively. In the observed data, long rates are higher than short rates for most of the sample. Both the reduced form and structural model with preference shocks can replicate this fact, hence the upward sloping yield curves in Figure 4. Models without preference shocks (bottom left) fail to capture this basic feature.

5.4 How structural parameters enter the bond loadings

We elaborate more on the mechanism of how structural parameters enter the loadings, and hence influence the slope of the yield curve. We use Gaussian models with no preference shock for intuition. First, consider the unconditional mean of (36), and write the loadings

Figure 5: Yields for Model #1

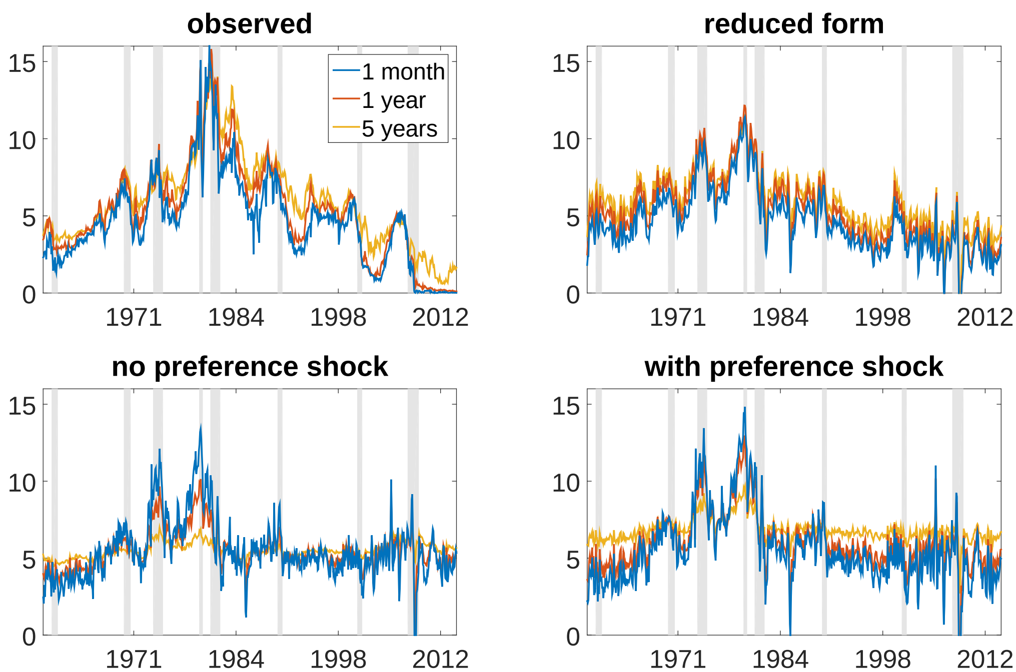


Top left: Observed Fama-Bliss yields. Top right: reduced form yields. Bottom left: model implied yields from Model #1 without preference shocks. Bottom right: model implied yields from Model # 1 with preference shock λ_g.

as explicit functions of the underlying parameters: $\bar{y}^{\$} = A^{\$}(\beta, \psi, \gamma, \theta^r) + B_g^{\$}(\psi, \theta^r)\,\bar{\mu}_g$. The unconditional mean of the Gaussian factors are positive empirically $\bar{\mu}_g > 0$. The loadings $B_g^{\$}$ on consumption growth only depend on ψ, more precisely, the initial bond loading $b_{1,g}^{\$}$ on expected consumption growth always equals ψ^{-1} and the initial loading on expected inflation equals 1. As long as the Gaussian factors are modeled as a stationary (non-explosive) process, the loadings will decay as a function of maturity. Consequently, the Gaussian term $B_g^{\$}\bar{\mu}_g$ must be positive and decrease as a function of maturity. This means that in a Gaussian model with no preference shocks the loadings $A^{\$}$ must be upward sloping in order to match the unconditional yield curve.

Next, consider the impact each structural parameter has on $A^{\$}$, see the detailed expressions for the bond recursions in Appendix D.2. We focus on the space near our estimates with the feature $\kappa_1 \approx 1$, and $\partial\kappa_1/\partial\theta^u \approx 0$. The time discount parameter β only enters

Figure 6: Yields for Model #2



Top left: Observed Fama-Bliss yields. Top right: reduced form yields. Bottom left: model implied yields from Model # 2 without preference shocks. Bottom right: model implied yields from Model # 2 with preference shock λ_g.

the loadings $A^{\$}$ and it does so additively as $\ln(\beta)$. Increases or decreases in $\ln(\beta)$ shift all yields in parallel and have no impact on the unconditional slope of the yield curve, i.e., $\partial A^{\$}/\partial \ln(\beta) \approx -\iota$. The remaining parameters γ and ψ enter the loadings through more complicated functional forms. And these two parameters need to pin down the slope as well as level of $A^{\$}$, and $B_g^{\$}$ for consumption.

The two preference parameters that enter λ_g create additionally flexibility for the Gaussian bond loadings, $B_g^{\$}$. In a standard model, the slope of the bond loadings $B_g^{\$}$ is only a function of the autocovariance matrix Φ_g under the $\mathbb{P}$ measure. The cross-section of yields and the time series of the factors are strongly linked. In the data, the autocorrelations of consumption growth are typically quickly mean reverting while monthly inflation reverts to its mean more slowly, see the estimates in Table 1. The preference parameters λ_g allow the slope of the bond loadings to differ from the autocorrelations of consumption growth and inflation. The bond loadings can even increase as a function of maturity which is an important feature of the data, especially inflation.

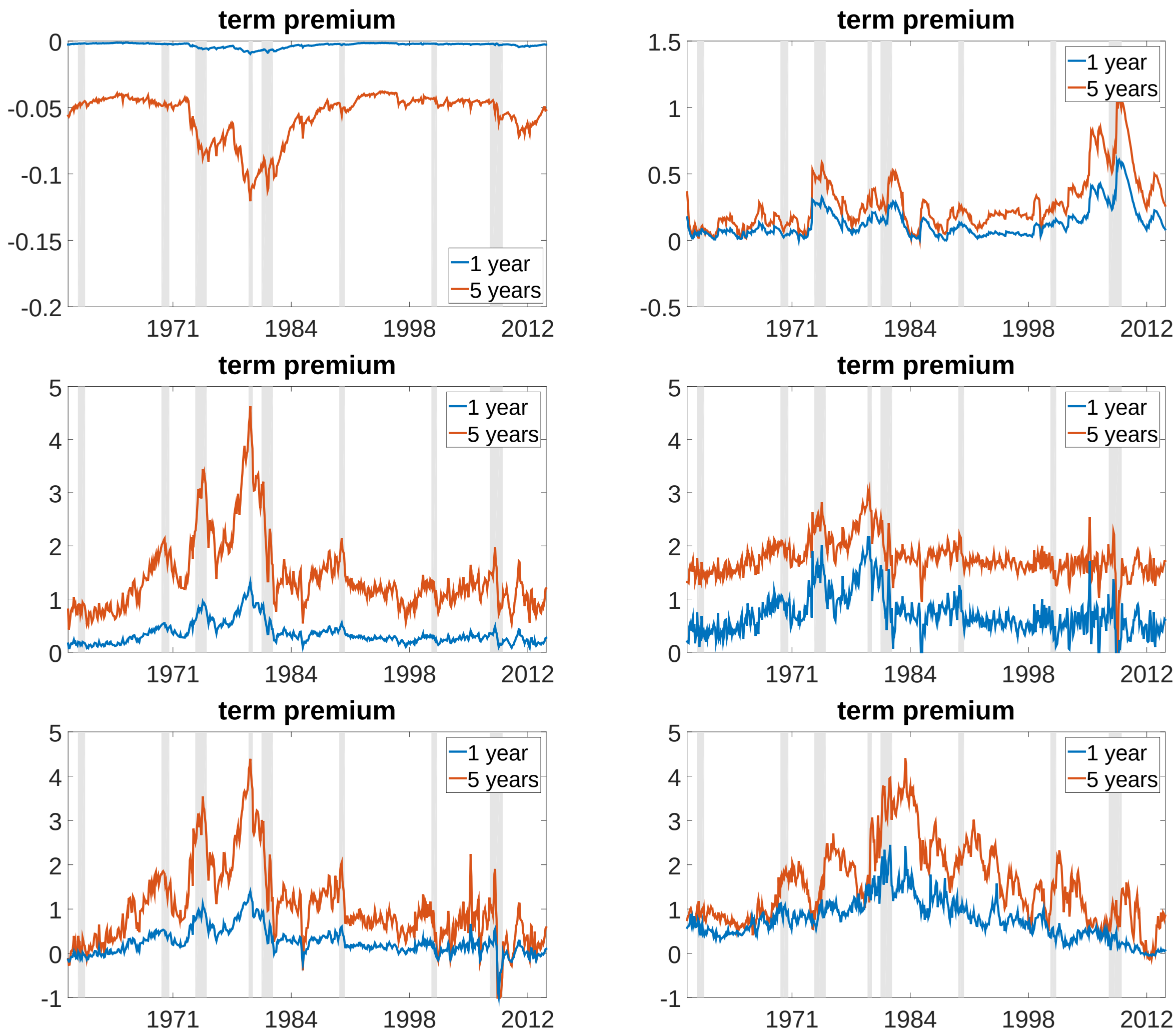
6 Term premia

The term premium is a crucial object to understand in order for central banks to implement monetary policy. The primary tool central banks rely on to estimate term premia are Gaussian ATSMs, see Wright(2011) and Bauer, Rudebusch, and Wu(2012). The only economic structural imposed within a ATSM is a condition guaranteeing the absence of arbitrage. The minimal amount of structure allows these models to fit the yield curve well, hence produce plausible term premia for monetary authorities. At the same time, the lack of economic structure and dependence on latent factors make it hard to interpret what economic mechanism ultimately drives the term premium. Conversely, structural models only have a few structural parameters. Consequently, they have trouble matching even the unconditional moments of term premia, not to mention the dynamics; for a discussion see Rudebusch and Swanson(2008) and Rudebusch and Swanson(2012).

Our new model brings economic structure and flexibility together by introducing preference shocks to an equilibrium model. It is able to produce realistic term premia and provide an economic interpretation to its movements. Our model allows for flexibility in both time-varying prices and quantities of risk. Hence, it allows us to disentangle the driving force of term premia, and determine which is a more plausible explanation for its variation: time-varying prices or quantities of risk.

First, we shut down the preference shocks, but allow stochastic volatility to replicate models in the literature. Model #1 with long run risk is similar to the model estimated by Bansal and Shaliastovich(2013). We plot term premia for this model in the top row left panel of Figure 7, for maturities of one (blue) and five (red) years. For comparison, we also

Figure 7: Term premia from Models #1, #2 and a Gaussian ATSM.



Estimated 1 and 5 year term premia from alternative models. Top left: Model #1 with SV and no preference shocks; Top right: Model #2 with SV and no preference shocks; Second row left: Model #1 with SV and preference shocks λ_g; Second row right: Model #2 with SV and preference shocks λ_g; Bottom left: Model #1 without SV and with preference shocks. Bottom right: reduced-form 3 factor Gaussian ATSM.

plot estimates from a three-factor Gaussian ATSM in the bottom right panel. The term premia generated by Model #1 are economically insignificant: the one year term premium is close to zero the whole time, and the five year term premium peaks at about 10 basis points. Moreover, the term premia generated by this model are negative. This is counter-intuitive and implausible as the model produces the wrong sign.

In the top row on the right, we change dynamics to a VARMA, Model #2, also with no preference shocks. We can see that the two plots give completely different pictures of how term premia behave. Model #2 does produce positive term premia. The size of the term premium is still smaller compared to the reduced form evidence in the lower right panel. Moreover, the dynamics of term premia are counter-intuitive as well: the term premium achieves its maximum during the Great Recession, as opposed to in the 1980s as suggested by ATSM. This is driven by the fact that term premia comove only with volatility, primarily inflation volatility, see right panel of Figure 3.

Next, we add preference shocks to both models, with term premia in the middle panels of Figure 7. Once we add the preference shock, the resulting term premia look closer to the reduced form term premia from the Gaussian ATSM, and each other. The general pattern is the term premia rise for the first half of the sample, drop sharply in the early 1980s, and then keep steady between 1% and 2% afterwards for the 5 year maturity. They also resemble the cyclical pattern discussed in Bauer, Rudebusch, and Wu(2012). Term premia increase in anticipation of recessions, and drop post recession. For the Great Recession, term premia drop sharply due to the flight to quality argument.

The contrast between the first two rows in Figure 7 demonstrate that adding preference shocks is crucial to produce meaningful dynamics for term premia. With stochastic volatility being present in all these pictures, the question is does stochastic volatility play a significant part in explaining variation in term premia? We investigate this question by looking into a Gaussian model with preference shocks. The term premia from Model #1 are in the lower left panel. Interestingly, the term premia in this panel look almost identical to those in the panel right above it. This illustrates that all the meaningful variation in the term premia is driven by time variation in the price of risk through preference shocks associated with inflation. In contrast, time variation in stochastic volatility can in theory generate time variation for term premia, but in practice, it is not the primary source of its fluctuations.

