Tractable Term Structure Models–A New Approach

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Motivation

1. Interest rates are close to or have reached their lower bound across several markets globally.

2. Bounded positive interest rates imply large tractability or flexibility costs within the existing DTSM framework.

3. These costs are especially acute when exploring the volatility of yields over the cycle. As the level and slope of the yield curve evolves,
   - How does the volatility of bond yields evolve throughout the cycle?
   - How does the (hump-shaped) term structure of yield volatility evolve throughout the cycle?
   - How does volatility of the expectation and risk premium components evolve throughout the cycle? (Cieslak and Povala, 2015)

4. Contribution: we introduce Tractable Term Structure Models (TTSMs) to answer these questions.
Examples

Models with positive yields are restrictive:

1. Positive affine DTSM models
   - Restrictions on the correlation structure (only positive).
   - Restrictions to accommodate macro variables that changes signs.
   - Restrictions on the risk premium (Dai and Singleton, 2002; Joslin and Le, 2013).

2. Quadratic DTSM models or Black’s DTMS
   - Tractable?
   - Limited to simple Gaussian state dynamics.
Motivation

- DTSMs are based on the fundamental theorems of asset pricing to ensure the Absence of Arbitrage.
- The focus is on the subset of “realistic” SDFs $M_t > 0$ such that:
  
  
  
  \[
  P_{1,t} = E_t[M_{t+1}] \text{ is closed form,}
  \]
  
  \[
  P_{2,t} = E_t[M_{t+1}M_{t+2}] \text{ is closed form,}
  \]
  
  \[
  \ldots
  \]
  
  \[
  P_{n,t} = E_t[M_{t+1}M_{t+2}\ldots M_{t+n}] \text{ is closed form}
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Question: Can we bypass specifying the SDF to retain tractability and flexibility yet producing bond prices that are “close” to AOA?
1. Our construction of bond prices

Assumption (1)

- The n-period bond price $P_n$ is given recursively by

\[
P_0(X_t) \equiv 1, \quad \forall X_t
\]
\[
P_n(X_t) = P_{n-1}(g(X_t)) \times \exp(-m(X_t)),
\]

- given some state $X_t$ with support $X$,
- and some functions $m(\cdot), g(\cdot)$ where $g(X_t) \in X$ for every $X_t \in X$.

Assumption 1 guarantees pricing tractability.
1. Our construction of bond prices

- Example n=1:

\[ P_1(X_t) = P_0(g(X_t)) \times \exp(-m(X_t)) = \exp(-m(X_t)) \]  

- \( m(\cdot) \) gives the one-period rate
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- Example n=1:

\[ P_1(X_t) = P_0(g(X_t)) \times \exp(-m(X_t)) = \exp(-m(X_t)) \]  (3)

  \( m(\cdot) \) gives the one-period rate

- Example n=2:

\[ P_2(X_t) = P_1(g(X_t)) \times \exp(-m(X_t)) \\
= \exp(-m(g(X_t))) \times \exp(-m(X_t)) \]  (4)

  \( g(\cdot) \) lets us price \( P_n(\cdot) \) given \( P_{n-1}(\cdot) \).
1. Properties of bond prices

Assumption (2)

P1 — Positivity \( P_n(X_t) \leq 1 \ \forall X \in X \) or equivalently \( y_{n,t} \geq 0 \);
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P3 — Invertibility \( \exists u(\cdot) : \mathbb{R} \to \mathbb{R} \) such that \( u^{-1}(f_{n,t}) = a_n + b_nX_t \) \( \forall n \).
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The following choices of functions \( m(\cdot) \), \( g(\cdot) \) guarantee Properties P1-P3:

1. \( m(\cdot) \) is continuous and monotonic with \( m(X) \geq 0 \ \forall X \in X \),
2. \( g(X) \) is a contraction with unique fixed-point \( g(X^*) = X^* \),
3. \( g(X) = kX \).
1. Time series dynamics

Assumption (3)

The time series dynamics of $X_t$ admits $X$ as support and is such that yields for all maturities $y_{n,t} \equiv -\log(P_n(X_t))/n$ have a joint distribution that is stationary and ergodic.

