Bond Positions, Expectations, And The Yield Curve[*]

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Abstract

This paper implements a structural model of the yield curve with data on nominal positions and survey forecasts. Bond prices are characterized in terms of investors' current portfolio holdings as well as their subjective beliefs about future bond payoffs. Risk premia measured by an econometrician vary because of changes in investors' subjective risk premia, identified from portfolios and subjective beliefs, but also because subjective beliefs differ from those of the econometrician. The main result is that investors' systematic forecast errors are an important source of business-cycle variation in measured risk premia. By contrast, subjective risk premia move less and more slowly over time.

I   Introduction

Asset pricing theory says that the price of an asset is equal to the expected present value of its payoff, less a risk premium. Structural models relate the risk premium to the covariance between the asset payoff and investors' marginal utility. Quantitative evaluation of asset pricing models thus requires measuring investors' expectations of both asset payoffs and marginal utility. The standard approach is to estimate a time series model for these variables, and then invoke rational expectations to argue that investors' beliefs conform with the time series model. The standard approach has led to a number of asset pricing puzzles. For bonds, the puzzle is that long bond prices are much more volatile than expected long bond payoffs, and that returns on long bonds appear predictable.

This paper considers a model of bond returns without assuming rational expectations. Instead, we use survey forecasts and data on investor asset positions to infer investors' *subjective* distribution of asset payoffs and marginal utility. The basic asset pricing equation–price equals expected payoff plus risk adjustment–holds in our model, even though investors do not have rational expectations. Consider the price $P_t^{(n)}$ of a zero-coupon bond of maturity n. In particular, there is a pricing kernel M^*, such that

$$P_t^{(n)} = \frac{1}{R_t} E_t^* P_{t+1}^{(n-1)} + cov_t^* \left(M_{t+1}^*, P_{t+1}^{(n-1)} \right),$$

where R_t is the riskless rate. The expected payoff from a zero coupon bond next period is the expected price next period, when the bond will have maturity $n-1$. The risk premium depends on how that price covaries with M^*. The key difference to the standard approach is that both moments are computed under a subjective probability distribution, indicated by an asterisk. This subjective distribution is estimated using survey data, and can be different from the *objective* distribution implied by our own time series model.

Asset pricing puzzles arise because observed bond prices are more volatile than present values of expected payoffs computed *under an objective distribution used by the researcher*, $E_t P_{t+1}^{(n-1)}$ say. To see how our model speaks to the puzzles, rewrite the equilibrium price as

(1) $$P_t^{(n)} = \frac{1}{R_t} E_t P_{t+1}^{(n-1)} + cov_t^* \left(M_{t+1}^*, P_{t+1}^{(n-1)} \right) + \frac{1}{R_t} \left(E_t^* P_{t+1}^{(n-1)} - E_t P_{t+1}^{(n-1)} \right).$$

The equilibrium price can deviate from objective expected payoffs not only because of (subjective) risk premia, but also because subjective and objective expectations differ. Under the rational expectations approach, the second effect is assumed away. Structural models then fail because they cannot generate enough time variation in risk premia to account for *all* price volatility in excess of volatility in (objectively) expected payoffs.

This paper explores asset pricing under subjective expectations in three steps. First, we document properties of subjective expectations of interest rates using survey data over the last four decades. Here we show that the last term in (1) is not zero, and moves systematically over the business cycle. Second, we estimate a reduced form model that describes jointly the distribution of interest rates and inflation and investors' subjective beliefs about these variables. Since we impose the absence of arbitrage opportunities in bond markets, we can recover a subjective pricing kernel M^* implied by survey forecasts. This allows us to quantify the contribution of forecast errors to the excess volatility and predictability of bond prices. Third, we derive subjective risk premia in a representative agent model, where M^* reflects the investor's marginal rate of substitution.

The first step combines evidence from the Blue Chip survey, available since 1982, as well as its precursor, the Goldsmith-Nagan survey, available since 1970. We establish two stylized facts. First, there are systematic differences in subjective and objective interest-rate expectations, and hence bond prices and excess returns on bonds. We compare expected excess returns on bonds implied by predictability regressions that are common in the literature to expected excess returns on bonds perceived by the median survey investor. We find survey expected excess returns to be both smaller on average and less countercyclical than conventional measures of expected excess returns. In particular, conventionally measured expected returns appear much higher than survey expected excess returns during and after recessions.

The second stylized fact is that there are systematic biases in survey forecasts. In particular, with hindsight, survey forecasters do a bad job forecasting mean reversion in yield spreads after recessions This finding says that the first stylized fact is not driven by our choice of a particular, and thus perhaps inadequate, objective model. To the contrary, the better the forecasting performance of the time series model used to construct objective forecasts, the larger will be the difference between those forecasts and survey forecasts.

In the second part of the paper, we estimate a reduced form model using interest rate data for many different maturities as well as forecast data for different maturities and forecast horizons. To write down a parsimonious model that nevertheless uses all the information we have, we employ techniques familiar from the affine term structure literature. There, actual interest rates are represented as conditional expectations under a "risk neutral probability", which are in turn affine functions of a small number of state variables. The Radon-Nikodym derivative of the risk neutral probability with respect to the objective probability that governs the evolution of the state variables then captures risk premia. Our exercise also represents subjective forecasts as conditional expectations under a subjective probability, again written as an affine function of the state variables. The Radon-Nikodym derivative of this subjective probability with respect to the objective probability then captures forecast biases, and can be identified from survey data.

The main result from the estimated reduced form model is that survey forecasters perceive both the level and the slope of the yield curve to be more persistent than they are under an objective statistical model. For example, in the early 1980s when the level of the yield curve was high, survey forecasters expected all interest rates to remain high whereas statistical models predicted faster reversion to the mean. In addition, at the end of recessions when the slope of the yield curve was high, survey forecasters believed spreads on long bonds to remain high, whereas statistical models predicted faster mean reversion in yield spreads.

By equation (1), both biases have important implications for the difference between objective and subjective risk premia. First, both biases imply that survey forecasters predict lower bond prices, and hence lower excess returns than statistical models when either the level or the slope of the yield curve is high. However, times of high level or high slope are precisely the times when objective premia are high. This explains why we find subjective premia that are significantly less volatile than objective premia: the volatility of 1-year holding premia is reduced by 40%-60%, depending on the maturity of the bond.

A second implication is that subjective and objective premia have different qualitative properties. For long bonds (such as a 10 year bond), yields move relatively more with the slope of the yield curve as opposed with the level. As a result, the bias in spread forecasts is more important. This goes along with the fact that objective premia on long bonds also move more with the slope of

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the yield curve. Subjective premia of long bond are thus not only less volatile, but also much less cyclical. The lesson we draw for the structural modelling of long bonds is that the puzzle is less that premia are cyclical, but rather that they were much higher in the early 1980s than at other times. For medium term bonds (such as a two year bond), which relatively more with the level of the yield curve, the bias in level forecasts is more important. It implies that subjective premia are somewhat lower than objective premia in the 1980s, but both premia share an important cyclical component.

The third part of the paper proposes a new way to empirically evaluate a structural asset pricing model, using data on not only survey expectations, but also investor asset positions. The theoretical model is standard: we consider a group of investors who share the same Epstein-Zin preferences. We assume that these investors hold the same subjective beliefs – provided by our reduced form model – about future asset payoffs. To evaluate the model quantitatively, we work out investors' savings and portfolio choice problem given the beliefs to derive asset demand, for every period in our sample. We then find prices that make asset demand equal to investors' observed asset holdings in the data. We thus arrive at a sequence of model-implied bond prices of the same length as the sample. The model is "successful" if the sequence of model-implied prices is close to actual prices.

Since there is a large variety of nominal instruments, an investor's "bond position" is in principle a high-dimensional object. To address this issue, we use the subjective term-structure model to replicate positions in many common nominal instruments by portfolios that consist of only three zero coupon bonds. Three bonds work because a two-factor model does a good job describing quarterly movements in the nominal term structure. The replicating portfolios shed light on properties of bonds outstanding in the US credit market. One interesting fact is that the relative supply of longer bonds declined before 1980, as interest rate spreads were falling, but saw a dramatic increase in the 1980s, a time when spreads were extraordinarily high.

We illustrate our asset pricing approach by presenting an exercise where investors are assumed to be "rentiers", that is, they hold only bonds. Rentiers' bond portfolios are taken to be proportional to those of the aggregate US household sector, and we choose preference parameters to best match the mean yield curve. This leads us to consider relatively patient investors with low risk aversion. Our model then allows a decomposition of "objective" risk premia as measured under the objective

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statistical model of yields into their three sources of time variation. We find that subjective risk premia are small and vary only at low frequencies. This is because both measured bond positions, and the hedging demand for long bonds under investors' subjective belief move slowly over time. In contrast, the difference in subjective and objective forecasts is a source of large time variation in risk premia at business cycle frequencies.

We build on a small literature which has shown that measuring subjective beliefs via surveys can help understand asset pricing puzzles. Froot (1989) argued that evidence against the expectations hypothesis of the term structure might be due to the failure of the (auxiliary) rational expectations assumption imposed in the tests rather than to failures of the expectations hypothesis itself. He used the Goldsmith-Nagan survey to measure interest rate forecasts and found that the failure of the expectations hypothesis for long bonds can be attributed to expectational errors. The findings from our reduced form model confirm Froot's results while including the BlueChip data set that allows for a longer sample as well as more forecast horizons and maturities. Moreover, our estimation jointly uses all data and recovers and characterizes the entire subjective kernel M^*.[1] Several authors have explored the role of expectational errors in foreign exchange markets. Frankel and Froot (1989) show that much of the forward discount can be attributed to expectational errors. Gourinchas and Tornell (2004) use survey data to show that deviations from rational expectations can rationalize the forward premium and delayed overshooting puzzles. Bacchetta, Mertens and van Wincoop (2008) study expectational errors across a large number of asset markets.

The rest of the paper is structured as follows. Section II introduces the modelling framework. Section III documents properties of survey forecasts. Section IV describes estimation results for the reduced form model. Section V explains how we replicate nominal position by simple portfolios. Section VI reports results from the structural model.

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[1] Kim and Orphanides (2007) estimate a reduced-form term structure model using data on both interest rates and interest rate forecasts. They show that incorporating survey forecasts into the estimation sharpens the estimates of risk premia in small samples. In our language, they obtain more precise estimates of "objective premia"; they are not interested in the properties of subjective risk premia for structural modelling.

II Setup

Investors have access to two types of assets. *Bonds* are nominal instruments that promise dollar-denominated payoffs in the future. In particular, there is a one period bond – from now on, the short bond – that pays off one dollar at date $t+1$; it trades at date t at a price e^{-i_t}. Its real payoff is $e^{-\pi_{t+1}}$, where π_t is (log) inflation.[2] In addition to short bonds, there are long zero-coupon bonds for all maturities; a bond of maturity n trades at log price $p_t^{(n)}$ at date t and pays one dollar at date $t+n$. The log excess return over the short bond from date t to date $t+1$ on an n-period bond is defined as $x_{t+1}^{(n)} = p_{t+1}^{(n-1)} - p_t^{(n)} - i_t$. In some of our exercises, we also allow investors to trade a *residual asset*, which stands in for all assets other than bonds. The log real return from date t to date $t+1$ is r_{t+1}^{res}, so that its excess return over the short bond is $x_{t+1}^{res} = r_{t+1}^{res} - i_t - \pi_{t+1}$.

A. Reduced form model

We describe uncertainty about future returns with a state space system. The basic idea is to start from an objective probability P, provided by a system with returns and other variables that fits the data well from our (the modeler's) perspective. A second step then uses survey expectations to estimate the state space system under the investor's subjective probability, denoted P^*.

The state space system is for an S-vector of observables h_t which contains all variables that are needed to describe the statistical properties of nominal returns and inflation (so that investors can compute real returns.) Under the objective probability P, the state space system is

$$
\begin{aligned}
(2) \qquad\qquad\qquad\qquad h_t &= \mu_h + \eta_h s_{t-1} + e_t \\
s_t &= \phi_s s_{t-1} + \sigma_s e_t,
\end{aligned}
$$

where s_t and e_t are S-vectors of state variables and i.i.d. zero-mean normal shocks with $E e_t e_t^\top = \Omega$, respectively. The first component of h_t is always the short interest rate $i_t^{(1)}$ and the first state

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[2] This is a simple way to capture that the short (1 period) bond is denominated in dollars. To see why, consider a nominal bond which costs $P_t^{(1)}$ dollars today and pays of \$1 tomorrow, or $1/p_{t+1}^c$ units of numeraire consumption. Now consider a portfolio of p_t^c nominal bonds. The price of the portfolio is $P_t^{(1)}$ units of consumption and its payoff is $p_t^c/p_{t+1}^c = 1/\pi_{t+1}$ units of consumption tomorrow. The model thus determines the price $P_t^{(1)}$ of a nominal bond in \$.
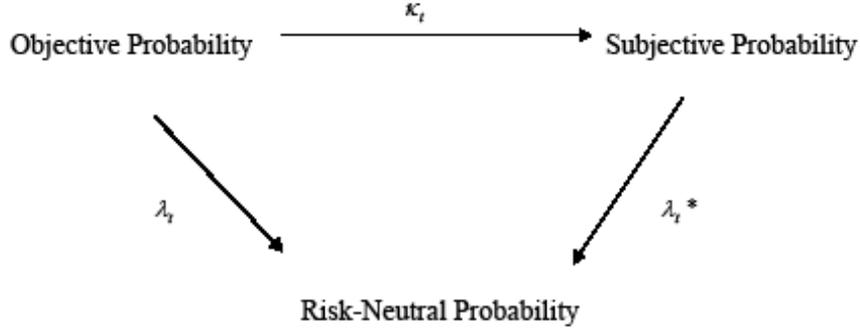
Figure 1: Relationship between the different probability measures

variable is the demeaned short interest rate, that is, $s_{t,1} = i_t^{(1)} - \mu_1$.

From objective to subjective probability

We assume that investors' beliefs are also described by the state space system, but with different coefficients. To define investors' subjective beliefs, we represent the Radon-Nikodym derivative of investors' subjective belief P^* with respect to the objective probability P by a stochastic process ξ_t^*, with $\xi_1^* = 1$ and

$$(3) \qquad \frac{\xi_{t+1}^*}{\xi_t^*} = \exp\left(-\frac{1}{2}\kappa_t^\top \Omega \kappa_t - \kappa_t^\top e_{t+1}\right).$$

Since e_t is i.i.d. mean-zero normal with variance Ω under the objective probability P, ξ_t^* is a martingale under P. Since e_t is the error in forecasting h_t, the process κ_t can be interpreted as investors' bias in their forecast of h_t. The forecast bias is affine in state variables, that is

$$\kappa_t = k_0 + k_1 s_t.$$

Standard calculations now deliver that $e_t^* = e_t + \Omega \kappa_t$ is i.i.d. mean-zero normal with variance matrix Ω under the investors' belief P^*, so that the dynamics of h_t under P^* can be represented

by

$$(4) \qquad \begin{aligned} h_t &= \mu_h - \Omega k_0 + (\eta_h - \Omega k_1)s_{t-1} + e_t^* := \mu_h^* + \eta_h^* s_{t-1} + e_t^* \\ s_t &= -\sigma_s \Omega k_0 + (\phi_s - \sigma_s \Omega k_1)s_{t-1} + \sigma_s e_t^* := \mu_s^* + \phi_s^* s_{t-1} + \sigma_s e_t^* \end{aligned}$$

The vector k_0 thus affects investors' subjective mean of h_t and also the state variables s_t, whereas the matrix k_1 determines how their forecasts of h deviate from the objective forecasts as a function of the state s_t.

Other bond prices

Our description of subjective beliefs needs to deal with the fact that we allow investors to invest in bonds of all maturities. As a consequence, the number of bond returns (and state variables) in the state space system (4) can become very large. By assuming the absence of arbitrage opportunities in bond markets, we can obtain a more parsimonious system. Another advantage of this approach is that a small number of bonds will be enough to span bond markets, which makes the portfolio choice problem of investors manageable.

An implication of the no-arbitrage assumption is that there exists a "risk neutral" probability Q under which bond prices are discounted present values of bond payoffs. In particular, the prices $P^{(n)}$ of zero-coupon bonds with maturity n satisfy the recursion

$$P_t^{(n)} = e^{-i_t} E_t^Q \left[P_{t+1}^{(n-1)} \right]$$

with terminal condition $P_t^{(0)} = 1$.

As illustrated in Figure 1, we specify the Radon-Nikodym derivative ξ_t^Q of the risk neutral probability Q with respect to the objective probability P by $\xi_1^Q = 1$ and

$$\frac{\xi_{t+1}^Q}{\xi_t^Q} = \exp\left(-\frac{1}{2} \lambda_t^\top \Omega \lambda_t - \lambda_t^\top e_{t+1} \right),$$

where λ_t is an S-vector which contains the "market prices of risk" associated with innovations e_{t+1}.

