Staying at Zero with Affine Processes
An Application to Term Structure Modelling

Alain Monfort\textsuperscript{1,2} Fulvio Pegoraro\textsuperscript{1,2}
Jean-Paul Renne\textsuperscript{2} Guillaume Roussellet\textsuperscript{1,2,3}

\textsuperscript{1}CREST
\textsuperscript{2}Banque de France
\textsuperscript{3}Dauphine University

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Zero lower bound (ZLB)

Several of the major central banks now face the ZLB

![Policy Rates (in %)](chart)

- U.S. Fed
- Bank of Japan
- Bank of England
- ECB
Stylized facts to match

- The short-term nominal rate can stay at the ZLB for several periods...

- and in the meantime, longer-term yields can show substantial fluctuations [JGB yields from June 1995 to May 2014]
Closed-form pricing

- Gaussian ATM
- CIR
  - QTSM

Positivity

- Shadow rate

Can stay at 0
Closed-form pricing

- Gaussian ATSM
- CIR
- QTSM
- Shadow rate

Positivity

This Paper

Can stay at 0
Our ZLB model: a primer

We introduce a new **affine** process:

![Simulation of an ARGo process](image1)

![Cumulative distribution function](image2)

- $P(R=0) = 0.6$
What we do in this paper

- **We derive** Non-negative Affine processes staying at 0 (ARG$_0$ processes) to build a Term Structure Model which is:
  - providing positive yields for all maturities;
  - consistent with the ZLB (a short-rate experiencing prolonged periods at 0) **WHILE** long-term rates still fluctuates;
  - affine: thus closed-form formulas for bond-pricing and lift-off probabilities are available.

- Empirical assessment on JGB yields (June 1995 to May 2014). **Good performance** of our model in terms of:
  - fitting yield levels and conditional variances;
  - calculating Risk-Neutral *and* Historical lift-off probabilities.
Related literature


- **Lift-off probabilities:** Bauer & Rudebusch (2013), Swanson & Williams (2013)
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Defining the **Gamma-Zero distribution**

We construct a new distribution in two steps:

- $Z \sim \mathcal{P}(\lambda) \implies Z(\omega) \in \{0, 1, 2, \ldots\}$ and $\mathbb{P}(Z = 0) = \exp(-\lambda)$.

- We define $X|Z \sim \gamma_Z(\mu)$, which implies:
  1. If $Z = 0$, $X$ is a Dirac point mass at 0.
  2. If $Z > 0$, $X$ is Gamma-distributed (continuous on $\mathbb{R}^+$).

**Definition**

The non-negative r.v. $X \sim \gamma_0(\lambda, \mu), \lambda > 0$ and $\mu > 0$, if

$$X \mid Z \sim \gamma_Z(\mu) \quad \text{with} \quad Z \sim \mathcal{P}(\lambda)$$

$$\implies \mathbb{P}(X = 0) = \mathbb{P}(Z = 0) = \exp(-\lambda).$$
A mixture of affine distributions

A mixture distribution

In other words, $X \sim \gamma_0(\lambda, \mu)$ if its (complicated) p.d.f. is:

$$f_X(x; \lambda, \mu) = \sum_{z=1}^{+\infty} \left[ \frac{\exp(-x/\mu) x^{z-1}}{(z-1)! \mu^z} \times \frac{\exp(-\lambda) \lambda^z}{z!} \right] 1_{\{x>0\}} + \exp(-\lambda) 1_{\{x=0\}}$$

However, simple Laplace transform:

$$\varphi_X(u; \lambda, \mu) := \mathbb{E} [\exp(uX)] = \exp \left[ \lambda \frac{u \mu}{1 - u \mu} \right] \text{ for } u < \frac{1}{\mu}.$$
A mixture of affine distributions

A mixture distribution

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$\Rightarrow$ Exponential-affine in $\lambda$. 


A mixture of affine distributions

A mixture distribution

In other words, \( X \sim \gamma_0(\lambda, \mu) \) if its (complicated) p.d.f. is:

\[
f_X(x ; \lambda, \mu) = \sum_{z=1}^{+\infty} \left[ \frac{\exp(-x/\mu) x^{z-1}}{(z-1)! \mu^z} \times \frac{\exp(-\lambda) \lambda^z}{z!} \right] \mathbb{1}_{\{x > 0\}} + \exp(-\lambda) \mathbb{1}_{\{x = 0\}}
\]