- Virtually any time series dynamics is acceptable in our framework and will not affect any of our earlier results.
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- Virtually any time series dynamics is acceptable in our framework and will not affect any of our earlier results.
- This means that our framework is flexible enough to accommodate:
  - GARCH-like or stochastic volatility
  - DCC-like or stochastic correlation
  - Unspanned macro variables
  - Long or infinite lag structure
  - Shifting endpoints and unit roots.
  - ...

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Tractable Term Structure Modeling: A New Approach

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2. How close are we to AOA?

**Theorem 1: Nelson-Siegel Yield Curve**

Bond prices generated using

\[
m(X_t) = \begin{bmatrix} 1 & \frac{1-e^{-\lambda}}{\lambda} & \frac{1-e^{-\lambda}}{\lambda} - e^{-\lambda} \end{bmatrix} X_t,
\]

\[
g(X_t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-\lambda} & \lambda e^{-\lambda} \\ 0 & 0 & e^{-\lambda} \end{bmatrix} X_t,
\]

have yields-to-maturity with Nelson-Siegel (1987) loadings.
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\]

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1. Implementations of the Nelson-Siegel model are not strictly free of arbitrage (Bjork and Christensen; Filipovic) and the same applies here.

2. Nevertheless, the empirical literature has long concluded that not much distinguishes NS from a fully-fledged DTSM implementation. (Diebold and Li; Christensen, Diebold and Rudebusch).

3. We also clarify how close TTSM are to strict AOA.
2. How close are we to AOA?

**Theorem 2: No Dominant Trading Strategy**
Our bond price construction allows no dominant trading strategies.
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**Figure:** Prices of portfolios with strictly positive payoffs.
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**Theorem 3: Self-Financing Arbitrage**

Portfolios with non-negative payoffs cannot have negative price.
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![Diagram of No Arbitrage Strategies: prices of portfolios with non-negative payoffs.](image)

**Figure:** No Arbitrage Strategies: prices of portfolios with non-negative payoffs.
2. How close are we to AOA?

**Theorem 4: Transaction Costs**
Our bond price construction allows no arbitrage opportunities in presence of transaction costs (however small)
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Our bond price construction allows no arbitrage opportunities in presence of transaction costs (however small)

- How reasonable/important for us to think about transaction costs?
- Strictly speaking, we only need to invoke the transaction costs for self-financing portfolios. These must involve costly short positions.
  - See evidence in e.g., Duffie (1996); Krishnamurthy (2002); Vayanos and Weill (2008); and Banerjee and Graveline (2012)
3. Specification—*f* and *g* functions

- Choose \( g(X) = KX \) and \( m(X) = u(\theta, X) \) such that
  1. limit (i): Black’s \( \max(0, \delta_0 + \delta_1' X_t) \) with \( \theta_1 \to 0 \),
  2. limit (ii): linear \( \delta_0 + \delta_1' X \) with \( \theta_1 \to \infty \).

- Guarantees positivity in spirit of \( \max(0, \delta_0 + \delta_1' X_t) \) but remains invertible:
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![Graph of max function and different shapes of the short-rate function $u(\theta, s)$](image-url)

**Figure:** The max function and different shapes of the short-rate function $u(\theta, s)$
3. Specification—\( f \) and \( g \) functions

- Analytical yields/ forwards:
  \[ f_{n,t} = u(\theta, \delta_0 + \delta'_1 K^n X_t) \]  
  (7)

- We can work with transformed forwards \( \tilde{f}_{n,t} \),
  \[ \tilde{f}_{n,t} \equiv u^{-1}(\theta, f_{n,t}) = \delta_0 + \delta'_1 K^n X_t \]  
  (8)