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We assume that risk premia are linear in the state vector, that is,

$$\lambda_t = l_0 + l_1 s_t,$$

for some $S \times 1$ vector l_0 and some $S \times S$ matrix l_1. With this specification, $e_t^Q = e_t + \Omega \lambda_t$ is an i.i.d. zero-mean normal innovation under Q. The state vector dynamics under Q are

$$s_t = -\sigma_s \Omega l_0 + (\phi_s - \sigma_s \Omega l_1) s_{t-1} + \sigma_s e_t^Q := \mu_s^Q + \phi_s^Q s_{t-1} + \sigma_s e_t^Q.$$

We select a subset of the state variables as term-structure factors $f_t = \eta_f s_t$ using the selection matrix $\eta = [I_{F \times F} \ 0_{F \times S - F}]$. The advantage of this approach is that we can describe the time-t price for a bond with any maturity n as an exponential-linear function of only F factors. Standard calculations deliver that the price is

$$(5) \qquad\qquad P_t^{(n)} = \exp\left(A_n + B_n^\top f_t\right)$$

where A_n is a scalar and B_n is an $F \times 1$ vector of coefficients that depend on maturity n.

Given these formulas for bond prices, interest rates $i_t^{(n)} = -\ln P_t^{(n)}/n$ are also linear functions of the factors with the coefficients $a_n = -A_n/n$ and $b_n = -B_n/n$. Moreover, the objective expectation of the excess (log) return on an n-period bond is

$$(6) \qquad\qquad E_t\left[x_{t+1}^{(n)}\right] + \frac{1}{2} Var_t\left[x_{t+1}\right] = B_{n-1}^\top \sigma_f \Omega \lambda_t.$$

This affine function in the state vector has a constant which depends on the parameter l_0 and a slope coefficient which is driven by l_1. The expression shows that the expectations hypothesis holds under the objective probability if $l_1 = 0$.

Investors have beliefs P^* rather than P, and so their subjective market prices of risk are not equal to λ_t. Instead, their subjective market price of risk process is $\lambda_t^* = \lambda_t - \kappa_t$, so that the bond prices computed earlier are also risk-adjusted present discounted values of bond payoffs under the

subjective belief P^*:

$$P_t^{(n)} = e^{-i_t} E_t^Q \left[P_{t+1}^{(n-1)} \right] = e^{-i_t} E_t^* \left[\exp \left(-\frac{1}{2} \lambda_t^{*\top} \Omega \lambda_t^* - \lambda_t^{*\top} e_{t+1} \right) P_{t+1}^{(n-1)} \right].$$

Investors have subjective excess (log) returns expectations

$$E_t^* \left[x_{t+1}^{(n)} \right] + \frac{1}{2} Var_t^* [x_{t+1}] = B_{n-1}^\top \sigma_f \Omega \lambda_t^*,$$

which depend on the parameters of the subjective market price of risk process. The expectations hypothesis holds under their beliefs provided that $l_1^* = 0$.

B. Structural model

A large number of identical investors live forever. Their preferences over consumption plans are represented by Epstein-Zin utility with unitary intertemporal elasticity of substitution. The utility u_t of a consumption plan $(C_\tau)_{\tau=t}^\infty$ solves

(7)
$$u_t = (1 - \beta) \log C_t + \beta \log E_t^* \left[e^{(1-\gamma)u_{t+1}} \right]^{\frac{1}{1-\gamma}},$$

where E^* denotes the expectation operator based on the subjective probability measure P^*. Investors' ranking of certain consumption streams is thus given by discounted logarithmic utility. At the same time, their attitude towards atemporal lotteries is determined by the risk aversion coefficient γ. We focus below on the case $\gamma > 1$, which implies an aversion to persistent risks (as discussed in Piazzesi and Schneider 2006).

Investors start a trading period t with initial wealth $\overline{W}_t$. They decide how to split this initial wealth into consumption as well as investment in $F + 2$ assets: the residual asset, F long bonds, and the short bond. These $F + 1$ bonds are enough to span bond markets, because bond prices (5) depend on F factors. We collect the log nominal prices of the F long bonds – which we refer to as *spanning bonds* – at date t in a vector $\hat{p}_t$, and we collect their log nominal payoffs[3] at date $t + 1$ in

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[3]This notation is convenient to accomodate the fact that the maturity of zero-coupon bonds changes from one date to the next. For example, assume that there is only one long bond, of maturity n, and let $i_t^{(n)}$ denote its yield

a vector $\hat{p}_{t+1}^{+1}$. The log excess returns over the short bond from date t to date $t+1$ on these bonds can thus be written $\widehat{x}_{t+1} = \hat{p}_{t+1}^{+1} - \hat{p}_t - i_t$.

We denote by ω_t^{res} the portfolio weight on the residual asset (that is, the fraction of savings invested in that asset), and we collect the portfolio weights on the spanning bonds in an F-dimensional vector $\widehat{\omega}_t$, so that $1 - \omega_t^{res} - \iota^\top \widehat{\omega}_t$ is the weight on the short bond, where ι denotes a vector of ones. Therefore, the one-period return on wealth from date t to date $t+1$ is $R_{t+1}^W = \sum_{i=1}^{F+2} \omega_t^i \exp\left(r_{t+1}^i\right)$, where ω_t^i is the date-t portfolio weight on asset i and r_{t+1}^i is its log real return from t to $t+1$. The household's sequence of budget constraints can then be written as[4]

$$
\begin{aligned}
\overline{W}_{\tau+1} &= R_{\tau+1}^w \left(\overline{W}_\tau - C_\tau\right), \\
(8) \qquad R_{\tau+1}^w &= \left(1 + \omega_\tau^{res}\left(\exp\left(x_{\tau+1}^{res}\right) - 1\right) + \widehat{\omega}_\tau^\top\left(\exp\left(\widehat{x}_{\tau+1}\right) - \iota\right)\right)\exp\left(i_\tau - \pi_{\tau+1}\right); \quad \tau \geq t.
\end{aligned}
$$

The household problem at date t is to maximize utility (7) subject to (8), given initial wealth $\overline{W}_t$ as well as subjective beliefs about returns. These beliefs are based on current bond prices $\hat{p}_t$, the current short rate i_t, as well as the conditional distribution of the vector $\left(r_\tau^{res}, i_\tau, \pi_\tau, \hat{p}_\tau, \hat{p}_\tau^{+1}\right)_{\tau > t}$, that is, the return on the residual asset, the short interest rate, the inflation rate and the prices and payoffs on the long spanning bonds. We denote this conditional distribution by P_t^*.

We now relate bond prices to positions and subjective expectations using investors' optimal policy functions. Since preferences are homothetic and all assets are tradable, optimal consumption and investment plans are proportional to initial wealth. The optimal portfolio weights on long bonds and the residual asset thus depend only on subjective beliefs about returns and can be written as $\widehat{\omega}_t\left(i_t, \hat{p}_t, P_t^*\right)$ and $\omega_t^{res}\left(i_t, p_t, P_t^*\right)$, respectively. Moreover, with an intertemporal elasticity of substitution of one, the optimal consumption rule is $C_t = (1 - \beta)\overline{W}_t$. Now suppose we observe investors' bond positions: we write B_t for the total dollar amount invested in bonds at date t, and we collect investors' holdings of the F long bonds in the vector $\widehat{B}_t$.

We perform two types of exercises. Consider first a class of investors who invest only in bonds; there is no residual asset. These "rentier" investors have wealth B_t and so their portfolio weights

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to maturity. The long bond trades at date t at a log price $\hat{p}_t = -ni_t^{(n)}$, and it promises a log payoff at date $t+1$ of $\hat{p}_{t+1}^{+1} = -(n-1)i_{t+1}^{(n-1)}$.

[4]Here $\exp\left(\widehat{x}_t\right)$ is an F-vector with the jth element equal to $\exp\left(\widehat{x}_{t,j}\right)$.

on long bonds are

$$(9) \qquad \widehat{\omega}_t \left(i_t, \hat{p}_t, P_t^* \right) = \frac{\hat{B}_t}{B_t}.$$

These equations can be solved for long bond prices $\hat{p}_t$ as a function of the short rate i_t, bond positions $(B_t, \hat{B}_t)$ and subjective expectations P_t^*. We can thus characterize yield spreads in terms of these variables.

Second, suppose there is a residual asset. Since investors' wealth equals their savings $\overline{W}_t - C_t = \beta \overline{W}_t = \frac{\beta}{1-\beta} C_t$, we must have

$$(10) \qquad \begin{aligned} \widehat{\omega}_t \left(i_t, \hat{p}_t, P_t^* \right) &= \frac{1-\beta}{\beta} \frac{\hat{B}_t}{C_t}, \\ \omega_t^{res} \left(i_t, \hat{p}_t, P_t^* \right) &= 1 - \frac{1-\beta}{\beta} \frac{B_t}{C_t}. \end{aligned}$$

These equations can be solved for long bond prices $\hat{p}_t$ and the short rate i_t, as a function of bond positions $(B_t, \hat{B}_t)$, consumption C_t and expectations P_t^*. This characterizes both short and long yields in terms of positions and subjective expectations.

III Preliminary evidence about subjective beliefs

We measure subjective expectations of interest rates with survey data from two sources. Both sources conduct comparable surveys that ask approximately 40 financial market professionals for their interest-rate expectations at the end of each quarter and record the median survey response. Our first source are the Goldsmith-Nagan surveys that were started in mid-1969 and continued until the end of 1986. These surveys ask participants about their one-quarter ahead and two-quarter ahead expectations of various interest rates, including the 3-month Treasury bill, the 12-month Treasury bill rate, and a mortgage rate. Our second source are Bluechip Financial Forecasts, a survey that was started in 1983 and continues until today. This survey asks participants for a wider range of expectation horizons (from one to six quarters ahead) and about a larger set of interest rates. The most recent surveys always include 3-month, 6-month and 1-year Treasury bills,

the 2-year, 5-year, 10-year and 30-year Treasury bonds, and a mortgage rate.[5]

To measure objective interest-rate expectations, we estimate unrestricted VAR dynamics for a vector of interest rates with quarterly data over the sample 1964:1-2007:4 and compute their implied forecasts. Later, in Section IV, we will impose more structure on the VAR by assuming the absence of arbitrage and using a lower number of variables in the VAR, and thereby check the robustness of the empirical findings we document here. The vector of interest rates Y includes the 1-year, 2-year, 3-year, 4-year, 5-year, 10-year and 20-year zero-coupon yields. We use data on nominal zero-coupon bond yields with longer maturities from the McCulloch file available from the website http://www.econ.ohio-state.edu/jhm/ts/mcckwon/mccull.htm. The sample for these data is 1952:2 - 1990:4. We augment these data with the new Gurkaynak, Sack, and Wright (2006) data. We compute the forecasts by running OLS directly on the system $Y_{t+h} = \mu + \phi Y_t + \varepsilon_{t+h}$, so that we can compute the h-horizon forecast simply as $\mu + \phi Y_t$.

Deviations of subjective expectations from objective expectations of interest rates have consequences for expected excess returns on bonds. We write the (log) excess return on an n-period bond for a h-period holding period as the log-return from t to $t+h$ on the bond in excess of the h-period interest rate, $x_{t+h}^{(n,h)} = p_{t+h}^{(n-h)} - p_t^{(n)} - h i_t^{(h)}$. The objective expectation E of an excess returns can be decomposed as follows:

$$
(11) \qquad \underbrace{E_t\left[x_{t+h}^{(n,h)}\right]}_{} = E_t^*\left[x_{t+h}^{(n,h)}\right] + E_t\left[p_{t+h}^{(n-h)}\right] - E_t^*\left[p_{t+h}^{(n-h)}\right]
$$

$$
= \underbrace{E_t^*\left[x_{t+h}^{(n,h)}\right]}_{} + (n-h)\underbrace{\left(E_t^*\left[i_{t+h}^{(n-h)}\right] - E_t\left[i_{t+h}^{(n-h)}\right]\right)}_{}
$$

$$
\text{objective premium} \quad = \quad \text{subjective premium} \quad + \text{ subj. - obj. interest-rate expectation}
$$

This expression shows that, if subjective expectations E^* of interest rates deviate from their objective expectations E, the objective premium is different from the subjective premium. In particular, if the difference between objective and subjective beliefs changes in systematic ways over time, the objective premium may change over time even if the subjective premium is constant.

We can evaluate equation (11) based on our survey measures of subjective interest-rate ex-

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[5] The survey questions ask for constant-maturity Treasury yield expectations. To construct zero-coupon yield expectations implied by the surveys, we use the following approximation. We compute the expected change in the n-year constant-maturity yield. We then add the expected change to the current n-year zero-coupon yield.

pectations $E_t^*\left[i_{t+h}^{(n-h)}\right]$ and the VAR measures of objective expectations $E_t\left[i_{t+h}^{(n-h)}\right]$ for different maturities n and different horizons h. Figure 2 plots the left-hand side of equation (11), expected excess returns under objective beliefs as a black line, and the second term on the right-hand side of the equation, the difference between subjective and objective interest-rate expectations, as a gray line. For the short post-1983 sample for which we have Bluechip data, we have data for many maturities n and many forecasting horizons h. The lower two panels of Figure 2 use maturities $n = $ 3 years and 11 years and a horizon of $h = 1$ year, so that we deal with expectations of the $n - h$ $= 2$ year and 10 year interest rate. These combinations of n and h are in the Bluechip survey, and the VAR includes these two maturities as well so that the computation of objective expectations is easy.

For the long post-1970 sample, we need to combine data from the Goldsmith-Nagan and Bluechip surveys. The upper left panel shows the $n = 1.5$ year bond and $h = 6$ month holding period. from the estimated VAR (which includes the $n - h = 1$ year yield.) This works, because both surveys include the $n - h = 1$ year interest rate and a $h = 6$-month horizon. The VAR delivers an objective 6-month ahead expectation of the 1-year interest rate. For long bonds, we do not have consistent survey data over this long sample. To get a rough idea of long-rate expectations during the Great Inflation, we take the Goldsmith-Nagan data on expected mortgage-rate changes and the Bluechip data on expected 30-year Treasury-yield over the next $h = 2$ quarters and add them to the current 20-year zero-coupon yield. The VAR produces a $h = 2$ quarter ahead forecast of the 20-year yield.

Figure 2 also shows NBER recessions as shaded areas. The plots indicate that expected excess returns under objective beliefs and the difference between subjective and objective interest-rate expectations have common business-cycle movements. The patterns appear more clearly in the lower panels which use longer (1 year) horizons. This is not surprising in light of the existing predictability literature which documents that expected excess returns on bonds and other assets are countercyclical when we look at longer holding periods, such as one year (e.g., Cochrane and Piazzesi 2005.) In particular, expected excess returns are high right after recession troughs. The lower panels show indeed high values for both series around and after the 1991 and 2001 recessions. The series are also high in 1984 and 1996, which are years of slower growth (as indicated, for
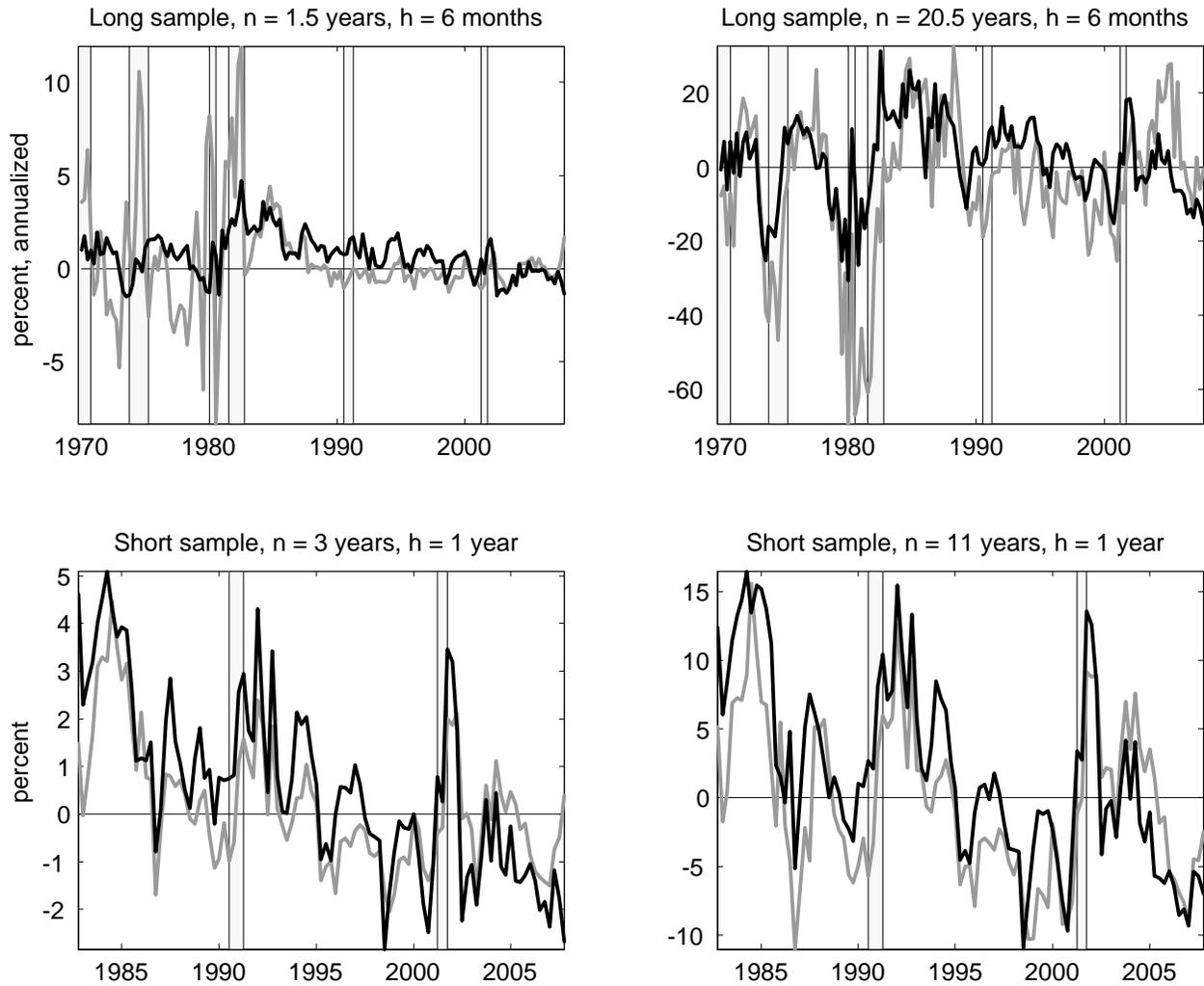
Figure 2: Each panel shows objective expectations of excess returns in black (the left-hand side of equation (11)) and the difference between subjective and objective interest-rate expectations in gray (the second term on the right-hand side of the equation) for the indicated bond maturity n and holding period/forecast horizon h. Shaded areas indicate NBER recessions. The numbers are annualized and in percent. The upper panels show data over a longer sample than the lower panels.

example, by employment numbers) although they were not classified as recessions.

