Discount rates and employment fluctuations

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September 27, 2017

Preliminary draft.

Abstract

We study the role of fluctuations in discount rates for the joint dynamics of expected returns in the stock market and employment dynamics. We construct a non-parametric bound on the predictability and time-variation in conditional volatility of the firm's profit flow that must be met to rationalize the observed business-cycle fluctuations in hiring. A stochastic discount factor consistent with conditional moments of the risk-free rate and expected returns on risky assets alleviates the need for an excessively volatile model of the expected profit flow.

1 Introduction

Stock market valuation and the dynamics of labor market variables are highly volatile relative to conventional measures of macroeconomic risk and strongly correlated with the business cycle, while the real risk-free rate is smooth. The volatility in the stock market variables in excess of the subsequent movements in dividends is the basis of the Shiller (1981) excess volatility puzzle in the asset pricing literature. On the other hand, Shimer (2005) identified the discrepancy between the volatility in labor market tightness and measures of labor productivity as an excess volatility puzzle in search and matching models of the labor market.

Campbell and Shiller (1988) and the subsequent literature use linear approximations of present value budget constraints to argue that variation in price-dividend ratios unexplained by movements in subsequent dividends have to be attributed to fluctuations of discount rates applied to these dividends. However, given the smoothness of yields on risk-free assets, in particular the real risk-free rate, this fluctuation in discount rates has to come mainly in the form of time-varying compensation for risk, manifested in fluctuations in the conditional volatility of the stochastic discount factor.

The search and matching literature dealt with the Shimer (2005) puzzle by introducing mechanisms that increase the volatility and cyclicality of the profit flow received by the firm from hiring the marginal worker (Hall and Milgrom (2008), Hagedorn and Manovskii (2008), Christiano et al. (2015), and others). However, the preferences used in these papers imply minimal compensations for risk, which are not able to rationalize standard asset pricing facts including large risk premia for risky cash flows and fluctuations in the valuation of the stock market.

In this paper, we provide a quantitative connection between these economic forces through business cycle fluctuations in discount rates that investors and firms use to discount future risky cash flows. In particular, we construct a non-parametric bound on two moments of firm's profit dynamics that emerges from the optimality condition on firm's hiring.

One of the moments is the volatility of the conditional expectation of firm's profits. The standard approach in the search and matching literature dictates to make this volatility high, so that business-cycle fluctuations in expected profits are able to explain changes in incentives of firms to hire workers. The other moment is the average conditional volatility of the profit flow. Business cycle fluctuations in the conditional volatility of innovations to profits that are correlated with innovations to the stochastic discount factor lead to fluctuations in risk premia associated with the firm's profit flow and again generate procyclical incentives to hire workers.

The bound that we construct reveals that a successful model of the labor market dynamics that relies on the optimality condition for hiring must satisfy this condition through a combination of the above two channels. Introducing a properly modeled stochastic discount factor than alleviates the burden imposed by the optimality condition on generating an excessively volatile expected profit flow, and allows to shift part of the weight on the risk compensation channel. This approach has been effectively followed by some of the recent work in the asset pricing literature (Petrosky-Nadeau et al. (2015), Favilukis and Lin (2016), Kuehn et al. (2014) Favilukis et al. (2015), Belo et al. (2014), Donangelo et al. (2016), Kilic and Wachter (2015) and others).

We then direct our attention to modeling these risk compensations using structural restrictions on the form of the stochastic discount factor. We endow a representative investor with recursive preferences introduced by Epstein and Zin (1989, 1991). In order to allow for fluctuations in risk compensation, we allow the risk aversion parameter in the preferences to be time varying. While this adds flexibility to the preference structure, we discipline the time-variation in the risk aversion parameter by making it consistent with the dynamics of the equity premium in the economy. We call the innovations to the risk aversion parameter *discount rate shocks*.

Finally, we use the inferred stochastic discount factor to price cash flows that firms receive. We model firms as operating in a labor market with search and matching frictions. Firms incur cost of post vacancies that are necessary to hire workers. In equilibrium with free entry, the cost of vacancy posting has to be equal to the value of a job to the firm, given by the present discounted value of the surplus the firm earns from hiring a marginal worker. Fluctuations in the value of the job lead to time-variation in incentives of firms to hire workers, and hence to fluctuations in employment. In the environment, we quantify the contribution of discount rate shocks to the fluctuations in the labor market variables.

Our paper is not the first to consider discount rates as a plausible explanation. Perhaps the closest paper to ours is Hall (2017), who highlights the economic forces that discount rate shocks play in the search and matching framework, and provides valuable insights into the interactions between the stochastic discount factor and profit flow. However, his model of preferences implies that most of the time-variation in discount rates is manifested as time-variation in the mean, as opposed to the dispersion, of the stochastic discount factor, and hence implies quantitatively implausible movements in the risk-free rate.

2 Theoretical framework

We consider a discrete-time environment with optimizing investors in financial markets and firms hiring workers in a frictional labor market. In this environment, optimality conditions are described by two types of forward-looking restrictions. In the financial market, these restrictions have the form of Euler equations for the valuation of returns on alternative assets. In the labor market, optimal choice of hiring implies an intertemporal condition that equalizes the cost of hiring the marginal worker with the present discounted value of profits generated by employing this worker. We then introduce a criterion that will be useful for the characterization of profit flow processes consistent with this intertemporal condition.

2.1 Restrictions

Investors have unconstrained access to a vector of n securities with one-period real returns R^i_{t+1}, $i = 1, \ldots, n$. Investor optimization and absence of arbitrage imply that there exists a stochastic discount factor process S_t such that

$$E_t\left[s_{t+1}R^i_{t+1}\right] = 1, \qquad i = 1, \ldots, n. \tag{1}$$

{eq:EsR}

where $s_{t+1} \doteq S_{t+1}/S_t$ is the investors' marginal rate of substitution.

Firms in the production sector hire workers in a Diamond–Mortensen–Pissarides labor market. Firms post vacancies at a per-period cost κ, and the vacancies are filled at an equilibrium rate q_t. Free entry and optimal hiring choice imply that the value to the firm of hiring a marginal worker must be equal to the expected cost of hiring the marginal worker, κ/q_t. This implies the Euler equation

$$\frac{\kappa}{q_t} = E_t\left[s_{t+1}\pi_{t+1}\right] + E_t\left[s_{t+1}\left(1-\delta_{t+1}\right)\frac{\kappa}{q_{t+1}}\right] \tag{2}$$

{eq:hiring_

where π_{t+1} is the firm's profit flow from the marginal worker, and δ_{t+1} the probability of match separation. The equation dictates that observed fluctuations in vacancy filling rates q_t and separation rates δ_{t+1} have to be rationalized through fluctuations in firms' marginal profit flow π_{t+1} and the stochastic discount factor s_{t+1}. In expansions, the vacancy filling rate q_t is low, and hence $E_t\left[s_{t+1}\pi_{t+1}\right]$ needs to be high. Writing the expected profit flow as

$$E_t\left[s_{t+1}\pi_{t+1}\right] = E_t\left[s_{t+1}\right]E_t\left[\pi_{t+1}\right] + Cov_t\left(s_{t+1},\pi_{t+1}\right) \tag{3}$$

{eq:Espi}

uncovers three sources of variation. When agents are risk-neutral, as in much of the search and matching literature, we have $s_{t+1} = \beta$ and observed differences in vacancy filling rates must be rationalized through fluctuations in firm's expected profits $E_t\left[\pi_{t+1}\right]$. In a more general environment, fluctuations in the conditional mean of the stochastic discount factor $E_t\left[s_{t+1}\right]$ and time-variation in the covariance $Cov_t\left(s_{t+1},\pi_{t+1}\right)$ contribute to the variation in expected discounted profit flow as well.

Our goal is to provide a stochastic discount factor that is consistent with evidence on cross-sectional differences and time-variation in expected returns, and use it to infer characteristics of a class of processes π_t that are consistent with optimal hiring choice (2). Denoting

$$g_{t+1} = \frac{\kappa}{q_t} - s_{t+1}\left(1-\delta_{t+1}\right)\frac{\kappa}{q_{t+1}} \tag{4}$$

{eq:g_expre

we have

$$0 = E_t\left[s_{t+1}\pi_{t+1} - g_{t+1}\right]$$

In order to empirically implement the conditional restriction (2), we instrument the equation as in Hansen and Singleton (1982) with a vector of variables z_t^{π}, and get

$$0 = E\left[z_t^{\pi}\left(s_{t+1}\pi_{t+1} - g_{t+1}\right)\right]. \tag{5}$$

{eq:Espi_in

The vector of instruments z_t^{π} contains business cycle variables that serve as predictors of future state of the labor market. Equation (5) states that the Euler equation errors $s_{t+1}\pi_{t+1} - g_{t+1}$ are not systematically related to these predictors.

We proceed in an analogous fashion with the vector of asset pricing restrictions (1), using a vector of instruments z_t^{s}. This implies the set of restrictions[1]

$$0 = E\left[z_t^{s}\left(s_{t+1}R_{t+1}^{i} - 1\right)\right], \qquad i = 1,\ldots n. \tag{6}$$

{eq:EsR_ins

2.2 Objective

There will typically be many processes π_t consistent with the set of restrictions (5), and both theoretical modeling as well as empirical measurement of π_t are challenging. Theory dictates that the Euler equation (2) holds for the profit flow associated with the *marginal* worker but this profit flow is not directly obtainable from data. The theoretical literature then frequently sidesteps this issue by introducing environments where the average and marginal workers are identical.

Rather than explicitly modeling the profit flow, we aim at asking what are the minimal requirements that every profit flow process consistent with (2) must satisfy. Since the literature has traditionally struggled to generate profit processes that are sufficiently volatile to rationalize fluctuations in the vacancy filling rates, we are looking for the lower bound on

[1] We can write this set of restrictions in concise form as $0 = E\left[z_t^{s} \otimes \left(s_{t+1}R_{t+1} - \mathbf{1}\right)\right]$ where $\otimes$ is the Kronecker product and $\mathbf{1}$ is a column vector of ones of length n. For two column vectors $x = \left(x^i\right)_{i=1}^{k}$ and $y = \left(y^j\right)_{j=1}^{n}$, $x \otimes y$ is a column vector of length $k \times n$ with $\left(x \otimes y\right)^{(i-1)n+j} = x^i y^j$.

dispersion of the profit process for (5) to hold.

Since we can write (3) as

$$E_t\left[s_{t+1}\pi_{t+1}\right]=E_t\left[s_{t+1}\right]E_t\left[\pi_{t+1}\right]+\rho_t\left(s_{t+1},\pi_{t+1}\right)\sigma_t\left(s_{t+1}\right)\sigma_t\left(\pi_{t+1}\right)$$

where ρ_t is the conditional correlation and σ_t the conditional standard deviation, the two key moments that can offset fluctuations in q_t are $E_t\left[\pi_{t+1}\right]$ and $\sigma_t\left(\pi_{t+1}\right)$. This motivates the objective

$$L_\alpha=\alpha Var\left[E_t\left[\pi_{t+1}\right]\right]+(1-\alpha)\,E\left[Var_t\left[\pi_{t+1}\right]\right]$$

As a function of α, this objective generates a lower bound on the combinations of variance of the conditional mean and average conditional variance of the profit process such that it is consistent with the restrictions above. When $\alpha=\frac{1}{2}$, then the objective is equal to $\frac{1}{2}Var\left[\pi_{t+1}\right]$. The objective L_α can therefore be thought of as a generalization of the Law of Total Variance.

When $\alpha\searrow 0$, the objective emphasizes the minimization of average conditional variance $E\left[Var_t\left[\pi_{t+1}\right]\right]$. Then, in order to rationalize (2), the process π_t has to become highly predictable and exhibit large fluctuations in the conditional mean $E_t\left[\pi_{t+1}\right]$. This is the only way how to resolve the unemployment volatility puzzle when $s_{t+1}=\beta$.

On the other hand, when $\alpha\nearrow 1$, the objective L_α stresses the minimization of the variation in $E_t\left[\pi_{t+1}\right]$. From equation (3), we infer that the resolution of the unemployment volatility puzzle must arise from sufficiently large business cycle fluctuations in $Cov_t\left(s_{t+1},\pi_{t+1}\right)$. These can emerge from sufficiently large fluctuations in the conditional volatility of the stochastic discount factor but if these are not large enough, conditional volatility of the profit process will have to fluctuate as well.

3 Bounds

For a given model of a stochastic discount factor and data on separation rates and vacancy filling probabilities, the problem is

$$\min_{\pi_{t+1}} L_\alpha \quad \text{subject to} \quad 0=E\left[z_t^\pi\left(s_{t+1}\pi_{t+1}-g_{t+1}\right)\right]. \tag{7}$$ {eq:profit_

When $\alpha=\frac{1}{2}$, the problem reduces to finding the minimum variance profit process, and the solution corresponds to the lowest point on the Hansen and Jagannathan (1991) bound.

Proposition 3.1. *The solution to problem (7) satisfies*

$$\pi_{t+1} = E_t\left[\pi_{t+1}\right] + \frac{1}{2\left(1-\alpha\right)}\left(s_{t+1} - E_t\left[s_{t+1}\right]\right)\left(z_t^{\pi}\right)' \lambda^{\pi} \tag{8}$$ {eq:profit_

$$E_t\left[\pi_{t+1}\right] = \bar{\pi} + \frac{1}{2\alpha}\left(z_t^{\pi} E_t\left[s_{t+1}\right] - E\left[z_t^{\pi} s_{t+1}\right]\right)' \lambda^{\pi} \tag{9}$$ {eq:profit_

where the vector of loadings λ^{π} is given by

$$\lambda^{\pi} = \left(V_{\alpha}\right)^{-1}\left(E\left[z_t^{\pi} g_{t+1}\right] - E\left[z_t^{\pi} s_{t+1}\right]\bar{\pi}\right)$$

with

$$V_{\alpha} = \frac{1}{2\alpha} Var\left[z_t^{\pi} E_t\left[s_{t+1}\right]\right] + \frac{1}{2\left(1-\alpha\right)} E\left[Var_t\left[z_t^{\pi} s_{t+1}\right]\right]$$

$$\bar{\pi} = \frac{\left(E\left[z_t^{\pi} s_{t+1}\right]\right)' V_{\alpha}^{-1} E\left[z_t^{\pi} g_{t+1}\right]}{\left(E\left[z_t^{\pi} s_{t+1}\right]\right)' V_{\alpha}^{-1} E\left[z_t^{\pi} s_{t+1}\right]}. \tag{10}$$ {eq:pibar}

Innovations to the profit process are conditionally linear in innovations to the stochastic discount factor, while deviations of the conditional mean of the profit process from unconditional mean are conditionally linear in the corresponding deviations of the stochastic discount factor. When $\alpha \nearrow 1$, innovations in (8) become large, and the profit process exhibits large conditional variance $Var_t\left[\pi_{t+1}\right]$. On the other hand, $\alpha \searrow 0$ implies large fluctuations in the conditional mean $E_t\left[\pi_{t+1}\right]$ in (9).