7 Conclusion

We developed a model of recursive preferences to capture time variation in term premia through two sources: time-varying prices and quantities of risk. We introduced time-varying prices of risk through a preference shock that comoves with consumption growth and inflation. This generates a time-varying risk premia even when the shocks are homoskedastic. We found that the time varying prices of risk driven by expected inflation is the primary channel empirically. On the contrary, once the preference shock is present, with or without stochastic volatility does not alter the economic implication of the dynamics of term premia. Conversely, a stochastic volatility model without preference shocks cannot match the upward sloping unconditional yield curve, the fundamental moment in the term structure. Adding preference shock solves this problem as well.

Empirical implementation of recursive preferences requires careful attention when solving for the stochastic discount factor. A solution does not exist for certain combinations of structural parameters. Our paper provided conditions that guaranteed the existence of a solution. We use these conditions to provide guidelines for empirical implementation.

Several authors have studied term structure models with recursive preferences in DSGE models, e.g. Rudebusch and Swanson(2008), Rudebusch and Swanson(2012), van Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez(2012), and Dew-Becker(2014). How to introduce our technology of capturing realistic dynamics into a DSGE framework remains an open question, and logical next step for the literature.

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Appendix A Dynamics of the state vector

Appendix A.1 General Model

The dynamics of the Gaussian state vector g_t driving Δc_t and π_t are

$$\begin{aligned}
g_{t+1} &= \mu_g + \Phi_g g_t + \Phi_{gh} h_t + \Sigma_{gh} \varepsilon_{h,t+1} + \Sigma_{g,t} \varepsilon_{g,t+1}, \qquad \varepsilon_{g,t+1} \sim \mathrm{N}(0, I), \\
\Sigma_{g,t} \Sigma'_{g,t} &= \Sigma_{0,g} \Sigma'_{0,g} + \sum_{i=1}^{H} \Sigma_{i,g} \Sigma'_{i,g} h_{it}, \\
\varepsilon_{h,t+1} &= h_{t+1} - \mathrm{E}_t [h_{t+1} | h_t],
\end{aligned}$$

where the dynamics of volatility are

$$\begin{aligned}
h_{t+1} &= \Sigma_h w_{t+1}, \\
w_{i,t+1} &\sim \mathrm{Gamma}(\nu_{h,i} + z_{i,t+1}, 1), \qquad i = 1, \ldots, H \qquad \text{(A.1)} \\
z_{i,t+1} &\sim \mathrm{Poisson}\left(\mathrm{e}'_i \Sigma_h^{-1} \Phi_h \Sigma_h w_t\right), \qquad i = 1, \ldots, H. \qquad \text{(A.2)}
\end{aligned}$$

This is a discrete-time, multivariate Cox, Ingersoll, and Ross(1985) process. To guarantee positivity and existence of h_t, the process requires $\Sigma_h > 0$, $\Sigma_h^{-1} \Phi_h \Sigma_h > 0$ and the Feller condition $\nu_{h,i} > 1$ for $i = 1, \ldots, H$. The conditional mean and variance of the process are

$$\begin{aligned}
\mathrm{E}_t [h_{t+1} | h_t] &= \Sigma_h \nu_h + \Phi_h h_t, \qquad \text{(A.3)} \\
V_t [h_{t+1} | h_t] &= \Sigma_{h,t} \Sigma'_{h,t} \\
&= \Sigma_h \mathrm{diag}(\nu_h) \Sigma'_h + \Sigma_h \mathrm{diag}\left(2 \Sigma_h^{-1} \Phi_h h_t\right) \Sigma'_h, \qquad \text{(A.4)}
\end{aligned}$$

where Σ_h is a $H \times H$ matrix of scale parameters, Φ_h is a $H \times H$ matrix of autoregressive parameters and the intercept is equal to $\Sigma_h \nu_h$. The unconditional mean is $\bar{\mu}_h = (I_H - \Phi_h)^{-1} \Sigma_h \nu_h$. The transition density is

$$\begin{aligned}
p(h_{t+1} | h_t, \nu_h, \Phi_h, \Sigma_h) &= |\Sigma_h^{-1}| \prod_{i=1}^{H} \left(\mathrm{e}'_i \Sigma_h^{-1} h_{t+1}\right)^{\frac{\nu_{h,i}-1}{2}} \left(\mathrm{e}'_i \Sigma_h^{-1} \Phi_h h_t\right)^{-\frac{\nu_{h,i}-1}{2}} \\
&\quad \exp\left(-\sum_{i=1}^{H} \mathrm{e}'_i \Sigma_h^{-1} h_{t+1} + \mathrm{e}'_i \Sigma_h^{-1} \Phi_h h_t\right) I_{\nu_{h,i}-1}\left(2 \sqrt{\left(\mathrm{e}'_i \Sigma_h^{-1} h_{t+1}\right)\left(\mathrm{e}'_i \Sigma_h^{-1} \Phi_h h_t\right)}\right) \qquad \text{(A.5)}
\end{aligned}$$

where $I_\nu(x)$ is the modified Bessel function. The Laplace transform needed to solve the model with recursive preferences and for pricing assets is

$$\mathbb{E}_t\left[\exp\left(u' h_{t+1}\right)\right] = \exp\left(\sum_{i=1}^{H} \frac{\mathrm{e}_i'\Sigma_h' u}{1-\mathrm{e}_i'\Sigma_h' u}\mathrm{e}_i'\Sigma_h^{-1}\Phi_h h_t - \sum_{i=1}^{H}\nu_{h,i}\log\left(1-\mathrm{e}_i'\Sigma_h' u\right)\right),$$

which exists only if $\mathrm{e}_i'\Sigma_h' u < 1$ for $i = 1, \ldots, H$. Further properties of the univariate process are developed by Gouriéroux and Jasiak(2006).

Appendix A.2 Risk neutral $\mathbb{Q}^{\$}$ dynamics

After solving for the short rate $r_t^{\$}$, we can deduce the implied dynamics of the state variables under the nominal risk neutral measure

$$p\left(g_{t+1}|g_t, h_{t+1}, h_t; \theta, \mathbb{Q}^{\$}\right) p\left(h_{t+1}|h_t; \theta, \mathbb{Q}^{\$}\right) = \exp\left(r_t^{\$}\right) M_{t+1}^{\$} p\left(g_{t+1}|g_t, h_{t+1}, h_t; \theta, \mathbb{P}\right) p\left(h_{t+1}|h_t; \theta, \mathbb{P}\right)$$

The dynamics of the state vector under the nominal risk-neutral measure are

$$\begin{aligned}
g_{t+1} &= \mu_g^{\mathbb{Q},\$} + \Phi_g^{\mathbb{Q},\$} g_t + \Phi_{gh}^{\mathbb{Q},\$} h_t + \Sigma_{gh}\varepsilon_{h,t+1}^{\mathbb{Q},\$} + \Sigma_{g,t}\varepsilon_{g,t+1}^{\mathbb{Q},\$} \qquad \varepsilon_{g,t+1}^{\mathbb{Q},\$} \sim \mathrm{N}(0, I) \\
\Sigma_{g,t}\Sigma_{g,t}' &= \Sigma_{0,g}\Sigma_{0,g}' + \sum_{i=1}^{H}\Sigma_{i,g}\Sigma_{i,g}' h_{it} \\
\varepsilon_{h,t+1}^{\mathbb{Q},\$} &= h_{t+1} - \mathrm{E}_t^{\mathbb{Q},\$}\left[h_{t+1}|h_t\right] \\
h_{t+1} &\sim \mathrm{NCG}\left(\nu_h, \Phi_h^{\mathbb{Q},\$}, \Sigma_h^{\mathbb{Q},\$}\right)
\end{aligned}$$

For the stochastic volatility dynamics, the relationship between $\mathbb{P}$ and $\mathbb{Q}^{\$}$ is

$$\begin{aligned}
\Phi_h^{\mathbb{Q},\$} &= \Sigma_h\left(I_H - \mathrm{diag}\left(\Sigma_h'\left[\Sigma_{gh}'\left(\vartheta Z_\upsilon - \gamma Z_c - Z_\pi + (\vartheta-1)\kappa_1 D_g\right) - \vartheta\lambda_h + (\vartheta-1)\kappa_1 D_h\right]\right)\right)^{-2}\Sigma_h^{-1}\Phi_h \\
\Sigma_h^{\mathbb{Q},\$} &= \Sigma_h\left(I_H - \mathrm{diag}\left(\Sigma_h'\left[\Sigma_{gh}'\left(\vartheta Z_\upsilon - \gamma Z_c - Z_\pi + (\vartheta-1)\kappa_1 D_g\right) - \vartheta\lambda_h + (\vartheta-1)\kappa_1 D_h\right]\right)\right)^{-1}
\end{aligned}$$

For the Gaussian dynamics, the relationship between the $\mathbb{P}$ and $\mathbb{Q}^{\$}$ parameters is

$$\begin{aligned}
\mu_g^{\mathbb{Q},\$} &= \mu_g - \Sigma_{0,g}\Sigma_{0,g}'\left(\gamma Z_c + Z_\pi\right) + \kappa_1 \frac{(\eta-\gamma)}{(1-\eta)}\Sigma_{0,g}\Sigma_{0,g}' D_g + \vartheta \Sigma_{0,g}\Sigma_{0,g}' Z_v - \eta\vartheta\lambda_0 \\
&\quad -\Sigma_{gh}\left(\Sigma_h - \Sigma_h^{\mathbb{Q},\$}\right)\nu_h \\
\Phi_g^{\mathbb{Q},\$} &= \Phi_g - \eta\vartheta\lambda_g \\
\Phi_{gh}^{\mathbb{Q},\$} &= \Phi_{gh} - \left(\left[\gamma Z_c + Z_\pi\right]\otimes I_G\right)' \tilde{S}_g + \kappa_1 \frac{(\eta-\gamma)}{(1-\eta)}\left(D_g \otimes I_G\right)' \tilde{S}_g \\
&\quad +\vartheta\left(Z_v \otimes I_G\right)' \tilde{S}_g - \Sigma_{gh}\left(\Phi_h - \Phi_h^{\mathbb{Q},\$}\right) - \eta\vartheta\lambda_{gh}
\end{aligned}$$

The parameters of the real risk neutral measure $\mathbb{Q}$ can be determined by setting $Z_\pi = 0$.

Appendix A.3 Model #1

The model with long-run risk to consumption growth and trend inflation has four stochastic volatility factors given by $h_t = (h_{t,c_1}, h_{t,c_2}, h_{t,\pi_1}, h_{t,\pi_2})'$. We assume that these evolve independently of one another as

$$\begin{aligned}
h_{t+1,\pi_1} &\sim \text{NCG}\left(\nu_{h,\pi_1}, \phi_{h,\pi_1}, \sigma_{h,\pi_1}\right) \\
h_{t+1,c_1} &\sim \text{NCG}\left(\nu_{h,c_1}, \phi_{h,c_1}, \sigma_{h,c_1}\right) \\
h_{t+1,\pi_2} &\sim \text{NCG}\left(\nu_{h,\pi_2}, \phi_{h,\pi_2}, \sigma_{h,\pi_2}\right) \\
h_{t+1,c_2} &\sim \text{NCG}\left(\nu_{h,c_2}, \phi_{h,c_2}, \sigma_{h,c_2}\right)
\end{aligned}$$

The models can be placed in the notation of the paper as

$$g_t = \begin{pmatrix} \pi_t \\ \Delta c_t \\ \bar{\pi}_t \\ \bar{c}_t \end{pmatrix} \quad Z_c = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad Z_\pi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Phi_g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \phi_\pi & 0 \\ 0 & 0 & 0 & \phi_c \end{pmatrix} \quad \mu_g = \begin{pmatrix} 0 \\ 0 \\ \mu_\pi \\ \mu_c \end{pmatrix}$$

$$\Phi_{gh} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Sigma_{gh} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Sigma_{0,g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Sigma_{1,g} = \begin{pmatrix} \frac{1}{\sqrt{1000}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Sigma_{2,g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1000}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Sigma_{3,g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{1000}} & 0 \\ 0 & 0 & \frac{\sigma_{c,\pi}}{\sqrt{1000}} & 0 \end{pmatrix} \quad \Sigma_{4,g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{1000}} \end{pmatrix}$$

We have scaled these matrices by $1/\sqrt{1000}$ so that the volatility factors h_t are roughly the same magnitude as the Gaussian factors g_t. For the volatility processes, the matrices are

$$\nu_h = \begin{pmatrix} \nu_{h,\pi_1} \\ \nu_{h,c_1} \\ \nu_{h,\pi_2} \\ \nu_{h,c_2} \end{pmatrix} \quad \Phi_h = \begin{pmatrix} \phi_{\pi_1} & 0 & 0 & 0 \\ 0 & \phi_{c_1} & 0 & 0 \\ 0 & 0 & \phi_{\pi_2} & 0 \\ 0 & 0 & 0 & \phi_{c_2} \end{pmatrix} \quad \Sigma_h = \begin{pmatrix} \sigma_{h,\pi_1} & 0 & 0 & 0 \\ 0 & \sigma_{h,c_1} & 0 & 0 \\ 0 & 0 & \sigma_{h,\pi_2} & 0 \\ 0 & 0 & 0 & \sigma_{h,c_2} \end{pmatrix}$$