For shorter holding periods, the patterns are also there in the data but they are much weaker. However, the upper panels show additional recessions where similar patterns appear. For example, the two series in both panels are high in the 1970, 1974, 1980 and 1982 recessions or shortly afterwards. (As we can see in the upper panels, expected excess returns for short holding periods are large when annualized. Of course, the risk involved in these investment strategies is high, and

so they are not necessarily attractive.)

Table 1 shows summary statistics of subjective beliefs measured from surveys. During the short Bluechip sample, the average difference between realized interest rates and their one-quarter ahead subjective expectation is negative for short maturities and close to zero, or slightly positive for longer maturities. The average forecast error is -15 basis points for the 3-quarter interest rate and -45 basis points for the 6-quarter interest rates. These two mean errors are the only ones that are statistically significant, considering the sample size of 98 quarters (which means that the ratio of mean to standard deviation needs to be multiplied by roughly 10 to arrive at the relevant t-statistic.) There is stronger evidence of bias at the 1-year horizon, where on average subjective interest-rate expectations are above subsequent realizations for all maturities. During the long combined Goldsmith-Nagan and Bluechip sample, the average 2-quarter ahead forecast errors are -54 and -27 basis points for the 3-month and 1-year yields. The average 1-quarter forecast errors are also negative for these maturities. The column for the 30 year yield in Table 1 includes the average forecast errors for the constructed long bond. The roughly 10 bp errors for this long interest rate needs to be viewed with some caution due to how we constructed the survey series for this bond (as explained earlier.)

The upward bias in subjective expectations may partly explain why we observe positive average excess returns on bonds. The right-hand side of equation (11) shows why: if objective expectations are unbiased, then $E_t^* i_{t+h}^{(n-h)} > E_t i_{t+h}^{(n-h)}$ on average, which raises the value of the left-hand side of the equation. The magnitude of the bias is also economically significant. For example, the -56 basis-point bias in subjective expectations of the 1-year interest rate contributes 56 basis points to the objective premium on the 2-year bond. For higher maturities, we need to multiply the subjective bias by $n-1$ as in equation (11). For example, the -52 basis point bias in 2-year interest rate expectations multiplied by $n - h = 2$ contributes 1.04 percentage points to the objective premium.

TABLE 1: SUBJECTIVE BIASES AND OBJECTIVE BOND PREMIA

subj. bias	horizon h	maturity n								
		3 qtr	6 qtr	1 year	2 year	3 year	5 year	7 year	10 year	30 year
		Short Bluechip sample 1983:1 - 2007:3								
average	1 qrt	-0.17	-0.47	-0.17	-0.13	-0.10	-0.02	0.03	0.11	0.00
	1 year	-0.58	-0.86	-0.58	-0.54	-0.49	-0.39	-0.33	-0.24	-0.30
std	1 qrt	0.58	0.83	0.80	0.77	0.77	0.74	0.71	0.67	0.60
	1 year	1.40	1.62	1.65	1.54	1.46	1.46	1.27	1.21	1.11
		Long Combined Goldsmith-Nagan and Bluechip Sample 1970:1-2007:3								
average	1 qrt	-0.11		-0.02						0.10
	2 qrt	-0.55		-0.29						0.10
stdev	1 qrt	1.31		1.84						0.66
	2 qrt	1.77		2.27						0.88

Note: The table reports summary statistics of subjective expectational errors computed as $i_{t+h}^{n-h} - E_t^* i_{t+h}^{(n-h)}$ for the indicated horizon h and maturity n. The numbers are annualized and in percent.

When we match up these numbers, it is important to keep in mind that subjective biases and objective premia are measured imprecisely, because they are computed with small data samples. In particular, over most of the Bluechip sample, interest rates were declining.

A potential concern with Bluechip forecast data is that the survey is not anonymous, and so career concerns of survey respondents may matter. To address this concern, we also measure subjective interest-rate expectations using the Survey of Professional Forecasters. Starting in 1992, the SPF reports median interest-rate forecasts for the 10-year Treasury bond over various forecast horizons. We find that median forecasts from the SPF are similar to those from the Bluechip survey. Importantly, the differences between SPF forecasts and objective expectations show the same patterns as those documented in Figure 2.

To sum up, the evidence presented in this section suggests that subjective interest-rate expectations deviate from the objective expectations that we commonly measure from statistical models. Table 1 suggests that these deviations may account for average objective premia. Figure 2 suggest that these deviations may also be responsible for the time-variation in objective bond premia.

IV Modeling subjective beliefs

The previous section has documented some properties of survey forecasts of interest rates. In order to implement our asset pricing model, we need investors' subjective conditional distributions over future asset returns. Subsection A. describes our approach to construct such distributions. In subsection B., we report estimation results for a specific model of beliefs.

A. Estimation

Our baseline state space system is a version of (2) with two term structure factors – the short rate and one yield spread – as well as expected inflation as a third state variable. In terms of the general notation from in section II, we have $S = 3$ and $F = 2$. The observables vector h_t contains the one quarter rate, the spread between the five year and one quarter rate, and CPI inflation. While expected inflation is not itself a term structure factor, it can affect expected excess returns to the extent that it helps forecast the factors. We choose this baseline because (i) two term structure factors are known to fit well the dynamics of yields at the quarterly frequency and (ii) we would like inflation in the system in order to describe real returns. We describe the baseline results in detail in this section.

We also consider the robustness of our conclusions to our choice of system. The appendix estimates four alternative systems. On the one hand, we want to know how the presence of inflation affects the results. Model 2 is a "yields only" two factor model ($S = F = 2$), with only the one quarter rate and the 5-year-1-quarter-spread. Model 3 is a "yields only" 3 factor model which has the 5-year-4-year spread as an additional state variable and factor. Model 4 is Model 3 with inflation: it has $S = 4 > F = 3$. Finally, Model 5 is a three factor model, where inflation serves as a factor along with the 1 quarter rate and the 5-year-1-quarter spread ($S = F = 3$).

Estimation

The sample for estimation is 1964:1:2007:3. We follow the literature in not using yields before 1964 for data quality reasons (Fama and Bliss 1987). The data on zero-coupon interest rates and survey forecasts are the same as in section III. Moreover, we measure inflation with quarterly

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data on the GDP deflator from the NIPA tables. We also use measures of subjective inflation expectations from the Survey of Professional Forecasters. This survey is conducted at a quarterly frequency during the years 1968:4-2007:3.

Step 1: State space system

The estimation proceeds in three steps. First, we estimate the state space system (2) under the objective probability P by maximum likelihood. We need to restrict the parameter matrices to ensure identification. We would also like to impose that those term structure factors that are based on yields are contained in both the vector of observables h_t and in the state vector s_t. Let $s_t = (s_t^y, s_t^o)$ and $h_t = (h_t^y, h_t^o)$, where $s_t^y = h_t^y$ contains the term structure factors based on yields. We then write the system (2) as

$$
\begin{pmatrix} h_t^y \\ h_t^o \end{pmatrix} = \begin{pmatrix} \mu^y \\ \mu^o \end{pmatrix} + \begin{pmatrix} \phi_y & \\ 0 & I_{(S-Y)} \end{pmatrix} \begin{pmatrix} s_{t-1}^y \\ s_{t-1}^o \end{pmatrix} + e_t,
$$

$$
\begin{pmatrix} s_t^y \\ s_t^o \end{pmatrix} = \begin{pmatrix} \phi_y \\ \phi_o \end{pmatrix} s_{t-1} + \begin{pmatrix} I_Y & 0 \\ & \sigma_o \end{pmatrix} e_t,
$$

We have imposed two restrictions here. First, for the variables based on yields, the first Y observation equations must be copies of the first Y state equations. Second, for all other observables h_t^o, the state variables s_t^o are the expected values of the h_t^o. For example, in our baseline model, $s_t^y = h_t^y$ contains the short rate and spread, while h_t^o is inflation and s_t^o is expected inflation. The system is more general than simply a VAR in the three observables, because it allows additional MA style dynamics in inflation. Altogether, there are $\frac{3}{2}S + \frac{5}{2}S^2 - Y(S-Y)$ parameters. The likelihood is formed recursively, starting the system at $s_0 = 0$. The estimation then also recovers a sequence of estimates $\hat{s}_t$ of the state vector, including the unobservable components. For example, in the baseline case we recover an expected inflation series.

Step 2: Objective risk premia

The second step is to estimate the parameters l_0 and l_1 that describe the objective risk premia λ_t. Here we take as given the dynamics of the state variables s_t under the probability P delivered

by step 1. A distribution of s_t under P plus a set of parameters l_0 and l_1 give rise to a distribution of s_t under the risk neutral probability Q. From the bond price equation (5), we then obtain interest rate coefficients (a^n, b^n) implied by the absence of arbitrage. We can thus form a sample of predicted zero coupon yields

$$
\text{(12)} \qquad\qquad i_t^{(n)} = a_n + b_n^\top \hat{f}_t,
$$

where $\hat{f}_t = \eta_f \hat{s}_t$ is a sample of factor realizations derived from the backed out state realizations $\hat{s}_t$ from step 1.

We estimate l_0 and l_1 by minimizing, for a set of maturities, the sum of squared fitting errors, that is, differences between actual yields and predicted yields computed from (12). We use yields of maturities 1, 10, 15, 20 and 30 years. We also impose the constraint that that the short rate and the spreads that serve as factors are matched exactly. This constraint amounts to imposing $a_1 = 0$ and $b_1 = e_1$, and $a_{20} = 0$ and $b_{20} = e_2$ on the yield coefficients from equation (12), where e_i is an F-dimensional vector of zeros with 1 as ith entry. We also have to restrict the parameter matrix l_1 to ensure that the term structure factors $f_t = \eta_f s_t$ are Markov under the risk neutral probability Q. The distribution of the term structure factors under Q can be represented by

$$
\text{(13)} \qquad\qquad f_t = -\eta_f \sigma_s \Omega l_0 + \eta_f \left(\phi_s - \sigma_s \Omega l_1 \right) s_{t-1} + \eta_f \sigma_s e_t^Q
$$

A matrix l_1 is admissible only if there exists an $F \times F$ matrix ϕ_f^Q such that $\eta_f \left(\phi_s - \sigma_s \Omega l_1 \right) = [\phi_f^Q \quad 0]$. If this is true, then we can indeed represent f_t as a VAR(1) under Q: we have

$$
f_t = \mu_f^Q + \phi_f^Q f_{t-1} + \sigma_f e_t^Q,
$$

with $\sigma_f = \eta_f \sigma_s$ and $\mu_f^Q = -\sigma_f \Omega l_0$. The condition is not restrictive if $F = S$. More generally, it says that the top right hand $F \times (S - F)$ block of the matrix $\sigma_f \Omega l_1$ must equal the top right hand $F \times (S - F)$ block of ϕ_s.

With the term structure factors Markov under Q, the bond price coefficients can be computed

from the difference equations

$$A_{n+1} = A_n + B_n^\top \mu_f^Q + \frac{1}{2} B_n^\top \sigma_f \Omega \sigma_f^\top B_n - \mu_1$$
$$B_{n+1}^\top = B_n^\top \phi_f^Q - e_1^\top$$

where initial conditions are given by $A_0 = 0$ and $B_0 = 0_{F\times 1}$. The coefficients for the short (one-period) bond are thus $A_1 = -\mu_1$ and $B_1 = -e_1$.

The coefficient formulas show that the estimation cannot identify all the parameters in l_0 and l_1 if $F < S$. Indeed, the parameters appear always as part of the terms $\sigma_f \Omega l_0$ and $\sigma_f \Omega l_1$, which have dimension $F \times 1$ and $F \times (S - F)$, respectively. Intuitively, since bond prices depend only on shocks to the term structure factors, only risk premia that compensate for term structure factor shocks can be identified from bond price data. In other words, while the vector λ_t represents market prices of risk for the innovations e_t, bond prices will reflect only factor market prices of risk $\sigma_f \Omega \lambda_t$. The latter are sufficient to describe expected excess returns on all bonds, as in equation (6) above.

Step 3: Subjective state space system

The third step is to estimate the parameters k_0 and k_1 that govern the Radon-Nikodym derivative (3) and thus the change of measure from objective beliefs to subjective beliefs described in equations (4). Here we take as given the dynamics of the state variables s_t under the probability P delivered by step 1 and the interest rate coefficients (a^n, b^n) delivered by step 2. A distribution of s_t under P plus a set of parameters k_0 and k_1 give rise to a distribution of s_t under the subjective probability P^*. We can thus form a sample of subjective interest rate forecasts

$$(14) \qquad E_t^* \left[i_{t+h}^{(n)} \right] = a_n + b_n^\top E_t^* \left[f_{t+h} \right] = a_n + b_n^\top \eta_f \left[\left(I - \phi_s^{*h} \right) \left(I - \phi_s^* \right)^{-1} \mu_s^* + \phi_s^{*h} \, \hat{s}_t \right]$$

where $\hat{s}_t$ is the sample of backed out state variables from step 1. Similarly, we can form a sample of subjective inflation forecasts

$$(15) \qquad E_t^* \left[\pi_{t+h} \right] = \eta_\pi \left[\left(I - \phi_s^{*h} \right) \left(I - \phi_s^* \right)^{-1} \mu_s^* + \phi_s^{*h} \, \hat{s}_t \right],$$

where η_π is the row of η_h that corresponds to the inflation rate.

We estimate k_0 and k_1 by minimizing a sum of squared fitting errors, that is, differences between median survey forecasts and subjective model forecasts computed as in (14)-(15). The maturities and horizons for the interest rate forecasts differ by sample period. For 1970:1-1982:2, we use Goldsmith-Nagan data and consider a horizon of 2 quarters and maturities 1 year and 20 years. For 1982:3-2007:3, we Bluechip data and consider horizons of 2 and 4 quarters, and maturities of 1 quarter, as well as 1, 2, 3, 5, 7, 10 and 30 years. For inflation forecasts, we use a horizon of 4 quarters over the sample 1968:2-2007:3.

B.   Results

The appendix reports our estimates for objective and subjective beliefs. To understand the estimated objective dynamics, we report covariance functions which completely characterize the Gaussian state space system. Figure X plots covariance functions computed from the objective state space system and the raw data. At 0 quarters, these represent variances and contemporaneous covariances. The black lines from the system match the gray lines in the data quite well. To interpret the units, consider the upper left panel. The quarterly variance of the short rate is 0.54 in model and data which amounts to $\sqrt{0.54 \times 4} = 1.47$ percent annualized volatility. Figure x shows that all three state variables are positively autocorrelated. For example, the covariance $\mathrm{cov}\left(i_t^{(1)}, i_{t-1}^{(1)}\right) = \rho \, \mathrm{var}\left(i_t^{(1)}\right) = \rho \times 0.54 = 0.53$ which implies that the first-order autocorrelation is 0.98.