However, simple Laplace transform:

\[
\phi_X(u ; \lambda, \mu) := \mathbb{E} [\exp(uX)] = \exp \left[ \lambda \frac{u\mu}{(1 - u\mu)} \right] \quad \text{for} \quad u < \frac{1}{\mu}.
\]

\( \implies \) Exponential-affine in \( \lambda \).
Introducing dynamics: the ARG\(_0\) process

**Main goal:** Build a dynamic **affine** process with **zero point mass**.

**Definition**

\((X_t)\) is a ARG\(_0(\alpha, \beta, \mu)\) if \((X_{t+1}|X_t)\) is Gamma-zero distributed:

\[(X_{t+1}|X_t) \sim \gamma_0(\alpha + \beta X_t, \mu) \quad \text{for} \quad \alpha \geq 0, \; \mu > 0, \; \beta > 0.\]

Again, simple conditional LT, exponential-affine in \(X_t\):

\[
\varphi_{X,t}(u; \alpha, \beta, \mu) := E_t [\exp(uX_{t+1})] \\
= \exp \left[ \frac{u\mu}{1 - u\mu} (\alpha + \beta X_t) \right], \quad \text{for} \quad u < \frac{1}{\mu}.
\]
Key properties:

- **Non-negative** process.
- **Affine** process: the conditional Laplace transform is exp-affine.
  \[
  \varphi_{X,t}(u; \alpha, \beta, \mu) := \mathbb{E}_t [\exp(uX_{t+1})] = \exp [a(u)X_t + b(u)]
  \]
- **Staying at zero** with probability:
  \[
  \mathbb{P}(X_{t+1} = 0 | X_t = 0) = \exp(-\alpha) \neq 0.
  \]
  \[\square\] \(\alpha \neq 0 \implies\) zero is not absorbing.
  \[\square\] in our multivariate yield curve model this probability will be time-varying, function of all date-\(t\) factors;
- **Closed-form moments** (affine conditional cumulants).
We extend the ARG\(_0(\alpha, \beta, \mu)\) process to the more general ARG\(_\nu(\alpha, \beta, \mu)\) case:

\[
X_t \text{ follows an ARG}_{\nu}(\alpha, \beta, \mu) \text{ process if:} \\
X_{t+1} \mid Z_{t+1} \sim \gamma_{\nu + Z_{t+1}}(\mu) \text{ with } Z_{t+1} \mid X_t \sim \mathcal{P} (\alpha + \beta X_t)
\]

- \(\nu = 0 \implies \text{ARG}_0 \text{ process.}\)
- \(\nu > 0, \alpha = 0 \implies \text{ARG process of Gouriéroux and Jasiak (2006).}\)

<table>
<thead>
<tr>
<th>(\nu = 0)</th>
<th>(\nu &gt; 0)</th>
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<tbody>
<tr>
<td>Positivity</td>
<td>Yes</td>
</tr>
<tr>
<td>Affine</td>
<td>Yes</td>
</tr>
<tr>
<td>Zero point mass</td>
<td>Yes</td>
</tr>
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Stylized facts to match (1)

- short-term nominal rate at the ZLB for several periods
- longer-term yields showing substantial fluctuations [JGB yields from June 1995 to May 2014]
The state of the economy is defined by a \( n \)-dimensional vector \( X_t \). These factors follow a \( \text{VARG}_\nu \) process under \( Q \) (the same under \( P \)).

\( \text{VARG}_\nu \) processes

\( X_t \) follows a \( \text{VARG}_\nu(\alpha, \beta, \mu) \) if, \( \forall t, \forall i: \)

- \( Z_{i,t+1}|X_t \sim \mathcal{P}(\alpha_i + \beta_i'X_t) \).
- \( X_{i,t+1}|Z_{i,t+1} \sim \gamma Z_{i,t+1} + \nu_i(\mu_i) \) cond. indep across \( i \).

Conditional \( Q \)-moments (same formulas under \( P \)):

\[
\mathbb{E}_t^Q(X_{t+1}) = \mu^Q \odot (\alpha^Q + \beta^Q' X_t + \nu) \\
\mathbb{V}_t^Q(X_{t+1}) = \text{diag} \left[ \mu^Q \odot \mu^Q \odot \left( \nu + 2\alpha^Q + 2\beta^Q' X_t \right) \right]
\]

\textit{Note:} Conditional correlations can be allowed.
Short-rate specification and the affine framework

Short-rate specification

- The vector of factors $X_t$ is split into two: $X_t = (X_t^{(1)'}, X_t^{(2)'})'$

where:

(i) All components of $X_t^{(1)}$ have $\nu_j = 0$ (point mass at 0).