The form of the process g_{t+1} in (4) implies that the average level of firm's profit will be proportional to the vacancy cost κ, which can be directly inferred from the solution (10). In the absence of a widely accepted value for this parameter, we normalize the results by the average profit $E\left[\pi_{t+1}\right]$. Including information on the vacancy parameter κ together with average profitability $\bar{\pi}$, instead of optimizing over $\bar{\pi}$ in (10), would further tighten the bound.

3.1 Stochastic discount factor

Given observable data on vacancy filling rates and separation probabilities, we can empirically implement problem (7) if we can observe realizations of the stochastic discount factor. One way is to construct a structural model of the stochastic discount factor that is based on observable quantities. Another way is to find a stochastic discount factor that is consistent with the restrictions (6), without imposing a particular model of preferences.

Given problem (7), it is natural to follow the latter path. Since there are again typically many stochastic discount factors consistent with (6), we choose one with the lowest variance.

This stochastic discount factor is obtained by solving

$$\min_{s_{t+1}} \frac{1}{2} E\left[(s_{t+1} - E[s_{t+1}])^2\right] \quad \text{subject to} \quad 0 = E\left[z_t^s \left(s_{t+1} R_{t+1}^i - 1\right)\right], \qquad i = 1, \ldots n. \quad (11)$$ {eq:SDF_bou

The variance-minimizing stochastic discount factor lies at the lowest point of the Hansen–Jagannathan bound and the solution has a structure analogous to that for the profit process π_{t+1} in Proposition 3.1.[2]

Proposition 3.2. *The minimum-variance stochastic discount factor that solves problem (11) is given by*

$$s_{t+1} = \bar{s} + \left(z_t^s \otimes R_{t+1} - E\left[z_t^s \otimes R_{t+1}\right]\right)' \lambda^s$$

where the vector of loadings λ^s and the mean of the stochastic discount factor are

$$\lambda^s = \left(Var\left[z_t^s \otimes R_{t+1}\right]\right)^{-1} \left(E\left[z_t^s\right] - E\left[z_t^s \otimes R_{t+1}\right] \bar{s}\right)$$
$$\bar{s} = \frac{\left(E\left[z_t^s \otimes R_{t+1}\right]\right)' \left(Var\left[z_t^s \otimes R_{t+1}\right]\right)^{-1} E\left[z_t^s\right]}{\left(E\left[z_t^s \otimes R_{t+1}\right]\right)' \left(Var\left[z_t^s \otimes R_{t+1}\right]\right)^{-1} E\left[z_t^s \otimes R_{t+1}\right]}.$$

The stochastic discount factor is therefore a conditionally linear function of returns, and variation in instruments serves as fluctuations in the its conditional mean and conditional volatility.

3.2 Interpretation

It is useful to highlight the differences of the constructed weighted variance bound on firm's profit relative to the familiar Hansen and Jagannathan (1991) and other similar bounds.

Proposition 3.1 derives the weighted variance bound on firm's profit for a *given* model of the stochastic discount factor. The construction can be implemented for any particular model of the stochastic discount factor, as long as we can provide an observable counterpart to the path s_{t+1}. For instance, the realized path of an Epstein–Zin stochastic discount factor can be constructed using realizations of consumption growth and a proxy for the return on aggregate wealth.

Instead, we choose to construct the stochastic discount factor non-parametrically as the minimum variance stochastic discount factor consistent with asset pricing restrictions (6). This particular stochastic discount factor, which achieves the minimum of the Hansen and Jagannathan (1991) bound, is derived Proposition 3.2. However, there are other stochastic

[2]Problem 11 ignores the usual no-arbitrage restriction that the stochastic discount factor has to remain strictly positive. See Hansen and Jagannathan (1991) and Hansen et al. (1995) for analysis with restrictions that assure positivity of the solution. We abstract from this restriction for reasons of analytical tractability.

discount factors consistent with the same restrictions that can be written as

$$s_{t+1}^{\varepsilon} = s_{t+1} + \varepsilon_{t+1}$$

where ε_{t+1} is an error term unconditionally orthogonal to $z_t^s \otimes R_{t+1}$.

The critical assumption for the constructed variance bound to truly be a lower bound on the variance of firm's profit is that the stochastic discount factor s_{t+1} encompasses all relevant economic risks that affect firm's profits. Otherwise, we could potentially be able to explain fluctuations in firms' hiring rates partly through covariance between ε_{t+1} and the profit process, which could be achieved at a lower overall variance of firm's profits.

This issue is not specific to the minimum variance stochastic discount factor but applies to structural models of stochastic discount factors as well. Put differently, by choosing a particular stochastic discount factor from which we infer the variance bound on profits, we are taking a stand on the relevant risks that can affect firms' hiring. In our case, we are making the assumption that the vector of returns R_{t+1}, instrumented by the vector of instruments z_t^s fully captures these risks, so that perturbing s_{t+1} by including an orthogonal component ε_{t+1} is immaterial for the labor market dynamics.

The second main difference arises from the fact that in order to compute the matrix V_α in Proposition 3.1, we need to compute the conditional mean and variance of the stochastic discount factor. In order to do that, we run a predictive regression on the set of the inferred realizations of the stochastic discount factor on the set of instruments. This corresponds to taking a stand on the relevant information set available to agents in the model, an assumption that we exploit further.

3.3 Implementation

Propositions 3.1 and 3.2 provide observable counterparts of paths of stochastic discount factor and profit processes satisfying the restrictions (5) and (6). These paths are conditionally linear functions of asset returns but the procedure does not recover their conditional distributions that are necessary in order to compute conditional expectations of variables that are of our interest, for example, the implied valuation of a worker to the firm. We need to impose additional structure.

We will therefore assume that the economy is driven by a Markov state described by an n-dimensional vector of variables X_t with law of motion

$$X_{t+1} = \psi_x X_t + \psi_w W_{t+1}$$

where $W_{t+1} \sim N(0, I_n)$ and ψ_w is a full rank matrix. Preempting the results that will emerge from the nonparametric analysis, we specify the model for the stochastic discount factor, profit flow and VAR and the separation rate as follows

$$
\begin{aligned}
s_{t+1} &= b_s^{mean} X_t + \left(b_s^{vol} X_t\right)\left(b_s^w W_{t+1}\right) \\
\pi_{t+1} &= b_\pi^{mean} X_t + \left(b_\pi^{vol} X_t\right)\left(b_\pi^w W_{t+1}\right) \\
\delta_{t+1} &= b_\delta^{mean} X_t + \left(b_\delta^{vol} X_t\right)\left(b_\pi^w W_{t+1}\right)
\end{aligned}
\tag{12}
$$

with appropriate normalizations of the parameter vectors b. With this structure at hand, we can then compute the value of a marginal worker to the firm $J_t = J(X_t) = \kappa / q_t$ by solving for a fixed point in the recursion (2):

$$
J(X_t) = E_t\left[s_{t+1}\pi_{t+1}\right] + E_t\left[s_{t+1}\left(1-\delta_{t+1}\right) J(X_{t+1})\right]. \tag{13}
$$

{eq:JXt}

4 Data

We use financial and macroeconomic data for the period 1952Q2–2015Q1. We download the market return, three-month T-bill rate, and the returns on book-to-market and size-sorted portfolios from Kenneth French's website. We use the price-dividend ratio as constructed by Robert Shiller (downloaded from his website).

Vacancy data have been provided to us by Nicolas Petrosky-Nadeau, who combined several data sources to create a consistent time-series of vacancies for the 1929–2016 period; the details are explained in Petrosky-Nadeau and Zhang (2013). We download other macroeconomic variables from FRED, their names are provided in parenthesis. We follow Shimer (2012) in constructing the job-finding rate and job separation rate using data on civilian employment level (CE16OV), unemployment level (UNEMPLOY) and number of civilians unemployed for less than 5 weeks (UEMPLT5). We use data on unemployment, vacancies and job-finding rates to construct the vacancy-filling rate as $q_t = f_t/\theta_t$, where θ_t is the vacancy-unemployment ratio and f_t is job-finding rate. We note that the job finding and job separation rates correspond to movements between employment and unemployment.

5 Results

We start by extracting the path of realizations of the stochastic discount factor using Proposition 3.2. Given time series data on realized returns R_{t+1} and instruments z_t^s, we can construct sample analogs of the mean vectors and covariance matrices in Proposition 3.2

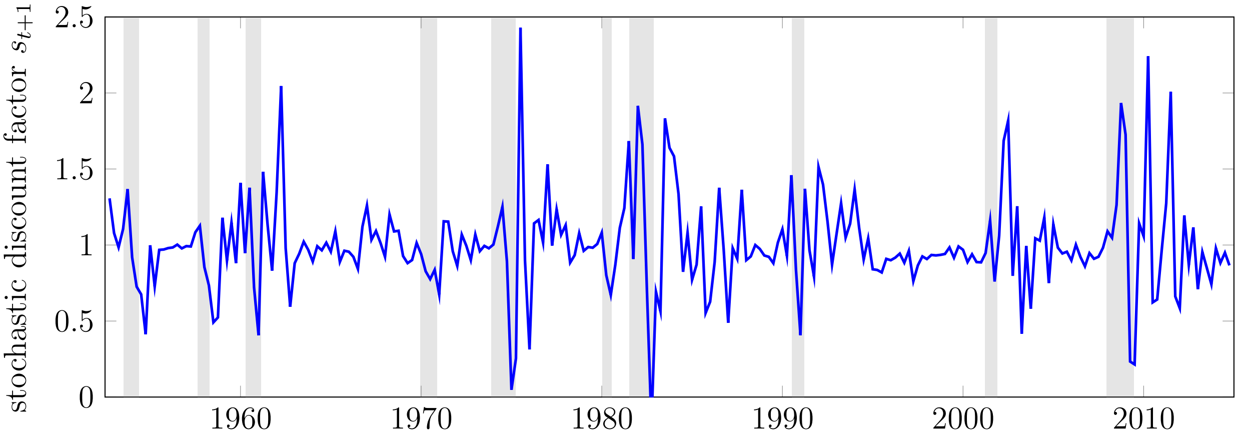

Figure 1: Extracted path of quarterly realizations of the minimum variance stochastic discount factor. NBER recessions shaded. The vector of returns used in the construction includes the S&P500 market return, 3M T-bill rate and returns on book-to-market and size-sorted portfolios from Kenneth French's website. The vector of instruments z_t^s includes market tightness, vacancy filling rate, price-dividend ratio, output growth and the risk-free rate.

and construct the path for s_{t+1}. This path is depicted in Figure 1.

The path of the stochastic discount factor exhibits substantial countercyclical fluctuations in conditional variance, with negligible movements in conditional mean. The stochastic process for the stochastic discount factor needs to jointly rationalize countercyclical risk premia and the relatively stable real risk-free rate. The countercyclical conditional variance is consistent with structural models of the stochastic discount factor that involve stochastic volatility in the consumption process at the business cycle frequency, or models with a countercyclical price of risk, and additional restrictions are needed at this point to distinguish between these interpretations.

5.1 Weighted variance bound

Instead, we use this constructed path of the stochastic discount factor as an input into the construction of the variance bound for the firm's profit process. We can now implement the problem (7) using observable data for vacancy filling rates, separation rates and the path of the stochastic discount factor. For alternative values of $\alpha \in (0,1)$, we obtain a locus of points in the space of the average conditional variance $E\left[Var_t\left[\pi_{t+1}\right]\right]$ and variance of the conditional expectation $Var\left[E_t\left[\pi_{t+1}\right]\right]$ that form a bound above which all profit processes satisfying restrictions (5) must lie.

Figure 2 depicts these bounds for several alternative stochastic discount factors. In the

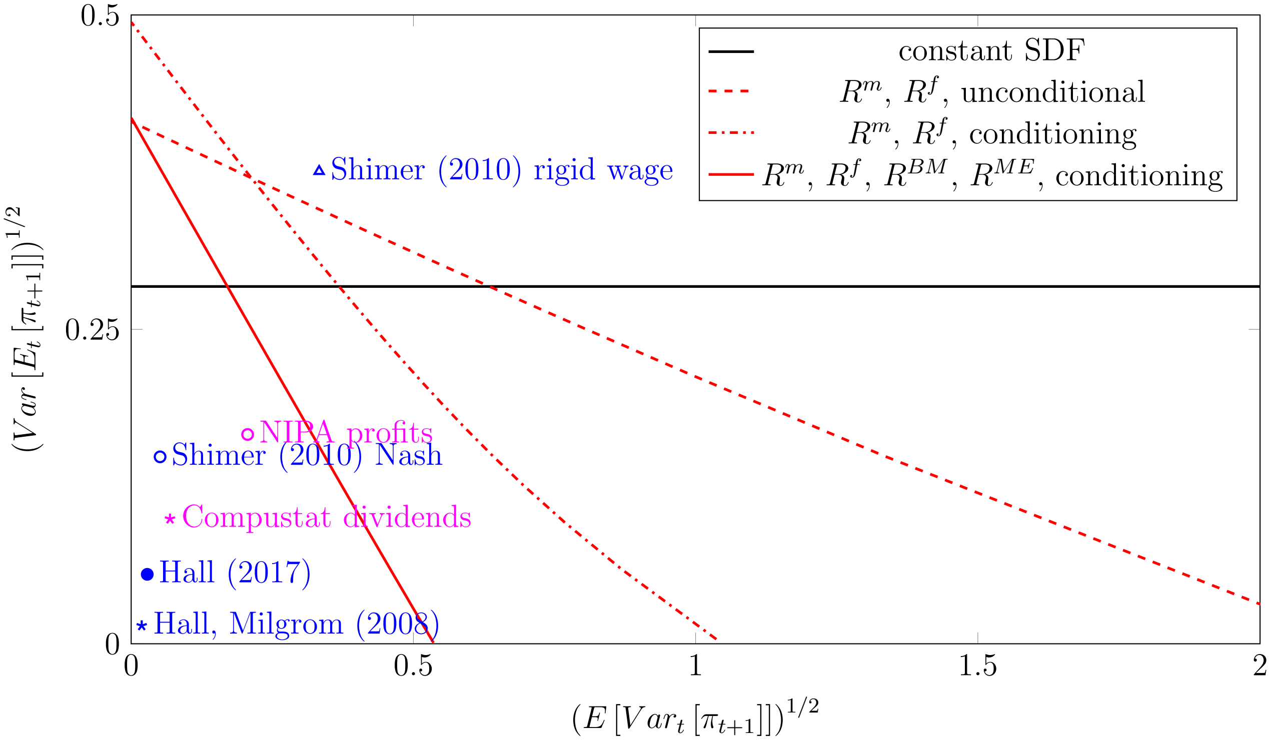

Figure 2: Normalized weighted variance bound for the profit process. The horizontal axis represents the square root of the average conditional variance of the profit process, while the vertical axis the square root of the variance of its conditional expectation. Both quantities are normalized by the average profit $E\,[\pi_{t+1}]$. Alternative lines represent bounds constructed for different extracted paths of stochastic discount factors. Blue dots are model-implied moments calculated from models used in the labor literature. The magenta dots represent empirical proxies for the moments.

absence of an agreed upon value of the vacancy posting cost κ, we impose a normalization and plot the average conditional variance $E\,[Var_t\,[\pi_{t+1}]]$ and variance of the conditional expectation $Var\,[E_t\,[\pi_{t+1}]]$ normalized by the mean profit $E\,[\pi_{t+1}]$.