For Gaussian models with no stochastic volatility, we set the scale matrix equal to

$$\Sigma_{0,g} = \begin{pmatrix} \sigma_{\pi_1} & 0 & 0 & 0 \\ 0 & \sigma_{c_1} & 0 & 0 \\ 0 & 0 & \sigma_{\pi_2} & 0 \\ 0 & 0 & \sigma_{c,\pi} & \sigma_{c_2} \end{pmatrix}.$$

Our economic restrictions on the prices of risk imply that $\Sigma_{g,t}^{-1}\lambda_g$ needs to be diagonal mathematically. This amounts to the following constraint for $\tilde{\lambda}_g = \tilde{\Sigma}_g^{-1}\lambda_g$ with two free parameters, where $\tilde{\Sigma}_g = \sum_{j=0}^{H}\Sigma_{j,g}$:

$$\tilde{\lambda}_g = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_\pi & 0 \\ 0 & 0 & 0 & \lambda_c \end{pmatrix}$$

For better numerical behavior, in practice, we allow two free parameters in $\Phi_g^{\mathbb{Q}}$, the (3,3) and (4,4) element, and the following relationship translates these two free parameters back into the parameters in $\tilde{\lambda}_g$

$$\begin{aligned} \lambda_\pi &= \frac{\Phi_{g,33} - \Phi_{g,33}^{\mathbb{Q},\$}}{\eta\vartheta\tilde{\Sigma}_{g,33}} = \frac{\phi_\pi - \phi_\pi^{\mathbb{Q},\$}}{\eta\vartheta\tilde{\Sigma}_{g,33}} \\ \lambda_c &= \frac{\Phi_{g,44} - \Phi_{g,44}^{\mathbb{Q},\$}}{\eta\vartheta\tilde{\Sigma}_{g,44}} = \frac{\phi_c - \phi_c^{\mathbb{Q},\$}}{\eta\vartheta\tilde{\Sigma}_{g,44}} \end{aligned}$$

Appendix A.4 Model #2

The VARMA(1,1) model has two stochastic volatility factors given by $h_t = (h_{t,\pi}, h_{t,c})'$. We assume that these evolve independently of one another as

$$
\begin{aligned}
h_{t+1,\pi} &\sim \mathrm{NCG}\left(\nu_{h,\pi}, \phi_{h,\pi}, \sigma_{h,\pi}\right) \\
h_{t+1,c} &\sim \mathrm{NCG}\left(\nu_{h,c}, \phi_{h,c}, \sigma_{h,c}\right)
\end{aligned}
$$

The models can be placed in the notation of the paper as

$$
g_t = \begin{pmatrix} \pi_t \\ \Delta c_t \\ \varepsilon_{\pi,t} \\ \varepsilon_{c,t} \end{pmatrix} \quad
Z_c = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad
Z_\pi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad
\Phi_g = \begin{pmatrix} \phi_\pi & \phi_{\pi,c} & \psi_\pi & 0 \\ \phi_{c,\pi} & \phi_c & 0 & \psi_c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad
\mu_g = \begin{pmatrix} \mu_\pi \\ \mu_c \\ 0 \\ 0 \end{pmatrix}
$$

$$
\Phi_{gh} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad
\Sigma_{gh} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad
\Sigma_{0,g} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad
\Sigma_{1,g} = \begin{pmatrix} \frac{1}{\sqrt{1000}} & 0 \\ \frac{\sigma_{c,\pi}}{\sqrt{1000}} & 0 \\ \frac{1}{\sqrt{1000}} & 0 \\ \frac{\sigma_{c,\pi}}{\sqrt{1000}} & 0 \end{pmatrix} \quad
\Sigma_{2,g} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{1000}} \\ 0 & 0 \\ 0 & \frac{1}{\sqrt{1000}} \end{pmatrix}
$$

For the volatility processes, the matrices are

$$
\nu_h = \begin{pmatrix} \nu_{h,\pi} \\ \nu_{h,c} \end{pmatrix} \quad
\Phi_h = \begin{pmatrix} \phi_\pi & 0 \\ 0 & \phi_c \end{pmatrix} \quad
\Sigma_h = \begin{pmatrix} \sigma_{h,\pi} & 0 \\ 0 & \sigma_{h,c} \end{pmatrix}
$$

For Gaussian models with no stochastic volatility, we set the scale

$$
\Sigma_{0,g} = \begin{pmatrix} \sigma_\pi & 0 \\ \sigma_{c,\pi} & \sigma_c \\ \sigma_\pi & 0 \\ \sigma_{c,\pi} & \sigma_c \end{pmatrix}.
$$

Our economic restrictions on prices of risk imply that $\Sigma_{g,t}^{-1}\lambda_g$ needs to be diagonal mathematically. This amounts to the following constraint for $\tilde{\lambda}_g = \tilde{\Sigma}_g^{-1}\lambda_g$ with two free parameters, where $\tilde{\Sigma}_g \;=\; \sum_{j=0}^{H}\Sigma_{j,g}$:

$$\tilde{\lambda}_g = \begin{pmatrix} \lambda_\pi & 0 & 0 & 0 \\ 0 & \lambda_c & 0 & 0 \end{pmatrix}$$

For better numerical behavior, in practice, we allow two free parameters in $\Phi_g^{\mathbb{Q}}$, the (3,3) and (4,4) element, and the following relationship translates these two free parameters back into the parameters in $\tilde{\lambda}_g$

$$\begin{aligned} \lambda_\pi &= \frac{\Phi_{g,33} - \Phi_{g,33}^{\mathbb{Q},\$}}{\eta\vartheta\tilde{\Sigma}_{g,33}} = \frac{\phi_\pi - \phi_\pi^{\mathbb{Q},\$}}{\eta\vartheta\tilde{\Sigma}_{g,33}} \\ \lambda_c &= \frac{\Phi_{g,44} - \Phi_{g,44}^{\mathbb{Q},\$}}{\eta\vartheta\tilde{\Sigma}_{g,44}} = \frac{\phi_c - \phi_c^{\mathbb{Q},\$}}{\eta\vartheta\tilde{\Sigma}_{g,44}} \end{aligned}$$

Appendix B Stochastic discount factor

The Euler equation can be shown to be

$$1 = \beta^\vartheta \mathrm{E}_t\left[\left(\frac{\Upsilon_{t+1}}{\Upsilon_t}\right)^\vartheta \left(\frac{C_{t+1}}{C_t}\right)^{-\eta\vartheta} R_{c,t+1}^\vartheta\right]$$

implying a pricing kernel of the form

$$M_{t+1} = \beta^\vartheta \left(\frac{\Upsilon_{t+1}}{\Upsilon_t}\right)^\vartheta \left(\frac{C_{t+1}}{C_t}\right)^{-\eta\vartheta} R_{c,t+1}^{\vartheta-1}$$

Let $\upsilon_{t+1} = \ln\Upsilon_{t+1} - \ln\Upsilon_t$. The log SDF is

$$m_{t+1} \;=\; \vartheta\ln(\beta) + \vartheta\upsilon_{t+1} - \eta\vartheta\Delta c_{t+1} + (\vartheta-1)\,r_{c,t+1}$$

The constant is

$$\bar{\Lambda} \;=\; \frac{\vartheta\eta^2}{2}\left(\lambda_0 + \lambda_g\bar{\mu}_g + \lambda_{gh}\bar{\mu}_h\right)'\left(\lambda_0 + \lambda_g\bar{\mu}_g + \lambda_{gh}\bar{\mu}_h\right) + \frac{\vartheta\eta^2}{2}\mathrm{tr}\left(\lambda_g'\lambda_g\tilde{\Sigma}_g\tilde{\Sigma}_g'\right) + \frac{\vartheta\eta^2}{2}\mathrm{tr}\left(\lambda_{gh}'\lambda_{gh}\tilde{\Sigma}_h\right)$$

where μ_h and $\tilde{\Sigma}_h$ are the unconditional mean and variance of h_t.

Appendix C Recursive preferences model solution

Appendix C.1 Model solution

In order to simplify the expressions, we introduce the following notation

$$
\begin{aligned}
Z_1 &= Z_v + (1-\eta) Z_c + \kappa_1 D_g \\
Z_2 &= \vartheta Z_v - \gamma Z_c + (\vartheta - 1) \kappa_1 D_g \\
Z_3 &= \Sigma_{gh}' (Z_v + (1-\eta) Z_c + \kappa_1 D_g) - \lambda_h + \kappa_1 D_h \\
&= \Sigma_{gh}' Z_1 - \lambda_h + \kappa_1 D_h \\
Z_4 &= \Sigma_{gh}' (\vartheta Z_v - \gamma Z_c + (\vartheta - 1) \kappa_1 D_g) - \vartheta \lambda_h + (\vartheta - 1) \kappa_1 D_h \\
&= \Sigma_{gh}' Z_2 - \vartheta \lambda_h + (\vartheta - 1) \kappa_1 D_h \\
Z_5 &= Z_4 - \Sigma_{gh}' Z_\pi
\end{aligned}
$$

where the vectors Z_c, Z_π and Z_v are selection vectors and the vectors D_g and D_h are part of the price to consumption ratio $pc_t = D_0 + D_g' g_t + D_h' h_t$. We use this notation throughout the appendix.

Appendix C.1.1 Step 1: Campbell-Shiller approximation

Let $pc_t = \ln\left(\frac{P_t}{C_t}\right)$ be the log price to consumption ratio. The return on the consumption asset is

$$
\begin{aligned}
r_{c,t+1} &\equiv \ln\left(\frac{P_{t+1} + C_{t+1}}{P_t}\right) = \ln(C_{t+1}) + \ln\left(\frac{P_{t+1} + C_{t+1}}{C_{t+1}}\right) - \ln(P_t) \\
&= \ln(C_{t+1}) - \ln(C_t) + \ln\left(1 + \frac{P_{t+1}}{C_{t+1}}\right) - \ln(P_t) + \ln(C_t) = \Delta c_{t+1} - pc_t + \ln(1 + \exp(pc_{t+1})).
\end{aligned}
$$

Take a first order Taylor expansion of the function $f(x) = \ln(1 + \exp(x))$ around $\bar{z}$.

$$
\begin{aligned}
r_{c,t+1} &\approx \Delta c_{t+1} - pc_t + \ln(1 + \exp(\bar{pc})) + \frac{\exp(\bar{pc})}{1 + \exp(\bar{pc})} (pc_{t+1} - \bar{z}) && \text{(C.6)} \\
&= \kappa_0 + \kappa_1 pc_{t+1} - pc_t + \Delta c_{t+1} && \text{(C.7)}
\end{aligned}
$$

where $\kappa_0 = \ln(1 + \exp(\bar{pc})) - \kappa_1 \bar{pc}$ and $\kappa_1 = \frac{\exp(\bar{pc})}{1+\exp(\bar{pc})}$.

Appendix C.1.2 Step 2: Solve for the price/consumption ratio

From the affine dynamics of the model, we can solve for the price to consumption ratio.

$$
\begin{aligned}
1 &= \mathrm{E}_t\left[\exp\left(m_{t+1}+r_{c,t+1}\right)\right] = \mathrm{E}_t\left[\exp\left(\vartheta\ln\left(\beta\right)+\vartheta\upsilon_{t+1}-\eta\vartheta\Delta c_{t+1}+\vartheta r_{c,t+1}\right)\right] \\
&= \exp\left(\vartheta\ln\left(\beta\right)+\vartheta\kappa_0-\vartheta pc_t\right)\mathrm{E}_t\left[\exp\left(\vartheta\upsilon_{t+1}+\vartheta\left(1-\eta\right)\Delta c_{t+1}+\vartheta\kappa_1 pc_{t+1}\right)\right]
\end{aligned}
$$

where we have used (C.7). Conjecture a solution for the price to consumption ratio

$$
pc_t = D_0 + D_g' g_t + D_h' h_t
$$

for unknown coefficients D_0, D_g and D_h. Substitute the guess into the problem

$$
\begin{aligned}
1 = {} & \exp\left(\vartheta\ln\left(\beta\right)+\vartheta\kappa_0+\vartheta\kappa_1 D_0-\vartheta pc_t+\vartheta\Lambda_1\left(g_t,h_t\right)-\vartheta\Lambda_3'\left(\Sigma_h\nu_h+\Phi_h h_t\right)\right) \\
& \exp\left(\vartheta Z_1'\left(\mu_g+\Phi_g g_t+\Phi_{gh}h_t-\Sigma_{gh}\left(\Sigma_h\nu_h+\Phi_h h_t\right)\right)\right) \\
& \mathrm{E}_t\left[\exp\left(\left(\vartheta\Lambda_2\left(g_t,h_t\right)+\vartheta\Sigma_{g,t}' Z_1\right)'\varepsilon_{g,t+1}\right)\right]\mathrm{E}_t\left[\exp\left(\vartheta Z_3' h_{t+1}\right)\right]
\end{aligned}
$$

Calculate the expectations using the Laplace transform

$$
\begin{aligned}
0 = {} & \vartheta\ln\left(\beta\right)+\vartheta\kappa_0+\vartheta\kappa_1 D_0-\vartheta pc_t+\vartheta\Lambda_1\left(g_t,h_t\right)-\vartheta\Lambda_3'\left(\Sigma_h\nu_h+\Phi_h h_t\right) \\
& +\vartheta Z_1'\left(\mu_g+\Phi_g g_t+\Phi_{gh}h_t-\Sigma_{gh}\left(\Sigma_h\nu_h+\Phi_h h_t\right)\right)+\frac{\vartheta^2}{2}\left(\Lambda_2\left(g_t,h_t\right)+\Sigma_{g,t}' Z_1\right)'\left(\Lambda_2\left(g_t,h_t\right)+\Sigma_{g,t}' Z_1\right) \\
& -\sum_{i=1}^{H}\nu_{h,i}\ln\left(1-\mathrm{e}_i'\Sigma_h'\vartheta Z_3\right)+\sum_{i=1}^{H}\frac{\mathrm{e}_i'\Sigma_h'\vartheta Z_3}{1-\mathrm{e}_i'\Sigma_h'\vartheta Z_3}\mathrm{e}_i'\Sigma_h^{-1}\Phi_h h_t
\end{aligned}
$$

The solution exists if $\mathrm{e}_i'\Sigma_h'\vartheta Z_3 < 1$ for $i = 1, \ldots, H$.