(ii) All components of $X_t^{(2)}$ have $\nu_j > 0$ (no point mass).

(iii) $\mu_j^P = 1$, $\beta_P^P$ and $\beta_Q^Q$ lower-triangular (identification).

- The short-term rate $r_t$ is given by:

$$r_t = \delta' X_t^{(1)} \quad (= r_{min} + \delta' X_t^{(1)}, \text{ if } LB \neq 0)$$

1. **Key Property**

   $\{\text{Eq.(1) + (i)}\} \Rightarrow r_t$ has a zero point mass.
Other Properties:

\{\text{Eq.}(1) + (iii)\}:

\[
\begin{align*}
    r_t &= \delta' X_t^{(1)} \\
    \begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix} &= \text{constant} + \begin{pmatrix} \beta_{11}^Q & \beta_{12}^Q \\ 0 & \beta_{22}^Q \end{pmatrix} \begin{pmatrix} X_{t-1}^{(1)} \\ X_{t-1}^{(2)} \end{pmatrix} + \xi_t^Q
\end{align*}
\]

- We have $X^{(2)} \overset{G.C.}{\rightarrow} X^{(1)}$
- and thus $X_t^{(2)}$ appears in the short rate conditional $\mathbb{Q}$-expectations (hence in long rates).

$\implies$ long-term yields can move during the ZLB.
Pricing Formulas

The model belongs to the class of ATSM:

- Explicit closed-form bond-pricing
- Yields are affine in the factors for all maturities:

\[
R_t(h) = -\frac{1}{h} (A_h'X_t + B_h) = \bar{A}_hX_t + \bar{B}_h.
\]

- Recursive pricing formulas:

\[
A_h = -\delta + \beta^Q \left( \frac{A_{h-1} \circ \mu^Q}{1 - A_{h-1} \circ \mu^Q} \right)
\]

\[
B_h = B_{h-1} + \alpha^{Q'} \left( \frac{A_{h-1} \circ \mu^Q}{1 - A_{h-1} \circ \mu^Q} \right) - \nu' \log \left( 1 - A_{h-1} \circ \mu^Q \right)
\]
The historical dynamics

- The SDF is exp-affine with market price of risk vector $\theta$:

$$\frac{dP_{t,t+1}}{dQ_{t,t+1}} = \exp \left[ \theta' X_{t+1} - \psi^Q_t(\theta) \right]$$

Change of measure property

$X_t$ follows a $\text{VARG}_\nu(\alpha^P, \beta^P, \mu^P)$ process under the historical measure $P$.

$$\alpha^P_j = \frac{\alpha^Q_j}{1 - \theta_j \mu^Q_j}, \quad \beta^P_j = \frac{1}{1 - \theta_j \mu^Q_j} \beta^Q_j, \quad \mu^P_j = \frac{\mu^Q_j}{1 - \theta_j \mu^Q_j}.$$

Rk: $\nu$ is the same under both measures.
Stylized facts to match (2)

Conditional volatilities: time-varying and maturity-dependent.
How to treat it

- Conditional variance of yields:

\[ \mathbb{V}_t^P [R_{t+1}(h)] \]

\[ = \bar{A}_h' \mathbb{V}_t^P (X_{t+1}) \bar{A}_h \]

\[ = \bar{A}_h' \left\{ \text{diag} \left[ \mu^P \odot \mu^P \odot \left( \nu + 2\alpha^P + 2\beta^P' X_t \right) \right] \right\} \bar{A}_h \]

- Time-varying and maturity-dependent.
Advantages of an affine framework

**NATSM properties**

- Yields $R_t(h)$ are non-negative;
- Long-term yields can move while $r_t = 0$ for several periods;
- Unconditional first two moments are available in closed-form;
- Conditional first two moments of yields are affine in $X_t$ (available in closed-form);
- Yields forecasts are explicitly affine in $X_t$;
State-space formulation

Estimation technique

State vector \( Y_t = (R'_t, V'_t, S'_t)' \) affine in \( X_t \):

- \( R_t \) = yield levels (6 maturities);
- \( V_t \) = 2- and 10-y yield conditional (EGARCH) variance;
- \( S_t \) = SPF for 3-m and 1-y ahead 10-y yield;
- prelim. estimations have suggested \( \text{dim}(X_t^{(1)}) = 1 \), \( \text{dim}(X_t^{(2)}) = 3 \) and \( \nu = 0 \);