The objective dynamics of the state variables are persistent. The largest eigenvalues of the matrix ϕ_s are complex with a modulus of 0.95, while the third eigenvalue is 0.71. In Figure X, the autocovariance functions of the short rate and inflation are flatter than that of the spread, which indicates that they are more persistent. The short rate and the spread are contemporaneously negatively correlated and the spread is negatively correlated with the short rate lagged less than year, and positively correlated with longer lags of the short rate. The short rate is negatively correlated with the spread lagged less than three years, with weak correlation for longer lags.

To understand the implications of the estimated parameters l_0 and l_1, we report the properties of excess returns expected under the objective probability. Table 2 reports the loadings of these

conditional expected values on the state variables. For a 1-quarter holding period, these loadings are $B_{n-1}^\top \sigma_f \Omega l_1$. For longer holding periods, we can compute $E_t\left[x_{t+h}^{(n,h)}\right] + \frac{1}{2} Var_t\left[x_{t+h}^{(n,h)}\right]$ using the recursions for the coefficients A_n and B_n. The results in Table 2 indicate that the expected excess return on a 2-year bond is high in periods with high short rate, high spreads or high expected inflation. For example, a 1-percent increase in the spread leads to a 2.22 percent increase in the objective premium. This dependence on the spread captures that objective premia are countercyclical. For each 1-percent increase in the short rate, the premium increases by 1.53 percent. This dependence on the short rate induces some low-frequency movements in expected excess returns. The premium on the 10-year bond has larger loadings on all state variables. Both the spread coefficient and the short rate coefficient for the 10 year bond are roughly 10 times higher than for the 2-year bond.

TABLE 2: ESTIMATION OF OBJECTIVE MODEL

Panel A: Loadings of expected excess returns on state variables

		horizon $h = 1$ year		
	n	short rate	spread	expected inflation
maturity of the bond	2 year	1.53	2.22	0.09
	10 year	8.84	22.4	0.15

Panel B: Fitting errors for bond yields (annualized)

				maturity			
	1 qrt	1 year	5 year	10 year	15 year	20 year	30 year
mean absolute errors (in %)	0	0.30	0	0.24	0.36	0.42	0.45

NOTE: Panel A reports the model-implied loadings of the function $E_t\left[x_{t+h}^{(n,h)}\right] = A_{n-h} + B_{n-h}^\top E_t\left[f_{t+h}\right] - A_n - B_n^\top f_t + A_h + B_h^\top f_t$ on the current factors f_t for a holding period of $h = 1$ year and bond maturities of $n = 2$ years, 10 years. Panel B reports mean absolute model fitting errors for yields.

Panel B of Table 2 reports by how much the model-implied yields differ from observed yields on average. By construction, the model hits the 1-quarter and 5-year interest rates exactly, because these rates are included as factors. For intermediate maturities, the error lies within the $24 - 45$ basis points range. We will see below that these errors are sufficiently small for our purposes.

Subjective vs. objective dynamics

Subjective forecasts are on average very close to the forecasts from our statistical model: the unconditional means of the factors under P^* are very small. At the same time, the factors are more persistent under P^*. The largest eigenvalues of the two matrices

$$\phi_s = \begin{pmatrix} 0.88 & 0.12 & 0.13 \\ 0.05 & 0.76 & -0.06 \\ -0.04 & -0.08 & 0.97 \end{pmatrix}, \quad \phi_s^* = \begin{pmatrix} 0.99 & 0.10 & 0.05 \\ -0.04 & 0.92 & -0.03 \\ -0.10 & -0.08 & 0.86 \end{pmatrix}.$$

are 0.95 and 0.97, respectively. Other things equal, a one-percent increase in the short rate (spread) increases the forecast of the short rate (spread) next period by 99% (92%), as opposed to 88% (76%) under the objective model.

The estimated market prices of risk λ_t^* imply that subjective risk premia are less cyclical than objective premia. The subjective loadings on the spread in Table 3 are smaller than those in Table 2. For long bonds, the loading on expected inflation increases under subjective beliefs. This indicates that subjective premia on long bonds will reflect some of the low-frequency movements in expected inflation.

TABLE 3: ESTIMATION OF SUBJECTIVE MODEL

Panel A: Loadings of expected excess returns on state variables

		horizon $h = 1$ year		
	n	short rate	spread	expected inflation
bond maturity	2 year	0.35	1.27	-0.15
	10 year	6.34	4.86	4.58

Panel A: Mean absolute fitting errors for yield forecasts (% p.a.)

maturity

	subjective model					objective model				
	Bluechip sample, maturity of forecasted yield									
horizon	1 qrt	1 year	3 year	5 year	10 year	1 qrt	1 year	3 year	5 year	10 year
2 quarter	0.28	0.39	0.26	0.25	0.35	0.42	0.60	0.39	0.28	0.34
1 year	0.33	0.36	0.25	0.24	0.33	0.62	0.75	0.60	0.51	0.54

	Combined sample, maturity of forecasted yield			
	1 year	20 year	1 year	20 year
2 quarter	1.81	0.46	1.94	0.64

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Panel B of Table 3 reports mean absolute distances between the survey forecasts and model-implied forecasts, for both the subjective belief and the objective statistical model. Comparison of these errors provides a measure of how well the change of measure works to capture the deviation of survey forecasts from statistical forecasts.
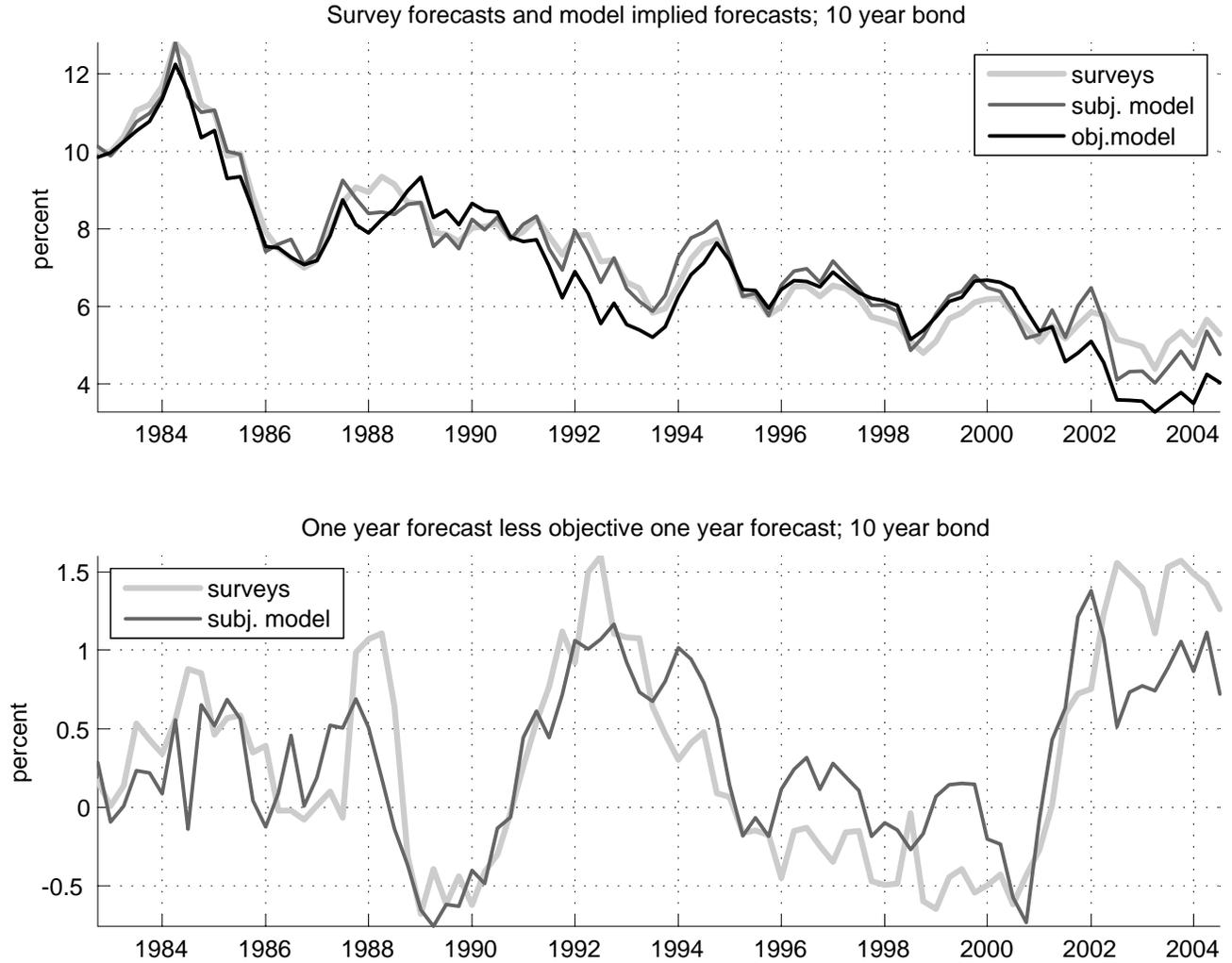

Figure 3: The top panel shows one-year ahead forecasts of the 10-year zero coupon rate constructed from survey data in Section III, together with the corresponding forecasts from our objective and subjective models. The bottom panel shows the difference between the survey forecast and the objective model forecast, as well as the difference between the subjective and objective model forecasts.

The results show that the improvement is small for short-horizon forecasts of short yields. However, there is a marked reduction of errors for 1-year forecasts, especially for the 10-year bond. Figure 3 shows where the improvements in matching the long-bond forecasts come from. The top

panel shows one-year ahead forecasts of the 10-year zero coupon rate constructed from survey data in Section III, together with the corresponding forecasts from our objective and subjective models, for the sample 1982:4-2007:3. All forecasts track the actual 10-year rate over this period, which is natural given the persistence of interest rates. The largest discrepancies between the survey forecasts and the subjective model on the one hand, and the objective model on the other hand, occur during and after the recessions of 1990 and 2001. In both periods, the objective model quickly forecasts a drop in the interest rate, whereas investors did not actually expect such a drop. The subjective model captures this property.

For our asset pricing application, we are particularly interested in how well the subjective model captures deviations of survey forecasts of long interest rates from their statistical forecasts over the business cycle. As discussed in Section III, this forecast difference is closely related to measured expected excess returns. The bottom panel of the figure focuses again on forecasting a 10-year rate over one year, and plots the difference between the survey forecast and the objective model forecast, as well as the difference between the subjective and objective model forecasts. It is apparent that both forecast differences move closely together at business cycle frequencies, increasing during and after recessions. We thus conclude that the subjective model is useful to capture this key fact about subjective forecasts that matters for asset pricing.

C.   Subjective risk premia

We now compare our estimated subjective risk premia to common statistical measures of objective risk premia. The motivation is that measures of objective premia provide stylized facts that rational expectations asset pricing models try to match. In particular, empirical evidence of predictability of excess returns from standard predictability regressions has led to a search for sources of time varying risk or risk aversion. The preliminary results of section III suggest that less time variation in risk premia is required once investors' forecast errors are taken into account. Here we quantify how much time variation in expected excess returns is left once we move to subjective beliefs.

We focus on 1-year holding period returns on bonds with 2 and 10 years maturity. We compare our subjective premia to three statistical measures of objective premia. The first is the fitted value

of a regression of excess returns on a single yield spread, the 5-year-1-quarter spread, denoted the YS measure. In other words, we regress the excess return $x_{t+4}^{(n,4)}$ on $i_t^{(20)} - i_t^{(1)}$ and a constant. This regression is closely related to that in the classic Fama-Bliss study of bond return predictability, which uses the forward-spot spread. The second measure (labelled CP measure) is the fitted value from a regression on five yields with maturities 1,2,3, 4 and 5 years. This follows Cochrane and Piazzesi (2005) who showed that this approach leads to substantially higher R^2s. The third measure is, for each subjective model specification, the forecast from the corresponding objective model which provides the conditional expectation of the excess return under P.

Table 4 summarizes the properties of the regression based measures of objective premia. According to these measures, the volatility of the predictable part of 1-year holding period returns is around 1% per year for the 2-year bond, and around 6% for the 10-year bond. The regression based on five yields naturally delivers a higher R^2, over 30% on both bonds. We are also interested in the frequency properties of premia. We use a band pass filter to decompose premia into three orthogonal components, a low frequency "trend" component (period > 8 years), a "cycle" component (period between 1.5 and 8 years), as well as high frequency noise. The columns labelled "trend" and "cycle" show the percentage of variance contributed by the respective components. Since the yield spread is a key business cycle indicator, the YS measure is particularly cyclical. The CP measure improves in part by including a larger trend component.

TABLE 4: OBJECTIVE PREMIA FROM REGRESSIONS

maturity 2 years				maturity 10 years			
Regression on yield spread (YS measure)							
volatility	% trend	% cycle	R^2	volatility	% trend	% cycle	R^2
0.72	12	57	.14	6.10	12	57	.28
Regression on five yields (CP measure)							
volatility	% trend	% cycle	R^2	volatility	% trend	% cycle	R^2
1.08	29	40	0.31	6.29	24	50	0.30

Table 5 presents a set of comparison statistics. The first line shows the properties of the subjective premium itself. For the baseline model, the standard deviation of the one-year premium on a 2-year bond is 43 basis points; it is 3.6% on the 10-year bond. These volatilities are substantially smaller than those of regression measures of premia. The frequency properties are also different:

subjective premia tend to have larger trend components and smaller cyclical components than regression based premia. This is particularly pronounced for the longer 10 year bond: in regression models at least one half of the time variation in premia is cyclical, whereas for subjective premia the share of cyclical variation is only 17%.

TABLE 5: SUBJECTIVE RISK PREMIA

	maturity 2 years			maturity 10 years		
	Baseline Model					
	volatility	% trend	% cycle	volatility	% trend	% cycle
	0.43	39	41	3.63	65	17
	volatilities relative to measures of objective premia					
	total	trend	cycle	total	trend	cycle
YS measure	0.60	1.08	0.51	0.59	1.39	0.32
CP measure	0.40	0.46	0.41	0.58	0.96	0.33
P-system	0.44	0.37	0.60	0.49	0.59	0.34

The other rows in the table consider directly the change in volatility as one moves from objective to subjective premia. All numbers are ratios of standard deviations, subjective divided by objective. The columns labelled "total" report the volatility of a subjective premium as a fraction of the volatility of the objective premium for the different measures. They range between 40 and 60 percent. Since the numerator is always the same, the volatility ratios are lower the better the comparison objective model predicts excess returns. The largest ratios arise for the YS measure which generates the least time variation in objective premia. The columns labelled "trend" and "cycle" report volatility ratios for the trend and cycle components across models. For the regression based measures, the reduction in volatility is driven primarily by a reduction in the volatility of the cyclical component. In fact, for the YS measure, the trend component is less volatile than that of the subjective premium.

The row labelled "P-system" compares the subjective premium to the premium from the state space system under the objective probability P. For the 10-year bond, this system is a better predictor of expected excess returns than even the CP regression based measure. The reason is that the expected inflation state variable helps forecast returns. For the P-system, the move to subjective premia implies a substantial reduction in the volatility of the trend component. The

P-system measure differs from the regression based measures in that it generates premia with larger trend components; the shares of the trend components is 55% for the 2 year bond and 58% for the 10 year bond. These components are due to the presence of expected inflation in the system.

Figure 4 plots subjective premia on the two bonds together with the respective CP measures as well as the objective premia from the state space system under P. The properties from the table are also visible to the naked eye. Consider first the long, (10 year) bond in the bottom panel. Both measures of objective premia show substantial cyclical movements: most recessions during the sample period can be identified as upward spikes in objective premia, for example in 1970, 1991 and 2001. The CP measure also spikes in 1974 and 1979. Here the P-system measure responds less to the business cycle; this is because it is driven more by expected inflation which lowers premia during this period. In contrast to both measures of objective premia, the subjective premium on the long bond responds only weakly to recessions. The bulk of the movement in the subjective premium is at low frequencies: it was high in the late 1970s and early 1980s when the level of the yield curve was high, and low towards the beginning and end of our sample.

Consider next the medium (2 year) bond. It is clear again that the subjective premium is less volatile than the measures of objective premia. At the same time, recession periods now register as upward spikes in all of the displayed lines. The main difference between objective and subjective premia for the medium bond is in the trend: the subjective premium is much smaller in the late 1970s and early 1980s then the objective premia. Comparing the objective premia across maturities (across panels), it is also apparent that objective premia on the long bond are more cyclical and exhibit less trend than objective premia on the medium bond. For subjective premia, the situation is the reverse.