- We are back to the linear space: restate the model in terms of portfolios \( P_t = W \tilde{f}_{n,t} \) and proceed with preferred estimation method.
3. Specification

1. Joint VAR dynamics for yield portfolios $\mathcal{P}_t$ and unspanned macro variables $U_t$:

$$
\begin{pmatrix}
\mathcal{P}_{t+1} \\
U_{t+1}
\end{pmatrix} = K_0^P + K_1^P \begin{pmatrix}
\mathcal{P}_t \\
U_t
\end{pmatrix} + \sqrt{\Sigma_t} \begin{pmatrix}
\varepsilon_{\mathcal{P}, t+1} \\
\varepsilon_{U, t+1}
\end{pmatrix},
$$

(9)

2. The innovations $\varepsilon_t \equiv (\varepsilon_{\mathcal{P}, t+1}, \varepsilon_{U, t+1})'$ $\sim N(0, \Sigma_t)$

3. $\Sigma_t$ combines EGARCH(1,1) and DCC dynamics.

4. Yields: GSW forward rates from GSW; 1990 and 2015; quarterly maturities between 3 months and 10 years.

5. Macro: Survey forecasts of inflation and gdp 1-year ahead (Blue Chips Financials).
4. Results—Model nomenclature

1. $A_0(3)$ Gaussian DTSM $\rightarrow A$
2. Affine TTSM $\rightarrow AT$
3. Affine TTSM with Volatility dynamics $\rightarrow ATV$
4. Positive TTSM $\rightarrow PT$
5. Positive TTSM with Volatility dynamics $\rightarrow PTV$

Here: focus on cyclical volatility variations

In the paper: also check that pricing errors, forecasts, liftoff time, risk premium and Sharpe ratios are identical between models
4. Results—Sharpe Ratios

Figure: 2-year bond annual Sharpe ratio; 1990-2008

Essentially no differences between model-implied Sharpe ratios.
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Figure: 2-year bond annual Sharpe ratio; 2008-2015

Essentially no differences between model-implied Sharpe ratios.
4. Results—Conditional volatility

**Figure:** 1-Year Yield Conditional Volatility 1990-2008

Volatility peaks in recession, adding to risk.
4. Results—Conditional volatility

Need volatility compression to capture short-term volatility near the lower bound.
4. Results—Conditional volatility

Figure: 10-year Yield Conditional Volatility (2008-2015)

Still need time-varying factor volatility to match volatility of long-term yields near the lower bound
4. Results—Conditional volatility

Figure: Volatility hump: Difference between 12-month and 1-month ahead volatility.

Volatility term structure downward-sloping in recession.
4. Results—The changing role of level and slope

Figure: Principal components $R^2$s from yields’ conditional correlation matrix.
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Figure: Principal components $R^2$'s from yields’ conditional correlation matrix.

slope factor

![Graph showing time series of correlation factors with lines for ATV, PT, and PTV over time from 1990 to 2015.](image-url)
4. Results—The changing role of level and slope

Figure: Principal components $R^2$s from yields’ conditional correlation matrix.

 curvature factor

Correlation

\begin{center}
\begin{tabular}{c c c c c c c c c c c}
ATV & PT & PTV \\
\end{tabular}
\end{center}

Time

Correlation
Conclusion

1. Propose Tractable Term Structure Models (TTSMs).
   ▶ We specify bond prices directly without imposing a parametric SDF.
   ▶ Like Nelson-Siegle curves, bond prices are nearly but not strictly AOA.
   ▶ Imposition of lower bound is straightforward without giving away flexibility, tractability and ease of implementation.

2. Empirically:
   ▶ DTSM and TTSM risk premium and Sharpe ratios are essentially the same away from the lower bound.
   ▶ TTSM can match volatility dynamics both near and away from the lower bound.
   ▶ The relative importance of level risk and slope risk changes plays a key role.