Solve for pc_t as

$$
\begin{aligned}
pc_t = {} & \ln\left(\beta\right)+\kappa_0+\kappa_1 D_0+\Lambda_1\left(g_t,h_t\right)-\Lambda_3'\left(\Sigma_h\nu_h+\Phi_h h_t\right) \\
& +Z_1'\left(\mu_g+\Phi_g g_t+\Phi_{gh}h_t-\Sigma_{gh}\left(\Sigma_h\nu_h+\Phi_h h_t\right)\right) \\
& +\frac{\vartheta}{2}\left(\Lambda_2\left(g_t,h_t\right)+\Sigma_{g,t}' Z_1\right)'\left(\Lambda_2\left(g_t,h_t\right)+\Sigma_{g,t}' Z_1\right) \\
& -\frac{1}{\vartheta}\sum_{i=1}^{H}\nu_{h,i}\log\left(1-\mathrm{e}_i'\Sigma_h'\vartheta Z_3\right)+\sum_{i=1}^{H}\frac{\mathrm{e}_i'\Sigma_h' Z_3}{1-\vartheta\mathrm{e}_i'\Sigma_h' Z_3}\mathrm{e}_i'\Sigma_h^{-1}\Phi_h h_t
\end{aligned}
$$

Plug in the risk sensitivity functions and cancel terms

$$
\begin{aligned}
pc_t &= \ln(\beta) + \kappa_0 + \kappa_1 D_0 + \lambda_h' (\Sigma_h \nu_h + \Phi_h h_t) \\
&\quad + Z_1' (\mu_g + \Phi_g g_t + \Phi_{gh} h_t - \Sigma_{gh} (\Sigma_h \nu_h + \Phi_h h_t)) \\
&\quad + \bar{\Lambda} - \eta\vartheta Z_1' (\lambda_0 + \lambda_g g_t + \lambda_{gh} h_t) \\
&\quad + \frac{\vartheta}{2} Z_1' \Sigma_{g,t} \Sigma_{g,t}' Z_1 - \frac{1}{\vartheta} \sum_{i=1}^{H} \nu_{h,i} \log(1 - \mathrm{e}_i' \Sigma_h' \vartheta Z_3) + \sum_{i=1}^{H} \frac{\mathrm{e}_i' \Sigma_h' Z_3}{1 - \vartheta \mathrm{e}_i' \Sigma_h' Z_3} \mathrm{e}_i' \Sigma_h^{-1} \Phi_h h_t
\end{aligned}
$$

We now solve for the coefficients. Both D_0 and D_g are analytical

$$
\begin{aligned}
D_0 &= \frac{1}{(1-\kappa_1)} \Big[\ln(\beta) + \kappa_0 + \bar{\Lambda} + \lambda_h' \Sigma_h \nu_h + Z_1' (\mu_g - \Sigma_{gh} \Sigma_h \nu_h - \eta\vartheta\lambda_0) \\
&\quad - \frac{1}{\vartheta} \sum_{i=1}^{H} \nu_{h,i} \ln(1 - \mathrm{e}_i' \Sigma_h' \vartheta Z_3) + \frac{\vartheta}{2} Z_1' \Sigma_{0,g} \Sigma_{0,g}' Z_1 \Big] \\
D_g &= \left(I_G - \kappa_1 (\Phi_g - \eta\vartheta\lambda_g)' \right)^{-1} (\Phi_g - \eta\vartheta\lambda_g)' (Z_v + (1-\eta) Z_c)
\end{aligned}
$$

An identifying assumption is that $(I_G - \kappa_1 (\Phi_g - \eta\vartheta\lambda_g))$ is invertible. The vector D_h is the solution to the system of equations

$$
D_h = \Phi_h' \lambda_h + (\Phi_{gh} - \Sigma_{gh} \Phi_h - \eta\vartheta\lambda_{gh})' Z_1 + \frac{\vartheta}{2} (\iota_H \otimes Z_1)' \tilde{\Sigma}_g \tilde{\Sigma}_g' (I_H \otimes Z_1) + \sum_{i=1}^{H} \frac{\mathrm{e}_i' \Sigma_h' Z_3}{1 - \vartheta \mathrm{e}_i' \Sigma_h' Z_3} \Phi_h' \Sigma_h^{-1,\prime} \mathrm{e}_i
$$

where $\tilde{\Sigma}_g \tilde{\Sigma}_g'$ is a $GH \times GH$ block diagonal matrix with $\Sigma_{i,g} \Sigma_{i,g}'$ along the diagonal. This cannot be solved in closed-form in the general case. However, if Σ_h and Φ_h are lower triangular, then it can be calculated in closed-form recursively for $i = 1, \ldots, H$. We discuss the analytical solution of this equation in more detail in Appendix C.2.

Appendix C.1.3 Step 3: Solve for the fixed-point

During estimation, we determine the value of $\bar{pc}$ and the log-linearization constants κ_0 and κ_1 as a function of the model parameters by solving the fixed-point problem

$$
0 = \bar{pc} - D_0(\bar{pc}) - D_g(\bar{pc})' \bar{\mu}_g - D_h(\bar{pc})' \bar{\mu}_h
$$

where the coefficients D_0, D_g and D_h are functions of $\bar{pc}$ through κ_0 and κ_1. The parameters $\bar{\mu}_g$ and $\bar{\mu}_h$ are the unconditional means of g_t and h_t.

Appendix C.1.4 Step 4: Substitute the solution into the SDF

We can now write the log-SDF as a function of the r.v.'s $\varepsilon_{g,t+1}$ and h_{t+1} as

$$\begin{aligned}
m_{t+1} &= \vartheta \ln(\beta) + (\vartheta - 1)(\kappa_0 - (1-\kappa_1) D_0) \\
&\quad - (\vartheta - 1) D_g' g_t - (\vartheta - 1) D_h' h_t + \vartheta \Lambda_1 (g_t, h_t) - \vartheta \Lambda_3' (\Sigma_h \nu_h + \Phi_h h_t) \\
&\quad + Z_2' (\mu_g + \Phi_g g_t + \Phi_{gh} h_t - \Sigma_{gh} (\Sigma_h \nu_h + \Phi_h h_t)) + \left(\vartheta \Lambda_2 (g_t, h_t) + \Sigma_{g,t}' Z_2\right)' \varepsilon_{g,t+1} + Z_4' h_{t+1}
\end{aligned}$$

Appendix C.2 Analytical solution of D_h

The $H \times 1$ vector of loadings D_h are a system of H equations in H unknowns. They can be solved analytically when both Φ_h and Σ_h are lower triangular by recursively solving one equation after another. We will consider the simpler case when they are both diagonal. Under this assumption, each equation is independent of one another and they simplify to

$$D_{h,i} = \bar{D}_i + \frac{\left(\bar{Z}_{3,i} + \kappa_1 D_{h,i}\right) \Phi_{h,i}}{1 - \vartheta \Sigma_{h,i} \left(\bar{Z}_{3,i} + \kappa_1 D_{h,i}\right)} \qquad i = 1, \ldots, H \tag{C.8}$$

where $\Phi_{h,i}$ and $\Sigma_{h,i}$ are the i-th diagonal elements and

$$\begin{aligned}
\bar{D} &= \Phi_h' \lambda_h + (\Phi_{gh} - \Sigma_{gh} \Phi_h - \eta \vartheta \lambda_{gh})' Z_1 + \frac{\vartheta}{2} (\iota_H \otimes Z_1)' \tilde{\Sigma}_g \tilde{\Sigma}_g' (I_H \otimes Z_1) \\
\bar{Z}_3 &= \Sigma_{gh}' Z_1 - \lambda_h
\end{aligned}$$

and $\bar{Z}_{3,i}$ and $\bar{D}_i$ are the i-th element of these $H \times 1$ vectors.

Each loading (C.8) for $i = 1, \ldots, H$ is a quadratic equation

$$0 = \kappa_1 \vartheta \Sigma_{h,i} D_{h,i}^2 + D_{h,i} \left(\kappa_1 \Phi_{h,i} - \kappa_1 \vartheta \Sigma_{h,i} \bar{D}_i - 1 + \vartheta \Sigma_{h,i} \bar{Z}_{3,i}\right) + \bar{D}_i \left(1 - \vartheta \Sigma_{h,i} \bar{Z}_{3,i}\right) + \bar{Z}_{3,i} \Phi_{h,i}$$

The solutions are

$$\begin{aligned}
D_{h,i} &= \frac{-\left(\kappa_1 \Phi_{h,i} - \kappa_1 \vartheta \Sigma_{h,i} \bar{D}_i - 1 + \vartheta \Sigma_{h,i} \bar{Z}_{3,i}\right)}{2 \kappa_1 \vartheta \Sigma_{h,i}} \\
&\quad \pm \frac{\sqrt{\left(\kappa_1 \Phi_{h,i} - \kappa_1 \vartheta \Sigma_{h,i} \bar{D}_i - 1 + \vartheta \Sigma_{h,i} \bar{Z}_{3,i}\right)^2 - 4 \kappa_1 \vartheta \Sigma_{h,i} \left[\bar{D}_i \left(1 - \vartheta \Sigma_{h,i} \bar{Z}_{3,i}\right) + \bar{Z}_{3,i} \Phi_{h,i}\right]}}{2 \kappa_1 \vartheta \Sigma_{h,i}}
\end{aligned}$$

When a real solution exists, it has two solutions. Only one solution leads to a sensible value. This is the value with a negative sign, see also Campbell, Giglio, Polk, and Turley(2014) for the ICAPM model.

Appendix C.3 Model solution when $\eta = 1$ and no preference shocks

When the inverse of the elasticity of intertemporal substitution equals one $\eta = 1$ and there are no preference shocks, recursive preferences can be solved in closed form; see, e.g. Tallarini(2000) and Hansen, Heaton, and Li(2008). This provides a benchmark for comparison.

Let $vc_t = \ln\left(\frac{V_t}{C_t}\right)$. From Hansen, Heaton, and Li(2008), we know

$$vc_t = \frac{\beta}{1-\gamma}\ln\left(\mathrm{E}_t\left[\exp\left((1-\gamma)\left[vc_{t+1}+\Delta c_{t+1}\right]\right)\right]\right)$$

Guess that the solution is $vc_t = E_0 + E_g' g_t + E_h' h_t$ for some unknown coefficients E_0, E_g and E_h. Plugging in the guess and calculating the expectation, we find

$$\begin{aligned} vc_t &= \beta\left(E_0 + (E_g+Z_c)'\left(\mu_g + \Phi_g g_t + \Phi_{gh}h_t - \Sigma_{gh}\left(\Sigma_h\nu_h + \Phi_h h_t\right)\right)\right) + \beta\left(\frac{(1-\gamma)}{2}(E_g+Z_c)'\,\Sigma_{g,t}\Sigma_{g,t}'\,(E_g+Z_c)\right) \\ &+\frac{\beta}{1-\gamma}\left[-\sum_{i=1}^{H}\nu_{h,i}\log\left(1-(1-\gamma)\,\mathrm{e}_i'\Sigma_h'\left[\Sigma_{gh}'(E_g+Z_c)+E_h\right]\right)\right] \\ &+\beta\left[\sum_{i=1}^{H}\frac{\mathrm{e}_i'\Sigma_h'\left[\Sigma_{gh}'(E_g+Z_c)+E_h\right]}{\left(1-(1-\gamma)\,\mathrm{e}_i'\Sigma_h'\left[\Sigma_{gh}'(E_g+Z_c)+E_h\right]\right)}\mathrm{e}_i'\Sigma_h^{-1}\Phi_h h_t\right]. \end{aligned}$$