The intuition for these results comes from the differences between survey forecasts and objective forecasts. Under a statistical model, both the slope and the level are indicators of high expected excess returns. For example, Figure 4 shows that measures of objective premia comove positively with both slope and level. We have seen in the previous section that survey forecasters treat both the level and the slope of the yield curve as more persistent than what they are under an objective model. This difference between survey forecasts and objective forecasts of future bond prices then weakens the effect of both indicators.
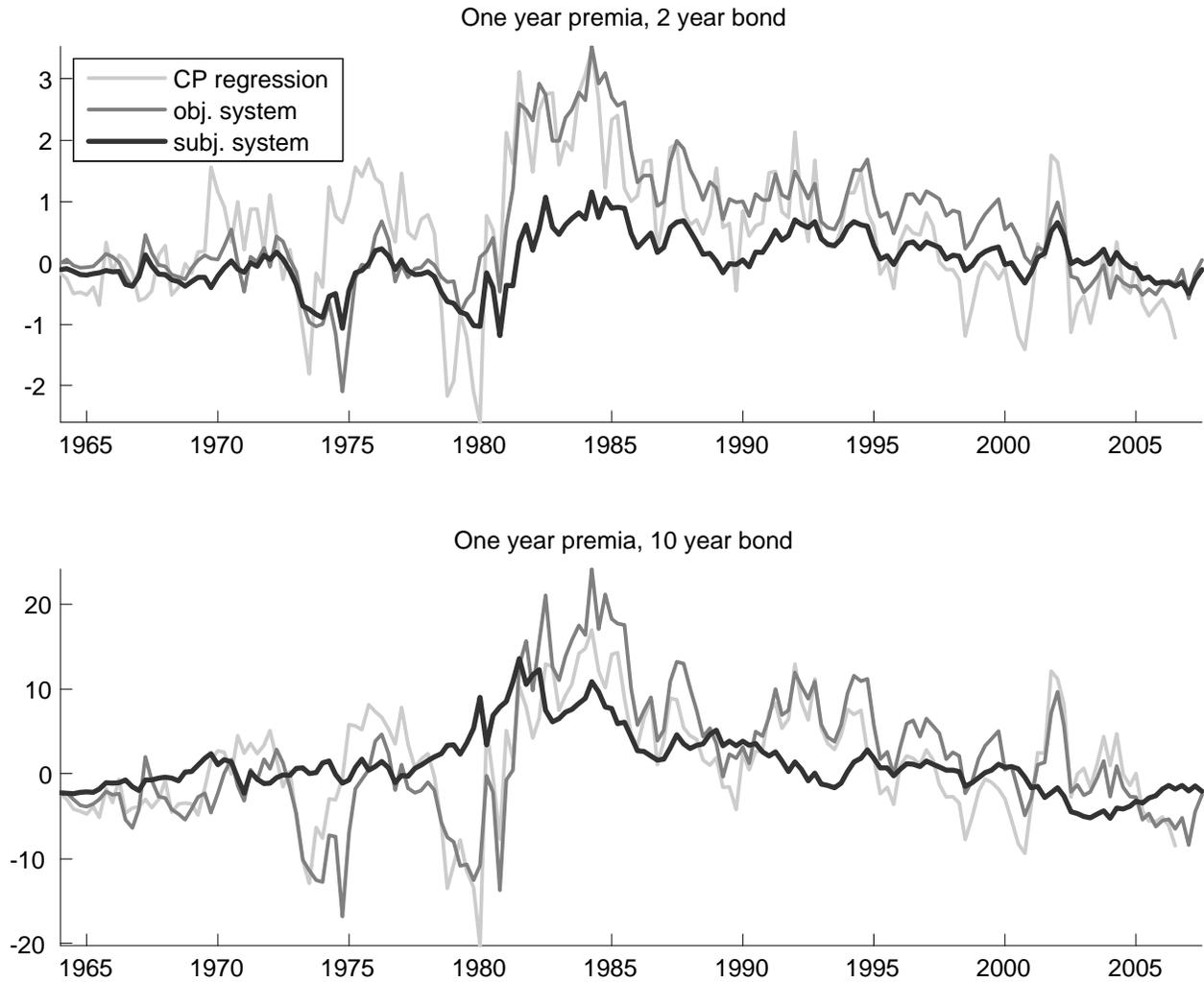
Figure 4: Subjective premia compared with measures of objective premia. Premia are expected excess holding period returns over one year, for a 2 year bond (top panel) and a 10 year bond (bottom panel). In both panels, black lines are estimated subjective premia, light gray lines are fitted values from CP regression, dark gray lines are objective premia from state space system.

If the level of the yield curve is high, survey forecasters, who view the level as more persistent than does an objective model, expect higher interest rates, and hence lower prices, than the objective model. Lower expected prices means lower expected excess returns. Similarly, if the slope is high, survey forecasters, who view the slope as more persistent, expect higher spreads, and hence higher long interest rates, and lower long bond prices, than the objective model. Again, lower expected prices means lower expected excess returns. Both situations (high level, high spread) which lead objective models to indicate high expected excess returns thus induce survey forecasters to predict

lower prices and returns than the objective models. This generates the overall reduction in volatility.

The higher persistence of level and slope perceived by survey forecasters also helps explain the different frequency properties of premia on medium (for example 2 year) and long (for example 10 year) bonds. The level of the yield curve is always relatively more important for short bonds rather than for long bonds. This is true not only for yields themselves, but also for measures of objective premia: those measures are driven relatively more by the slope for long bonds and relatively more by the level for medium bonds. A move to subjective premia weakens the effect of both indicators of high premia, the slope and the level. For a given maturity, it tends to weaken more the effect of the indicator that is more important. The move thus makes the premium on long bonds less responsive to the slope and it makes the premium on medium bonds less responsive to the level. This is the effect displayed in the figure.

V   Bond positions

In this section, we use the (subjective) term structure model estimated in the previous section to represent the universe of bonds available to investors in terms of a small number of "spanning bonds". In subsection A., we construct, for every zero-coupon bond, a portfolio of three bonds – a short bond and two long bonds – that replicates closely the return on the given zero-coupon bond. In subsection B., we then show how statistics on bond positions in the US economy can be converted into a time series of positions in the three bonds.

A.   Replicating Zero Coupon Bonds

According to the term structure model, the price $P_t^{(n)}$ of a zero coupon bond of maturity n at date t is well described by $\exp\left(A_n + B_n' f_t\right)$. We now select N long bonds, zero-coupon bonds with maturity greater than one, and stack their coefficients in a vector $\widehat{a}$ and a matrix $\widehat{b}$. Our goal is to construct a portfolio containing the long bonds and the short bond such that the return on the portfolio replicates closely the return on any other zero-coupon bond with maturity n. We use a discretization of continuous time returns, similar to those used above for the return on wealth. We

approximate the change in price on an n-period bond by

$$P_{t+1}^{(n-1)} - P_t^{(n)} \approx P_t^{(n)} \left(A_{n-1} - A_n + B_{n-1}'(f_{t+1} - f_t) + (B_{n-1} - B_n)'f_t + \frac{1}{2}B_{n-1}'\sigma_f\sigma_f'B_{n-1}' \right)$$

$$= P_t^{(n)} \left(A_{n-1} - A_n + B_{n-1}'\mu_f^* + B_{n-1}'(\phi_f^* - I)f_t + (B_{n-1} - B_n)'f_t + \frac{1}{2}B_{n-1}'\sigma_f\sigma_f'B_{n-1}' \right)$$

$$+ P_t^{(n)}B_{n-1}'\sigma_f\varepsilon_{t+1}^*$$

$$(16) \qquad = \quad : a_t^{(n)} + b_t^{(n)}\sigma_f\varepsilon_{t+1}^*$$

Conditional on date t, we thus view the change in value of the bond as an affine function in the shocks to the factors $\sigma_f\varepsilon_{t+1}^*$. Its distribution is described by $N+1$ time-dependent coefficients: the constant $a_t^{(n)}$ and the loadings $b_t^{(n)}$ on the N shocks. In particular, we can calculate coefficients $\left(a_t^{(1)}, b_t^{(1)}\right)$ for the short bond, and we can arrange coefficients for the N long bonds in a vector $\hat{a}_t$ and a matrix $\hat{b}_t$. Now consider a portfolio that contains θ_1 units of the short bond and $\hat{\theta}_i$ units of the ith long bond. The change in value of this portfolio is also an affine function in the factor shocks and we can set it equal to the change in value of any n-period bond:

$$(17) \qquad \left(\theta_1 \ \hat{\theta}' \right) \begin{pmatrix} a_t^{(1)} & b_t^{(1)} \\ \hat{a}_t & \hat{b}_t \end{pmatrix} \begin{pmatrix} 1 \\ \sigma_f\varepsilon_{t+1}^* \end{pmatrix} = \left(a_t^{(n)} \ b_t^{(n)} \right) \begin{pmatrix} 1 \\ \sigma_f\varepsilon_{t+1}^* \end{pmatrix}.$$

Since the $(N+1)\times(N+1)$ matrix of coefficient on the left hand side is invertible for a nondegenerate term structure model, we can select the portfolio $\left(\theta_1 \ \hat{\theta}' \right)$ to make the conditional distribution of the value change in the portfolio equal to that of the bond.

Replicating portfolios based on a two-factor model

When stated in terms of units of bonds $\left(\theta_1, \hat{\theta}\right)$, the replicating portfolio for the n-period bond answers the question: how many spanning bonds are equivalent to one n-period bond? For our work below it is more convenient to define portfolio weights that answer the question: how many dollars worth of spanning bonds are equivalent to one dollar worth of invested in the n-period bonds? The answer to this question can be computed using the units $\left(\theta_1, \hat{\theta}\right)$ and the prices of spanning bonds. Figure 5 provides the answer computed from the two-factor term structure model estimated above. Since the term structure model is stationary, these weights do not depend on calendar time.
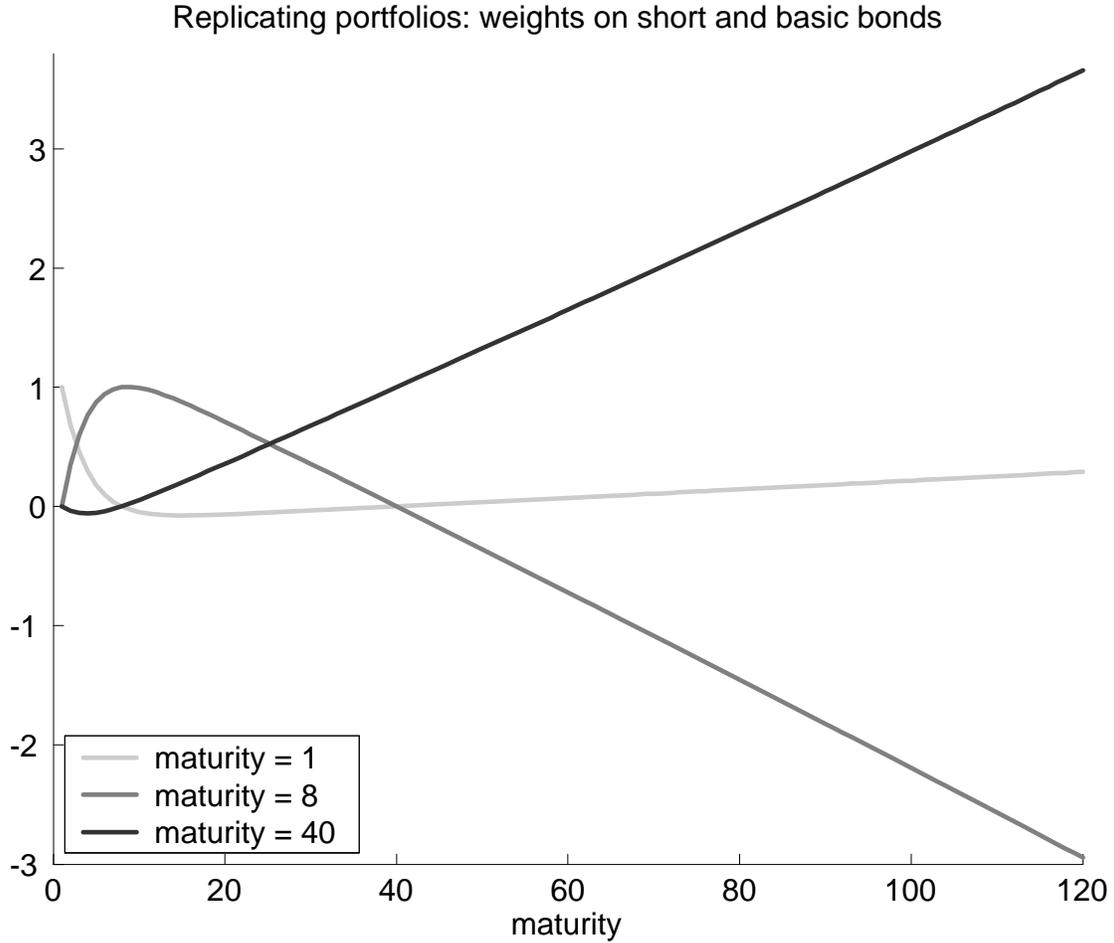
Replicating portfolios: weights on short and basic bonds

Figure 5: Replicating portfolios; the maturity in quarters of the bond to be replicated is measured along the x-axis.

The maturity of the n-period bond to be replicated is measured along the horizontal axis. The three lines are the portfolio weights $\theta_i P_t^{(i)}/P_t^{(n)}$ on the different spanning bonds i; they sum to one for every maturity n. As spanning bonds i, we have selected the 1-, 8- and 20-quarter bonds. For simplicity, we refer to the long spanning bonds as the middle and the long bond, respectively. The figure shows that the spanning bonds are replicated exactly by portfolio weights of one on themselves. More generally, the replicating portfolios for the most average neighboring bonds. For example, most of the bonds with maturities in between the 1-quarter and 8-quarter bond are generating by simply mixing these two bonds, although there is also a small short position in the long bond. Similarly, most of the bonds with maturities in between the middle and long bond are generating by mixing those two bonds. Intuitively, mixing of two bonds will lead to expected

returns that are linear in maturity, whereas adding a third bond helps generate curvature.

Quality of the approximation

We now provide some evidence that the approximation of a zero-coupon bond by a portfolio of spanning bonds is decent for our purposes. There are two dimensions along which we would like to obtain a good approximation. First, we would like the value of the approximating portfolio to be the same as the value of the zero-coupon bond. This is relevant for measuring the supply of bonds: below we will take existing measures of the quantity of zero-coupon-bonds held by households and convert them into portfolios of the small number of spanning bonds that are tradable by agents in our model. Along this dimension, the approximation is essentially as good as the term structure model itself. For a replicating portfolio defined by (17), the portfolio value $e^{-i_t}\theta_1 + \hat{P}_t'\hat{\theta}$ differs from the bond value $P_t^{(n)}$ only to the extent that the term structure model does not fit bonds of maturity n. The additional approximation error introduced by the matching procedure is less than .0001 basis points.

Second, we would like the conditional distribution of bond returns to be the same as that of the portfolio return. This is important because we would like agents in our model to have bond investment opportunity sets that are similar to those of actual households who trade bonds of many more maturities. Figure 6 gives an idea about the goodness of the approximation (17) by comparing statistics of the actual return implied by the term structure model and the approximating return. All statistics are unconditional moments computed from our sample, using the realizations of the term structure factors. For example, to obtain the difference in means in the top panel, we compute (i) quarterly returns from the term structure model using the formula $\exp\left(A_n + B_n'f_t\right)$ for prices, and (ii) quarterly returns based on the approximate formula for price changes (16) and subtract the mean of (i) from the mean of (ii).