Estimation technique

Linear Kalman-filter-based QML:

\[
\begin{align*}
X_{t+1} & = m + MX_t + \Sigma_t^{1/2} \varepsilon_{t+1} \\
Y_t & = \Gamma_0 + \Gamma_1 X_t + \Omega \eta_t 
\end{align*}
\]
Filtered factors

![Graph showing filtered factors from 1995 to 2015](image-url)
Factor loadings of yields and conditional variances

(a) Factor loadings of yields

(b) Factor loadings of conditional variances
Fit of Conditional Variances and SPFs

- 2-year yield
- 10-year yield
- 3-month ahead 10-year yield
- 12-month ahead 10-year yield

Conditional variance proxy

Forecast Surveys (in %, annual basis)
Fit of Yields

Estimation results

Dates

Yield in %, annual basis


6-month yield

2-year yield

7-year yield

10-year yield

Dates

1-year yield

4-year yield

10-year yield

observed

fitted

-0.00
-0.50
-1.00
-1.50
-2.00
0.00
0.50
1.00
1.50
2.00

0.00
0.25
0.50
0.75
1.00

0.00
0.5
1.0
1.5
2.0

0.0
0.5
1.0
1.5
2.0
3.0

0.0
0.5
1.0
1.5
2.0
3.0

Yields (in %, annual basis)
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Lift-off probability dates under $\mathbb{P}$ and $\mathbb{Q}$

We calculate the following probabilities:

- $\mathbb{P}(r_{t+k} = 0 \mid X_t)$ and $\mathbb{Q}(r_{t+k} = 0 \mid X_t)$;
- $\mathbb{P}(r_{t+k} < 25 \text{ bps.} \mid X_t)$ and $\mathbb{Q}(r_{t+k} < 25 \text{ bps.} \mid X_t)$.

Useful formula

If $z \in \mathbb{R}^+$ and $\varphi_z(u)$ its Laplace transform.

$$
\mathbb{P}(z = 0) = \lim_{u \to -\infty} \varphi_z(u).
$$

Next two plots ($\mathbb{Q}$ is the black solid line):

- *Time-series dimension*: $t$ varies ($k = 2\text{yrs and 5yrs}$).
- *Horizon dimension*: $k$ varies ($t = 11/30/07$ and $05/30/14$).
The results are presented in terms of probabilities for lift-off dates 2−years and 5−years ahead. Two different cases are considered:

1. **lambda = 0**: This scenario represents the baseline case with no external shocks.
2. **lambda = 25 bps**: This case incorporates a moderate level of external shocks.

The graphs display the evolution of these probabilities over time, with dates marked from 1995 to 2015. The y-axis represents probabilities ranging from 0 to 1, while the x-axis shows the years from 1995 to 2015 in increments of 5 years.

The graphs illustrate how the probabilities of lift-off dates change over time under both scenarios, highlighting the impact of external shocks on the assessment of lift-off dates.
Horizon dimension of probabilities

![Graph showing probabilities over forecast horizon for two different lambda values: lambda = 0 and lambda = 25 bps. The bars represent probabilities at different forecast horizons, with lines indicating the progression over time. The x-axis represents the forecast horizon in years, ranging from 1 to 5. The y-axis represents the probabilities ranging from 0.00 to 1.00. Two sets of probabilities are displayed: Q probability and P probability. The graph includes data points for dates 2007-11-30 and 2014-05-30.]
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Summary and further research

We have derived **affine non-negative processes staying at 0** and built an affine term-structure model (**NATSM**) gathering:

- a **short-rate consistent with the ZLB** experiencing periods at 0 while **long-run rates still fluctuates**;
- **closed-form formulas** for bond-pricing and lift-off probabilities.

An empirical assessment showed performance of our model for:

- **fitting yield levels and conditional variances**;
- calculating risk-neutral *and* historical **lift-off probabilities**.

**Further research:** Empirical comparison of NATSMs, derivatives pricing.
Thank you for your attention.
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Table: Parameter estimates

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<th>$P$-parameters</th>
<th>$Q$-parameters</th>
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<td>$\beta_{1,1}$</td>
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<tr>
<td>$\beta_{2,2}$</td>
<td>0.9978</td>
<td>0.0005</td>
</tr>
<tr>
<td>$\beta_{3,3}$</td>
<td>0.9486</td>
<td>0.0044</td>
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<tr>
<td>$\beta_{4,4}$</td>
<td>0.9967</td>
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<tr>
<td>$\beta_{2,1}$</td>
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<td>0.0041</td>
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<td>0.1094</td>
<td>0.0059</td>
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<td>$\beta_{4,3}$</td>
<td>$3.88 \cdot 10^{-4}$</td>
<td>$2.28 \cdot 10^{-5}$</td>
</tr>
<tr>
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</tr>
<tr>
<td>$\mu_2$</td>
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<td>0.0005</td>
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<td>$\mu_3$</td>
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<td>$\mu_4$</td>
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Other Parameters