The black solid line corresponds to a stochastic discount factor of a risk-neutral agent, $s_{t+1} = \beta$. This stochastic discount factor is clearly inconsistent with the asset pricing restrictions (6) but much of the search and matching literature makes this assumption. In this case

$$E_t\,[s_{t+1}\pi_{t+1}] = \beta E_t\,[\pi_{t+1}]$$

and the covariance term in (3) does not contribute to fluctuations in expected discounted profit. Consequently, the conditional variance of the profit $Var_t\,[\pi_{t+1}]$ is irrelevant for the hiring equation (2), and the bound represented by the black line must be flat.

The dashed, dash-dotted and solid red lines represent the profit variance bounds constructed using alternative stochastic discount factors extracted by solving problem (11) for different sets of returns that must be priced and different assumptions on conditioning. The bounds our downward sloping, implying that the bound can be satisfied using a profit process with high conditional volatility, or a high volatility of the conditional mean (i.e., a process that is highly predictable), or a combination of both, as we move along the bound.

The bounds are showing that as we include additional returns that the stochastic discount factor must price, and include relevant conditioning representing fluctuations in conditional expected returns over the business cycle, the stochastic discount factor captures more of the relevant macroeconomic risks, and the bound can be satisfied with a lower conditional volatility of the profit flow $E\left[Var_t\left[\pi_{t+1}\right]\right]$. This is representing by the bounds becoming steeper in Figure 2.

Interestingly, the intersects of the bounds with the vertical axis also lie higher when the stochastic discount factor prices relevant asset returns. The reason has to do with the procyclicality of the real interest rate. Consider the case when we want to satisfy the bound with a profit flow that is perfectly predictable and hence $Var_t\left[\pi_{t+1}\right] = 0$. Such a profit low will lie on the vertical axis in Figure 2. In that case

$$E_t\left[s_{t+1}\pi_{t+1}\right] = E_t\left[s_{t+1}\right]E_t\left[\pi_{t+1}\right].$$

When $E_t\left[s_{t+1}\right] = \beta$, as in the case of the black line, the conditional expectation of the profit flow $E_t\left[\pi_{t+1}\right]$ must be sufficiently procyclical to rationalize the evolution of hiring over the business cycle. At the same time, when the stochastic discount is supposed to correctly price the risk-free rate, which is mildly procyclical, $E_t\left[s_{t+1}\right]$ must be mildly countercyclical. This implies that $E_t\left[\pi_{t+1}\right]$ must be even more procyclical than in the case when $E_t\left[s_{t+1}\right] = \beta$, and the bound tightens.

Figure 3 displays two alternative extracted paths of firms' profit flow that satisfy the weighted variance bound. The underlying stochastic discount factor correctly prices the risk-free rate, the market return and the portfolios sorted on size and on book-to-market, including the conditional moments captured by the vector in included instruments. In Figure 2, this corresponds to the red solid line.

The top panel in Figure 3 depicts the path for weight $\alpha = 0.001$, which corresponds to the bottom right end of the bound. The path is visibly heteroskedastic, with no clear predictable pattern for its conditional mean. On the other hand, the bottom panel shows the path for $\alpha = 0.999$, which corresponds to the top left end of the bound from Figure 2. The trajectory now exhibits low conditional volatility and a high predictability over the business cycle.

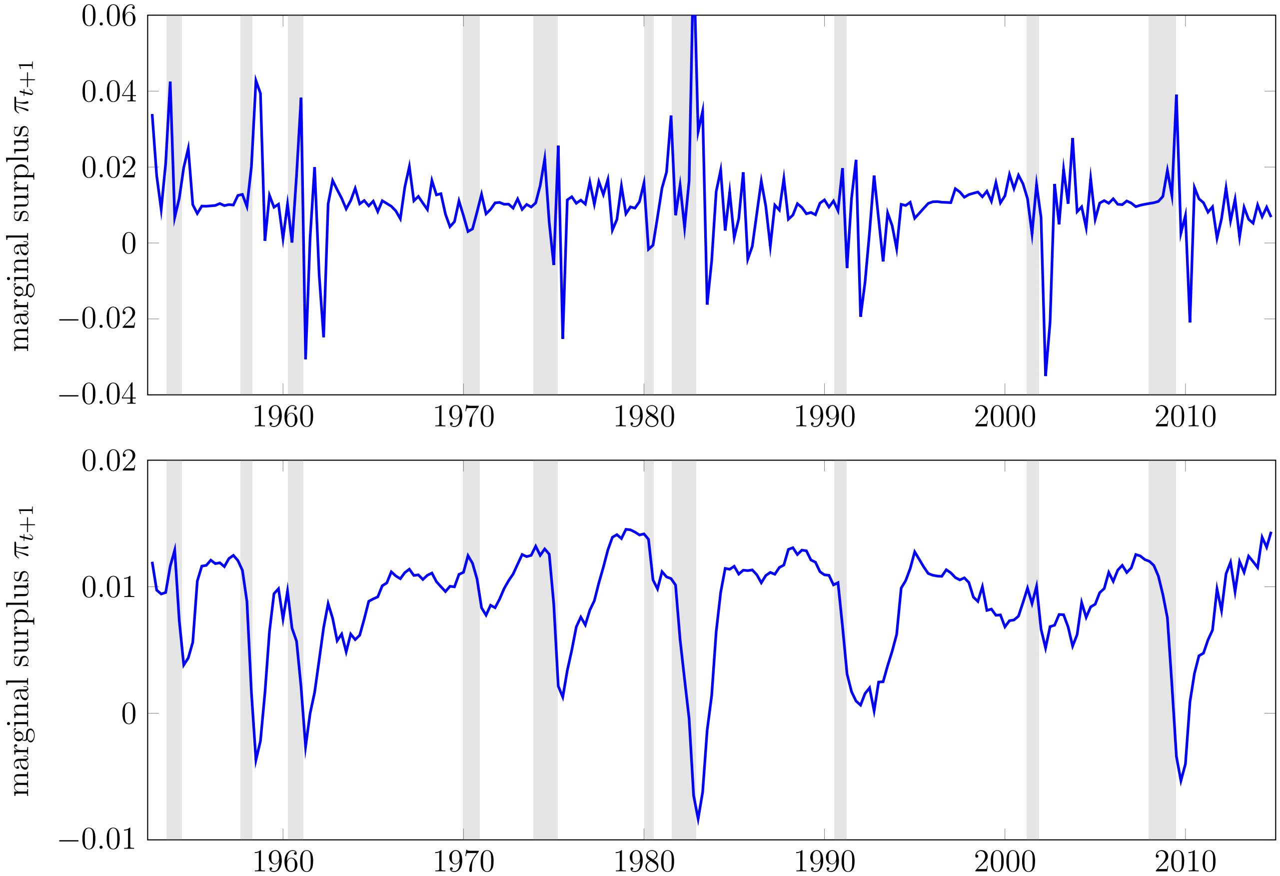

Figure 3: Alternative profit flow paths satisfying the weighted variance bounds. *Top panel*: $\alpha = 0.001$: high weight on minimizing variance of conditional expectation. *Bottom panel*: $\alpha = 0.999$, high weight on minimizing conditional variance. NBER recessions shaded.

To recapitulate, each of these two trajectories for firms' profit flow, together with the model of the stochastic discount factor, satisfy the satisfy the instrumented Euler equation for hiring (5), as well as the set of asset pricing restrictions (6). The bound thus provides guidance for alternative ways in which modeled or measured cash flow processes can be consistent with the dynamics of hiring in the labor market.

5.2 Empirical and model implied measures

We will now study alternative empirical proxies for the profit flow as well as theoretical processes for this profit flow implied by alternative models used in the labor market literature, and confront them with the bounds we inferred in the previous section. These results are depicted in Figure 2.

First, we generate time-series data from several leading models in the labor literature. We start with the model in Shimer (2010), Section 3.3, which extends the standard search model by including capital and considering shocks to productivity as well as to the separation rate. Importantly, the wage is determined by Nash bargaining so this model does not generate enough business cycle movements in employment. In that respect, it is similar to Shimer (2005). We further consider the model in Section 4.3 in Shimer (2010) which replaces the Nash bargaining mechanism with a backward-looking wage setting. The actual wage is a weighted average of last period's wage, and the current target wage, where the target wage is the one which solves the Nash bargaining problem. This is an ad-hoc wage setting rule but the goal is to study whether this helps to amplify employment fluctuations.

Two other models we examine are Hall and Milgrom (2008) and Hall (2017) which both aim at generating large fluctuations in employment. Hall and Milgrom (2008) introduce wage rigidity into an otherwise standard search model, and Hall (2017) adds movements in the stochastic discount factor on top of the wage rigidity.

For each of these models, we generate the time-series of profit flow π_t and all state variables. We then construct $E[Var_t[\pi_{t+1}]]$ and $Var[E_t[\pi_{t+1}]]$ by forecasting π_{t+1} using the time-t state variables implied by the model dynamics. We normalize both objects by the mean value of the profit flow, $E[\pi_{t+1}]$.

Figure 2 shows the results, each blue dot representing one these models. Three out of four analyzed models lie below all displayed bounds. First, these three models lie below the black line representing the constant SDF case. Indeed, Shimer (2010) uses a stochastic discount factor that is essentially constant and shows that the model with Nash bargaining does not generate enough employment fluctuations. Figure 2 not only confirms this finding but also shows that if we kept fixed the stochastic process for profit flow obtained from the Shimer (2010) model with Nash bargaining, and coupled it with alternative stochastic discount factors that we constructed previously, the model would still not be able to rationalize the Euler equation for hiring (2), and therefore would continue to understate the magnitude of employment fluctuations.

The same conclusion holds for the Hall and Milgrom (2008) model. On the other hand, the version of Shimer (2010) with backward-looking wage setting rules lies above the constructed bounds, meaning that the modeled profit flow process combines enough volatility through $E\left[Var_t\left[\pi_{t+1}\right]\right]$ and $Var\left[E_t\left[\pi_{t+1}\right]\right]$ to satisfy the hiring Euler equation, and therefore has the potential to match the magnitude of employment fluctuations observed in the data.

Interestingly, the model from Hall (2017) lies below all displayed bounds, which leads us to conclude that the model enough movements in the profit flow to satisfy (2). This is not the conclusion that the author reaches in that paper, though. The reason for this discrepancy is that we require the stochastic discount factor to be consistent with the cyclical properties of the risk-free rate. Hall (2017) does not impose such restriction and the fluctuations in $E_t\left[s_{t+1}\pi_{t+1}\right]$ are rationalized by highly procyclical fluctuations in $E_t\left[s_{t+1}\right]$ that generate a counterfactually large countercyclical volatility in the real risk-free rate. We will return to this issue in Section 5.4 where we provide an alternative interpretation of the profit flow process π_{t+1} that potentially reconciles these differences.

Next we construct a pair of empirical proxies for the profit flow. First, we construct real profits per worker using after-tax corporate profits from NIPA (variable CP in FRED), divided by the GDP deflator (GDPDEF) and employment (CE16OV). We detrend the time series by real potential GDP per capita (GDPPPOT/CNP16OV).

We also measure real per capita profits in Compustat. We use operating income before depreciation (OIBDPQ), deflated by the GDP deflator and divided by employment as measured in Compustat. Employment in Compustat is available only at annual frequency, and we linearly interpolate the data to obtain a quarterly data series. We again detrend by dividing the time series by real potential GDP per capita.

For both proxies of the profits, we compute $E[Var_t[\pi_{t+1}]]$ and $Var[E_t[\pi_{t+1}]]$ using a projection on the vector of instruments used in equations (5) and (6). We again normalize both objects by the mean value of the profit flow, $E[\pi_{t+1}]$. Magenta dots in Figure 2 represent these data.

It is important to stress the discrepancy between these measures of profit per worker and the theoretical object we are interested in. The theoretical model predicts that π_{t+1} corresponds to the profit flow from the marginal worker. Constructed empirical proxies, however, represents average profits per worker. Hence a failure of an empirical proxy to meet the bound can potentially be attributable to the discrepancy between average and marginal profits from a worker. In absence of empirical measures of marginal profits, one potential avenue is to use the distance between the data points and the bound as a constraint on the wedge implied by a theoretical model that distinguishes between average and marginal worker profit.

5.3 Predicted unemployment path

We further examine whether fluctuations in the value of the job can explain fluctuations in the unemployment rate. To do so, we construct a time path for the value of a worker $J\left(X_t\right)$ and feed the path into the model in order to generate a predicted time path for the unemployment rate.

For this exercise, we need to choose the functional form for the matching function. We assume a constant returns to scale matching function $q\left(\theta\right) = B\theta^{-\eta}$. We choose the elasticity

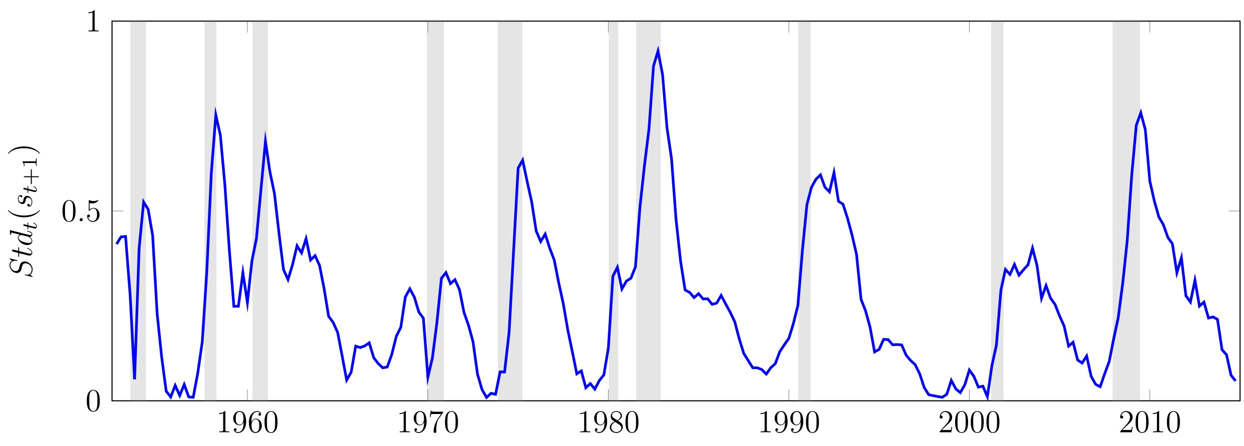

Figure 4: SDF volatility NBER recessions shaded.

$\eta = 0.72$ following Shimer (2005), and normalize B to match the mean of the unemployment rate in the simulated path with the mean in the data.

We now proceed as follows. First, we estimate the model (12) using the observable time series for s_{t+1}, π_{t+1} and δ_{t+1}. Figure 4 plots the conditional volatility of the stochastic discount factor $\left(b_s^{vol} X_t\right) |b_s^w|$ and confirms the substantial heteroskedasticity implied by the need to rationalize the time-variation in risk premia. Next, we solve equation (13) for $J\left(X_t\right)$. Figure 5 plots the time series for $J\left(X_t\right)$ constructed using estimated models for π_{t+1} from (12), estimated using the two extracted time series for firms' profit flow that satisfy the weighted variance bound for $\alpha = 0.001$ and $\alpha = 0.999$, respectively. In both cases, these values fall dramatically during recessions, although for different reasons. In the top panel, $J\left(X_t\right)$ falls because the conditional covariance between s_{t+1} and π_{t+1} becomes more negative. In the bottom panel, $J\left(X_t\right)$ falls because the conditional mean $E_t\left[\pi_{t+1}\right]$ falls substantially.