On average, the two return distributions are quite similar for all maturities. The mean returns differ by less than 10 basis points for all bonds shorter than 30 years. The difference in variance is at most 30 basis points. The approximation error increases with maturity, as do the mean return and the variance of returns themselves. The approximate mean return on bonds is always within 5% of the true mean return, while the approximate variance is within 5-15% of the true variance.
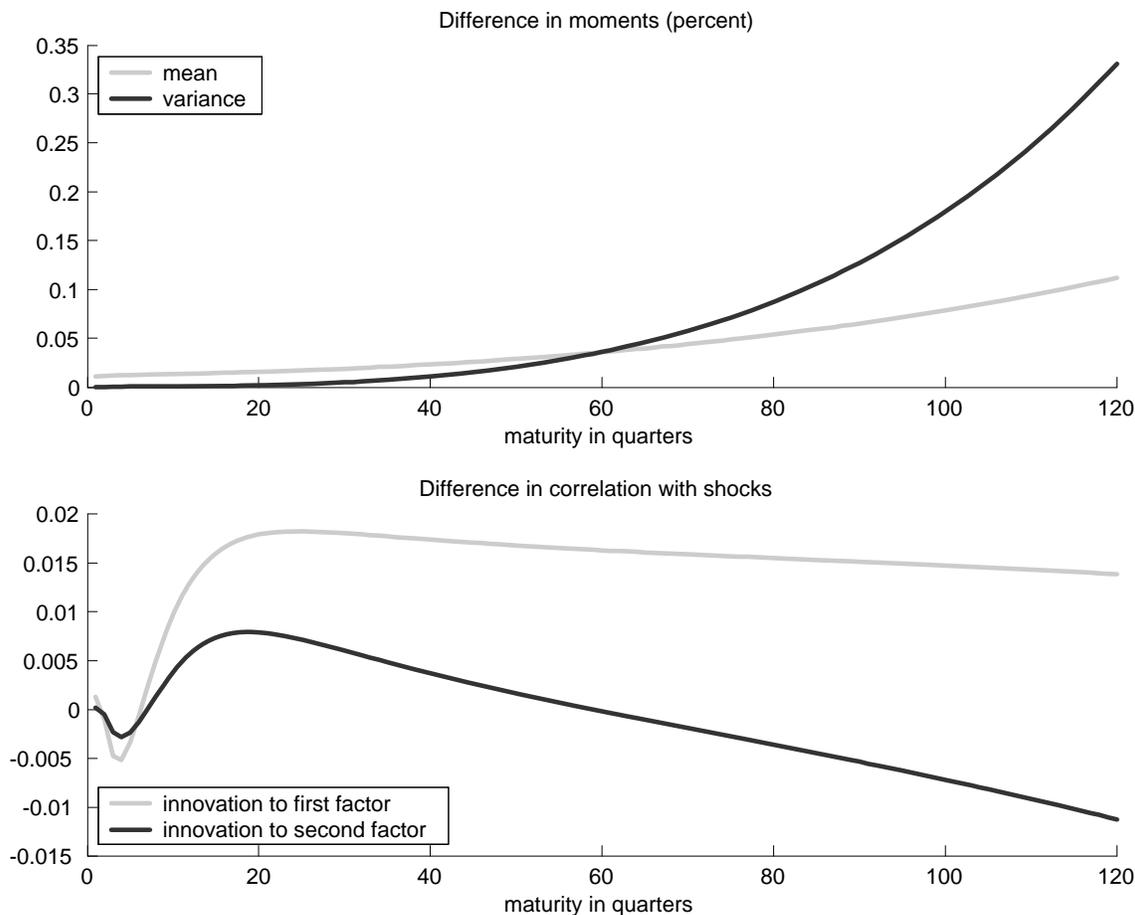
Figure 6: Approximation errors

Larger errors tend to arise for longer bonds. In addition to the univariate distribution of a return, it is also of interest how it covaries with other returns. If the term structure model is correct, then a parsimonious way to check this is to consider the correlation with the two factor innovations. The lower panel of the figure reports the difference between the correlation coefficients of the true and approximate returns with the factor innovations. These differences are very small.

B. Replicating nominal instruments in the US economy

We now turn to more complicated fixed-income instruments. The Flow-of-Funds (FFA) provides data on book value for many different types of nominal instruments. Doepke and Schneider (2006; DS) sort these instrument into several broad classes, and then use data on interest rates, maturities

and contract structure to construct, for every asset class and every date t, a certain net payment stream that the holders of the asset expect to receive in the future. Their procedure takes into account credit risk in instruments such as corporate bonds and mortgages. They use these payment streams to restate FFA positions at market value and assess the effect of changes in inflation expectations on wealth.

Here we determine, for every broad asset class, a replicating portfolio that consists of spanning bonds. For every asset class and every date t, DS provide a certain payment stream, which we can view as a portfolio of zero-coupon bonds. By applying equation (17) to every zero-coupon bond, and then summing up the resulting replicating portfolios across maturities, we obtain a replicating portfolio for the asset class at date t. Figure 7 illustrates replicating portfolios for Treasury bonds and mortgages. The top panel shows how the weights on the spanning bonds in the replicating portfolio for Treasury bonds have changed over the postwar period.

The reduction of government debt after the war went along with a shortening of maturities: the weight on the longest bond declined from over 60% in 1952 to less than 20% in 1980. This development has been somewhat reversed since 1980.[6] The bottom panel shows that the effective maturity composition of mortgages was very stable before the 1980s, with a high weight on long bonds. The changes that apparent since the 1980s are driven by the increased use of adjustable rate mortgages.

We do not show replicating portfolios for Treasury bills, municipal bonds and corporate bonds, since the portfolio weights exhibit few interesting changes over time. All three instruments are represented by essentially constant portfolios of only two bonds: T-bills correspond to about 80% short bonds and 20% middle bonds, that for corporate bonds corresponds to about 60% middle bonds and 40% long bonds, and the replicating portfolio for municipal bonds has 70% long bonds and 30% middle bonds. The final asset class is a mopup group of short instruments, which we replicate by a short bond.

Replicating aggregate FFA household sector positions

We compute measure aggregate household holdings in the FFA at date t. To derive their

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[6] The figure shows only the portfolios corresponding to outstanding Treasury *bonds*, not including bills. DS use data from the CRSP Treasury data base to construct a separate series for bills.
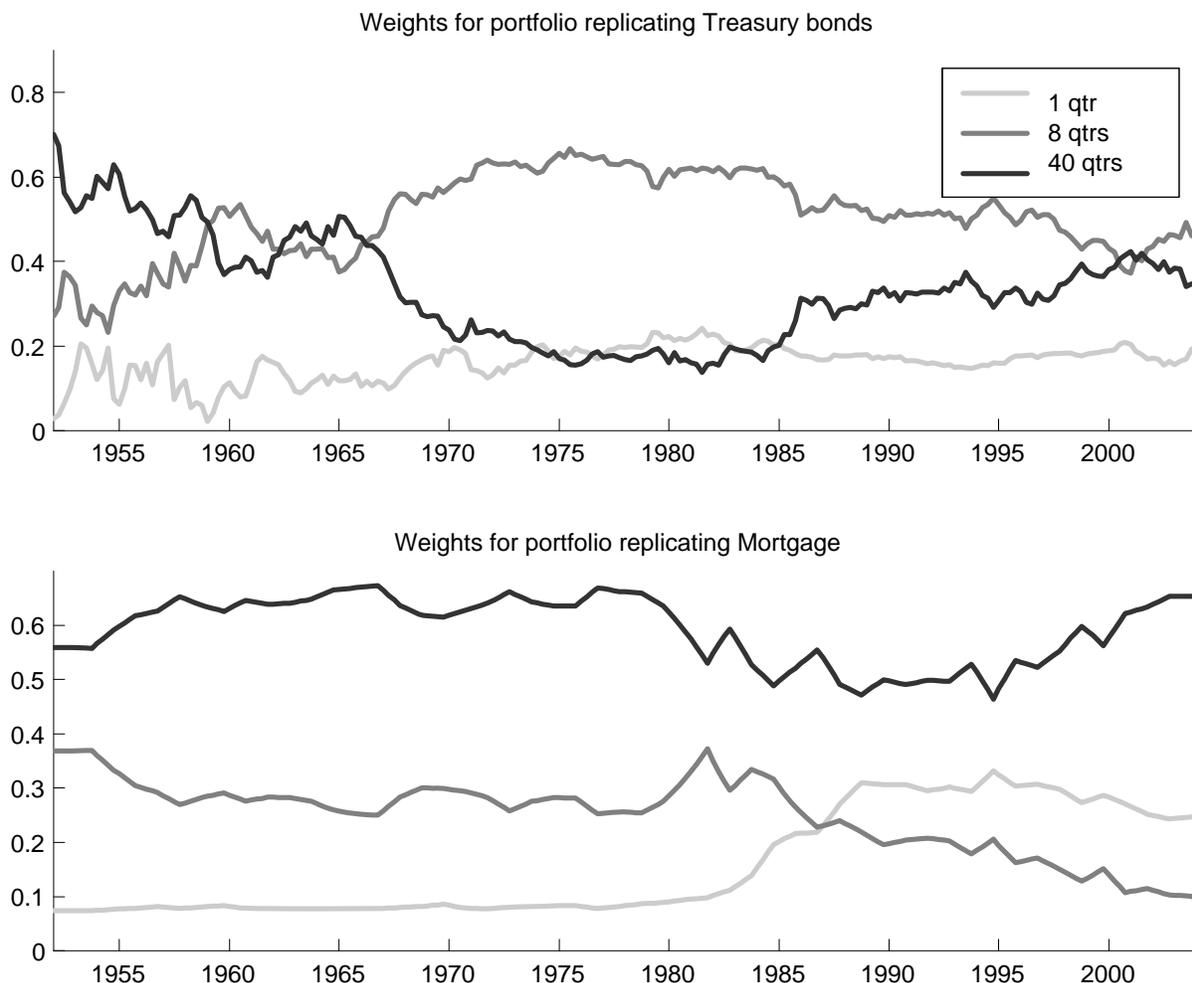
Figure 7: Replicating weights for Treasury bonds and mortgages

positions in spanning bonds, we compute replicating portfolios for household positions in the FFA. One important issue is how to deal with indirect bond positions, such as bonds held in a pension plan or bonds held by a mutual fund, the shares in which are owned by the household sector. Here we make use of the calculations in DS who consolidate investment intermediaries in the FFA to arrive at effective bond positions.

Applying the replicating portfolios for the broad asset classes to FFA household sector positions delivers three time series for holdings in spanning bonds, which are plotted in Figure 8. It is apparent that the early 1980s brought about dramatic changes in US bond portfolios. Until then, the positions in short bonds had been trending slightly upwards, whereas the positions in long and middle bonds had been declining. This pattern was reversed during the 1980s and early 1990s. The
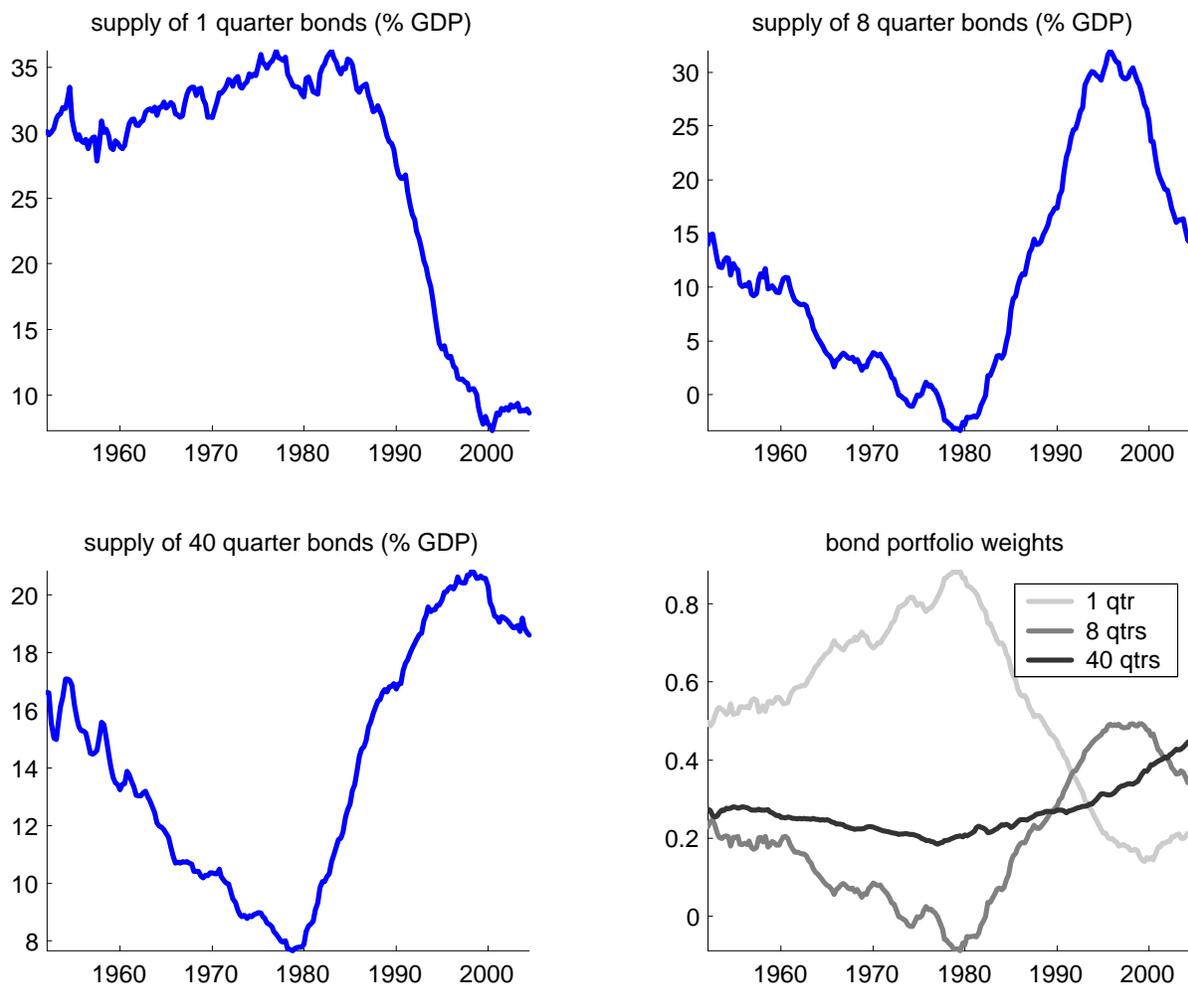
Figure 8: US household sector positions in the bond market

increase in the share of long bonds was partly due to changes in the composition of Treasury debt, as seen in Figure 7.

VI Equilibrium interest rates: a rentier model

In this section we study quantitatively the rentier model introduced in Section II. The starting point is a series of bond portfolio holdings. We consider an investor who invests only in bonds, and whose holdings are proportional to the given series of holdings. Since we do not observe the consumption of these bond investors, we do not allow for a residual asset, take the short rate as exogenous and only derive model-implied long bond prices. The idea behind the exercise is to explore what risk

premia look like if there is a subset of investors for whom bonds are very important.

Formally, the input to the rentier exercise is a sequence of portfolio weights $\widehat{\omega}_t = \hat{B}_t / B_t$ that represents the class of bonds considered in terms of our spanning bonds, a sequence of observed short-term interest rates, as well as a sequence of subjective beliefs P_t^* about future interest rates and inflation that is needed to solve the portfolio choice problem. The latter sequence of beliefs consists of a sequence of conditionals from the subjective probability distribution over yields and inflation estimated in subsection B.. The exercise proceeds by deriving portfolio demand and solving out for equilibrium prices from equation (9).

Portfolio choice with subjective return distributions driven by a normal VAR

When subjective beliefs are represented by a linear system with normal shocks, tractable approximate formulas for investors' portfolio policies and equilibrium prices are available. Subjective beliefs about term structure factors and inflation are represented by (4). The log short interest rate i_{t+1} and log inflation π_{t+1} are linear functions of h_{t+1} and log excess returns $\hat{x}_{t+1}$ are linear functions of s_{t+1} and s_t. The real return on the short bond $i_{t+1} - \pi_{t+1}$ and the excess returns on the other bonds $\hat{x}_{t+1}$ can thus also be represented as a state space system with state vector s_t. The state vector for the household problem is therefore $\left(\overline{W}_t, s_t \right)$.

We use the approximation method proposed by Campbell, Chan and Viceira (2003). The basic idea is that the log return on a portfolio in a discrete time problem is well approximated by a discretized version of its continuous-time counterpart. In our setup, the log return on wealth is approximated by

$$(18) \qquad \log R_{t+1}^w \approx i_t - \pi_{t+1} + \hat{\omega}_t^\top \hat{x}_{t+1} + \frac{1}{2} \hat{\omega}_t^\top \left(\operatorname{diag}\left(\Sigma_{\hat{x}\hat{x}} \right) - \Sigma_{\hat{x}\hat{x}} \hat{\omega}_t \right),$$

where $\Sigma_{\hat{x}\hat{x}}$ is the one-step-ahead conditional covariance matrix of excess returns x_{t+1}, and $\hat{\omega}_t$ is the vector of portfolio weights on the long bonds.

If this approximation is used for the return on wealth, the investor's value function can be written as $v_t\left(\overline{W}_t, s_t \right) = \log \overline{W}_t + \tilde{v}(s_{t+1})$, where $\tilde{v}$ is linear-quadratic in the state vector s_t. Moreover, the

optimal portfolio is

$$(19) \qquad \omega_t \quad \approx \quad \frac{1}{\gamma} \Sigma_{xx}^{-1} \left(E_t^* \left[x_{t+1} \right] + \frac{1}{2} \mathrm{diag} \left(\Sigma_{xx} \right) \right) + \left(1 - \frac{1}{\gamma} \right) \Sigma_{xx}^{-1} cov_t^* \left(x_{t+1}, \pi_{t+1} \right)$$
$$- \left(1 - \frac{1}{\gamma} \right) \Sigma_{xx}^{-1} cov_t^* \left(x_{t+1}, \tilde{v}(s_{t+1}) \right).$$

If $\gamma = 1$ – the case of separable logarithmic utility – the household behaves "myopically", that is, the portfolio composition depends only on the one-step-ahead subjective distribution of returns. More generally, the first line in (19) represents the myopic demand of an investor with one-period horizon and risk aversion coefficient γ. To obtain intuition, consider the case of independent returns, so that $\Sigma_{\hat{x}\hat{x}}$ is diagonal. The first term then says that the myopic investor puts more weight on assets with high expected returns and low variance, and more so when risk aversion is lower. The second term says that, if $\gamma > 1$, the investor also likes assets that provide insurance against inflation, and buys more such insurance assets if risk aversion is higher. For general $\Sigma_{\hat{x}\hat{x}}$, these statements must be modified to take into account correlation patterns among the individual assets.