Existence of the conditional expectation introduces the restriction that $(1-\gamma)\,\mathrm{e}_i'\Sigma_h'\left(\Sigma_{gh}'(E_g+Z_c)+E_h\right) \leq 1$ for $i = 1,\ldots,H$. The solutions for E_0, E_g and E_h are

$$\begin{aligned} E_0 &= \frac{\beta}{1-\beta}(E_g+Z_c)'\left(\mu_g - \Sigma_{gh}\Sigma_h\nu_h\right) + \frac{\beta}{1-\beta}\frac{(1-\gamma)}{2}(E_g+Z_c)'\,\Sigma_{0,g}\Sigma_{0,g}'\,(E_g+Z_c) \\ &-\frac{\beta}{(1-\beta)(1-\gamma)}\sum_{i=1}^{H}\nu_{h,i}\log\left(1-(1-\gamma)\,\mathrm{e}_i'\Sigma_h'\left[\Sigma_{gh}'(E_g+Z_c)+E_h\right]\right) \\ E_g &= \beta\left(I_G - \beta\Phi_g'\right)^{-1}\Phi_g' Z_c \\ E_h &= \beta\left(\Phi_{gh}-\Sigma_{gh}\Phi_h\right)'(E_g+Z_c) + \frac{\beta(1-\gamma)}{2}\left(I_H\otimes(E_g+Z_c)\right)'\tilde{\Sigma}_g\tilde{\Sigma}_g'\left(\iota_H\otimes(E_g+Z_c)\right) \\ &+\beta\sum_{i=1}^{H}\frac{\mathrm{e}_i'\Sigma_h'\left[\Sigma_{gh}'(E_g+Z_c)+E_h\right]}{\left(1-(1-\gamma)\,\mathrm{e}_i'\Sigma_h'\left[\Sigma_{gh}'(E_g+Z_c)+E_h\right]\right)}\Phi_h'\Sigma_h\mathrm{e}_i \end{aligned}$$

The solutions for E_0 and E_g are analytical. The solution for E_g exists as long as $\beta \neq 1$ if/when Φ_g has a unit root. The solution for E_h can be found using the same procedure as D_h. This introduces a restriction on the parameter space. Define the following notation for simplicity

$$\begin{aligned} E_1 &= (1-\gamma)\,E_g - \gamma Z_c - Z_\pi, \\ E_2 &= \Sigma_{gh}' E_1 + (1-\gamma)\,E_h. \end{aligned}$$

The nominal log-SDF can be written as

$$m_{t+1}^{\$} = \ln(\beta) + (1-\gamma)\left(E_0 + E_h' h_{t+1}\right) - \frac{(1-\gamma)}{\beta}\left(E_0 + E_g' g_t + E_h' h_t\right) + E_1' g_{t+1} \tag{C.9}$$

which can be used to solve for asset prices.

Appendix D Bond prices

Appendix D.1 Real bonds: general case

We will guess and verify that the solution for zero coupon bonds is $P_t^{(n)} = \exp\left(\bar{a}_n + \bar{b}_{n,g}' g_t + \bar{b}_{n,h}' h_t\right)$ for some unknown coefficients $\bar{a}_n$ and $\bar{b}_{n,g}$ and $\bar{b}_{n,h}$.

For a maturity $n=1$, the payoff is guaranteed to be $P_{t+1}^{(0)} = 1$ in which case $P_t^{(1)} = \mathrm{E}_t\left[M_{t+1}\right]$. Using standard techniques for affine bond pricing in discrete-time (see Creal and Wu(2015a)), we find that at maturity $n=1$ the bond loadings are

$$\begin{aligned} \bar{a}_1 &= \ln(\beta) + \lambda_h' \Sigma_h \nu_h + \bar{\Lambda} + \left(Z_\upsilon - \eta Z_c\right)' \left(\mu_g - \Sigma_{gh}\Sigma_h \nu_h - \eta\vartheta\lambda_0\right) \\ &\quad - \sum_{i=1}^{H} \nu_{h,i} \log\left(1 - \mathrm{e}_i' \Sigma_h' Z_4\right) + \frac{(\vartheta-1)}{\vartheta} \sum_{i=1}^{H} \nu_{h,i} \log\left(1 - \mathrm{e}_i'\Sigma_h' \vartheta Z_3\right) \\ &\quad - \frac{(\vartheta-1)\,\vartheta}{2} Z_1' \Sigma_{0,g} \Sigma_{0,g}' Z_1 + \frac{1}{2} Z_2' \Sigma_{0,g}\Sigma_{0,g}' Z_2 \\ \bar{b}_{1,g} &= \left(\Phi_g - \eta\vartheta\lambda_g\right)' \left(Z_\upsilon - \eta Z_c\right) \\ \bar{b}_{1,h} &= \left(\Phi_{gh} - \Sigma_{gh}\Phi_h - \eta\vartheta\lambda_{gh}\right)' \left(Z_\upsilon - \eta Z_c\right) + \Phi_h' \lambda_h \\ &\quad + \left(\sum_{i=1}^{H} \frac{\mathrm{e}_i'\Sigma_h' Z_4}{1 - \mathrm{e}_i'\Sigma_h' Z_4} \mathrm{e}_i' \Sigma_h^{-1} \Phi_h\right)' - (\vartheta-1)\left(\sum_{i=1}^{H} \frac{\mathrm{e}_i'\Sigma_h' Z_3}{1 - \vartheta \mathrm{e}_i'\Sigma_h' Z_3} \mathrm{e}_i' \Sigma_h^{-1} \Phi_h\right)' \\ &\quad + \frac{1}{2}\left(I_H \otimes Z_2\right)' \tilde{\Sigma}_g \tilde{\Sigma}_g' \left(\iota_H \otimes Z_2\right) - \frac{(\vartheta-1)\,\vartheta}{2}\left(I_H \otimes Z_1\right)' \tilde{\Sigma}_g \tilde{\Sigma}_g' \left(\iota_H \otimes Z_1\right) \end{aligned}$$

where bond prices only exist if $\mathrm{e}_i'\Sigma_h' Z_4 < 1$ for $i = 1, \ldots, H$. At maturity n, we use the fact that $P_t^{(n)} =$

$\mathrm{E}_t\left[\exp\left(m_{t+1}\right)P_{t+1}^{(n-1)}\right]$. The bond loadings are

$$
\begin{aligned}
\bar{a}_n &= \bar{a}_{n-1}+\bar{a}_1+\sum_{i=1}^{H}\nu_{h,i}\log\left(\frac{1-\mathrm{e}_i'\Sigma_h'Z_4}{1-\mathrm{e}_i'\Sigma_h'\left(\Sigma_{gh}'\bar{b}_{n-1,g}+\bar{b}_{n-1,h}+Z_4\right)}\right)\\
&\quad+\left(\mu_g-\Sigma_{gh}\Sigma_h\nu_h-\eta\vartheta\lambda_0\right)'\bar{b}_{n-1,g}+\frac{1}{2}\bar{b}_{n-1,g}'\Sigma_{0,g}\Sigma_{0,g}'\bar{b}_{n-1,g}+\bar{b}_{n-1,g}'\Sigma_{0,g}\Sigma_{0,g}'Z_2\\
\bar{b}_{n,g} &= \left(\Phi_g-\eta\vartheta\lambda_g\right)'\bar{b}_{n-1,g}+\bar{b}_{1,g}\\
\bar{b}_{n,h} &= \left(\Phi_{gh}-\Sigma_{gh}\Phi_h-\eta\vartheta\lambda_{gh}\right)'\bar{b}_{n-1,g}+\bar{b}_{1,h}\\
&\quad+\left(\sum_{i=1}^{H}\left(\frac{\mathrm{e}_i'\Sigma_h'\left(\Sigma_{gh}'\bar{b}_{n-1,g}+\bar{b}_{n-1,h}+Z_4\right)}{1-\mathrm{e}_i'\Sigma_h'\left(\Sigma_{gh}'\bar{b}_{n-1,g}+\bar{b}_{n-1,h}+Z_4\right)}-\frac{\mathrm{e}_i'\Sigma_h'Z_4}{1-\mathrm{e}_i'\Sigma_h'Z_4}\right)\mathrm{e}_i'\Sigma_h^{-1}\Phi_h\right)'\\
&\quad+\frac{1}{2}\left(I_H\otimes\bar{b}_{n-1,g}\right)'\tilde{\Sigma}_g\tilde{\Sigma}_g'\left(\iota_H\otimes\bar{b}_{n-1,g}\right)+\left(I_H\otimes Z_2\right)'\tilde{\Sigma}_g\tilde{\Sigma}_g'\left(\iota_H\otimes\bar{b}_{n-1,g}\right)
\end{aligned}
$$

Real yields are $y_t^{(n)}=a_n+b_{n,g}'g_t+b_{n,h}'h_t$ with $a_n=-\frac{1}{n}\bar{a}_n, b_{n,g}=-\frac{1}{n}\bar{b}_{n,g}$ and $b_{n,h}=-\frac{1}{n}\bar{b}_{n,h}$.

Appendix D.2 Nominal bonds: general case

The solution for zero coupon nominal bonds is $P_t^{\$,(n)}=\exp\left(\bar{a}_n^{\$}+\bar{b}_{n,g}^{\$,\prime}g_t+\bar{b}_{n,h}^{\$,\prime}h_t\right)$ for some unknown coefficients $\bar{a}_n^{\$}$ and $\bar{b}_{n,g}^{\$}$ and $\bar{b}_{n,h}^{\$}$. For a maturity $n=1$, the payoff is guaranteed to be $P_{t+1}^{\$,(0)}=1$ in which case $P_t^{\$,(1)}=\mathrm{E}_t\left[M_{t+1}^{\$}\right]$. The solutions are

$$
\begin{aligned}
\bar{a}_1^{\$} &= \ln(\beta)+\lambda_h'\Sigma_h\nu_h+\bar{\Lambda}+\left(Z_\upsilon-\eta Z_c-Z_\pi\right)'\left(\mu_g-\Sigma_{gh}\Sigma_h\nu_h-\eta\vartheta\lambda_0\right)\\
&\quad+\frac{(\vartheta-1)}{\vartheta}\sum_{i=1}^{H}\nu_{h,i}\log\left(1-\mathrm{e}_i'\Sigma_h'\vartheta Z_3\right)-\sum_{i=1}^{H}\nu_{h,i}\log\left(1-\mathrm{e}_i'\Sigma_h'Z_5\right)\\
&\quad-\frac{(\vartheta-1)\vartheta}{2}Z_1'\Sigma_{0,g}\Sigma_{0,g}'Z_1+\frac{1}{2}Z_2'\Sigma_{0,g}\Sigma_{0,g}'Z_2+\frac{1}{2}Z_\pi'\Sigma_{0,g}\Sigma_{0,g}'Z_\pi-Z_2'\Sigma_{0,g}\Sigma_{0,g}'Z_\pi\\
\bar{b}_{1,g}^{\$} &= \left(\Phi_g-\eta\vartheta\lambda_g\right)'\left(Z_\upsilon-\eta Z_c-Z_\pi\right)\\
\bar{b}_{1,h}^{\$} &= \left(\Phi_{gh}-\Sigma_{gh}\Phi_h-\eta\vartheta\lambda_{gh}\right)'\left(Z_\upsilon-\eta Z_c-Z_\pi\right)+\Phi_h'\lambda_h\\
&\quad-(\vartheta-1)\left(\sum_{i=1}^{H}\frac{\mathrm{e}_i'\Sigma_h'Z_3}{1-\vartheta\mathrm{e}_i'\Sigma_h'Z_3}\mathrm{e}_i'\Sigma_h^{-1}\Phi_h\right)'+\left(\sum_{i=1}^{H}\frac{\mathrm{e}_i'\Sigma_h'Z_5}{1-\mathrm{e}_i'\Sigma_h'Z_5}\mathrm{e}_i'\Sigma_h^{-1}\Phi_h\right)'\\
&\quad+\frac{1}{2}\left(I_H\otimes Z_\pi\right)'\tilde{\Sigma}_g\tilde{\Sigma}_g'\left(\iota_H\otimes Z_\pi\right)-\left(I_H\otimes Z_2\right)'\tilde{\Sigma}_g\tilde{\Sigma}_g'\left(\iota_H\otimes Z_\pi\right)\\
&\quad+\frac{1}{2}\left(I_H\otimes Z_2\right)'\tilde{\Sigma}_g\tilde{\Sigma}_g'\left(\iota_H\otimes Z_2\right)-\frac{(\vartheta-1)\vartheta}{2}\left(I_H\otimes Z_1\right)'\tilde{\Sigma}_g\tilde{\Sigma}_g'\left(\iota_H\otimes Z_1\right)
\end{aligned}
$$

where bond prices only exist if $\mathrm{e}_i'\Sigma_h' Z_5 < 1$ for $i = 1, \ldots, H$. At longer maturities n, we use the fact that $P_t^{\$,(n)} = \mathrm{E}_t\left[\exp\left(m_{t+1}^{\$}\right) P_{t+1}^{\$,(n-1)}\right]$. The bond loadings are