Finally, we use the first-order condition (2) for $J_t = \kappa/q_t$ together with the knowledge of the state variable u_t to compute v_t, by solving

$$\kappa \frac{v_t}{1-u_t} = B\left(\frac{v_t}{u_t}\right)^{-\eta} J_t.$$

We determine next period's unemployment rate using the law of motion for u_{t+1},

$$u_{t+1} = \left(1 - f_t\right) u_t + \delta_t e_t,$$

where $f_t = B\left(v_t/u_t\right)^{\eta}$. In this experiment, we take J_t and δ_t from the data, and generate f_t, q_t, v_t endogenously within the model.

Figure 6 shows the calculated paths for the unemployment rate constructed using the two

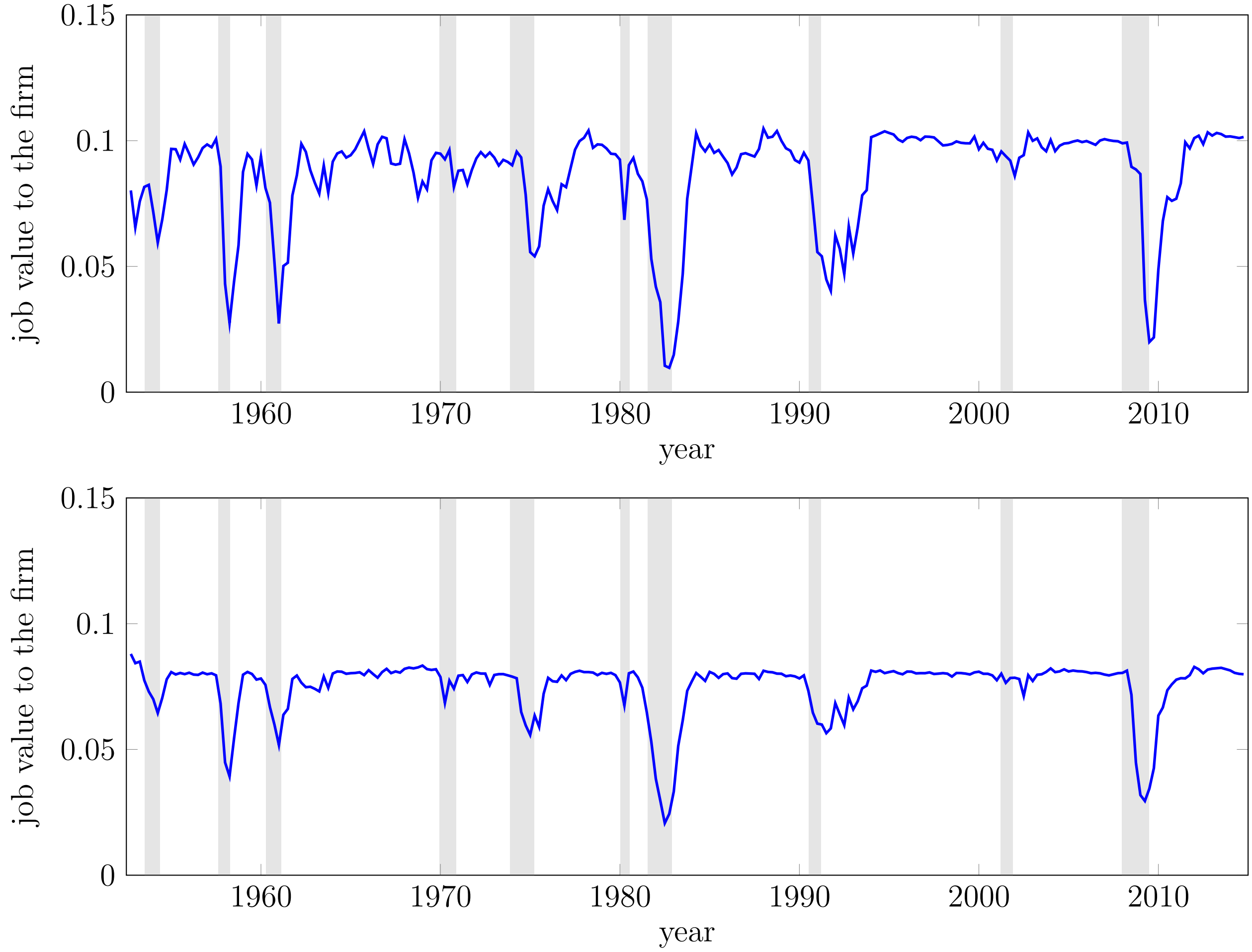

Figure 5: Value of the job using surplus from bounding exercise, Top: $\alpha = 0.001$: high weight on minimizing variance of conditional expectation. Bottom: $\alpha = 0.999$: high weight on minimizing the expected conditional variance.

alternative profit flow processes, both lying on the weighted variance bound. The top panel shows the result for the profit flow process for $\alpha = 0.001$, which generates the time-variation in incentives to hire through time-varying covariance with the stochastic discount factor. On the other hand, the profit flow process in the bottom panel, with $\alpha = 0.999$, generates fluctuations in the hiring rate through fluctuations in the expected profit flow.

Both these choices are highly successful in delivering a path for the unemployment rate that lines up well with data. We stress that this is not a mechanical consequence of the profit processes satisfying the weighted variance bound. While the profit process and the stochastic discount factor consistent with the bound must satisfy the instrumented Euler equation for hiring in (5) and for asset returns in (6), it is not guaranteed that these restrictions generate the right persistence in the time-variation of the first and second moment of the profit flow

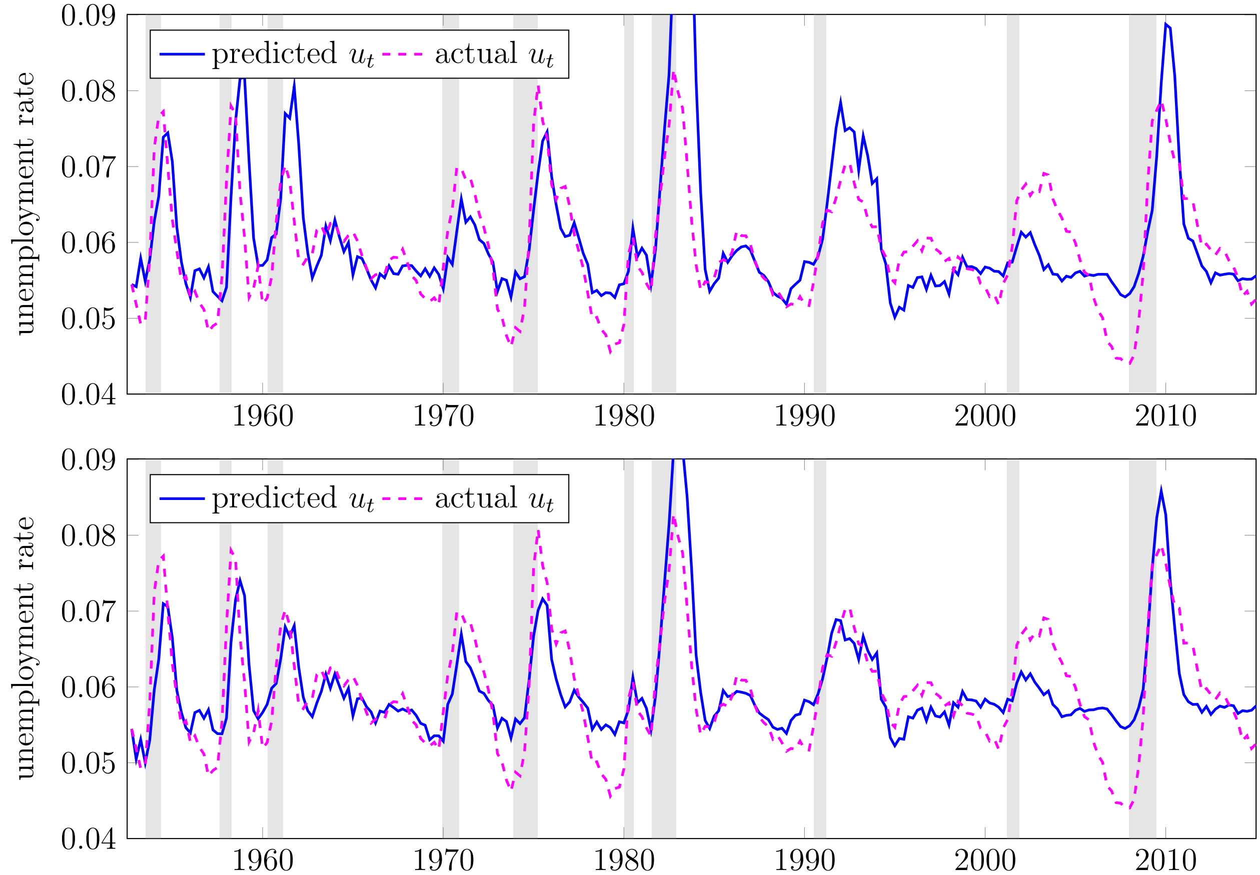

Figure 6: Predicted unemployment rate constructed using J_t generated using two alternative profit flow processes. *Top panel*: $\alpha = 0.001$: high weight on minimizing variance of conditional expectation. *Bottom panel*: $\alpha = 0.999$: high weight on minimizing the expected conditional variance.

and the stochastic discount factor that would generate the large and persistent increases in the unemployment rate observed in the data. The constructed implied dynamics of the unemployment rate thus constitute an informal test that the bound indeed is empirically relevant.

5.4 A broader interpretation of profit flow

Our focus lies in the study of the discounted profit flow $E_t[s_{t+1}\pi_{t+1}]$. So far, we have interpreted π_{t+1} as the net cash flow the firm earns from hiring the marginal worker, discounted by the stochastic discount factor inferred from financial market data, and assumed that firms have the same unconstrained access to financing as investors in financial market. However, if firms face financial constraints, then the corresponding discounted profit flow is given by

$$E_t\left[s_{t+1}\lambda_{t+1}\pi_{t+1}\right] \doteq E_t\left[s_{t+1}\widetilde{\pi}_{t+1}\right]$$

where $\widetilde{\pi}_{t+1}$ is the profit flow adjusted by the shadow prices of the borrowing constraints in individual future states relative to today, λ_{t+1}. These shadow prices emerge, for instance, in situations when firms need to borrow to finance the cost of hiring or to pre-pay wages. In such situations, an empirical proxy for profits constructed using NIPA or Compustat data would provide a biased measure of $\widetilde{\pi}_{t+1}$, and λ_{t+1} could explain, at least to some extent, the distance between the magenta points in Figure 2 and the weighted variance bounds. To answer this question, one needs to construct an empirical or theoretical measure of the fluctuations in these shadow prices λ_{t+1}.

This interpretation also provides a possible reconciliation between our bound and the results in the Hall (2017) model. If we write the discounted profit flow as $E_t\left[\widetilde{s}_{t+1}\pi_{t+1}\right]$ where $\widetilde{s}_{t+1} = s_{t+1}\lambda_{t+1}$, then $\widetilde{s}_{t+1}$ would not be the stochastic discount factor pricing assets in unconstrained markets but rather a constraint-adjusted stochastic discount factor that takes into account the role of pricing impact of constrained borrowing. We leave the exploration of this topic for future work.

6 Structural model

In the previous sections, we non-parametrically inferred the dynamics of a stochastic discount factor and a profit process that satisfied a lower bound on their second moments that made them consistent with restrictions on pricing of return and on hiring choices. We now impose more structure on the stochastic discount factor, and use an empirical counterpart of the profit flow series in order to investigate whether these restrictions are still consistent with the labor market dynamics.

The model economy is populated by a representative household and heterogeneous firms operating in frictional labor markets that are subject to financial constraints. Time is discrete, $t = 0, 1, 2, \ldots$, and we assume that aggregate uncertainty is driven by a state vector X_t that follows the vector-autoregression

$$X_{t+1} = \psi_x X_t + \psi_w W_{t+1} \qquad (14) \quad \texttt{\{eq:VAR\}}$$

where ψ_x and ψ_w are conforming matrices and $W_{t+1} \sim N(0, I)$ is an iid multivariate normal vector of innovations.

6.1 Household preferences

The representative household is endowed with Epstein and Zin (1989) preferences with unitary elasticity of substitution:[3]

$$U_t = (1-\beta)\log C_t - \frac{\beta}{\theta_t}\log E_t\left[\exp\left(-\theta_t U_{t+1}\right)\right] \tag{15}$$

{eq:EZ_pref

where β is the time-preference parameter and $\gamma_t \doteq \theta_t + 1$ is the time-varying risk-aversion parameter.[4] The preference structure (15) implies the one-period stochastic discount factor

$$\frac{S_{t+1}}{S_t} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-1}\frac{\exp\left(-\theta_t U_{t+1}\right)}{E_t\left[\exp\left(-\theta_t U_{t+1}\right)\right]}.$$

The time-variation in the risk-aversion parameter scales the volatility of the last term in the stochastic discount factor. This introduces time-varying prices of risk into our model, and hence fluctuations in discounting of risky cash flows. The dynamics of θ_t will be consistent with the dynamics of equilibrium returns, and we will infer it from fluctuations in risk premia.

In order to operationalize this model, we assume that the growth rate in aggregate consumption follows

$$\Delta c_{t+1} \doteq \log C_{t+1} - \log C_t = \iota_c' X_t. \tag{16}$$

{eq:consump

For instance, ι_c can be an indicator vector that selects Δc_t from the VAR. In Section 6.2, we show that under additional assumptions on the distribution of conditional returns, the dynamics of the risk aversion parameter θ_t and the continuation value U_t satisfy

$$\theta_t = \iota_\theta' X_t \tag{17}$$

{eq:theta_X

and

$$U_t = \log C_t + \bar{u}' X_t \tag{18}$$

{eq:EZ_pref

where ι_θ and $\bar{u}$ are vectors determined in equilibrium. The calculations are detailed in Appendix A.

[3] A generalization to non-unitary elasticity of substitution can be obtained by a linear expansion around the unitary case.

[4] The time-variation in the risk-aversion parameter offers alternative interpretations. Campbell and Cochrane (1999) is the seminal contribution in the context of an external habit model. Dew-Becker (2014) embeds this framework into the Epstein–Zin preference structure. See also Greenwald et al. (2014), Kozak (2015) and Kozak and Santosh (2016). Bhandari et al. (2016) interpret this time-variation in the context of robust preferences, see also Hansen and Sargent (2015).