For a long-lived household with $\gamma \neq 1$, asset demand also depends on the covariance of excess returns and future continuation utility $\tilde{v}(s_{t+1})$. Continuation utility is driven by changes in investment opportunities: a realization of s_{t+1} that increases $\tilde{v}$ is one that signals high returns on wealth ("good investment opportunities") in the future. Agents with $\gamma > 1$ prefer relatively more asset payoff in states of the world where investment opportunities are bad. As a result, an asset that pays off when investment opportunities are bad is attractive for a high-γ agent. He will thus demand more of it than a myopic agent.

Explicit price formulas

Substituting the portfolio policy (19) into (9), we can explicitly solve out for long bond prices. To clarify the effect of subjective expectations, we also add and subtract the objective price expectation $E_t \hat{p}_{t+1}$ to arrive at

$$\hat{p}_t \quad = \quad -i_t + E_t \hat{p}_{t+1} + \frac{1}{2} \mathrm{diag} \left(\Sigma_{\hat{x}\hat{x}} \right)$$
$$+ E_t^* \hat{p}_{t+1} - E_t \hat{p}_{t+1}$$
$$(20) \qquad\qquad - \gamma \Sigma_{\hat{x}\hat{x}} \widehat{\omega}_t + (\gamma - 1) \, cov_t^* \left(\hat{x}_{t+1}, \pi_{t+1} \right) - (\gamma - 1) \, cov_t^* \left(\hat{x}_{t+1}, \tilde{v}(s_{t+1}) \right).$$

According to the first line of this formula, the price of the long bond is determined as the (log) discounted expected future price (where the variance term appears because of Jensen's inequality). For a moment, suppose that the expressions in the other lines are constant. In this case, the prices are determined according to the expectations hypothesis: on average (up to a constant), buying a long bond at t and holding it to maturity should cost the same as buying a short bond at t, earning interest i_t on it and then buying the long bond only at $t + 1$.

The remaining lines of this formula may lead to deviations from the expectations hypothesis, and thus to time-varying expected excess returns. The two lines give two seperate seasons for such deviations. The second line in the formula says that we may observe deviations from the expectations hypothesis because investors may have expectations that are not rational. In this case, investors' subjective expectation of bond payoffs $E_t^* \widehat{p}_{t+1}$ differs from the objective expectation $E_t \widehat{p}_{t+1}$. The third line in the formula represents subjective risk premia, which may also change over time. In particular, they may change as the composition of investor portfolios ω_t changes over time. Below, we discuss these mechanism in detail.

Suppose that investors have expectations of future prices that are not rational so that subjective expectations $E_t^* \widehat{p}_{t+1}$ differ from objective expectations $E_t \widehat{p}_{t+1}$. This implies that even if subjective risk premia (in the third line of the formula) are constant, the modeler may be able to predict excess returns on long bonds with some variable known at time t. This predictability reflects the systematic forecast errors $E_t^* \hat{p}_{t+1} - E_t \hat{p}_{t+1}$ made by investors.

But even if expectations are rational, there are subjective risk premia which may change over time. These premia compensate investors for holding a risky long bond for three reasons. First, investors with Epstein-Zin utility demand compensation for holding assets whose returns covary with the return on wealth. This is captured by the conditional covariance $\Sigma_{\hat{x}\hat{x}} \widehat{\omega}_t$, which is the covariance of excess returns on bonds with the excess return on wealth, $\widehat{\omega}_t^\top \widehat{x}_{t+1}$ from equation (18). Since the composition of wealth $\widehat{\omega}_t$ changes over time, with changes in the relative amount of different bonds in $\hat{B}_t$, so there can be time variation in risk premia. In the case of log utility ($\gamma = 1$) this covariance represents the entire risk premium. More generally, a higher risk aversion coefficient drives up the compensation required for covariance with the return on wealth.[7]

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[7] Since consumption is proportional to savings, or wealth, the first term in the risk premium also represents the

Second, there is a premium for covariance with future investment opportunities. An asset that insures households against bad future investment opportunities – by paying off less when continuation utility $\tilde{v}_{t+1}$ is high – commands a lower premium. Hedging against future opportunities also changes the inflation risk premium. For $\gamma > 1$, households have to be compensated less to hold an asset that provides insurance against inflation.

A.   Numerical results

The inputs are a sequence of portfolio weights and a system for returns. For the weights, we have used US households' holdings of Treasury bonds, their holdings of all nominal assets except deposits, as well as US households' net position of nominal assets. The main lessons have been similar for all cases. In what follows, we report only the results for the total net position.

TABLE 6: SUBJECTIVE MOMENTS OF EXCESS RETURNS (% P.A.)

excess returns (%)	2 year	10 year
mean	0.80	1.16
cond. standard deviation	2.78	13.72
cond. correlation matrix	1	0.81
		1
correlation with short bond	0.19	0.08

Under the subjective system from Section IV, the short bond has a subjective mean real return of 1.80% and a one-quarter-ahead conditional volatility of 0.58%, both in annualized percentage terms. Table 6 summarizes other relevant moments of the subjective distribution of quarterly excess returns implied by the model for the middle and long spanning bonds (maturities of 2 and 10 years, respectively) In terms of conditional Sharpe ratios (mean excess return divided by standard deviation), the middle bonds looks more attractive than the long bond. It is also notable that the long and middle bond returns are highly positively correlated. No longer bond covaries much with the short bond return. We have also seen that the subjective belief generates a role for

covariance of returns with consumption growth, multiplied by risk aversion.

market timing: time variation in subjective premia over one year was illustrated in Figure 4 above.

Mean term premia and yield volatility

Table 7 reports means and standard deviations of equilibrium yields for three sets of parameters. As a benchmark, we include a log investor ($\gamma = 1$). In this case, the portfolio choice is myopic, so that equilibrium prices are independent of the discount factor β. The mean long bond yield is matched up to 4 basis points, while the mean middle bond yield is matched up to 21 basis points. By equation (20), the log investor model shuts down any time variation in subjective risk premia due to changes in changes in future investment opportunities. In addition, risk aversion is small, so that subjective consumption risk premia measured via bond portfolio weights are essentially zero. The model nevertheless generates positive term premia, because prices reflect investors' subjective expectations of future payoffs. For the same reason, model implied yields are almost as volatile as those in the data.

TABLE 7: MOMENTS OF EQUILIBRIUM YIELDS

nominal yields (% p.a.)		short bond	middle bond	long bond
Data	mean	5.27	5.90	6.53
	std. dev.	1.47	1.45	1.33
Log utility	mean		5.69	6.57
	std. dev.		1.37	1.28
$\gamma = 2$, $\beta = 0.97$	mean		5.67	6.53
	std. dev.		1.34	1.25
$\gamma = 40$, $\beta = 0.91$	mean		5.58	6.53
	std. dev.		1.32	1.22

For risk aversion different from one, equilibrium prices depend on both preference parameters γ and β. We have searched over a grid of values to explore the sensitivity of equilibrium yields. For every $\gamma \leq 150$, there exists β such that the model matches exactly the mean long bond yield. The required β is decreasing in γ. The table reports results for two other values of the risk aversion coefficient. Our leading example, to be explored further below, is the case $\gamma = 2$, a common choice

for risk aversion in macro models; matching the long rate then requires $\beta = 0.97$. The resulting summary statistics are quite close to the log investor case. Relative to the log case, the mean long rate falls, even though risk aversion has gone up. This is because a patient investor can hold the long bond to hedge against bad investment opportunities, that is, low future short interest rates. This reduces the required compensation for risk.

The table also provides a high risk aversion example: for $\gamma = 40$, we must pick $\beta = 0.91$ to match the mean long rate. As risk aversion increases, the desire to hedge becomes stronger which by itself would increase the demand for the long bond and lower the long rate. The reduction in β weakens the hedging motive to keep the mean long rate at its observed value. This tradeoff says that if high risk aversion of rentiers is to be consistent with yield spreads in the data, then the effective planning horizon of the rentier must be shorter than that of the typical RBC agent.

There does not exist β and γ in order to find values that match exactly the mean excess returns on both the 2 year and the 10 year yields. This is somewhat surprising, since the parameters β and γ do have different effects on portfolio demand. In particular, higher risk aversion γ lowers the myopic demand for bonds, but increases the hedging demand for assets that provide insurance against bad future investment opportunities. In contrast, the discount factor affects only the size of the hedging demand: provided that $\gamma \neq 1$, increasing β makes hedging demand more important relative to myopic demand. However, it appears that the myopic and hedging demands for the two bonds react to both parameters in a similar way, so that they cannot be identified from the two average premia. In fact, the mean spread between the long and middle bond reacts very little to changes in β and γ. The reason is that it is due in part to systematic differences in payoff expectations between the two bonds (induced by the subjective model), which affect prices independently of the risk premium terms that are responsive to γ and β.

Model-implied yield spreads

Figure 9 plots yield spreads from the data together with yield spreads implied by the model for the case $\gamma = 2$, $\beta = 0.97$. The middle panel shows that the model matches the 10-year spread quite well. The top and bottom panel show the two-year spread, as well as the spread of the ten year rate over the two year rate, respectively. Both panels reflects the fact that the mean 2-year
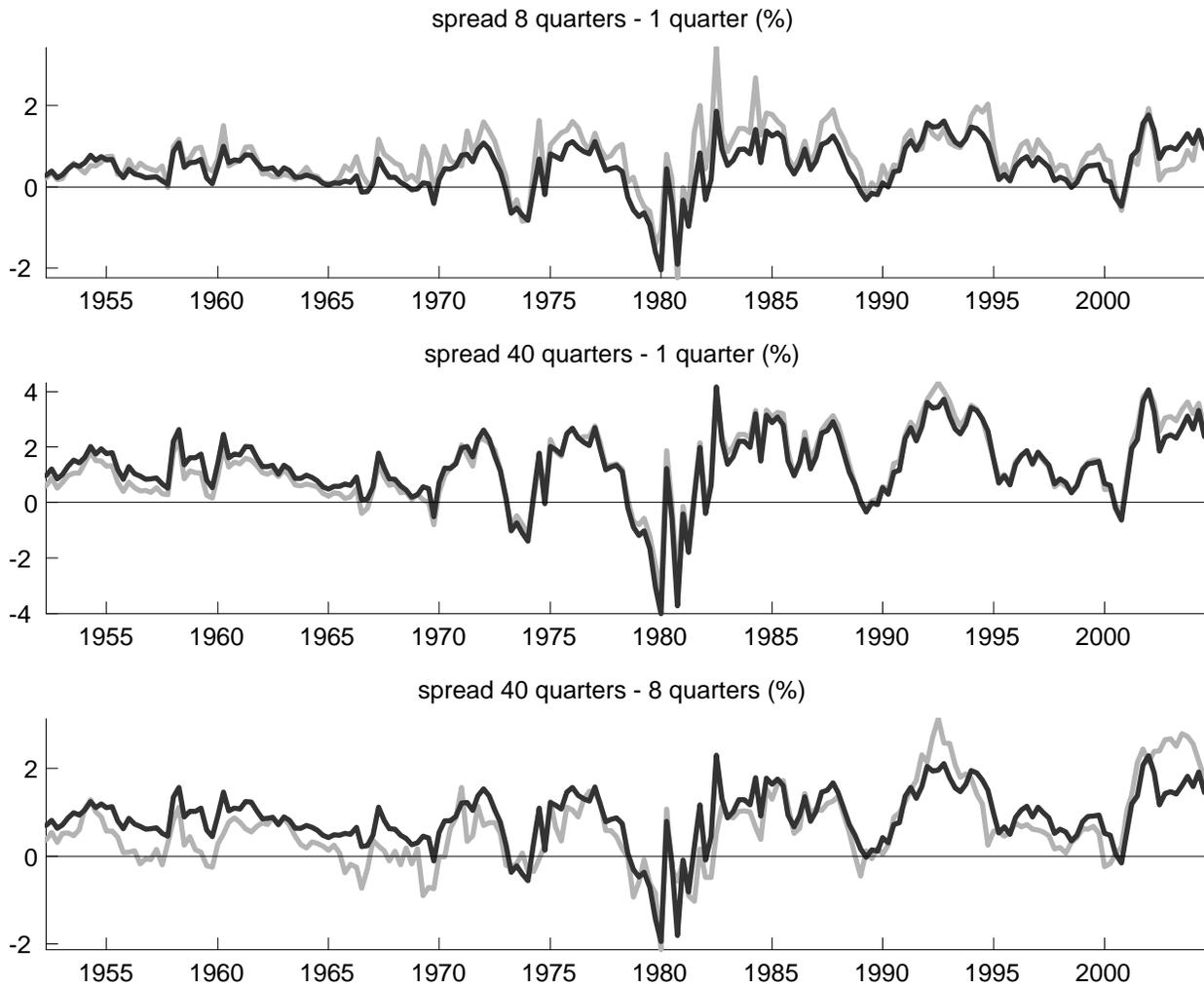
Figure 9: Yield spreads (as % p.a.) for the data sample (light lines) and the model-implied sample (dark lines).

rate is lower in the model than in the data. At the same time, the changes in model-implied and observed spreads track each other rather closely. The dynamics of spreads is driven in part by expectations of future yields. It is interesting to ask whether the model implied yields exhibit the same movements relative to statistical expected future yields as do their counterparts in the data. This is done in Figure 10, which reports risk premia relative to the expectations hypothesis for the 10 year yield.

The lines in the figure represent differences between a 10-year yield and its expectations hypothesis counterpart from the first line of (20), multiplied by n to make it consistent with our reporting

of premia in previous sections:

$$n \left(i_t^{(n)} - \frac{h}{n} i_t^{(h)} - \frac{(n-h)}{n} E_t i_{t+h}^{(n-h)} - \frac{1}{2} var_t \left(x_{t+1}^{(n,h)} \right) \right),$$

where $x_{t+1}^{(n)}$ is the 1-quarter excess holding period return on the n-period bond. In other words, the lines show the difference between the forward rate on a contract that promises an $(n-1)$-period bond one period from date t. The three lines differ in what forward rate is used, and how the expectation is formed. The light gray line labelled "data – obj. EH" shows the difference between the forward rate from the data and the expected rate under the objective probability. It is thus corresponds to measured expected excess returns, which tend to be high during and after recessions.

The black line represents a model implied objective risk premium – the difference between the forward rates implied by the model and the expected rates under the objective probability. It exhibits both a low frequency component and a business cycle component that comoves with the risk premium from the data. To illustrate the source of these movements, the dark gray lines show the subjective risk premia that is, differences between the forward rate implied by the model and the expected rate under the subjective probability. The subjective premia are smoother and contain more trend than the objective premia. Instead, those movements are due to the differences between subjective and objective forecasts documented in Sections III.

Investors' subjective perception of risk is reflected in the subjective risk premium, and is also responsible for the low frequency components in the model implied objective risk premium. For one quarter premia, equation (20) says that changes in the subjective premium must be due to changes in hedging opportunities or changes in bond positions. At the current parameter values, risk aversion is low and changes in positions do not have large effects. Instead, the V-shaped pattern in the subjective risk premium is due to changes in the covariance between long bond returns and future continuation utility. Intuitively, the insurance that long bonds provide to long horizon investors against drops in future short rates became more valuable in the 1980s when short rates were high. As a result, the compensation required for holding such bonds declined.

