Bond Risk Premia in Consumption-based Models

Drew D. Creal
Chicago Booth

Jing Cynthia Wu
Chicago Booth & NBER
Motivation

Chairman Janet Yellen: large-scale asset purchases  December, 2014, Press Conference

...we're reminding the public that we continue to hold a large stock of assets, and that is tending to push down term premiums in longer-term yields.

Chairman Ben Bernanke: decomposition  March, 2006, New York

To the extent that the decline in forward rates can be traced to a decline in the term premium... the effect is financially stimulative and argues for greater monetary policy restraint... However, if the behavior of long-term yields reflects current or prospective economic conditions, the implications for policy may be quite different—indeed, quite the opposite.

Chairman Alan Greenspan: conundrum  June, 2005, Beijing

That improved performance has doubtless contributed to lower inflation-related risk premiums, and the lowering of these premiums is reflected in significant declines in nominal and real long-term rates. Although this explanation contributes to an understanding of the past decade, I do not believe it explains the decline of long-term interest rates over the past year despite rising short-term rates.
Term premium: two models & two channels

- Gaussian ATSM:
  - benchmark model
  - time-varying term premia via price of risk
  - lack micro foundation

- structural models with recursive preferences
  - Gaussian: constant term premia
  - SV: time-varying term premia via SV
  - economic structure imposes restrictions

Goal of this paper: reconcile the two literatures
Contributions

- Introduce a new structural model with recursive preferences
  - preference shocks $\rightarrow$ time-varying prices of risk

- Bridge two literatures on bonds
  - GATSM and consumption based models.

- Representative agent’s problem must be well-posed
  - provide conditions guaranteeing a model solution
    - representative agent cannot be too patient
    - risk aversion cannot be too large
  - the geometry of the feasible regions makes estimation challenging
Results: term premia

TP: GATSM

TP: Gaussian w/ PS

TP: SV w/o PS

TP: SV w/ PS
Outline

1. Model
2. Estimation
3. Results
4. Model solution
Gaussian Model

Agent’s problem

\[ V_t = \max \left[ (1 - \beta) \ U_t C_t^{1-\eta} + \beta \left\{ E_t [ V_{t+1}^{1-\gamma} ] \right\}^{\frac{1-\eta}{1-\gamma}} \right]^{\frac{1}{1-\eta}} \]

s.t. \[ W_{t+1} = (W_t - C_t) R_{c,t+1} \]

Log stochastic discount factor

\[ m_{t+1} = \vartheta \ln (\beta) + \vartheta \Delta \nu_{t+1} - \eta \vartheta \Delta c_{t+1} + (\vartheta - 1) r_{c,t+1}, \]

\[ m^\$_{t+1} = m_{t+1} - \pi_{t+1} \]

where \( \vartheta = \frac{1-\gamma}{1-\eta} \)

State variables

\[ \Delta c_t = Z'_c g_t, \quad \pi_t = Z'_\pi g_t, \]

\[ g_{t+1} = \mu_g + \Phi_g g_t + \Sigma_{0,g} \varepsilon_{g,t+1} \quad \varepsilon_{g,t+1}, \sim N(0, I). \]

▶ It’s a companion form, nesting long-run risk & VARMA.
Preference shocks

\[ \Delta v_{t+1} = Z' g_{t+1} + \Lambda_1 (g_t) + \Lambda_2 (g_t)' \varepsilon_{g,t+1} \]

\[ \Lambda_2 (g_t) = -\eta \Sigma^{-1}_{0,g} (\lambda_0 + \lambda_g g_t) \]

- \( Z' g_{t+1} \): Albuquerque, Eichenbaum, & Rebelo (2014)
  - no time-varying price of risk.
- \( \lambda_g \neq 0 \Rightarrow \text{time-varying price of risk/ term premium} \)
Sources of risk premia

\[
m_{t+1} - E_t \left[ m_{t+1} \right] = -\lambda_{g,t}^s \varepsilon_{g,t+1}
\]

where the price of risk \( \lambda_{g,t}^s \) is

\[
\Sigma_{0,g} \lambda_{g,t}^s = \Sigma_{0,g} \Sigma'_{0,g} (\gamma Z_c + Z_{\pi}) - \kappa_1 \frac{(\eta - \gamma)}{(1 - \eta)} \Sigma_{0,g} \Sigma'_{0,g} D_g
\]

the price of risk only varies through time if \( \lambda_g \neq 0 \).