6.2 Asset pricing implications

The representative household invests its wealth into a portfolio of assets indexed by i, with gross returns R^i_{t+1}. Optimality conditions imply that asset returns have to satisfy the Euler equations

$$1 = E_t\left[\frac{S_{t+1}}{S_t}R^i_{t+1}\right].$$

Under the assumption of log-normally distributed returns, we show in the appendix that the expected excess return (risk premium) on asset i can be written as

$$E_t\left[r^i_{t+1}\right] + \frac{1}{2}Var_t\left[r^i_{t+1}\right] - r^f_t = (\theta_t + 1)\,Cov_t\left(r^w_{t+1}, r^i_{t+1}\right) + \theta_t Cov_t\left(\bar{u}'X_{t+1}, r^i_{t+1}\right) \qquad (19)$$ {eq:risk_pr

where $r^i_{t+1} = \log R^i_{t+1}$, r^f_t is the risk-free rate between periods t and $t+1$, and r^w_{t+1} is the return on household's wealth. Further assuming that we can identify the return on wealth with the return on the market portfolio, and the excess returns can be inferred from the VAR,

$$rx^w_{t+1} \doteq r^w_{t+1} - r^f_t = \iota'_{rxw}X_{t+1},$$

we can use the pricing equation (19) for $r^i_{t+1} = r^w_{t+1}$ to infer the path of θ_t:

$$\theta_t = \frac{\imath'_{rxw}\psi_x X_t - \frac{1}{2}Var_t\left[r^w_{t+1}\right]}{Var_t\left[r^w_{t+1}\right] + \bar{u}'\psi_w\psi'_w\imath_{rxw}} = \frac{\imath'_{rxw}\psi_x X_t - \frac{1}{2}\imath'_{rxw}\psi_w\psi'_w\imath_{rxw}}{\left(\imath_{rxw} + \bar{u}\right)'\psi_w\psi'_w\imath_{rxw}} \doteq \imath'_\theta X_t. \qquad (20)$$ {eq:itheta}

Equation (20) determines the loadings of the time-varying price of risk inferred from the equilibrium dynamics of returns in the VAR given by (14). This equation, together with equation (25) in the appendix, can be jointly solved for the vectors $\bar{u}$ and ι_θ.

6.3 Firms and labor market

We consider a model of a firm that faces financial constraints. The firm enters period t with a stock of capital k_t, labor force l_t, and debt b_t. Aggregate and firm-specific shocks are realized and production takes place. The firm pays out wages and interest on its borrowing. Some workers separate from the firm for exogenous reasons, and capital depreciates. The firm then posts vacancies ξ_t to hire new workers, invests into capital, pays out dividends and decides on new borrowing.

The financial position of the firm before the capital investment is captured by its net worth

$$n_t = y_t - w_t l_t - \kappa\left(\xi_t\right) + \left(1 - \delta^k\right)Q_t k_t - R^b_t b_t$$

where y_t is the produced output, $w_t l_t$ total wage bill, $\kappa(\xi_t)$ total vacancy cost, $\left(1-\delta^k\right) Q_t k_t$ market value of undepreciated capital, and $R_t^b b_t$ the market value of repayed debt. New capital k_{t+1} is financed by net worth after the divided payout d_t, and by issuance of new bonds b_{t+1}:

$$Q_t k_{t+1} = n_t - d_t + b_{t+1}. \tag{21}$$

{eq:balance

The laws of motion for firm's labor force and capital stock are given by

$$\begin{aligned} l_{t+1} &= q_t \xi_t + \left(1-\delta_t^l\right) l_t \\ k_{t+1} &= i_t + \left(1-\delta^k\right) k_t. \end{aligned}$$

Here, q_t is the probability of filling a vacancy posted by the firm, and δ_t^l the possibly time-varying worker separation rate. In the conventional Diamond–Mortensen–Pissarides search-and-matching framework, this vacancy filling probability is tied to aggregate labor market conditions.

The firm maximizes the present discounted value of dividends, implying that the value function is given by

$$V_t\left(k_t, l_t, b_t\right) = \max_{i_t, \xi_t, b_{t+1}, d_t} d_t + E_t\left[\frac{S_{t+1}}{S_t} V_{t+1}\left(k_{t+1}, l_{t+1}, b_{t+1}\right)\right]$$

where the expectations operator integrates over firm-specific and aggregate sources of uncertainty. We assume that firms are owned by a representative household, and hence they discount cash flows using the households' stochastic discount S. The stochastic discount factor is specified in Section 6.1.

Firms are subject to constraints that limit their ability to acquire outside financing to finance their labor costs and acquisition of new capital. The literature provides microfoundations for a whole variety of functional forms for the financial constraint that can be broadly written in the form

$$F\left(k_{t+1}, l_{t+1}, b_{t+1}, k_t, l_t, b_t\right) \geq 0.$$

As an illustration for the derivation of the optimality conditions, we specialize the constraint to the form

$$n_t - \phi^k Q_t k_{t+1} \geq 0. \tag{22}$$

{eq:financi

This is a leverage constraint, stating that the ratio of net worth to the market value of capital has to be at least ϕ^k. In order to limit the ability of the firm to outsave the financial constraint, we enforce an exogenous dividend policy

$$d_t = \bar{d} n_t.$$

Imposing Lagrange multipliers μ_t and λ_t on the balance sheet constraint (21) and financial constraint (22), respectively, we obtain the Euler equation for the hiring decision[5]

$$\frac{\kappa'(\xi_t)}{q_t} = E_t\left[\frac{S_{t+1}}{S_t}\frac{\overline{d}+\left(1-\overline{d}\right)\mu_{t+1}+\lambda_{t+1}}{\overline{d}+\left(1-\overline{d}\right)\mu_t+\lambda_t}\left(\frac{\partial y_{t+1}}{\partial l_{t+1}} - w_{t+1} + \left(1-\delta^l_{t+1}\right)\frac{\kappa'(\xi_{t+1})}{q_{t+1}}\right)\right]. \quad (23) \quad \{\text{eq:EE\_hiri}$$

The equation equalizes the marginal cost of hiring a worker, $\kappa'(\xi_t)/q_t$, with the expected discounted benefits to the firm next period from hiring this worker. The benefits consist of the period surplus accrued by the firm, given by the marginal product of the worker $\partial y_{t+1}/\partial l_{t+1}$ net of wage w_{t+1}, plus the saved cost from not having to hire the worker next period, in case the worker does not separate. The second term in brackets reflects the impact of the financial constraint. In case of an unconstrained firm with optimally chosen dividends, $\lambda_t = 0$ and $\mu_t = 1$. Otherwise financial constraints impact firm's hiring decisions. When financial constraints are tighter *relative to* next period, the second term in brackets is smaller than one, and effectively operates as an additional source of discounting of next period benefits.

6.4 Valuation of firms' cash flows

The Euler equation (23) can be solved forward for the value of the job as the present discounted value of future cash flows from the match acquired by the firm

$$\frac{\kappa'(\xi_t)}{q_t} = \sum_{j=1}^{\infty} E_t\left[\frac{S_{t+j}}{S_t}\frac{F_{t+j}}{F_t}\left(\prod_{i=1}^{j}\left(1-\delta^l_{t+i}\right)\right)G_{t+j}\right] \doteq J_t \quad (24) \quad \{\text{eq:J\_PDV}\}$$

where

$$\begin{aligned} G_{t+j} &= \frac{\partial y_{t+j}}{\partial l_{t+j}} - w_{t+j} \\ \frac{F_{t+j}}{F_t} &= \frac{\overline{d}+\left(1-\overline{d}\right)\mu_{t+1}+\lambda_{t+1}}{\overline{d}+\left(1-\overline{d}\right)\mu_t+\lambda_t}. \end{aligned}$$

[5]The remaining derivations are in Appendix A.

Similarly, we can denote

$$\frac{H_{t+j}}{H_t} = \prod_{i=1}^{j} \left(1 - \delta_{t+i}^{l}\right).$$

The current value of a job for the firm, J_t, is thus represented by its future cash flows, G_{t+j}, discounted by the household's stochastic discount factor, cumulative probability of future job separation, and the contribution from future financial constraints. Notice that tighter financial constraints in the future increase the value of the job, since an existing job reduces the need for costly vacancy posting in states when financial constraints are tight.

Since observed separation rates δ_t^l are high, the value of a job consists of cash flows that have on average shorter maturity than, for instance, cash flows received by shareholders. To the extent that shorter maturity cash flows appear to earn higher risk compensations (see van Binsbergen et al. (2013) and the subsequent quickly growing literature), we can plausibly expect the same for the cash flows constituting the value of the job for the firm.

In what follows, we will study the value of the job J_t constructed from estimated discounted cash flows. The empirical challenge in the search-and-matching literature is to make the value of the job sufficiently volatile so as to make it consistent with large cyclical fluctuations in vacancy posting rates ξ_t and job filling probabilities q_t, represented by the left-hand side of (24). We will study the separate contributions of discount rate fluctuations, captured by movements in θ_t, tightness in financial constraints, and separation rates on J_t.

7 Data and empirical implementation

In addition to the variables described in Section 4, we use additional macroeconomic variables to capture the dynamics of the state vector X_t. The names of these variables as they appear in FRED are in parenthesis.

The real per capita consumption growth is constructed from expenditures on nondurable goods (PCESV) and services (PCND), adjusted for inflation (CPIAUCNS) and population growth (CNP16OV). The real income growth corresponds to real personal disposable income (DPIC96), adjusted for population growth.

Firm's cash flow from hiring a worker equals marginal product of labor minus the wage. We measure the marginal product of labor as $(1-\alpha) Y_t/L_t$ where Y_t is real GDP (GDPC1), L_t is employment (CE16OV), and we choose $\alpha = 1/3$. To construct a quarterly measure of wages since 1952, we combine two variables: the index of real compensation per hour in non-farm business sector (COMPRNFB) and the average weekly wage for a given quarter (avg_wkly_wage) from the BLS website. While the time series for the former variable starts in 1947, it is only constructed as an index. On the other hand, the latter represents the wage level but only starts in 1975. We use the average weekly wage to convert the index to dollars.

8 Results

We start by estimating the VAR (14). Our choice of variables follows Bansal et al. (2014), extended by a measure of the aggregate separation rate. The state vector is given by

$$X_t = \left(1, \Delta c_t, rx_t^m, \Delta y_t, pd_t, r_t^f, \Delta h_t\right)$$

where Δc_t is real consumption growth per capita, rx_t^m market (S&P 500) excess return, Δy_t is the real disposable income growth per capita, pd_t the price-dividend ratio, Δg_t growth rate of firm surplus, r_t^f real risk-free rate, and $\Delta h_t \doteq \log\left(1 - \delta_t^l\right)$ the logarithm of one-period match survival rate.

From the estimated VAR, we can construct the paths for $u_t = \bar{u}' X_t$ and the discount rate shock $\theta_t = \iota_\theta' X_t$. These two time-series are plotted in Figure 7. The time series for θ_t is strongly countercyclical, implying a meaningful degree of predictability in aggregate returns.

8.1 Value of a job

The implied value of the job J_t is computed using equation (24). In the aggregate analysis, we abstract from the role of financial constraints, and set $F_{t+1}/F_t \equiv 1$. The exponential-quadratic framework delivers semi-analytical formulas formulas for individual terms in the sum (see Appendix B), making the present value straightforward to evaluate.

The extracted path for the job value is depicted in Figure 8. The upper panel shows J_t, scaled by aggregate potential GDP per worker. The job surplus flow G_t exhibits a trend (even after scaling by aggregate potential GDP), and J_t inherits this trend. While this secular evolution of the job surplus to the firm is an economically important phenomenon, our analysis does not speak to its causes. In the bottom panel, we therefore present the job value in the 'price-dividend ratio' form, i.e., job value divided by the current quarterly surplus flow, J_t/G_t.

The job value exhibits a strongly cyclical pattern, with particularly notable drops in the 1981–82 and 2007–09 recessions. The mean value of the surplus corresponds to 9.8 quarters of current surplus flow, and the decline in J_t/G_t at the trough of the 2007–09 recession reaches 13% of its average value.

Hall (2017) uses a version of the equilibrium condition (24) together with the assumption of a constant cost of posting a vacancy to infer the value of the job as $J_t = \bar{\kappa}/q_t$, and use

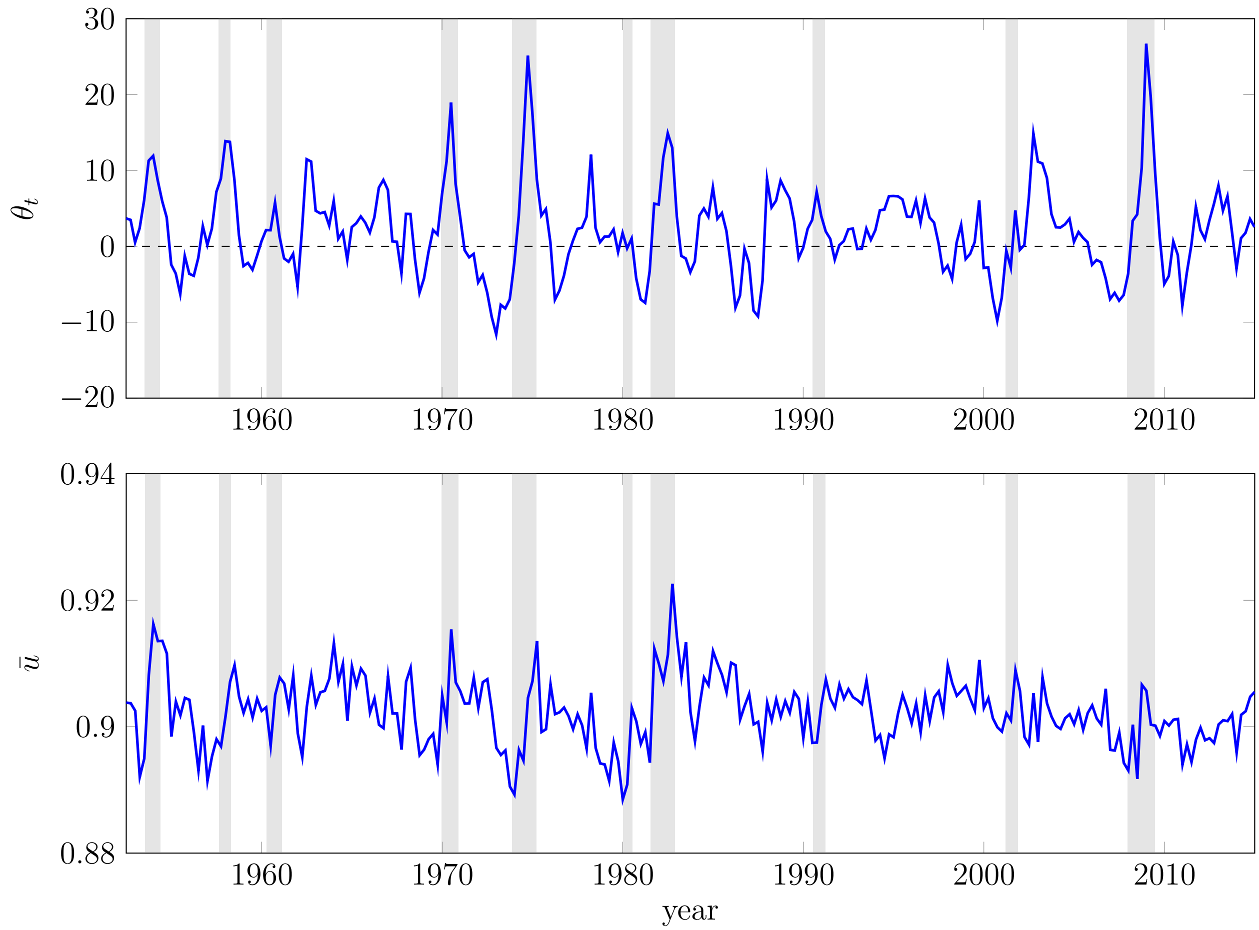

Figure 7: Paths of u_t and θ_t constructed from macroeconomic and financial data, using the Epstein–Zin preference model. NBER recessions shaded.