For longer horizon premia, the subjective expectation of the bond price at the end of the horizon reflects properties of interest rate expectations beyond the next quarter. Interestingly, the market
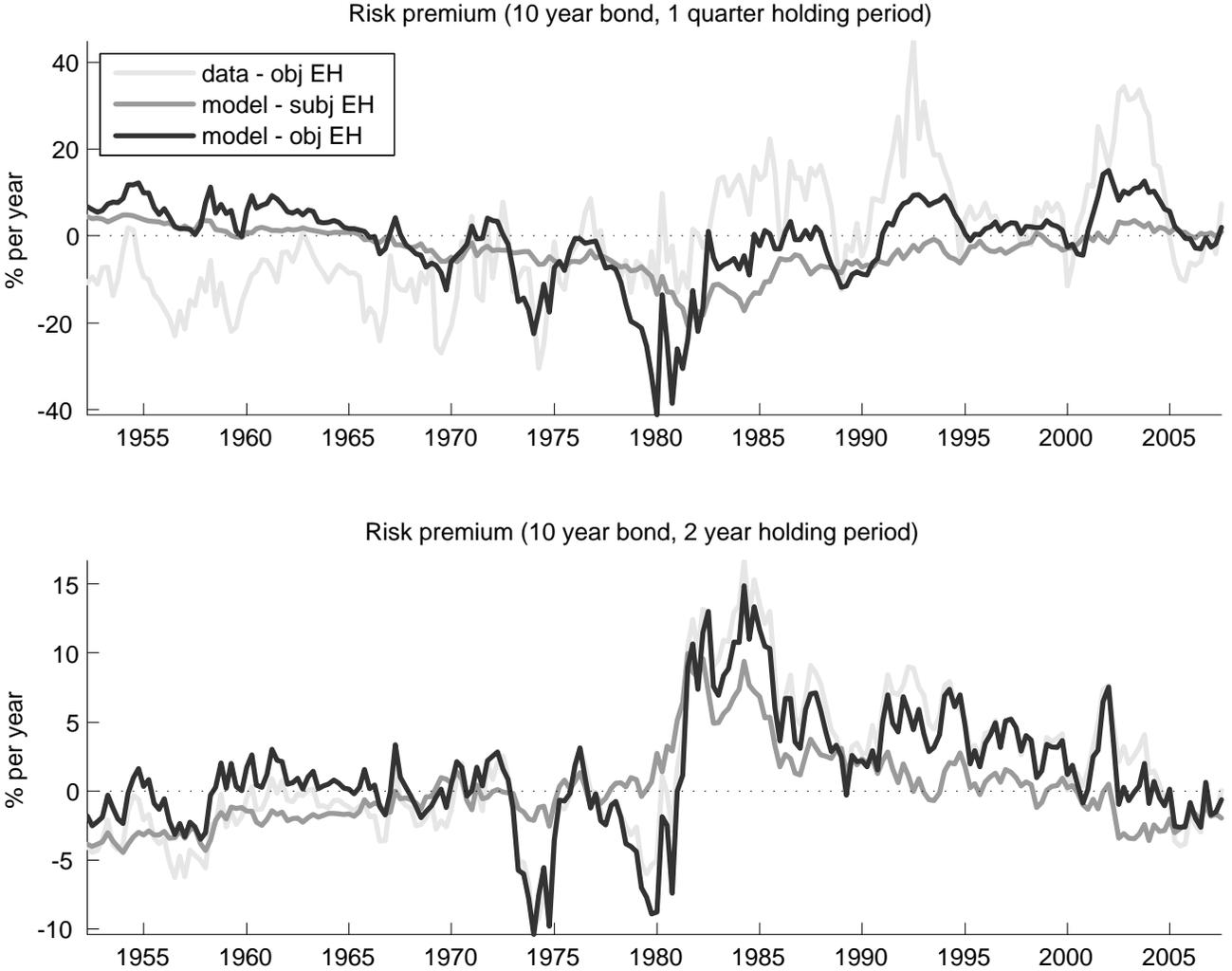
Figure 10: Risk premia (actual yield minus yield predicted under the expectations hypothesis) for (i) the data sample using predictions from the objective model (light gray), (ii) the model implied sample using predictions from the subjective model (black), and (iii) the model implied sample using predicitions from the subjective model.

timing opportunities at long horizons that were apparent in Figure 4 above do not seem to tempt the rentier, even though he is rather patient. As a result, equilibrium prices still reflect a lot of subjective predictability in returns.

To explore the role of changes in bond positions, the dark gray line in Figure 11 shows the component $\gamma\Sigma_{\hat{x}\hat{x}}\widehat{\omega}_t = \gamma\Sigma_{\hat{x}\hat{x}}\widehat{B}_t/B_t$ of the model-implied subjective risk premia for 2-year and 10-year bonds that are due to changes in positions. This compensation for risk is small and also moves slowly, at similar frequencies as the portfolio weights derived in Section 8. For comparison,

the figure also shows, at different scales, low frequency components of the respective term spreads that have been extracted using a band-pass filter. The increase in the spread in the 1980s thus went along with a shift towards longer bond positions that contributed positively to risk premia. However, the drop in subjective risk premia before 1980s did not go along with movements in the spread.

The quantitative lessons of this section are confirmed when we start the exercise from other classes of bond portfolios. Matching the long interest rate under the subjective model point us to small risk aversion coefficients, which in turn make the contribution of subjective risk premia, and especially the part due to bond positions, is quantitatively small. A qualitative difference is that when we eliminate deposits and liabilities from the portfolio, the middle and long bonds become relatively more important, and there is some evidence of business cycle variation in the model implied subjective risk premia. This suggests that a different model of conditional variance might lead to a larger role for subjective risk premia measured via bond positions.
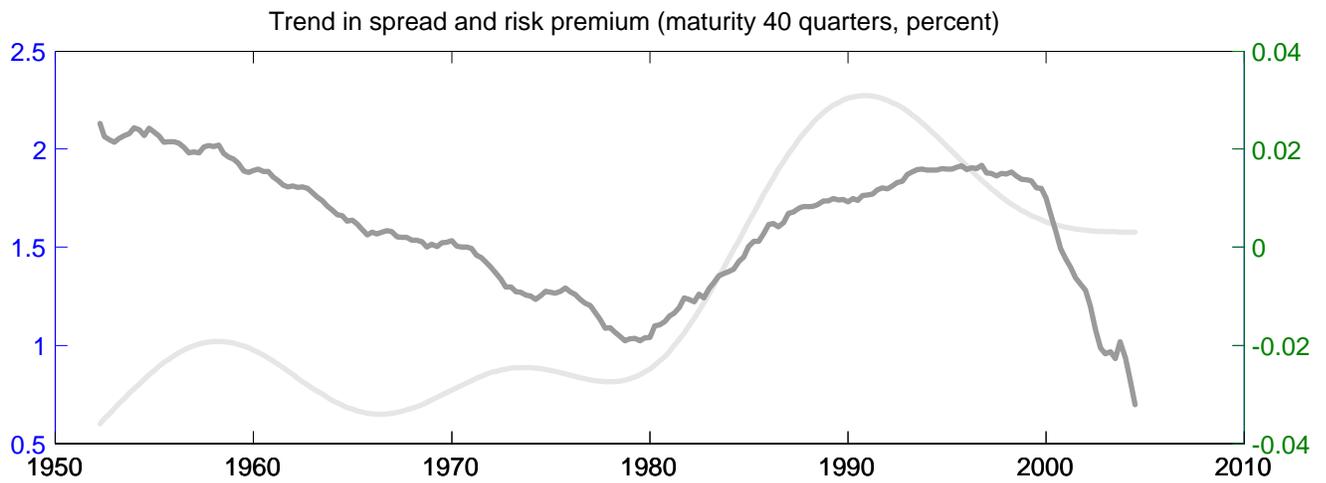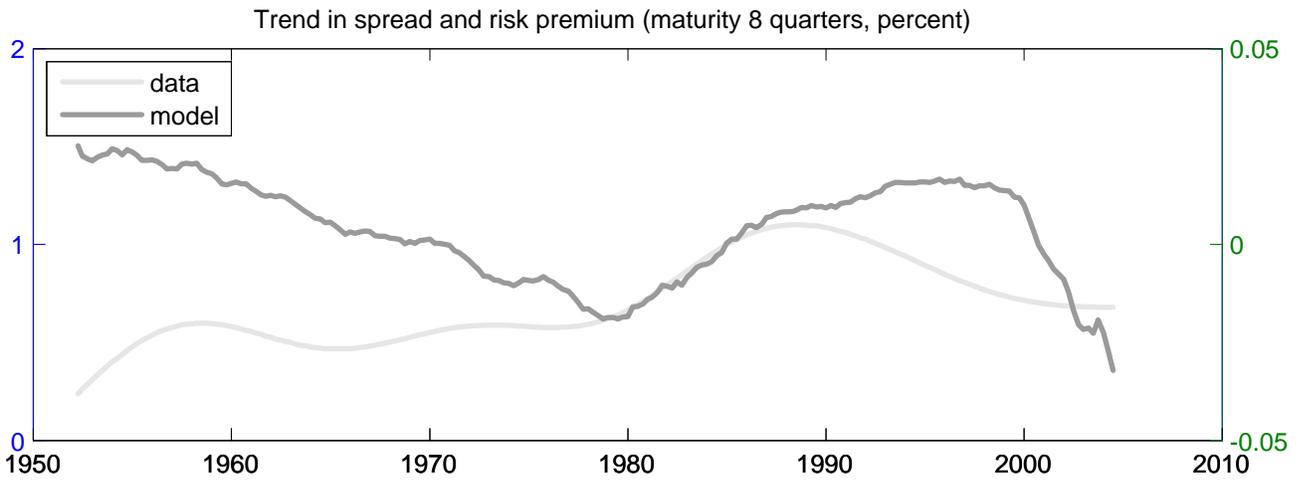
Figure 11: Trend in the term spread constructed with bandpass filter (light line, left scale) and model-implied risk premium (dark line, right scale).

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A   Appendix

Table 1 A reports maximum likelihood estimates of the objective state space system (2). The quarterly data sample is 1964:1-2007:4.

TABLE 1 A: ESTIMATION OF OBJECTIVE SYSTEM

	$\mu \times 100$	η_h			$\mathrm{chol}(\Omega)$		
short rate	1.47	0.88	0.12	0.13.	0.26		
	–	(0.04)	(0.08)	(0.05)	(0.01)		
spread	0.27	0.05	0.76	-0.06	-0.13	0.13	
	–	(0.03)	(0.06)	(0.03)	(0.01)	(0.01)	
exp. inflation	0.98	0	0	1	0.04	0.03	0.27
	–				(0.02)	(0.02)	(0.01)

	ϕ_s			σ_s		
short rate	0.88	0.12	0.13	1		
	(0.04)	(0.08)	(0.05)			
spread	0.05	0.76	-0.06	0	1	
	(0.03)	(0.06)	(0.03)			
exp. inflation	-0.04	-0.08	0.97	0.13	-0.09	0.53
	(0.03)	(0.06)	(0.04)	(0.11)	(0.16)	(0.09)

NOTE: This table contains maximum-likelihood estimates of the state space system

$$
\begin{aligned}
h_t &= \mu + \eta s_{t-1} + e_t \\
s_t &= \phi_s s_{t-1} + \sigma_s e_t.
\end{aligned}
$$

"chol(Ω)" denotes the Cholesky decomposition of var$(e_t) = \Omega$. Brackets indicate maximum-likelihood asymptotic standard errors computed from the Hessian.

TABLE 2A: ESTIMATION OF MARKET PRICES OF RISK

	λ_0	λ_1		
		short rate	spread	exp. inflation
short rate	-0.20	-0.13	-0.11	0.13
spread	0.09	0.05	-0.08	-0.06
	λ_0^*	λ_1^*		
short rate	-0.01	-0.01	-0.14	0.03
spread	-0.04	-0.03	0.08	-0.05

NOTE: This table contains nonlinear least-squares estimates of the first two rows of objective market price of risk parameters $\lambda_t = l_0 + l_1 s_t$ and their subjective counterparts.

B Appendix

The goal of this appendix is to describe an equilibrium model of the US economy. The equations about optimal behavior and model-implied prices described in the body of the paper (for the case with a residual asset) hold in this equilibrium model. The model goes beyond these equations because it (i) derives initial wealth of households as the (endogenous) value of exogenous quantities of asset endowments and (ii) and specifies trades between US households and other sectors of the economy, such as foreigners. The households in the model optimize for some given exogenous expectations. This specification allows expectations to be consistent with survey evidence or with some learning mechanism.

The model describes a single trading period t. In each period, the US household sector trades assets with another sector of the economy. We call the other sector the "rest of the economy" (ROE), which stands in for the government, business, and foreign sectors. Households enter the trading period t endowed with one unit of the residual asset. Their initial wealth $\overline{W}_t$ also comprises dividends from the residual asset as well as the value of all bonds written on the rest of the economy that households bought at date $t-1$, denoted $\bar{B}_t$:

(B-1) $$\overline{W}_t =: P_t^{res} + D_t + \bar{B}_t.$$

Households decide how to split this initial wealth into consumption as well as investment in the

$N+2$ assets. More specifically, the household problem at date t is to maximize utility (7) subject to the budget constraint (8), given initial wealth $\bar{W}_t$ as well as beliefs about (a conditional distribution of) the relevant future price variables $\left(x_\tau^{res}, i_\tau, \pi_\tau, \hat{p}_\tau, \hat{p}_\tau^{+1}\right)_{\tau>t}$: the excess return on the residual asset, the short interest rate, the inflation rate and the prices and payoffs on the long spanning bonds.

The trading strategy of the rest of the economy is exogenous. In particular, at date t the ROE sector sells residual assets worth $P^{res}f_t$. This trade captures, for example, the construction of new houses, and the net issuance of new equity. The ROE also trades in the bond market. It is convenient to summarize bond trades in terms of the value of outstanding bonds written on the ROE. At date t, the ROE redeems all short bonds issued at date $t-1$, and it also buys back all outstanding long bonds, at a total cost of $\bar{B}$. Moreover, the ROE issues new bonds worth B_t, so that its net sale of bonds is $B_t - \bar{B}_t$, which could be positive or negative.

To define new issues of individual bonds, we collect the values of new long bonds in a vector $\widehat{B}_t$. The value of outstanding short bonds is then $B_t - \iota_t'\widehat{B}$, where ι is an N-vector of ones. The ROE trading strategy is thus set up so that the same set of $N+1$ types of bonds – namely the short bond and the long bonds – are held by households at the end of every trading period. For example, suppose that the only long bond is a zero-coupon bond with a maturity of n periods. Between dates $t-1$ and t, households can then hold short (1-period) and long (n-period) bonds. At date t, the ROE buys back all long bonds (which now have maturity $n-1$), and again issues new 1-period short and n-period long bonds, and so on.

Equilibrium

We solve for a sequence of temporary equilibria. For each trading date t, we take as given (i) the strategy of the rest of the economy, summarized by its asset trades $\left(P^{res}f_t, \bar{B}_t, B_t, \widehat{B}_t\right)$, (ii) dividends D_t on the residual asset earned by households and (iii) household expectations about (that is, the conditional distribution of) the relevant price variables $\left(r_\tau^{res}, i_\tau, \pi_\tau, \hat{p}_\tau, \hat{p}_\tau^{+1}\right)_{\tau>t}$, which comprise the return on the residual asset, the short interest rate, the inflation rate and the prices and payoffs on the long bonds for all future periods. We characterize equilibrium prices as functions of these three inputs by equating household asset demand to the net asset supply provided by the

trading strategy of the ROE.

Formally, an equilibrium consists of sequences of short interest rates and (log) long bond prices $(i_t, \hat{p}_t)$ as well as optimal choices by households $(C_t, \alpha_t^{res}, \hat{\alpha}_t)$ such that, at every date t, all four asset markets clear:

$$\alpha_t^{res}\left(\overline{W}_t - C_t\right) = P_t^{res} + P^{res} f_t;$$
$$\hat{\alpha}_t\left(\overline{W}_t - C_t\right) = \widehat{B}_t;$$
(B-2)
$$\overline{W}_t - C_t = B_t + P_t^{res} + P^{res} f_t$$

Here the first equation clears the market for the residual asset, the second equation clears the markets for the long bonds and the last equation ensures that total savings equals the total value of outstanding assets, which implies that the market for short bonds also clears. This system of $N + 2$ equations determines the $N + 2$ asset prices $(P_t^{res}, i_t, \hat{p}_t)$. While the price of the residual asset P_t^{res} appears directly in (B-2), bond prices enter via the effect of bond returns on portfolio demand.

A sequence of temporary equilibria imposes weaker restrictions on allocations and prices than a standard rational expectations equilibrium. In particular, the definition above does not directly connect what happens at different trading periods. On the one hand, we do not require that the initial wealth of households is derived from its choices in the previous period. For example, the sequence $\left(\bar{B}_t\right)$ of payoffs from bonds bought earlier is an exogenous input to the model. On the other hand, we do not impose conditions relating return expectations at date t to model-implied realized (or expected) returns in future periods, as one would do when imposing rational expectations. At the same time, if there is a rational expectations equilibrium of our model that accounts for observed asset prices and household sector choices, then it also gives rise to a sequence of temporary equilibria.

The fact that our model allows for trades between the household sector and the rest of the economy distinguishes it from the endowment economies frequently studies in the asset pricing literature. In particular, our model accommodates nonzero personal savings. Combining (B-1) and

the last equation in (B-2), we obtain the flow-of-funds identity

(B-3)
$$C_t + P^{res} f_t + \left(B_t - \bar{B}_t \right) = D_t.$$

The dividend on the residual asset D_t corresponds to personal income less net personal interest income. As a result, $D_t - C_t$ is personal savings less net interest. It consists of $P^{res} f_t$ – net purchases of all assets other than bonds – and $B_t - \bar{B}_t$ – net purchases of bonds less net interest. A typical endowment economy model of the type studied by Lucas (1978) instead assumes that bonds are in zero net supply and that household net wealth is a claim on future consumption, so that $C_t = D_t$ and $P^{res} f_t = 0$.