<table>
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<th>Value</th>
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<tr>
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<td>$\theta_3$</td>
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<td>$\sigma_R$</td>
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<tr>
<td>$\sigma_V$</td>
<td>$3 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>$\sigma_S$</td>
<td>0.15</td>
</tr>
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</table>
ARG₀ Summary

\[ X_t \text{ realized} \]

\[ \alpha + \beta X_t \quad \rightarrow \quad Z_{t+1} | X_t \sim P(\alpha + \beta X_t) \]

\[ X_{t+1} | Z_{t+1} \sim \gamma Z_{t+1}(\mu) \]

\[ \text{time } t \quad \rightarrow \quad \text{time } t + 1 \]
Univariate case: lift-offs formulas

- $Z \in \mathbb{R}^+$ and $\varphi_Z(u)$ its Laplace transform.

$$
\mathbb{P}_Z\{0\} = \lim_{u \to -\infty} \varphi_Z(u).
$$

- Lift-off probabilities: $(X_t) \sim \text{ARG}_0(\alpha, \beta, \mu)$ and $\varphi_{t,h}(u_1, \ldots, u_h)$ its multi-horizon conditional Laplace transform.

  - $\mathbb{P}(X_{t+h} = 0 \mid X_t) = \lim_{u \to -\infty} \varphi_{t,h}(0, \ldots, 0, u)$
  - $\mathbb{P}[X_{t+1} = 0, \ldots, X_{t+h} = 0 \mid X_t] = \lim_{u \to -\infty} \varphi_{t,h}(u, \ldots, u) = \exp(-\alpha h - \beta X_t)$,
  - $\mathbb{P}[X_{t+1} = 0, \ldots, X_{t+h} = 0, X_{t+h+1} > 0 \mid X_t] = \exp[-\alpha h - \beta X_t] [1 - \exp(-\alpha)]$, $h > 1$. 
Multivariate Case

- \( Z \in \mathbb{R}_+^n \) and \( \varphi_Z(u_1, \ldots, u_n) \) its Laplace transform.

\[
\mathbb{P}_Z\{0, \ldots, 0\} = \lim_{u \to -\infty} \varphi_Z(u, \ldots, u).
\]

- **Notations:** Multi-horizon conditional LT.

\[
\varphi_{t,k}^{\mathbb{P}}(u_1, \ldots, u_k) = \mathbb{E}^{\mathbb{P}} \left[ \exp \left( u_1' X_{t+1} + \ldots + u_k' X_{t+k} \right) \bigg| X_t \right] \\
= \exp \left[ A_k' X_t + B_k \right] \\
\varphi_{R,t,k}^{(h)}(v_1, \ldots, v_k) = \mathbb{E} \left[ \exp \left( v_1 R_{t+1}(h) + \ldots + v_k R_{t+k}(h) \right) \bigg| X_t \right]
\]
Lift-offs

\( \mathbb{P} [ r_{t+k} = 0 \mid X_t ] = \lim_{u \to -\infty} \varphi_{R,t,k}^{(1)}(0, \ldots, 0, u) \)

\( \mathbb{P} [ r_{t+1} = 0, \ldots, r_{t+k} = 0 \mid X_t ] = \lim_{u \to -\infty} \varphi_{R,t,k}^{(1)}(u, \ldots, u) = p_{r,t,k} \) (say)

\( \mathbb{P} [ r_{t+1} = 0, \ldots, r_{t+k-1} = 0, r_{t+k} > 0 \mid X_t ] = p_{r,t,k-1} - p_{r,t,k} \)

\( \mathbb{P} \left[ v' R_{t+1}^{(t+k)} (h) > \lambda \mid X_t \right] = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{+\infty} \frac{\text{Im} \left[ \varphi_{R,t,k}^{(h)} (i v x) \exp(-i \lambda x) \right]}{x} \, dx \)
Useful remarks

**Remark 1**

Stationarity conditions are easily imposed:

\[ X_t \text{ stationary } \iff \forall j \in \{1, \ldots, n\}, \quad \rho_j := \mu_j \beta_{j,j} < 1. \]

**Remark 2**

The assumption of conditional independence can be relaxed keeping the affine structure of the multivariate process \( X_t \).

\[ \implies \text{Recursive discrete-time affine process (mimeo).} \]