JOLTS data to measure q_t. He documents a roughly 30% decline in $1/q_t$ during the 2007–09 recession. In the subsequent analysis, we discuss some of the challenges in generating such large declines in the present discounted value of the firm surplus.

Fluctuations in the job value are determined by three components and their interaction: the stochastic discount factor S_t, the cash-flow G_t and the match survival rate $H_t/H_{t-1} = 1 - \delta^l_t$. To understand the importance of these different forces, we provide two types of decompositions. In the first decomposition, we alternatively turn off fluctuations in one of the components and again compute the value of the job. The second decomposition looks closely at the term structure of the three components.

We start with the first decomposition. Let J^z_t denote the value of a job computed while keeping the growth rate of the component z constant at its mean value, $z \in \{S, G, H\}$. Figure 9 compares the individual paths for the whole time interval, and specifically for the last two recessions.

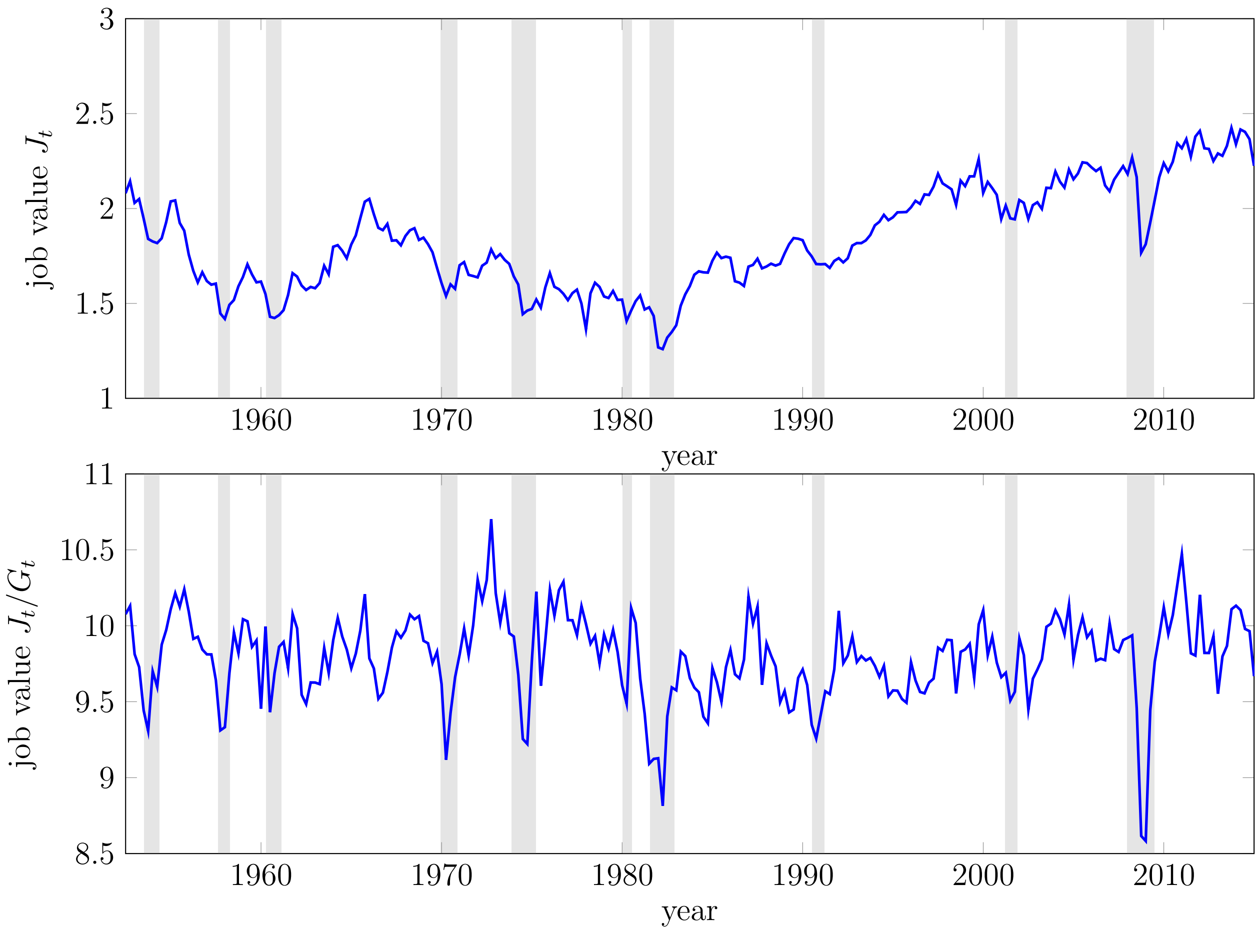

Figure 8: The extracted path for the value of job J_t to the firm. The top panel presents the job value scaled by quarterly potential real GDP. The bottom panel computes a measure of the 'price-dividend' ratio for the job value, scaling the job value by the current surplus flow G_t. NBER recessions shaded.

Comparing the fluctuations in J_t and J_t^S, we see that stochastic discounting more than doubles the time-series volatility of the present value of the surplus. Also, both the fluctuations in the separation rate as well as fluctuations in payoffs conditional on no separation contribute significantly to the volatility of the value of the job, with the former having a somewhat larger effect.

The bottom panel of Figure 9 compares the role of the stochastic discount factor in the fall of the job value during the past two recessions. The 2007–09 recession was characterized by a substantially larger increase in risk aversion and thus dispersion in the stochastic discount factor, which leads to a significantly larger risk adjustment in the value of the job during this recession. On the other hand, the increase in risk adjustment has a quantitatively small effect during the 2001 recession.

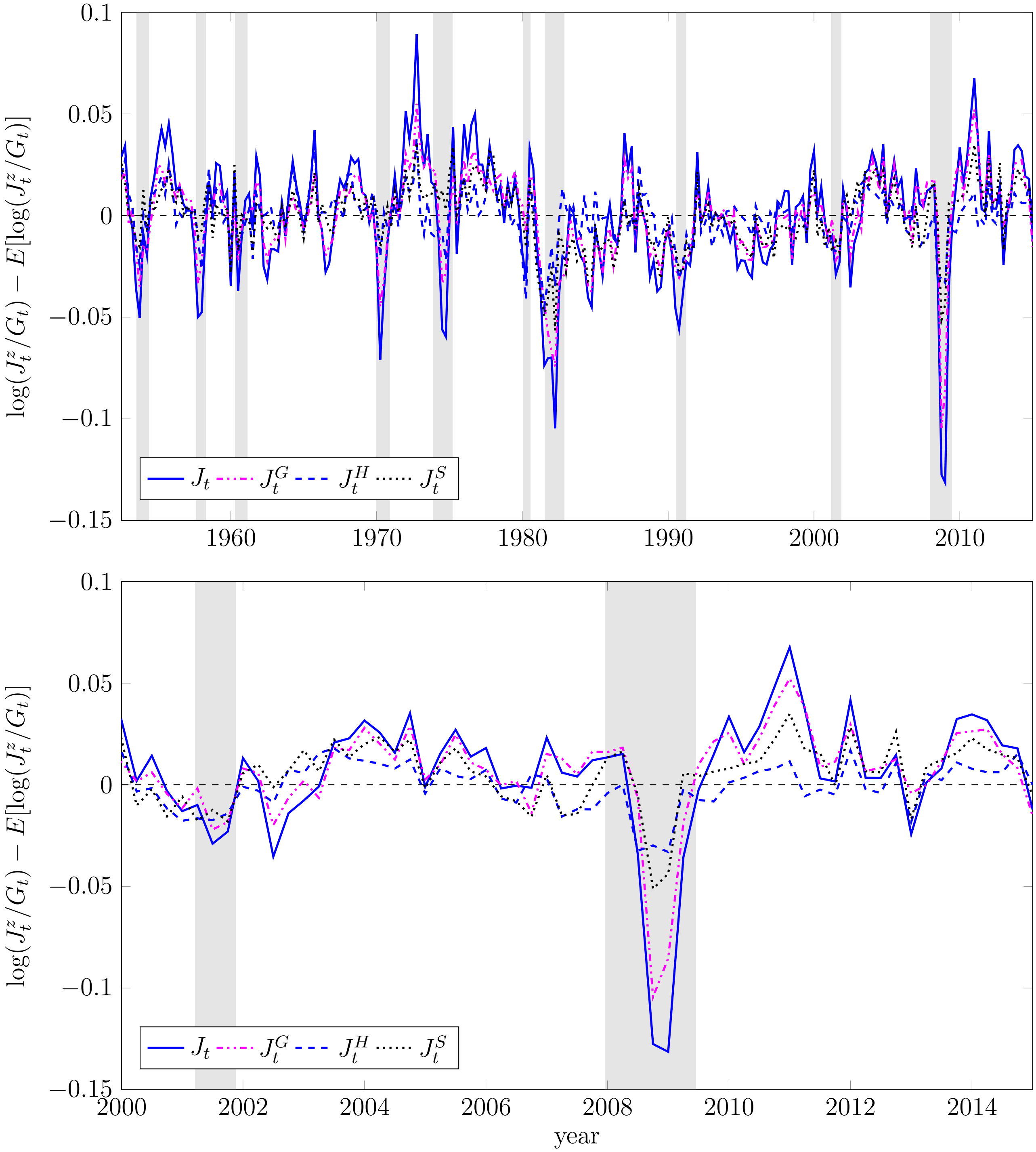

Figure 9: The extracted path for the value of job to the firm for alternative counterfactual experiments. The values of J_t^z/G_t, $z \in \{G, H, S\}$ correspond to trajectories of the job value holding the growth rate of z at its mean. The graph represents log deviations from long-run means. NBER recessions shaded.

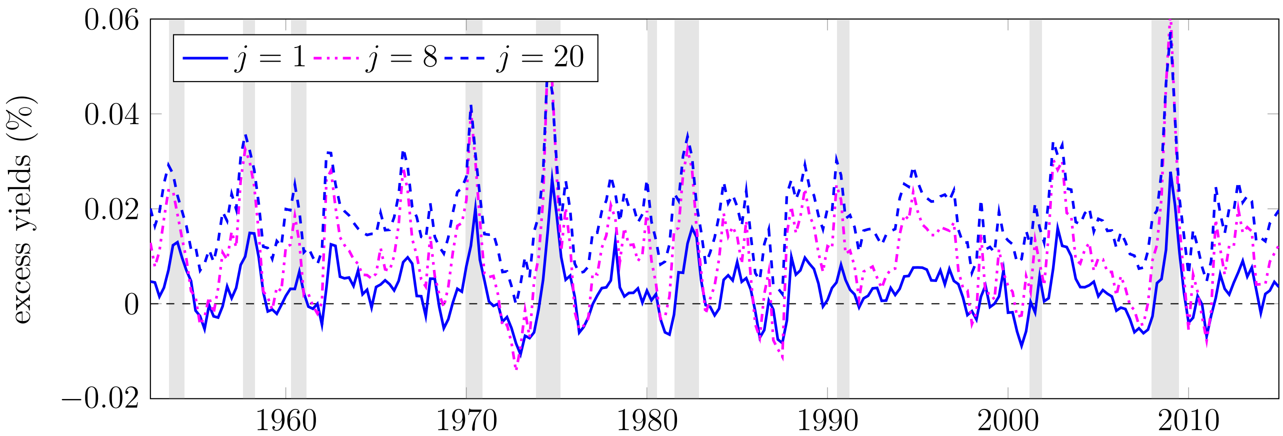

Figure 10: Term structure of risk premia (excess yields) on cash flows representing flow of surplus from a job. Yields are annualized and maturities in quarters. NBER recessions shaded.

8.2 Term structure of cash flow yields

A growing literature analyzes yields on risky cash flows of different maturities in order to establish the *term structure of risk premia*, the risky counterpart to the yield curve. See van Binsbergen et al. (2013) and others for empirical evidence, and Borovička et al. (2011), Borovička and Hansen (2014), or Borovička et al. (2014) for theoretical decomposition of the term structure of risk premia into risk exposures and prices of risk.

The value of a job J_t consists of a sum of the values of individual cash flow strips, G_{t+j}, $j \geq 1$, decaying at a rate determined by the separation probability, expressed in cumulative terms by H_{t+j}/H_t. This yields an effective payoff of $G_{t+j}\frac{H_{t+j}}{H_t}$. The yield on this cash flow strip is defined as the annualized logarithm of the ratio of the expected payoff and the current price:

$$y_t^{(j)}(GH) = \frac{1}{j}\left(\log E_t\left[\frac{H_{t+j}}{H_t}G_{t+j}\right] - \log E_t\left[\frac{S_{t+j}}{S_t}\frac{H_{t+j}}{H_t}G_{t+j}\right]\right) \qquad j \geq 1.$$

The term structure of yields $y_t^{(j)}(GH)$ in excess of the yields on maturity-matched zero-coupon bonds $y_t^{(j)}(1)$ is depicted in Figure 10. As expected, excess yields are volatile and cyclical, and the volatility is larger at longer maturities.

While the yields are volatile even at longer maturities, their ability to translate into large fluctuations in the value of the job J_t is limited by the short average duration of these cash flows. To understand this relationship, notice we can write the value of the horizon-j cash flow strip as

$$E_t\left[\frac{S_{t+j}}{S_t}\frac{H_{t+j}}{H_t}G_{t+j}\right] = \underbrace{E_t\left[\frac{S_{t+j}}{S_t}\right]}_{\text{risk-free discounting}} \underbrace{E_t\left[\frac{H_{t+j}}{H_t}G_{t+j}\right]}_{\substack{\text{expected}\\ \text{cash flow}}} + \underbrace{Cov_t\left(\frac{S_{t+j}}{S_t}, \frac{H_{t+j}}{H_t}G_{t+j}\right)}_{\text{risk compensation}}.$$

This formula decomposes the strip value into the contribution of risk-free discounting of expected cash flows (the first term on the right-hand side), and the risk-premium contribution, represented by the covariance term.

When shocks to the stochastic discount factor and cash flow growth rates are persistent, then their effects accumulate over time. Much of the search-and-matching literature focused on modeling the time-variation in expected cash flows, either through time-variation in match productivity, or alternative surplus sharing rules between the worker and the firm. Another possibility of generating fluctuations in the value of the cash-flow is through changes in the conditional risk-free rate, but given the smoothness of real risk-free rates in the data, this does not appear to be a quantitatively important channel.

The discount rate shock θ_t in our framework is a persistent shock that affects the dispersion of the stochastic discount factor, and hence the risk compensation represented by the covariance term above. Again, since the shock is persistent, its effects accumulate with maturity, and we may also expect larger effects for cash flows with longer maturities.