$$\begin{aligned}
\bar{a}_n^{\$} &= \bar{a}_{n-1}^{\$} + \bar{a}_1^{\$} + \left(\mu_g - \Sigma_{gh}\Sigma_h\nu_h - \eta\vartheta\lambda_0\right)' \bar{b}_{n-1,g}^{\$} \\
&\quad + \sum_{i=1}^{H} \nu_{h,i} \log\left(\frac{1 - \mathrm{e}_i'\Sigma_h' Z_5}{1 - \mathrm{e}_i'\Sigma_h'\left(\Sigma_{gh}'\bar{b}_{n-1,g}^{\$} + \bar{b}_{n-1,h}^{\$} + Z_5\right)}\right) \\
&\quad + \frac{1}{2}\bar{b}_{n-1,g}^{\$,\prime}\Sigma_{0,g}\Sigma_{0,g}'\bar{b}_{n-1,g}^{\$} + \bar{b}_{n-1,g}^{\$}\Sigma_{0,g}\Sigma_{0,g}'\left(Z_2 - Z_\pi\right) \\
\bar{b}_{n,g}^{\$} &= \left(\Phi_g - \eta\vartheta\lambda_g\right)' \bar{b}_{n-1,g}^{\$} + \bar{b}_{1,g}^{\$} \\
\bar{b}_{n,h}^{\$} &= \left(\Phi_{gh} - \Sigma_{gh}\Phi_h - \eta\vartheta\lambda_{gh}\right)' \bar{b}_{n-1,g}^{\$} + \bar{b}_{1,h}^{\$} \\
&\quad + \left(\sum_{i=1}^{H}\left(\frac{\mathrm{e}_i'\Sigma_h'\left(\Sigma_{gh}'\bar{b}_{n-1,g}^{\$} + \bar{b}_{n-1,h}^{\$} + Z_5\right)}{1 - \mathrm{e}_i'\Sigma_h'\left(\Sigma_{gh}'\bar{b}_{n-1,g}^{\$} + \bar{b}_{n-1,h}^{\$} + Z_5\right)} - \frac{\mathrm{e}_i'\Sigma_h' Z_5}{1 - \mathrm{e}_i'\Sigma_h' Z_5}\right)\mathrm{e}_i'\Sigma_h^{-1}\Phi_h\right)' \\
&\quad + \frac{1}{2}\left(I_H \otimes \bar{b}_{n-1,g}^{\$}\right)' \tilde{\Sigma}_g\tilde{\Sigma}_g'\left(\iota_H \otimes \bar{b}_{n-1,g}^{\$}\right) + \left(I_H \otimes \bar{b}_{n-1,g}^{\$}\right)' \tilde{\Sigma}_g\tilde{\Sigma}_g'\left(\iota_H \otimes \left(Z_2 - Z_\pi\right)\right)
\end{aligned}$$

Nominal yields are $y_t^{\$,(n)} = a_n^{\$} + b_{n,g}^{\$,\prime} g_t + b_{n,h}^{\$,\prime} h_t$ with $a_n^{\$} = -\frac{1}{n}\bar{a}_n^{\$}$, $b_{n,g}^{\$} = -\frac{1}{n}\bar{b}_{n,g}^{\$}$ and $b_{n,h}^{\$} = -\frac{1}{n}\bar{b}_{n,h}^{\$}$.

Appendix D.3 Nominal bonds: special case of $\eta = 1$ and no preference shocks

In the special case when recursive preferences can be solved analytically, we can solve for bond prices. Using the solution for the log-SDF given by (C.9), the loadings on nominal bonds are

$$\begin{aligned}
\bar{a}_1^{\$} &= \ln(\beta) + (1-\gamma)E_0 - \frac{(1-\gamma)}{\beta}E_0 + E_1'\left(\mu_g - \Sigma_{gh}\Sigma_h\nu_h\right) + \frac{1}{2}E_1'\Sigma_{0,g}\Sigma_{0,g}'E_1 \\
&\quad - \sum_{i=1}^{H}\nu_{h,i}\log\left(1 - \mathrm{e}_i'\Sigma_h' E_2\right) \\
\bar{b}_{1,g}^{\$} &= -\frac{(1-\gamma)}{\beta}E_g + \Phi_g' E_1 \\
\bar{b}_{1,h}^{\$} &= -\frac{(1-\gamma)}{\beta}E_h + \left(\Phi_{gh} - \Sigma_{gh}\Phi_h\right)' E_1 \\
&\quad + \frac{1}{2}\left(I_H \otimes E_1\right)' \tilde{\Sigma}_g\tilde{\Sigma}_g'\left(\iota_H \otimes E_1\right) + \sum_{i=1}^{H}\frac{\mathrm{e}_i'\Sigma_h' E_2}{1 - \mathrm{e}_i'\Sigma_h' E_2}\Phi_h'\Sigma_h^{-1,\prime}\mathrm{e}_i
\end{aligned}$$

where E_1 and E_2 are defined above. At higher maturities, we find

$$
\begin{aligned}
\bar{a}_n^{\$} &= \bar{a}_1^{\$} + \bar{b}_{n-1,g}^{\$,\prime}\left(\mu_g - \Sigma_{gh}\Sigma_h\nu_h + \Sigma_{0,g}\Sigma_{0,g}^{\prime}E_1\right) \\
&\quad + \frac{1}{2}\bar{b}_{n-1,g}^{\$,\prime}\Sigma_{0,g}\Sigma_{0,g}^{\prime}\bar{b}_{n-1,g}^{\$} + \sum_{i=1}^{H}\nu_{h,i}\log\left(1 - \mathrm{e}_i^{\prime}\Sigma_h^{\prime}E_2\right) \\
&\quad - \sum_{i=1}^{H}\nu_{h,i}\log\left(1 - \mathrm{e}_i^{\prime}\Sigma_h^{\prime}\left(\Sigma_{gh}^{\prime}\bar{b}_{n-1,g}^{\$} + \bar{b}_{n-1,h}^{\$} + E_2\right)\right) \\
\bar{b}_{n,g}^{\$} &= \Phi_g^{\prime}\bar{b}_{n-1,g}^{\$} + \bar{b}_{1,g}^{\$} \\
\bar{b}_{n,h}^{\$} &= \bar{b}_{1,h}^{\$} + \left(\Phi_{gh} - \Sigma_{gh}\Phi_h\right)^{\prime}\bar{b}_{n-1,g}^{\$} \\
&\quad - \sum_{i=1}^{H}\frac{\mathrm{e}_i^{\prime}\Sigma_h^{\prime}E_2}{1 - \mathrm{e}_i^{\prime}\Sigma_h^{\prime}E_2}\Phi_h^{\prime}\Sigma_h^{-1,\prime}\mathrm{e}_i + \sum_{i=1}^{H}\frac{\mathrm{e}_i^{\prime}\Sigma_h^{\prime}\left(\Sigma_{gh}^{\prime}\bar{b}_{n-1,g}^{\$} + \bar{b}_{n-1,h}^{\$} + E_2\right)}{1 - \mathrm{e}_i^{\prime}\Sigma_h^{\prime}\left(\Sigma_{gh}^{\prime}\bar{b}_{n-1,g}^{\$} + \bar{b}_{n-1,h}^{\$} + E_2\right)}\Phi_h^{\prime}\Sigma_h^{-1,\prime}\mathrm{e}_i \\
&\quad + \frac{1}{2}\left(I_H \otimes \left[\bar{b}_{n-1,g}^{\$} + E_1\right]\right)^{\prime}\tilde{\Sigma}_g\tilde{\Sigma}_g^{\prime}\left(\iota_H \otimes \left[\bar{b}_{n-1,g}^{\$} + E_1\right]\right) \\
&\quad - \frac{1}{2}\left(I_H \otimes E_1\right)^{\prime}\tilde{\Sigma}_g\tilde{\Sigma}_g^{\prime}\left(\iota_H \otimes E_1\right)
\end{aligned}
$$

Yields only exist if $\mathrm{e}_i^{\prime}\Sigma_h^{\prime}\left(\Sigma_{gh}^{\prime}\bar{b}_{n-1,g}^{\$} + \bar{b}_{n-1,h}^{\$} + E_2\right) < 1$ for $i = 1, \ldots, H$.

Appendix E Proof of Propositions

Appendix E.1 Gaussian models

Define the fixed point problem

$$
\begin{aligned}
\kappa_1 &= \frac{\exp\left(\bar{pc}\right)}{1 + \exp\left(\bar{pc}\right)} \\
\kappa_0 &= \ln\left(1 + \exp\left(\bar{pc}\right)\right) - \kappa_1\bar{pc} \\
D_g^{\prime} &= \left(Z_v + (1-\eta)Z_c\right)^{\prime}\left(\Phi_g - \vartheta\eta\lambda_g\right)\left(I - \kappa_1\left(\Phi_g - \vartheta\eta\lambda_g\right)\right)^{-1} \\
Z_1 &= Z_v + (1-\eta)Z_c + \kappa_1 D_g \\
D_0\left(1 - \kappa_1\right) &= \ln(\beta) + \kappa_0 + Z_1^{\prime}\mu_g + \bar{\Lambda} + \frac{1}{2}\vartheta Z_1^{\prime}\Sigma_{0,g}\Sigma_{0,g}^{\prime}Z_1 - \vartheta\eta Z_1^{\prime}\lambda_0 \\
\tilde{pc} &= D_0 + D_g^{\prime}\bar{\mu}_g
\end{aligned}
$$

which is solved if $\bar{pc} = \tilde{pc}$.

Proof of Proposition 1

First, the derivative $d(\bar{pc}-\tilde{pc})/d\bar{pc}$ is a finite number when $\bar{pc}$ is finite, given the eigenvalue of $(\Phi_g - \vartheta\eta\lambda_g)$ for consumption is smaller than 1. This proves the continuity of the function $\bar{pc}-\tilde{pc}$. Second, derive the limiting property for $\bar{pc}\to-\infty$: $\lim_{\bar{pc}\to-\infty}\kappa_1 = 0$ and $\lim_{\bar{pc}\to-\infty}\kappa_0 = 0$. In this case, $\tilde{pc}$ is finite, so $\lim_{\bar{pc}\to-\infty}(\bar{pc}-\tilde{pc}) \to -\infty$. Then, derive the limiting property for $\bar{pc}\to\infty$: $\lim_{\bar{pc}\to\infty}\kappa_1 = 1$ and $\lim_{\bar{pc}\to\infty}\kappa_0 = 0$. This implies D_g is finite as long as the eigenvalue of $(\Phi_g - \vartheta\eta\lambda_g)$ for consumption is smaller than 1. And $\lim_{\bar{pc}\to\infty} D_0 = \lim_{\bar{pc}\to\infty}\frac{1}{1-\kappa_1}(\ln(\beta) + Z_1'\mu_g + \bar{\Lambda} + \frac{1}{2}\vartheta Z_1'\Sigma_{0,g}\Sigma_{0,g}'Z_1 - \vartheta\eta Z_1'\lambda_0)$ is infinite due to $\lim_{\bar{pc}\to\infty}(1-\kappa_1) = 0$. The condition $\lim_{\bar{pc}\to\infty} D_0 \to -\infty$ implies $\lim_{\bar{pc}\to\infty}(\bar{pc}-\tilde{pc}) \to \infty$, which together $\lim_{\bar{pc}\to-\infty}(\bar{pc}-\tilde{pc}) \to -\infty$ and the continuity of the function guarantees there exists a solution for the fixed point problem.

With $\kappa_1 < 1$, the condition $\lim_{\bar{pc}\to\infty} D_0 \to -\infty$ is equivalent to

$$\beta < \lim_{\bar{pc}\to\infty} \exp\left(-Z_1'\left(\mu_g - \vartheta\eta\lambda_0\right) - \bar{\Lambda} - \frac{1}{2}\vartheta Z_1'\Sigma_{0,g}\Sigma_{0,g}'Z_1\right),$$

hence $\bar{\beta}(\psi,\gamma,\theta^{\mathbb{P}},\theta^{\lambda}) = \exp\left(-Z_1^{\infty\prime}\left(\mu_g - \vartheta\eta\lambda_0\right) - \bar{\Lambda} - \frac{1}{2}\vartheta Z_1^{\infty\prime}\Sigma_{0,g}\Sigma_{0,g}'Z_1^{\infty}\right)$, where $Z_1^{\infty} \equiv \lim_{\bar{pc}\to\infty} Z_1\left(\bar{pc}\right) = Z_\upsilon + (1-\eta)\,Z_c + D_g^{\infty}$ and $D_g^{\infty\prime} \equiv \lim_{\bar{pc}\to\infty} D_g\left(\bar{pc}\right)' = \left(Z_\upsilon + (1-\eta)\,Z_c\right)'\left(\Phi_g - \vartheta\eta\lambda_g\right)\left(I - \left(\Phi_g - \vartheta\eta\lambda_g\right)\right)^{-1}$.