Campbell & Cochrane (1999): Risk sensitivity function \( \lambda(s_t) \) creates time-varying price of risk

\[
m_{t+1} - E_t [m_{t+1}] = -\gamma \sigma_c (1 + \lambda(s_t)) \varepsilon_{c,t+1}
\]

Our model

\[\Lambda_2 (g_t)\] is (potentially) a function of any element of the state vector \( g_t \).

Bond prices are known in closed-form.
Nominal risk neutral measure $Q$

$$g_{t+1} = \mu_g^Q + \Phi_g^Q g_t + \Sigma_{0,g} \varepsilon_{g,t+1}, \quad \varepsilon_{g,t+1} \sim N(0, I).$$

where

$$\Sigma_{0,g} \lambda_{g,t} = (\mu_g - \mu_g^Q) + (\Phi_g - \Phi_g^Q) g_t$$

Therefore,

$$\Phi_g^Q = \Phi_g - \eta \nu \lambda_g$$

- Same form as a Gaussian ATSM.
- the price of risk only varies through time if

$$\lambda_g \neq 0 \Rightarrow \Phi_g \neq \Phi_g^Q$$

- Duffee (2002): important feature of the data
**Bond prices**

Bond prices

\[ P_{t,(n)}^\$ = E_t \left[ \exp \left( m_{t+1}^\$ \right) P_{t+1}^{\$, (n-1)} \right] \]

yields

\[ y_{t,(n)}^\$ \equiv -\frac{1}{n} \ln \left( P_{t,(n)}^\$ \right) = a_n^\$ + b_{n,g}^\$ g_t \]

- Bond loadings are similar to Gaussian ATSMs.
- They are functions of \((\beta, \gamma, \psi)\), where \(\psi = 1/\eta\)

**Consumption-inflation representation**

\[ y_{t,(n)}^\$ = -\ln(\beta) + \frac{\eta}{n} \sum_{j=0}^{n-1} E_t^{Q,\$} \Delta c_{t+j+1} + \frac{1}{n} \sum_{j=0}^{n-1} E_t^{Q,\$} \pi_{t+j+1} - \frac{1}{n} \sum_{j=0}^{n-1} E_t^{Q,\$} \Delta u_{t+j+1} + J.I. \]
SV model: dynamics

Stochastic volatility process from Creal & Wu (2015 JEconometrics)

\[ g_{t+1} = \mu_g + \Phi_g g_t + \Phi_{gh} h_t + \sum g_h \varepsilon_{h,t+1} + \sum g_t \varepsilon_{g,t+1} \quad \varepsilon_{g,t+1} \sim N(0, I) \]

\[ \Sigma_{g,t} \Sigma_{g,t}' = \Sigma_{0,g} \Sigma_{0,g}' + \sum_{i=1}^{H} \Sigma_{i,g} \Sigma_{i,g}' h_{it} \]

\[ h_{t+1} \sim \text{NCG}(\nu_h, \Phi_h, \Sigma_h) \]

\[ \varepsilon_{h,t+1} = h_{t+1} - \mathbb{E}_t[h_{t+1}|h_t] \]

Long run risk

\[ \pi_{t+1} = \bar{\pi}_t + \varepsilon_{\pi_1,t+1} \quad \varepsilon_{\pi_1,t+1} \sim N(0, h_{t,\pi_1}) \]

\[ \Delta c_{t+1} = \bar{c}_t + \varepsilon_{c_1,t+1} \quad \varepsilon_{c_1,t+1} \sim N(0, h_{t,c_1}) \]

\[ \bar{\pi}_{t+1} = \mu_{\pi} + \phi_{\pi} \bar{\pi}_t + \phi_{\pi,c} \bar{c}_t + \varepsilon_{\pi_2,t+1} \quad \varepsilon_{\pi_2,t+1} \sim N(0, h_{t,\pi_2}) \]

\[ \bar{c}_{t+1} = \mu_c + \phi_{c,\pi} \bar{\pi}_t + \phi_{c} \bar{c}_t + \sigma_{c,\pi} \varepsilon_{\pi_2,t+1} + \varepsilon_{c_2,t+1} \quad \varepsilon_{c_2,t+1} \sim N(0, h_{t,c_2}) \]
SV model: preference shock