Unfortunately, the impact of persistent shocks on J_t through fluctuations in the value of long-horizon cash flows is dampened by the decay rate H_{t+j}/H_t. Since separations are larger (of the order of 10% per quarter, implying an average maturity of the cash flows of 2.5 years), expected flow of surplus from long-maturity strips quickly becomes negligible as maturity grows. This is in contrast to, for instance, price-dividend ratios since dividends have a much higher average duration.

In this aggregate analysis, we abstract from the role of financial constraints, introduced in the formula (24). Financial constraints F_{t+1}/F_t would be manifested as another term in the above decomposition. We devote our attention to the role of financial constraints in our cross-sectional analysis.

8.3 Predicted unemployment path

We now repeat the exercise conducted in Section 5.3 and construct the inferred path for unemployment from the time series of the continuation values J_t, vacancy data, and the imposed matching function. In order to deal with the non-stationarity of the continuation

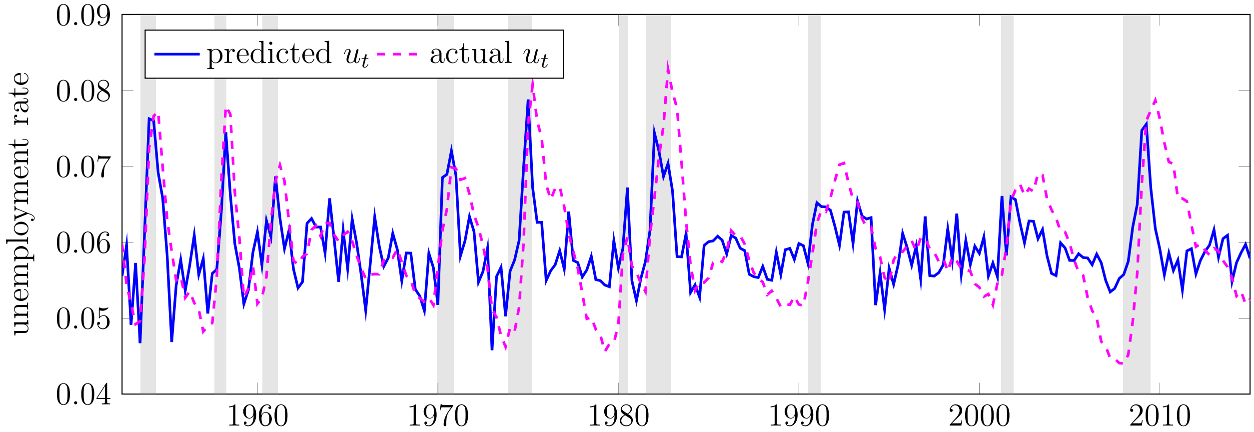

Figure 11: The predicted path for the unemployment rate. The solid line shows predictions from the model, the dashed line shows (detrended) unemployment rate in the data. NBER recessions shaded.

values. We scale the vacancy posting cost by the potential output per employee, $\bar{Y}_t$, measured from data. Hence we get

$$\kappa = q_t \frac{J_t}{\bar{Y}_t}.$$

Without this normalization, the real vacancy-posting costs would be declining over time as the economy grows richer. We normalize the level of κ to obtained the correct average level of unemployment rate.

Figure 11 shows the unemployment rate implied by the model and compares it to the (detrended) data. The model is successful at generating large increases in the unemployment rate at onset of recessions. The predicted unemployment rate is less persistent, and declines quickly after reaching its peak. The predicted unemployment rate shows considerable fluctuations, around 65 percent of those in the data: the standard deviation of the log deviations from trend is 0.129 in the data and 0.0845 in the model.

8.4 Worker's surplus

Our methodology allows us to compute worker's surplus from being employed. Denote W_t and U_t the value of being employed and unemployed, respectively. Then worker's surplus, S_t^W, can be recursively written as

$$S_{t+1}^W = w_t - b_t + E_t \left[\frac{S_{t+1}}{S_t} \left(1 - f_{t+1} - \delta_{t+1}^l\right) \left(w_{t+1} - b_{t+1}\right) \right]$$

where f_t is a job-finding probability and b_t is the flow value from unemployment. Then

$$S^W_{t+1} = \sum_{j=1}^{\infty} E_t \left[\frac{S_{t+j}}{S_t} \left(\prod_{i=1}^{j} \left(1 - f_{t+i} - \delta^l_{t+i}\right) \right) (w_{t+j} - b_{t+j}) \right].$$

This formula is analogous to equation (24), and we use the same methodology to compute the conditional expected values. Following Shimer (2005), we choose b_t equal to 40 percent of wage at time t. It is handy to introduce notation for the cash flow and survival rate, $G^W_t = w_t - b_t$, and $H^W_t = 1 - f_t - \delta_t$, and write

$$\frac{S^W_{t+1}}{G^W_t} = \sum_{j=1}^{\infty} E_t \left[\frac{S_{t+j}}{S_t} \frac{G^W_{t+j}}{G^W_t} \frac{H^W_{t+j}}{H^W_t} \right]$$

Figure 12 shows the extracted values for worker's surplus. The mean value of quarterly worker's surplus is \$1,030 (real 2009 dollars). This is rather small, and reflects the fact that due to the high job-finding probability, being unemployed is not very costly.

Worker's surplus is pro-cyclical, increasing sharply in downturns. Hence, recessions are times when it is very valuable to have a job. Golosov and Menzio (2015) come to the same conclusion, using a different methodology to evaluate worker's surplus.

Worker's surplus is again determined by three components, S_t, G^W_t and H^W_t. To understand the contribution of each of them to cyclicality of the surplus, we switch off fluctuations in each of them separately and compute the surplus under these counterfactual scenarios. This is the same type of decomposition we conducted with the value of the job. The bottom panel of 12 depicts the results, focusing on the last two recessions. We use $S^{W,x}_t$ to denote surplus with constant growth rate in component $x \in \left\{S, G^W, H^W\right\}$.

Comparing S^W_t to other three $S^{W,x}_t$, we see that H and S contribute greatly to the volatility of the surplus, while cash flow G^W does not. Indeed, worker's surplus barely changes if we turn off fluctuations in G^W. This contrasts with the contribution of H which is driven by the job-finding probability. Once we assume that it does not move over the cycle, worker's surplus $S^{W,H}_t$ actually decreases in recessions. Hence, the conclusion which immediately emerges from this analysis is that the main reason for a high worker's surplus in recessions is the very low value of unemployment, driven down by a drop in the job-finding rate.

Comparing S^W_t and $S^{W,S}_t$, we see that stochastic discounting decreases worker's surplus in recessions. Ignoring stochastic discounting, we would conclude that worker's surplus has increased by 30 percent in the great recession, while taking stochastic discounting into account, the increase is about half, 15 percent. Golosov and Menzio (2015) find an even

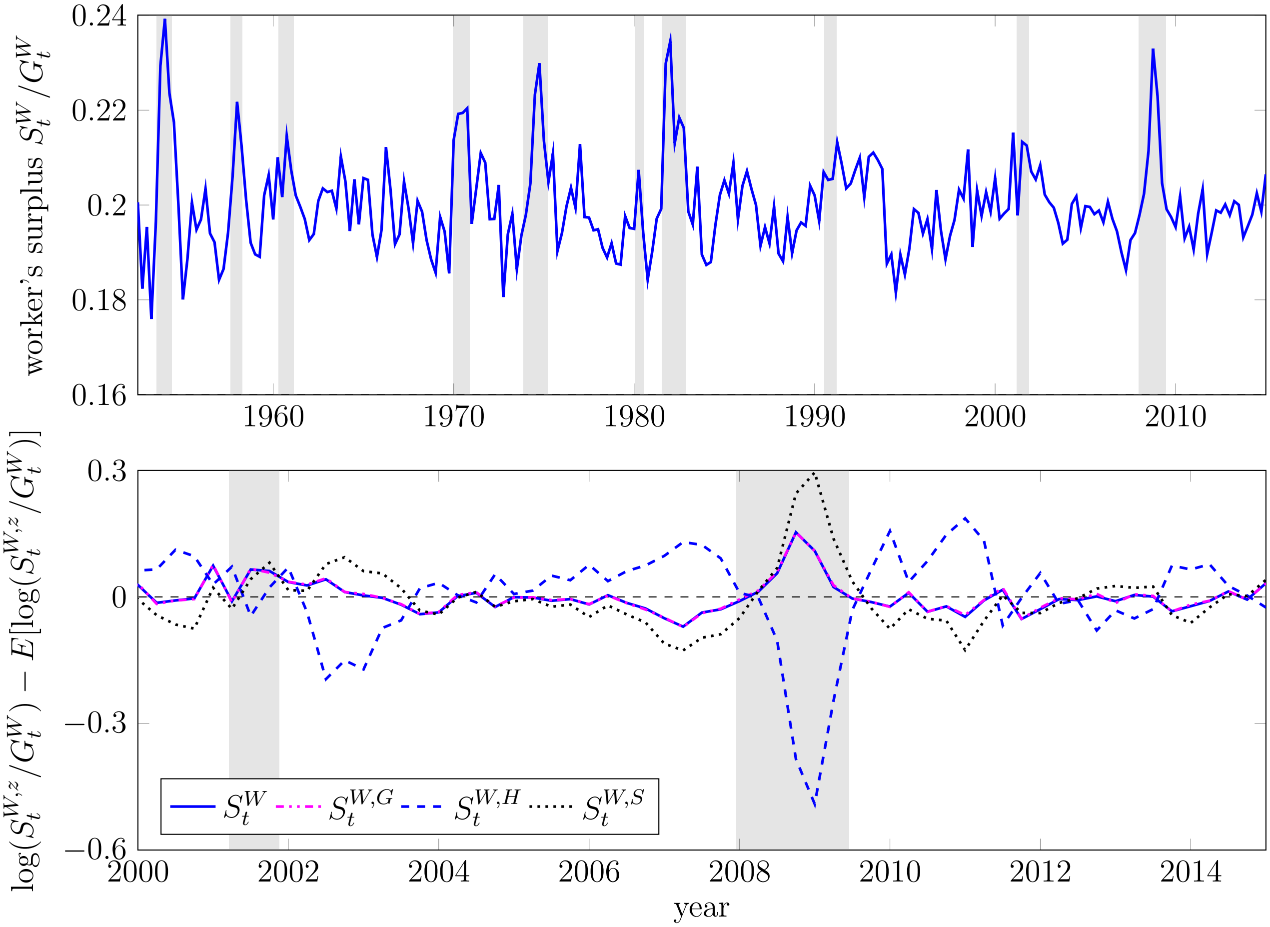

Figure 12: The extracted path for worker's surplus. The upper panel shows a measure of the "price-dividend" ratio for the worker's surplus, scaling the surplus by the current surplus flow G_t^W. The bottom panel shows worker's surplus for counterfactual experiments, plotting the values $W^{S,z}/G_t^W$ for $z \in \{G, H, S\}$. NBER recessions shaded.

higher increase, about 60 percent, because they use a constant discount factor and a non-stochastic environment.

Many search models use wage setting rules which result in worker's and firm's surplus being proportional to the total surplus of the match. This implies that correlation between the value of a job and worker's surplus is 1. Our results indicate that this feature of search model is not consistent with data, since the correlation is -0.48.

9 Directions for future research

In this paper, we connect the pricing of returns in the stock market with the pricing of cash flows accrued by the firm from worker-firm matches. We start by constructing a non-parametric lower bound on two moments of the profit flow the firm receives from the marginal worker that must be satisfied in order for the hiring Euler equation to hold. This weighted variance bound, while conservative, is able to discriminate among theoretical models used in the literature as well as among empirical proxies for the marginal profit flow. The bound is constructed conditional on a model of a stochastic discount factor that prices financial assets instrumented by a vector of variables that capture business-cycle variation in risk premia.

The properties of the profit flow and stochastic discount factor consistent with the bound and returns on financial assets lead us to construct a parametric model from which we infer fluctuations in the value of the worker to a firm. When constructed using inferred profit processes consistent with the weighted variance bound, these fluctuations int he value of the worker are able to match the business-cycle volatility of the unemployment rate.

Once we restrict the model of the stochastic discount factor to be consistent with the Epstein–Zin recursive preference specification with a time-varying price of risk, and inform the profit flow process using an empirical proxy, the ability of the model to generate empirically observed fluctuations in the unemployment rates decreases, although it remains substantial. We argue that the remaining wedge can be attributable to several factors, including the discrepancy between the profit from the average and the marginal worker, as well as shadow prices on financial constraints the firms are facing, relative to investors in financial markets. A further study of the role of these constraints is left for future work.

Similarly, it would be interesting to study the implications of this weighted variance bound for cross-sectional firm-level and industry-level data. If we interpret the wedge between the bound and the moments measured using observed cash flows as the contribution of financial constraints faced by different firms, then our framework allows to non-parametrically study the distribution of these wedges across firms and industries, allowing us to measure the cross-sectional heterogeneity in the impact of borrowing constraints for the unemployment dynamics.

Appendix

A Derivation of the model

In this appendix, we derive the asset pricing implications for the model of household preferences.

A.1 Preferences

The homogeneity of the Epstein–Zin preference structure (15) implies the conjecture (18). Substituting this expression and (16) into (15) yields the recursion

$$\begin{aligned} \bar{u}' X_t &= -\frac{\beta}{\theta_t} \log E_t \left[\exp\left(-\theta_t \left(\bar{u} + \iota_c \right)' X_{t+1} \right) \right] \\ &= \beta \left(\bar{u} + \iota_c \right)' \psi_x X_t - \frac{1}{2} \beta \theta_t \left(\bar{u} + \iota_c \right)' \psi_w \psi_w' \left(\bar{u} + \iota_c \right). \end{aligned} \tag{25}$$

Using (17), we obtain the preference recusion

$$\bar{u}' = \beta \left(\bar{u} + \iota_c \right)' \psi_x - \frac{1}{2} \beta \left(\bar{u} + \iota_c \right)' \psi_w \psi_w' \left(\bar{u} + \iota_c \right) \iota_\theta'$$

which, given ι_θ, is a Riccati equation for $\bar{u}$ that can be solved by iteration.