Proof of Corollary 1

For models with no preference shock $\lambda_0 = 0, \lambda_g = 0, \bar{\Lambda} = 0$, $Z_\upsilon = 0$, the condition becomes

$$\beta < \lim_{\bar{pc}\to\infty} \exp\left(-Z_1'\mu_g - \frac{1}{2}\vartheta Z_1'\Sigma_{0,g}\Sigma_{0,g}'Z_1\right), \tag{E.10}$$

1. The condition (E.10) is guaranteed by $Z_1^{\infty\prime}\mu_g \leq 0$ and $\vartheta < 0$.
2. If $\beta = 1$, then the condition can be simplified to

$$\begin{aligned} \gamma &> 1 + \frac{2Z_c'\left(I-\Phi_g\right)^{-1}\mu_g}{Z_c'\left(I-\Phi_g\right)^{-1}\Sigma_{0,g}\Sigma_{0,g}'\left(I-\Phi_g'\right)^{-1}Z_c}, \qquad \text{if} \quad \psi > 1 \\ \gamma &< 1 + \frac{2Z_c'\left(I-\Phi_g\right)^{-1}\mu_g}{Z_c'\left(I-\Phi_g\right)^{-1}\Sigma_{0,g}\Sigma_{0,g}'\left(I-\Phi_g'\right)^{-1}Z_c}, \qquad \text{if} \quad \psi < 1 \end{aligned}$$

   hence $1 + \bar{\gamma}(\theta^P) = \frac{2Z_c'(I-\Phi_g)^{-1}\mu_g}{Z_c'(I-\Phi_g)^{-1}\Sigma_{0,g}\Sigma_{0,g}'\left(I-\Phi_g'\right)^{-1}Z_c}$, does not depend on ψ.
3. We have $\frac{d\vartheta}{d\gamma} = -\frac{1}{1-\eta}$, $\frac{dD_g^{\infty\prime}}{d\gamma} = 0$ and $\frac{dZ_1^{\infty\prime}}{d\gamma} = \frac{dD_g^{\infty\prime}}{d\gamma} = 0$. Hence, the derivative of $\ln\bar{\beta}$ w.r.t. γ is

$$\frac{d\ln\bar{\beta}}{d\gamma} = \frac{1}{2(1-\eta)}Z_1^{\infty\prime}\Sigma_{0,g}\Sigma_{0,g}'Z_1^{\infty}$$

$\frac{d\ln\bar{\beta}}{d\gamma} = \frac{1}{\bar{\beta}}\frac{d\bar{\beta}}{d\gamma}$ implies that the two derivatives have the same sign. Therefore, for $\psi > 1$, then $\frac{d\bar{\beta}}{d\gamma} > 0$; for $\psi < 1$, then $\frac{d\bar{\beta}}{d\gamma} < 0$.

Appendix E.2 Stochastic volatility models

Define the fixed point problem

$$
\begin{aligned}
\kappa_1 &= \frac{\exp(\bar{pc})}{1+\exp(\bar{pc})} \\
\kappa_0 &= \ln(1+\exp(\bar{pc})) - \kappa_1 \bar{pc} \\
D_g' &= (Z_v + (1-\eta) Z_c)' (\Phi_g - \vartheta\eta\lambda_g)(I - \kappa_1(\Phi_g - \vartheta\eta\lambda_g))^{-1} \\
Z_1 &= Z_v + (1-\eta) Z_c + \kappa_1 D_g \\
Z_3 &= \Sigma_{gh}'(Z_v + (1-\eta) Z_c + \kappa_1 D_g) - \lambda_h + \kappa_1 D_h \\
D_h &= \Phi_h'\lambda_h + (\Phi_{gh} - \Sigma_{gh}\Phi_h - \eta\vartheta\lambda_{gh})' Z_1 + \frac{\vartheta}{2}(\iota_H \otimes Z_1)' \tilde{\Sigma}_g \tilde{\Sigma}_g' (I_H \otimes Z_1) + \sum_{i=1}^{H} \frac{\mathrm{e}_i'\Sigma_h' Z_3}{1-\vartheta \mathrm{e}_i'\Sigma_h' Z_3} \Phi_h' \Sigma_h^{-1,\prime} \mathrm{e}_i \\
(1-\kappa_1) D_0 &= \ln(\beta) + \kappa_0 + \bar{\Lambda} + \lambda_h'\Sigma_h\nu_h + Z_1'(\mu_g - \Sigma_{gh}\Sigma_h\nu_h - \eta\vartheta\lambda_0) \\
&\quad -\frac{1}{\vartheta}\sum_{i=1}^{H}\nu_{h,i}\ln(1-\mathrm{e}_i'\Sigma_h'\vartheta Z_3) + \frac{\vartheta}{2} Z_1'\Sigma_{0,g}\Sigma_{0,g}' Z_1 \\
\tilde{pc} &= D_0 + D_g'\bar{\mu}_g + D_h'\bar{\mu}_h
\end{aligned}
$$

which is solved if $\bar{pc} = \tilde{pc}$.

Conditions for Assumption 1

Each loading (C.8) for $i = 1, \ldots, H$ is a quadratic equation

$$
0 = \kappa_1\vartheta\Sigma_{h,i} D_{h,i}^2 + D_{h,i}\left(\kappa_1\Phi_{h,i} - \kappa_1\vartheta\Sigma_{h,i}\bar{D}_i - 1 + \vartheta\Sigma_{h,i}\bar{Z}_{3,i}\right) + \bar{D}_i\left(1 - \vartheta\Sigma_{h,i}\bar{Z}_{3,i}\right) + \bar{Z}_{3,i}\Phi_{h,i}
$$

When Σ_h and Φ_h are diagonal, the solutions are

$$
\begin{aligned}
D_{h,i} &= \frac{-\left(\kappa_1\Phi_{h,i} - \kappa_1\vartheta\Sigma_{h,i}\bar{D}_i - 1 + \vartheta\Sigma_{h,i}\bar{Z}_{3,i}\right)}{2\kappa_1\vartheta\Sigma_{h,i}} \\
&\quad \pm \frac{\sqrt{\left(\kappa_1\Phi_{h,i} - \kappa_1\vartheta\Sigma_{h,i}\bar{D}_i - 1 + \vartheta\Sigma_{h,i}\bar{Z}_{3,i}\right)^2 - 4\kappa_1\vartheta\Sigma_{h,i}\left[\bar{D}_i\left(1 - \vartheta\Sigma_{h,i}\bar{Z}_{3,i}\right) + \bar{Z}_{3,i}\Phi_{h,i}\right]}}{2\kappa_1\vartheta\Sigma_{h,i}}
\end{aligned}
$$

Therefore, the fixed point solution is only well-posed if for any value of κ_1 the parameters satisfy

$$
\left(\kappa_1\Phi_{h,i} - \kappa_1\vartheta\Sigma_{h,i}\bar{D}_i - 1 + \vartheta\Sigma_{h,i}\bar{Z}_{3,i}\right)^2 - 4\kappa_1\vartheta\Sigma_{h,i}\left[\bar{D}_i\left(1 - \vartheta\Sigma_{h,i}\bar{Z}_{3,i}\right) + \bar{Z}_{3,i}\Phi_{h,i}\right] \geq 0
$$

for $i = 1, \ldots, H$.

Conditions for Assumption 2

In order to solve for the condition asset pc_t, the conditional Laplace transform must exist. It exists if

$$\vartheta e_i' \Sigma_h' \left[\Sigma_{gh}' \left(Z_\upsilon + (1-\eta) Z_c + \kappa_1 D_g \right) - \lambda_h + \kappa_1 D_h \right] \quad < \quad 1 \qquad i = 1, \ldots, H$$

This is a joint restriction across all parameters of the model.

Proof of Proposition 2

First, derive the limiting property for $\bar{pc} \to -\infty$: $\lim_{\bar{pc} \to -\infty} \kappa_1 = 0$ and $\lim_{\bar{pc} \to -\infty} \kappa_0 = 0$. In this case, both D_0 and D_h are finite due to $\vartheta e_i' \Sigma_h' Z_3 < 1$ in Assumption 1. Therefore, $\tilde{pc}$ is finite, so $\lim_{\bar{pc} \to -\infty} (\bar{pc} - \tilde{pc}) \to -\infty$.

Next, derive the limiting property for $\bar{pc} \to \infty$: $\lim_{\bar{pc} \to \infty} \kappa_1 = 1$ and $\lim_{\bar{pc} \to \infty} \kappa_0 = 0$. This implies D_g is finite as long as the eigenvalue of $(\Phi_g - \vartheta \eta \lambda_g)$ for consumption is smaller than 1. D_h is finite due to $\vartheta e_i' \Sigma_h' Z_3 < 1$. And $\lim_{\bar{pc} \to \infty} (1 - \kappa_1) D_0 = \lim_{\bar{pc} \to \infty} \ln(\beta) + \bar{\Lambda} + \lambda_h' \Sigma_h \nu_h + Z_1' \left(\mu_g - \Sigma_{gh} \Sigma_h \nu_h - \eta \vartheta \lambda_0 \right) - \frac{1}{\vartheta} \sum_{i=1}^{H} \nu_{h,i} \ln\left(1 - \mathrm{e}_i' \Sigma_h' \vartheta Z_3\right) + \frac{\vartheta}{2} Z_1' \Sigma_{0,g} \Sigma_{0,g}' Z_1$. The right hand side is finite due to $\vartheta e_i' \Sigma_h' Z_3 < 1$. Therefore, $\lim_{\bar{pc} \to \infty} \kappa_1 = 1$ leads to an infinite D_0. The condition $\lim_{\bar{pc} \to \infty} D_0 \to -\infty$ implies $\lim_{\bar{pc} \to \infty} (\bar{pc} - \tilde{pc}) \to \infty$, which together $\lim_{\bar{pc} \to -\infty} (\bar{pc} - \tilde{pc}) \to -\infty$ guarantees there exists a solution for the fixed point problem.

With $\kappa_1 < 1$, the condition $\lim_{\bar{pc} \to \infty} D_0 \to -\infty$ is equivalent to

$$\beta < \lim_{\bar{pc} \to \infty} \exp\left[- \left(\bar{\Lambda} + \lambda_h' \Sigma_h \nu_h + Z_1' \left(\mu_g - \Sigma_{gh} \Sigma_h \nu_h - \eta \vartheta \lambda_0 \right) - \frac{1}{\vartheta} \sum_{i=1}^{H} \nu_{h,i} \ln\left(1 - \mathrm{e}_i' \Sigma_h' \vartheta Z_3\right) + \frac{\vartheta}{2} Z_1' \Sigma_{0,g} \Sigma_{0,g}' Z_1 \right) \right].$$

Therefore, the boundary condition is

$$\bar{\bar{\beta}} = \exp\left[- \left(\bar{\Lambda} + \lambda_h' \Sigma_h \nu_h + Z_1^{\infty\prime} \left(\mu_g - \Sigma_{gh} \Sigma_h \nu_h - \eta \vartheta \lambda_0 \right) - \frac{1}{\vartheta} \sum_{i=1}^{H} \nu_{h,i} \ln\left(1 - \mathrm{e}_i' \Sigma_h' \vartheta Z_3^\infty\right) + \frac{\vartheta}{2} Z_1^{\infty\prime} \Sigma_{0,g} \Sigma_{0,g}' Z_1^\infty \right) \right],$$

where

$$Z_1^\infty = Z_\upsilon + (1-\eta) Z_c + D_g^\infty$$

$$D_g^{\infty\prime} = \left(Z_\upsilon + (1-\eta) Z_c \right)' \left(\Phi_g - \vartheta \eta \lambda_g \right) \left(I - \left(\Phi_g - \vartheta \eta \lambda_g \right) \right)^{-1},$$

$$Z_3^\infty = \bar{Z}_3^\infty + D_h^\infty$$

$$\bar{Z}_3^\infty = \Sigma_{gh}' Z_1^\infty - \lambda_h$$

$$D_{h,i}^\infty = -\frac{1}{2} \left(\frac{\Phi_{h,i} - 1}{\vartheta \Sigma_{h,i}} - \bar{D}_i^\infty + \bar{Z}_{3,i}^\infty \right) - \sqrt{\frac{1}{4} \left(\frac{\Phi_{h,i} - 1}{\vartheta \Sigma_{h,i}} - \bar{D}_i^\infty + \bar{Z}_{3,i}^\infty \right)^2 - \frac{1}{\vartheta \Sigma_{h,i}} \left[\bar{D}_i^\infty \left(1 - \vartheta \Sigma_{h,i} \bar{Z}_{3,i}^\infty \right) + \bar{Z}_{3,i}^\infty \Phi_{h,i} \right]}$$

$$\bar{D}^{\infty} = \Phi_h' \lambda_h + \left(\Phi_{gh} - \Sigma_{gh}\Phi_h - \eta\vartheta\lambda_{gh}\right)' Z_1^{\infty} + \frac{\vartheta}{2}\left(\iota_H \otimes Z_1^{\infty}\right)' \tilde{\Sigma}_g \tilde{\Sigma}_g' \left(I_H \otimes Z_1^{\infty}\right)$$

Appendix F MCMC and particle filters

Appendix F.1 MCMC

Our MCMC algorithm is the particle Gibbs (PG) sampler. It iterates between two broad steps: (i) drawing the latent state variables $(g_{1:T}, h_{0:T})$ conditional on the model's parameters; and (ii) drawing the model's parameters $\theta^{\mathbb{P}}$ given the latent state variables. We make heavy use of the fact that the model is a conditionally linear Gaussian state space model.