Preference shock

\[ \Delta \nu_{t+1} = Z'_v g_{t+1} + \Lambda_1 (g_t, h_t) + \Lambda_2 (g_t, h_t)' \varepsilon_{g,t+1} + \Lambda_3 (h_t)' \varepsilon_{h,t+1} \]
\[ \Lambda_2 (g_t, h_t) = -\eta \sum_{g,t}^{-1} (\lambda_0 + \lambda_g g_t + \lambda_{gh} h_t), \quad \Lambda_3 (h_t) = -\lambda_h. \]

pricing kernel

\[ m^$_{t+1} - E_t \left[ m^$_{t+1} \right] = -\lambda_{g,t}^$ \varepsilon_{g,t+1} - \lambda_{h,t}^$ \tilde{\varepsilon}_{h,t+1} \]

risk premium

\[ \Sigma_{g,t} \lambda_g^$ = \Sigma_{0,g} \Sigma_{0,g}' (\gamma Z_c + Z_{\pi}) - \kappa_1 \frac{(\eta - \gamma)}{(1 - \eta)} \Sigma_{0,g} \Sigma_{0,g}' D_g - \vartheta \Sigma_{0,g} \Sigma_{0,g}' Z_{\nu} + \eta \vartheta \lambda_0 + \eta \vartheta \lambda_g g_t \]
\[ + ([\gamma Z_c + Z_{\pi}] \otimes I_G)' \tilde{S}_g h_t - \kappa_1 \frac{(\eta - \gamma)}{(1 - \eta)} (D_g \otimes I_G)' \tilde{S}_g h_t \]

power utility  recursive preferences

\[ -\vartheta (Z_{\nu} \otimes I_G)' \tilde{S}_g h_t + \eta \vartheta \lambda_{gh} h_t + \text{ leverage effects} \]

\[ \Sigma'_{h,t} \lambda_{h,t}^$ = \Sigma'_{gh} (\gamma Z_c + Z_{\pi}) - \kappa_1 \frac{(\eta - \gamma)}{(1 - \eta)} \left( \Sigma'_{gh} D_g + D_h \right) - \vartheta \left( \Sigma'_{gh} Z_{\nu} - \lambda_h \right) \]
Data

Monthly data from Feb. 1959 to June 2014

Yields

- Fama-Bliss zero-coupon yields from CRSP
- maturities: 3m, 1y, 2y, 3y, 4y, 5y

Inflation + Population

- FRED database at St. Louis FRB
- CPI inflation
- Civilian population over 16

Consumption

- U.S. Bureau of Economic Analysis
- non-durables + services


## Estimation approach

### Step 1: estimate the time series dynamics ($\theta^P, g_t, h_t$)

- use MCMC + particle filters $\rightarrow$ Particle Gibbs sampler.

### Step 2: estimate $\theta^u = (\beta, \gamma, \psi)$ and $\theta^\lambda$

- Run a cross-sectional regression on filtered estimates $\hat{g}_t$ and $\hat{h}_t$

\[
y_t^s = A^s \begin{pmatrix} \theta^u, \theta^\lambda \end{pmatrix} + B_g^s \begin{pmatrix} \theta^u, \theta^\lambda \end{pmatrix} \hat{g}_t + B_h^s \begin{pmatrix} \theta^u, \theta^\lambda \end{pmatrix} \hat{h}_t + \eta_t \quad \eta_t \sim N(0, \Omega)
\]

- to estimate $\hat{\zeta}^r = \begin{pmatrix} \hat{A}^{s,r}, \text{vec} \left( \hat{B}_g^{s,r} \right)^\prime, \text{vec} \left( \hat{B}_h^{s,r} \right)^\prime \end{pmatrix}^\prime$

- We use minimum-$\chi^2$ estimation of Hamilton & Wu (2012 JEconometrics)

\[
\arg\min_{\theta^u, \theta^\lambda} \ T \left( \hat{\zeta}^r - \zeta \begin{pmatrix} \theta^u, \theta^\lambda \end{pmatrix} \right)^\prime W \left( \hat{\zeta}^r - \zeta \begin{pmatrix} \theta^u, \theta^\lambda \end{pmatrix} \right)
\]
### Structural parameters

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<th>Gaussian+PS</th>
<th>SV</th>
<th>SV+PS</th>
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Unconditional yield curves
Why SV is doing worse?