Substituting this result into the formula for the stochastic discount factor, we get

$$\frac{S_{t+1}}{S_t} = \beta \exp\left(-\iota_c' X_{t+1} \right) \frac{\exp\left(-\theta_t \left(\bar{u} + \iota_c \right)' \psi_w W_{t+1} \right)}{E_t \left[\exp\left(-\theta_t \left(\bar{u} + \iota_c \right)' \psi_w W_{t+1} \right) \right]}$$

and hence

$$\begin{aligned} \Delta s_{t+1} &\doteq \log S_{t+1} - \log S_t = \\ &= \log \beta - \iota_c' \psi_x X_t - \frac{1}{2} \theta_t^2 \left(\bar{u} + \iota_c \right)' \psi_w \psi_w' \left(\bar{u} + \iota_c \right) - \left(\iota_c + \theta_t \left(\bar{u} + \iota_c \right) \right)' \psi_w W_{t+1}. \end{aligned}$$

This form of the stochastic discount factor fits into the class of exponential-quadratic models introduced in Appendix B, with

$$\begin{aligned} \log S_{t+1} - \log S_t &= \Gamma_{s,0} + \Gamma_{s,1} X_t + \Gamma_{s,3} \left(X_t \otimes X_t \right) + \\ &\quad + \Psi_{s,0} W_{t+1} + \Psi_{s,1} \left(X_t \otimes W_{t+1} \right) \end{aligned}$$

where we utilize the Kronecker product notation detailed in Appendix B, and

$$\begin{aligned} \Gamma_{s,0} &= \log \beta \qquad \Gamma_{s,1} = -\iota_c' \psi_x \qquad \Gamma_{s,3} = -\frac{1}{2} \left(\operatorname{vec} \left[\iota_\theta \left(\bar{u} + \iota_c \right)' \psi_w \psi_w' \left(\bar{u} + \iota_c \right) \iota_\theta' \right] \right)' \\ \Psi_{s,0} &= -\iota_c' \psi_w \qquad \Psi_{s,1} = -\left(\operatorname{vec} \left[\left(\iota_\theta \left(\bar{u} + \iota_c \right)' \psi_w \right)' \right] \right)'. \end{aligned}$$

A.2 Asset pricing implications

The representative household invests its wealth into a portfolio of assets indexed by i, with gross returns R_{t+1}^{i}. Optimality conditions imply that asset returns have to satisfy the Euler equations

$$1 = E_t\left[\frac{S_{t+1}}{S_t}R_{t+1}^{i}\right]. \tag{26}$$

{eq:EE_stan

The model of the stochastic discount factor implies the risk-free rate

$$\begin{aligned} r_t^f &= -\log E_t\left[\frac{S_{t+1}}{S_t}\right] = -\log\beta + \iota_c'\psi_x X_t + \frac{1}{2}\theta_t^2\bar{u}'\psi_w\psi_w'\bar{u} - \frac{1}{2}\left(\iota_c + \theta_t\bar{u}\right)'\psi_w\psi_w\left(\iota_c + \theta_t\bar{u}\right) \\ &= -\log\beta - \frac{1}{2}\iota_c'\psi_w\psi_w\iota_c + \iota_c'\left(\psi_x - \psi_w\psi_w\bar{u}\iota_\theta'\right)X_t \end{aligned}$$

The risk-free rate will be relatively smooth as long as the consumption process does not fluctuate much with the state variables, as captured by the selection vector ι_c.

Assuming that returns are log-normally distributed, we can use (26) to write

$$0 = \log E_t\left[\exp\left(\Delta s_{t+1} + r_{t+1}^{i}\right)\right] = E_t\left[\Delta s_{t+1} + r_{t+1}^{i}\right] + \frac{1}{2}Var_t\left[\Delta s_{t+1} + r_{t+1}^{i}\right]$$

Applying this result to the risk-free rate as well and substracting the two expressions, we obtain

$$E_t\left[r_{t+1}^{i}\right] + \frac{1}{2}Var_t\left[r_{t+1}^{i}\right] - r_t^f = -Cov_t\left(\Delta s_{t+1}, r_{t+1}^{i}\right)$$

where the left-hand side is the one-period risk-premium of asset i. Plugging in the model of the stochastic discount factor, we obtain (19).

B Conditional expectations in exponential-quadratic framework

In this appendix, we present formulas for conditional expectations in a general form of the exponential-quadratic model used in the main text. The results are derived in Borovička and Hansen (2014) where this model arises as an approximation for a class of dynamic stochastic general equilibrium models constructed using a second-order series expansion.

Let $X = (X_1', X_2')'$ be an $(n_1+n_2)\times 1$ vector of states, $W \sim N(0, I)$ a $k \times 1$ vector of independent Gaussian shocks, and $\mathcal{F}_t$ the filtration generated by $(X_0, W_1, \ldots, W_t)$. We will show that given the law of motion

$$\begin{aligned} X_{1,t+1} &= \Theta_{10} + \Theta_{11}X_{1,t} + \Lambda_{10}W_{t+1} \qquad (27) \\ X_{2,t+1} &= \Theta_{20} + \Theta_{21}X_{1,t} + \Theta_{22}X_{2,t} + \Theta_{23}\left(X_{1,t}\otimes X_{1,t}\right) + \\ &\quad + \Lambda_{20}W_{t+1} + \Lambda_{21}\left(X_{1,t}\otimes W_{t+1}\right) + \Lambda_{22}\left(W_{t+1}\otimes W_{t+1}\right) \end{aligned}$$

and a process $M_t = \exp(Y_t)$ whose additive increment is given by

$$\begin{aligned} Y_{t+1} - Y_t &= \Gamma_0 + \Gamma_1 X_{1,t} + \Gamma_2 X_{2,t} + \Gamma_3 (X_{1,t} \otimes X_{1,t}) + \\ &\quad + \Psi_0 W_{t+1} + \Psi_1 (X_{1,t} \otimes W_{t+1}) + \Psi_2 (W_{t+1} \otimes W_{t+1}), \end{aligned} \tag{28}$$

we can write the conditional expectation of M as

$$\log E[M_t \mid \mathcal{F}_0] = (\bar{\Gamma}_0)_t + (\bar{\Gamma}_1)_t X_{1,0} + (\bar{\Gamma}_2)_t X_{2,0} + (\bar{\Gamma}_3)_t (X_{1,0} \otimes X_{1,0}) \tag{29}$$ {eq:em_app}

where $(\bar{\Gamma}_i)_t$ are constant coefficients to be determined.

The dynamics given by (27)–(28) embed as a special case the VAR specification of the Markov dynamics. In this case the state vector X_t is represented by $X_{1,t}$ in (27), and the vector $X_{2,t}$ is empty.

B.1 Definitions

To simplify work with Kronecker products, we define two operators vec and $\text{mat}_{m,n}$. For an $m \times n$ matrix H, $\text{vec}(H)$ produces a column vector of length mn created by stacking the columns of H:

$$h_{(j-1)m+i} = [\text{vec(H)}]_{(j-1)m+i} = H_{ij}.$$

For a vector (column or row) h of length mn, $\text{mat}_{m,n}(h)$ produces an $m \times n$ matrix H created by 'columnizing' the vector:

$$H_{ij} = [\text{mat}_{m,n}(h)]_{ij} = h_{(j-1)m+i}.$$

We drop the m,n subindex if the dimensions of the resulting matrix are obvious from the context. For a square matrix A, define the sym operator as

$$\text{sym}(A) = \frac{1}{2}(A + A').$$

Apart from the standard operations with Kronecker products, notice that the following is true. For a row vector $H_{1\times nk}$ and column vectors $X_{n\times 1}$ and $W_{n\times 1}$

$$H(X \otimes W) = X' [\text{mat}_{k,n}(H)]' W$$

and for a matrix $A_{n\times k}$, we have

$$X'AW = (\text{vec}A')' (X \otimes W).$$

Also, for $A_{n\times n}$, $X_{n\times 1}$, $K_{k\times 1}$, we have

$$\begin{aligned} (AX) \otimes K &= (A \otimes K) X \\ K \otimes (AX) &= (K \otimes A) X. \end{aligned}$$

Finally, for column vectors $X_{n\times 1}$ and $W_{k\times 1}$,

$$(AX)\otimes(BW)=(A\otimes B)(X\otimes W)$$

and

$$(BW)\otimes(AX)=[B\otimes A_{\bullet j}]_{j=1}^{n}(X\otimes W)$$

where

$$[B\otimes A_{\bullet j}]_{j=1}^{n}=[B\otimes A_{\bullet 1}\ B\otimes A_{\bullet 2}\ \dots\ B\otimes A_{\bullet n}].$$

B.2 Conditional expectations

Notice that a complete-the squares argument implies that, for a $1\times k$ vector A, a $1\times k^2$ vector B, and a scalar function $f(w)$,

$$\begin{aligned}
&E\left[\exp\left(B\left(W_{t+1}\otimes W_{t+1}\right)+AW_{t+1}\right)f\left(W_{t+1}\right)\mid\mathcal{F}_t\right]= \qquad (30)\\
&\quad=E\left[\exp\left(\frac{1}{2}W_{t+1}'\left(\mathrm{mat}_{k,k}\left(2B\right)\right)W_{t+1}+AW_{t+1}\right)f\left(W_{t+1}\right)\mid\mathcal{F}_t\right]\\
&\quad=\left|I_k-\mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2B\right)\right]\right|^{-1/2}\exp\left(\frac{1}{2}A\left(I_k-\mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2B\right)\right]\right)^{-1}A'\right)\tilde{E}\left[f\left(W_{t+1}\right)\mid\mathcal{F}_t\right]
\end{aligned}$$

where $\tilde{\cdot}$ is a measure under which

$$W_{t+1}\sim N\left(\left(I_k-\mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2B\right)\right]\right)^{-1}A',\left(I_k-\mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2B\right)\right]\right)^{-1}\right).$$

We start by utilizing formula (30) to compute

$$\begin{aligned}
\bar{Y}\left(X_t\right)&=\log E\left[\exp\left(Y_{t+1}-Y_t\right)\mid\mathcal{F}_t\right]=\\
&=\Gamma_0+\Gamma_1X_{1,t}+\Gamma_2X_{2,t}+\Gamma_3\left(X_{1,t}\otimes X_{1,t}\right)+\\
&\quad+\log E\left[\exp\left(\left[\Psi_0+X_{1t}'\left[\mathrm{mat}_{k,n}\left(\Psi_1\right)\right]'\right]W_{t+1}+\frac{1}{2}W_{t+1}'\left[\mathrm{mat}_{k,k}\left(\Psi_2\right)\right]W_{t+1}\right)\mid\mathcal{F}_t\right]\\
&=\Gamma_0+\Gamma_1X_{1,t}+\Gamma_2X_{2,t}+\Gamma_3\left(X_{1,t}\otimes X_{1,t}\right)-\\
&\quad-\frac{1}{2}\log\left|I_k-\mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2\Psi_2\right)\right]\right|+\frac{1}{2}\mu'\left(I_k-\mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2\Psi_2\right)\right]\right)^{-1}\mu
\end{aligned}$$

with μ defined as

$$\mu=\Psi_0'+\left[\mathrm{mat}_{k,n}\left(\Psi_1\right)\right]X_{1,t}.$$

Reorganizing terms, we obtain

$$\bar{Y}\left(X_t\right)=\bar{\Gamma}_0+\bar{\Gamma}_1X_{1,t}+\bar{\Gamma}_2X_{2,t}+\bar{\Gamma}_3\left(X_{1,t}\otimes X_{1,t}\right)$$

where

$$\begin{aligned}
\bar{\Gamma}_0 &= \Gamma_0 - \frac{1}{2}\log\left|I_k - \mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2\Psi_2\right)\right]\right| + \frac{1}{2}\Psi_0\left(I_k - \mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2\Psi_2\right)\right]\right)^{-1}\Psi_0' \\
\bar{\Gamma}_1 &= \Gamma_1 + \Psi_0\left(I_k - \mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2\Psi_2\right)\right]\right)^{-1}\left[\mathrm{mat}_{k,n}\left(\Psi_1\right)\right] \qquad (31) \\
\bar{\Gamma}_2 &= \Gamma_2 \\
\bar{\Gamma}_3 &= \Gamma_3 + \frac{1}{2}\mathrm{vec}\left[\left[\mathrm{mat}_{k,n}\left(\Psi_1\right)\right]'\left(I_k - \mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2\Psi_2\right)\right]\right)^{-1}\left[\mathrm{mat}_{k,n}\left(\Psi_1\right)\right]\right]'.
\end{aligned}$$

For the set of parameters $\mathcal{P} = (\Gamma_0, \ldots, \Gamma_3, \Psi_0, \ldots, \Psi_2)$, equations (31) define a mapping

$$\bar{\mathcal{P}} = \bar{\mathcal{E}}\left(\mathcal{P}\right),$$

with all $\bar{\Psi}_j = 0$. We now substitute the law of motion for X_1 and X_2 to produce $\bar{Y}\left(X_t\right) = \tilde{Y}\left(X_{t-1}, W_t\right)$. It is just a matter of algebraic operations to determine that

$$\begin{aligned}
\tilde{Y}\left(X_{t-1}, W_t\right) &= \log E\left[\exp\left(Y_{t+1} - Y_t\right) \mid \mathcal{F}_t\right] = \\
&= \tilde{\Gamma}_0 + \tilde{\Gamma}_1 X_{1,t-1} + \tilde{\Gamma}_2 X_{2,t-1} + \tilde{\Gamma}_3\left(X_{1,t-1} \otimes X_{1,t-1}\right) \\
&\quad + \tilde{\Psi}_0 W_t + \tilde{\Psi}_1\left(X_{1,t-1} \otimes W_t\right) + \tilde{\Psi}_2\left(W_t \otimes W_t\right)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\Gamma}_0 &= \bar{\Gamma}_0 + \bar{\Gamma}_1\Theta_{10} + \bar{\Gamma}_2\Theta_{20} + \bar{\Gamma}_3\left(\Theta_{10} \otimes \Theta_{10}\right) \qquad (32) \\
\tilde{\Gamma}_1 &= \bar{\Gamma}_1\Theta_{11} + \bar{\Gamma}_2\Theta_{21} + \bar{\Gamma}_3\left(\Theta_{10} \otimes \Theta_{11} + \Theta_{11} \otimes \Theta_{10}\right) \\
\tilde{\Gamma}_2 &= \bar{\Gamma}_2\Theta_{22} \\
\tilde{\Gamma}_3 &= \bar{\Gamma}_2\Theta_{23} + \bar{\Gamma}_3\left(\Theta_{11} \otimes \Theta_{11}\right) \\
\tilde{\Psi}_0 &= \bar{\Gamma}_1\Lambda_{10} + \bar{\Gamma}_2\Lambda_{20} + \bar{\Gamma}_3\left(\Theta_{10} \otimes \Lambda_{10} + \Lambda_{10} \otimes \Theta_{10}\right) \\
\tilde{\Psi}_1 &= \bar{\Gamma}_2\Lambda_{21} + \bar{\Gamma}_3\left(\Theta_{11} \otimes \Lambda_{10} + \left[\Lambda_{10} \otimes \left(\Theta_{11}\right)_{\bullet j}\right]_{j=1}^{n}\right) \\
\tilde{\Psi}_2 &= \bar{\Gamma}_2\Lambda_{22} + \bar{\Gamma}_3\left(\Lambda_{10} \otimes \Lambda_{10}\right).
\end{aligned}$$

This set of equations defines the mapping

$$\tilde{\mathcal{P}} = \tilde{\mathcal{E}}\left(\bar{\mathcal{P}}\right).$$

B.3 Iterative formulas

We can write the conditional expectation in (29) recursively as

$$\log E\left[M_t \mid \mathcal{F}_0\right] = \log E\left[\exp\left(Y_1 - Y_0\right) E\left[\frac{M_t}{M_1} \mid \mathcal{F}_1\right] \mid \mathcal{F}_0\right].$$

Given the mappings $\bar{\mathcal{E}}$ and $\tilde{\mathcal{E}}$, we can therefore express the coefficients $\bar{\mathcal{P}}$ in (29) using the recursion

$$\bar{\mathcal{P}}_t = \bar{\mathcal{E}}\left(\mathcal{P} + \tilde{\mathcal{E}}\left(\bar{\mathcal{P}}_{t-1}\right)\right)$$

where the addition is by coefficients and all coefficients in $\bar{\mathcal{P}}_0$ are zero matrices.

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