Appendix F.1.1 Conditionally linear, Gaussian state space form

Conditional on $h_{0:T}$, the model is a linear, Gaussian state space model. We write the model using the state space form of Durbin and Koopman(2012) given by

$$Y_t = Z g_t + d + \eta_t^* \qquad \eta_t^* \sim \mathrm{N}(0, H), \tag{F.11}$$

$$g_{t+1} = T g_t + c_t + R\varepsilon_{t+1}^* \qquad \varepsilon_{t+1}^* \sim \mathrm{N}(0, Q_t), \tag{F.12}$$

where $Y_t = (\Delta c_t \ \pi_t)'$. The models in this paper can placed in this state space form as

$$Z = \begin{pmatrix} Z_c \\ Z_\pi \end{pmatrix} \qquad T = \Phi_g \qquad d = 0_{2\times 1} \qquad H = 0_{2\times 2}$$

$$c_t = \mu_g + \Phi_{gh} h_t + \Sigma_{gh}\varepsilon_{h,t+1} \qquad Q_t = \Sigma_{g,t}\Sigma_{g,t}'$$

For some models, there are free, estimable parameters in the matrices $(\mu_g, \Phi_{gh}, \Sigma_{gh})$. We can place these in the state vector. This allows any free parameters in $(\mu_g, \Phi_{gh}, \Sigma_{gh})$ to be drawn jointly with the state variables $g_{1:T}$. It also allows us to marginalize over them when drawing other parameters, see Creal and Wu(2015b) for discussion.

Appendix F.1.2 Drawing the state variables

We draw $(g_{1:T}, h_{0:T})$ from their full conditional distribution in two steps.

$$
\begin{aligned}
g_{1:T} &\sim p\left(g_{1:T}|Y_{1:T}, h_{0:T}, \theta^{\mathbb{P}}\right) \\
h_{0:T} &\sim p\left(h_{1:T}|Y_{1:T}, g_{1:T}, \theta^{\mathbb{P}}\right)
\end{aligned}
$$

We draw $g_{1:T}$ conditional on $h_{0:T}$ from the conditionally linear, Gaussian state space model (F.11) and (F.12) using a forward filtering backward sampling algorithm or simulation smoother; see, e.g. Durbin and Koopman(2002). Conditional on the draw for $g_{1:T}$, we draw $h_{0:T}$ using a particle Gibbs sampler.

There are two PG samplers developed in the literature. The original PG sampler of Andrieu, Doucet, and Holenstein(2010) with the backward-sampling pass developed by Whiteley(2010), see Creal and Tsay(2015). And, the PG sampler with ancestor sampling (PGAS) of Lindsten, Jordan, and Schön(2014). The former algorithm is simple to implement for Model #1. We describe its implementation here.

Let J be the total number of particles. In our work, we select $J = 100$. The PG sampler starts with a set of existing particles $h_{0:T}^{(1)}$ that were drawn from the previous iteration.

For $t = 1, \ldots, T$, run:

- For $j = 2, \ldots, J$, draw from a proposal: $(h_t, h_{t-1})^{(j)} \sim q\left(h_t, h_{t-1}|g_{t-1:t}, \theta^{\mathbb{P}}\right)$.
- For $j = 1, \ldots, J$, calculate the importance weight:

$$
w_t^{(j)} \quad \propto \quad \frac{p\left(g_t|g_{t-1}, h_t^{(j)}, h_{t-1}^{(j)}, \theta^{\mathbb{P}}\right) p\left(h_t^{(j)}|h_{t-1}^{(j)}, \theta^{\mathbb{P}}\right)}{q\left(h_t^{(j)}, h_{t-1}^{(j)}|g_{t-1:t}, \theta^{\mathbb{P}}\right)}
$$

- For $j = 1, \ldots, J$, normalize the weights: $\hat{w}_t^{(j)} = \frac{w_t^{(j)}}{\sum_{j=1}^{J} w_t^{(j)}}$.
- Conditionally resample the particles $\left\{h_t^{(j)}\right\}_{j=1}^{J}$ with probabilities $\left\{\hat{w}_t^{(j)}\right\}_{j=1}^{J}$. In this step, the first particle $h_t^{(1)}$ always gets resampled and may be randomly duplicated.

Implementation of the PG sampler is different than a standard particle filter due to the "conditional" resampling algorithm used in the last step. We use the conditional multinomial resampling algorithm from Andrieu, Doucet, and Holenstein(2010).

In the original PG sampler, the particles $\left\{h_t^{(j)}\right\}_{j=1}^{J}$ are stored for $t = 1, \ldots, T$ and a single trajectory is sampled using the probabilities from the last iteration $\left\{\hat{w}_T^{(j)}\right\}_{j=1}^{J}$. An important improvement upon the original PG sampler was introduced by Whiteley(2010), who suggested drawing the path of the state variables from the discrete particle approximation using the backwards sampling algorithm of Godsill, Doucet, and

West(2004). On the forwards pass, we store the normalized weights and particles $\left\{\hat{w}_t^{(m)}, h_{i,t}^{(m)}\right\}_{m=1}^{M}$ for $t = 1, \ldots, T$. We draw a path of the state variables $(h_1^*, \ldots, h_T^*)$ from this discrete distribution.

At $t = T$, draw a particle $h_T^* = h_T^{(j)}$ with probability $\hat{w}_T^{(j)}$.

For $t = T-1, \ldots, 0$, run:

- For $j = 1, \ldots, J$, calculate the backwards weights: $w_{t|T}^{(j)} \propto \hat{w}_t^{(j)} p\left(h_{t+1}^* | h_t^{(j)}, \theta\right)$
- For $j = 1, \ldots, J$, normalize the weights: $\hat{w}_{t|T}^{(j)} = \frac{w_{t|T}^{(j)}}{\sum_{j=1}^{J} w_{t|T}^{(j)}}$.
- Draw a particle $h_t^* = h_t^{(j)}$ with probability $\hat{w}_{t|T}^{(j)}$.

The draw $h_{0:T} = (h_0^*, \ldots, h_T^*)$ is a draw from the full-conditional distribution. In practice, when the dimension H of h_t is high, the number of particles J required for satisfactory performance can be quite large. In this case, we can separate each element of the state vector $h_{i,t}$ for $i = 1, \ldots, H$ and draw them one at a time.

Appendix F.1.3 Drawing the parameters

We block the parameters into groups that are highly correlated. These groups can be separated into parameters governing the dynamics of g_t and the parameters that enter the dynamics of volatility h_t.

1. **Drawing parameters in** $\mu_g, \Phi_{gh}, \Sigma_{gh}$: We place these parameters in the state vector and draw them jointly with the Gaussian state variables.

2. **Drawing parameters in** $\Phi_g, \Sigma_{0,g}$: We use the independence Metropolis Hastings algorithm. Conditional on the volatility state variables $h_{0:T}$, the model is a linear, Gaussian state space model (F.11) and (F.12). We maximize the likelihood using the Kalman filter and calculate the Hessian at the posterior mode. We then draw from a Student's t proposal distribution with mean equal to the posterior mode and covariance matrix equal to the inverse Hessian at the mode; see, e.g. Creal and Wu(2015b) for details.

3. **Drawing parameters of the volatility process** ν_h, Φ_h, Σ_h**:** We use an independence Metropolis-Hastings step. When drawing these parameters, we can marginalize out the Gaussian state variables using the Kalman filter. Conditional on the remaining parameters of the model (which we omit), the target distribution of ν_h, Φ_h, Σ_h can be written as

$$p\left(\nu_h, \Phi_h, \Sigma_h | Y_{1:T}, h_{0:T}\right) \quad \propto \quad p\left(Y_{1:T} | h_{0:T}, \nu_h, \Phi_h, \Sigma_h\right) p\left(h_{0:T} | \nu_h, \Phi_h, \Sigma_h\right) p\left(\nu_h, \Phi_h, \Sigma_h\right)$$

where $p\left(Y_{1:T} | h_{0:T}, \nu_h, \Phi_h, \Sigma_h\right)$ is the likelihood from the Kalman filter, $p\left(h_{0:T} | \nu_h, \Phi_h, \Sigma_h\right)$ is the transition density of the volatility process (A.5). We maximize this target density and calculate the Hessian at the posterior mode. We then draw from a Student's t proposal distribution with mean equal to the posterior mode and covariance matrix equal to the inverse Hessian at the mode.

For Gaussian models, we draw the free parameters in $\Sigma_{0,g}$ instead of ν_h, Φ_h, Σ_h.

Appendix F.2 Particle filter

To estimate the structural parameters (β, γ, ψ) and the preference parameters θ^{λ}, we run cross-sectional regressions on filtered estimates of the factors. In order to calculate the filtered estimates of the state variables, we use a particle filter. The particle filter we implement is the mixture Kalman filter of Chen and Liu(2000). Let $g_{t|t-1}$ denote the conditional mean and $P_{t|t-1}$ the conditional covariance matrix of the one-step ahead predictive distribution $p(g_t|Y_{1:t-1}, h_{0:t-1}; \theta)$ of a conditionally linear, Gaussian state space model. Similarly, let $g_{t|t}$ denote the conditional mean and $P_{t|t}$ the conditional covariance matrix of the filtering distribution $p(g_t|Y_{1:t}, h_{0:t}; \theta)$. Conditional on the volatilities $h_{0:T}$, these quantities can be calculated by the Kalman filter.

Let J denote the number of particles and let $Y_t = (\pi_t, \Delta c_t)$ be $N \times 1$. The particle filter then proceeds as follows:

At $t = 0$, for $i = 1, \ldots, J$, set $w_0^{(i)} = \frac{1}{J}$ and

- Draw $h_0^{(i)} \sim p(h_0; \theta)$ and calculate $\Sigma_{g,0}^{(i)} \Sigma_{g,0}^{(i),\prime}$.
- Set $g_{1|0}^{(i)} = \bar{\mu}_g + \Phi_{gh} \bar{h}_0^{(i)}$, $P_{1|0}^{(i)} = \Sigma_{g,0}^{(i)} \Sigma_{g,0}^{(i),\prime}$,
- Set $\ell_0 = 0$.

For $t = 1, \ldots, T$ do:

STEP 1: For $i = 1, \ldots, J$:

- Draw from the transition density: $h_{t+1}^{(i)} \sim p(h_{t+1}|h_t^{(i)}; \theta)$ given by:

$$
\begin{aligned}
z_{j,t+1}^{(i)} &\sim \text{Poisson}\left(e_j' \Sigma_h^{-1} \Phi_h h_t^{(i)}\right) \qquad j = 1, \ldots, H \\
w_{j,t+1}^{(i)} &\sim \text{Gamma}\left(\nu_{h,j} + z_{j,t+1}^{(i)}, 1\right) \qquad j = 1, \ldots, H \\
h_{t+1}^{(i)} &= \Sigma_h w_{t+1}^{(i)}
\end{aligned}
$$

- Calculate $c_t^{(i)}$ and $Q_t^{(i)}$ using $h_t^{(i)}$.

$$\begin{aligned} c_t^{(i)} &= \Phi_{gh} h_t^{(i)} + \Sigma_{gh} \varepsilon_{h,t+1}^{(i)} \\ Q_t^{(i)} &= \Sigma_{g,t}^{(i)} \Sigma_{g,t}^{(i),\prime} \end{aligned}$$

- Run the Kalman filter:

$$\begin{aligned} v_t^{(i)} &= Y_t - Z g_{t|t-1}^{(i)} - d \\ F_t^{(i)} &= Z P_{t|t-1}^{(i)} Z' + H \\ K_t^{(i)} &= P_{t|t-1}^{(i)} Z' \left(F_t^{(i)}\right)^{-1} \\ g_{t|t}^{(i)} &= g_{t|t-1}^{(i)} + K_t^{(i)} v_t^{(i)} \\ P_{t|t}^{(i)} &= P_{t|t-1}^{(i)} - K_t^{(i)} Z P_{t|t-1}^{(i)} \\ g_{t+1|t}^{(i)} &= T g_{t|t}^{(i)} + c_t^{(i)} \\ P_{t+1|t}^{(i)} &= T P_{t|t}^{(i)} T' + R Q_t^{(i)} R' \end{aligned}$$

- Calculate the weight: $\log\left(w_t^{(i)}\right) = \log\left(\hat{w}_{t-1}^{(i)}\right) - 0.5N\log(2\pi) - 0.5\log|F_t^{(i)}| - \frac{1}{2} v_t^{(i)\prime}\left(F_t^{(i)}\right)^{-1} v_t^{(i)}$.

STEP 2: Calculate an estimate of the log-likelihood: $\ell_t = \ell_{t-1} + \log\left(\sum_{i=1}^J w_t^{(i)}\right)$.

STEP 3: For $i = 1, \ldots, J$, calculate the normalized importance weights: $\hat{w}_t^{(i)} = \frac{w_t^{(i)}}{\sum_{j=1}^J w_t^{(j)}}$.

STEP 4: Calculate the effective sample size $E_t = \frac{1}{\sum_{j=1}^J \left(\hat{w}_t^{(j)}\right)^2}$.

STEP 5: If $E_t < 0.5J$, resample $\left\{g_{t+1|t}^{(i)}, P_{t+1|t}^{(i)}, h_{t+1}^{(i)}\right\}_{i=1}^J$ with probabilities $\hat{w}_t^{(i)}$ and set $\hat{w}_t^{(i)} = \frac{1}{J}$.

STEP 6: Increment time and return to STEP 1.

Within the particle filter, we use the residual resampling algorithm of Liu and Chen(1998). We set $J = 100000$.