- SV seems to be more flexible with $h_t$
- But there are more moments to match
- There are only 3 free parameters to match all
- It’s difficult to match both the average level, and slope

$$\psi = [0, 20]; \beta = [0.6, 1]; \gamma = [0.001, 1100]$$
Term premia: SV without preference shock

parameters from Bansal & Yaron (2004)
Comparison with the literature

Tension between fitting macroeconomic variables and yields

- Macro variables: $P$ parameters
- Cross section of yields: $Q$ parameters
- Term premia: the difference between $P$ and $Q$

In consumption-based models with recursive preferences, $\Phi_g = \Phi_Q^g$

- If we force the model to fit macro variables (*ours*), then
  - $\Phi_g$ determines the slope is downward
- If we force the model to fit the upward slope (*literature*), then
  - $\Phi_Q^g$ determines factor dynamics
  - macro factors mimic level, slope and curvature of yields

Autocovariance of volatility, $\Phi_h \neq \Phi_Q^h$, is not enough to break the tension

*Preference shock allows* $\Phi_g \neq \Phi_Q^g$
## Macro factors and yields

Regression $R^2$s of macro factors on yields

<table>
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<th>estimates w/o yields</th>
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<tr>
<td>consumption V</td>
<td>1</td>
<td>0.87</td>
</tr>
</tbody>
</table>
Slope and term premium in SV model
Adding preference shocks

- Preference shock breaks the strong tie between $P$ and $Q$
- With the preference shock, SV does not add more flexibility for term premia.
Model solution

- **Step #1**: log-linearize $r_{c,t+1}$ via Campbell & Shiller (1989)
  
  $$ r_{c,t+1} = \kappa_0 + \kappa_1 p c_{t+1} - p c_t + \Delta c_{t+1} $$

- **Step #2**: guess a solution
  
  $$ p c_t = D_0 + D'_g g_t + D'_h h_t $$

- **Step #3**: solve the fixed point problem.
  
  $$ \bar{p} c = D_0 (\bar{p} c) + D_g (\bar{p} c)' \bar{\mu}_g + D_h (\bar{p} c)' \bar{\mu}_h $$

- **Step #4**: plug the solution into the SDF.

- **Problem**: a solution to the fixed point problem does not always exist.
Solution to the fixed point problem

Proposition

There is a value $\bar{\beta}(\psi, \gamma, \theta^r)$ such that if $\beta < \bar{\beta}$, then there exists a solution for the fixed point problem.

Sketch of proof

Define $\tilde{\beta}(\bar{\beta}) = D_0(\bar{\beta}) + D_g(\bar{\beta})' \bar{\mu}_g + D_h(\bar{\beta})' \bar{\mu}_h$

The fixed point problem has a solution if $\bar{\beta} - \tilde{\beta}(\bar{\beta}) = 0$
Solution to the fixed point problem: Gaussian

Corollary

If there is no preference shock, then

1. If $Z_1^\infty \mu_g \leq 0$, then $\vartheta = \frac{1-\gamma}{1-\eta} < 0$ guarantees the existence of a solution.

2. If $\beta = 1$, there is a value $\bar{\gamma}(\theta^r)$ such that $\frac{\bar{\gamma}-\gamma}{1-\eta} < 0$ guarantees a solution.

3. For any $\psi$, $\bar{\beta}$ is monotonic in $\gamma$: for $\psi > 1$, then $\frac{d\bar{\beta}}{d\gamma} > 0$; for $\psi < 1$, then $\frac{d\bar{\beta}}{d\gamma} < 0$. 
Model #1: long-run risk, no SV, no preference shock

Gaussian model

SV model
Conclusion

- Build a new consumption-based asset pricing model
  - capture realistic dynamics for risk premia
  - preference shocks $\rightarrow$ flexible time-varying prices of risk
  - stochastic volatility $\rightarrow$ time-varying quantity of risk

- Empirical results:
  - Flexible time-varying price of risk is key
    - term premia fluctuate with expected inflation
  - Term premia from recursive preferences with stochastic volatility are implausible

- Provide conditions guaranteeing a solution for these models.
Literature:

**Consumption-based models**

- recursive preferences: *Piazzesi Schneider (07), Le Singleton (10)*
- recursive preferences + SV: *Bansal Yaron (04), Bansal Gallant Tauchen (07), Bansal Shaliastovich (13)*
- habit formation: *Wachter (06)*
- recursive preferences + SV in ICAPM: *Campbell Giglio et al. (14)*
- recursive preferences + preference shocks: *Albuquerque Eichenbaum Rebelo (14) Schorfheide Song Yaron (14)*

**DSGE models**

- habit formation: *Rudebusch Swanson (08)*
- recursive preferences: *Rudebusch Swanson (08), van Binsbergen et al. (12), Dew-Becker (14)*
- solution methods: *Caldara et al. (12)*
Literature:

**Term premium**
- **ATSM**: Duffee (02), Ang Piazzesi (03), Wright (11), Bauer Rudebusch Wu (12)

**Model solution**
- Hansen Scheinkman (12), Campbell Giglio et al. (14)
Gaussian bond loadings

\[ a_n = -\frac{1}{n} \bar{a}_n, \quad b_{n,g} = -\frac{1}{n} \bar{b}_{n,g} \]

\[ \bar{a}^g = \bar{a}^g_{n-1} + \bar{a}^g_1 + (\mu_g - \eta g \lambda_0 + \Sigma_{0,g} \Sigma'_{0,g} [Z_2 - Z_\pi])' \bar{b}^g_{n-1,g} \]
\[ + \frac{1}{2} \bar{b}^g_{n-1,g} \Sigma_{0,g} \Sigma'_{0,g} \bar{b}^g_{n-1,g} \]

\[ \bar{b}^g_{n,g} = (\Phi_g - \eta g \lambda_g)' \bar{b}^g_{n-1,g} + \bar{b}^g_{1,g} \]

where the initial conditions are

\[ \bar{a}^g_1 = \ln (\beta) + \bar{\Lambda} + (Z_\upsilon - \eta Z_c - Z_\pi)' (\mu_g - \eta g \lambda_0) - \frac{(\upsilon - 1) \upsilon}{2} Z_1' \Sigma_{0,g} \Sigma'_{0,g} Z_1 \]
\[ + \frac{1}{2} Z_2' \Sigma_{0,g} \Sigma'_{0,g} Z_2 + \frac{1}{2} Z_\pi' \Sigma_{0,g} \Sigma'_{0,g} Z_\pi - Z_2' \Sigma_{0,g} \Sigma'_{0,g} Z_\pi \]

\[ \bar{b}^g_{1,g} = (\Phi_g - \eta g \lambda_g)' (Z_\upsilon - \eta Z_c - Z_\pi) \]

where

\[ Z_1 = Z_\upsilon + (1 - \eta) Z_c + \kappa_1 D_g \]
\[ Z_2 = \upsilon Z_\upsilon - \gamma Z_c + (\upsilon - 1) \kappa_1 D_g \]
Particle Gibbs sampler

For $j = 1, \ldots, M$

$$(g_{1:T}, h_{0:T})^{(j)} \sim p \left( g_{1:T}, h_{0:T} \mid Y_{1:T}, \theta^{\text{IP}}, (j-1) \right)$$

$$\theta^{\text{IP},(j)} \sim p \left( \theta^{\text{IP}} \mid Y_{1:T}, g_{1:T}^{(j)}, h_{0:T}^{(j)} \right)$$

- Draw the state variables using the particle filter, see Andrieu, Doucet, Holenstein (10).

- Use independence Metropolis-Hastings to draw the parameters $\theta^{\text{IP}}$. 
Particle Gibbs sampler

For $t = 1, \ldots, T$, run:

- For $j = 2, \ldots, J$, draw from a proposal: $(h_t)^{(j)} \sim q \left( h_t | h_{t-1}^{(j)}, y_t, \theta \right)$.
- For $j = 1, \ldots, J$, calculate the importance weight:

$$w_t^{(j)} \propto p \left( y_t | h_t^{(j)}, \theta \right) \frac{p \left( h_t^{(j)} | h_t^{(j)}, \theta \right)}{q \left( h_t^{(j)} | h_{t-1}^{(j)}, y_t, \theta \right)}$$

- For $j = 1, \ldots, J$, normalize the weights: $\hat{w}_t^{(j)} = \frac{w_t^{(j)}}{\sum_{j=1}^{J} w_t^{(j)}}$.

- Conditionally resample the particles $\left\{ h_t^{(j)} \right\}_{j=1}^{J}$ with probabilities $\left\{ \hat{w}_t^{(j)} \right\}_{j=1}^{J}$. In this step, the first particle $h_t^{(1)}$ always gets resampled and may be randomly duplicated.

Key point: the particle approximation: $\left\{ \hat{w}_1:T, h_0:T \right\}_{1}^{J} \approx p \left( h_0:T | y_1:T, \theta \right)$.