Slow Moving Debt Crises*

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We study slow moving debt crises, self-fulfilling equilibria where high interest rates due to fears of higher future default leads to a gradual but faster accumulation of debt, ultimately validating investors' fears. We show that slow moving crises arise in a variety of settings, both when fiscal policy follows a given rule or when it is chosen by an optimizing government. We discuss how multiplicity is avoided for low debt levels, for sufficiently responsive fiscal policy rules, and for long enough debt maturities. When the equilibrium is unique, debt dynamics are characterized by a tipping point, below which debt fall and stabilizes and above which debt and default rates grow. We provide game-theoretic foundations for our approach.

1 Introduction

Yields on sovereign bonds for Italy, Spain and Portugal shot up dramatically in late 2010 with nervous investors suddenly casting the debt sustainability of these countries into doubt. An important concern for policy makers was the possibility that higher interest rates were self-fulfilling. High interest rates, the argument goes, contribute to the rise in debt over time, eventually driving countries into insolvency, thus justifying higher interest rates in the first place. News coverage illustrates how future debt dynamics were at center stage.[1] Thomson Reuters offered a simple web application, under the title "Ital-

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[1]For example, a *Financial Times*' report on the Italian bond market cites a pessimistic observer expecting "Italian bonds to perform worse than Spanish debt this summer, as investors focus on the sustainability of Italy's debt burden," given Italy's high initial debt-to-GDP ratio. A more optimistic investor in the same report argues that Italy "can cope with elevated borrowing costs for some time particularly when shorter-dated bond yields remain anchored" and that "it's critical to bring these yields down, but there is time for Italy to establish that its policies are working." (*"Investors wary of Italy's borrowing test"*, *Financial Times*, July 16, 2012.)

ian Debt Spiral", that computed the primary surplus needed to stabilize the debt-to-GDP ratio under different scenarios.[2]

Yields subsided in the late summer of 2012 after the European Central Bank's president, Mario Draghi, unveiled plans to purchase sovereign bonds to help sustain their market price. A view based on self-fulfilling crises was explicitly used to justify such interventions during Draghi's news conference announcing the Outright Monetary Transactions (OMT) bond-purchasing program (September 6th, 2012),

> "The assessment of the Governing Council is that we are in a situation now where you have large parts of the Euro Area in what we call a bad equilibrium, namely an equilibrium where you have self-fulfilling expectations. You may have self-fulfilling expectations that generate, that feed upon themselves, and generate adverse, very adverse scenarios. So there is a case for intervening to, in a sense, break these expectations [...]"

If this view is correct, a credible announcement to do "whatever it takes" is all it takes to rule out bad equilibria, no bond purchases need actually be carried out. To date, this is exactly how it seems to have played out: there have been no purchases by the ECB and no country has applied to the OMT program.

In this paper, we build a dynamic model that formalizes this multiple-equilibria view of debt crises. We then use this model to explore how the initial debt level, the fiscal policy regime, and the maturity of debt affect the vulnerability to such crises.

In our model, the government sells bonds to a large group of investors to finance shocks to its funding needs. Investors are risk neutral and price bonds according to their expected payoffs, forming expectations over future default probabilities. Given bond prices and fiscal policy, one can compute the path for debt. This, in turn, affects future default probabilities and, through investors' expectations, bond prices. This feedback loop between interest rates and debt accumulation opens the door to multiple equilibria.

When a crisis occurs—a switch from a good to a bad equilibrium path—bond prices jump in response to changes in future default probabilities, but the crisis can play out for some time before default actually occurs. We label this episode a slow moving crisis to distinguish it from a rollover crisis, which is essentially a run on the country's debt leading to a failed bond auction and immediate default. Both slow moving crises and rollover crises seem relevant to interpret observed turbulence in sovereign debt markets. However, the literature has mostly focused on the latter so our goal here is to highlight and study the slow moving mechanism.

[2] See the widget here.

Multiple equilibria arise due to a coordination problem across investors in sovereign debt markets. The precise way in which the borrower is modeled is not as crucial for multiplicity. In most of the paper, we simply take as given the behavior of the borrower, as summarized by a fixed fiscal rule that determines the primary surplus as a function of the debt level and shocks. Under a fiscal rule, default occurs mechanically when the borrower is unable to finance payments on the existing stock of debt. We also consider versions of our model where fiscal policy is endogenous, chosen by an optimizing government under discretion.

Multiple Equilibria. Our first contribution is a simple characterization of bond price schedules and debt dynamics when the stochastic process for the primary surplus is given by a fiscal policy rule. This construction extends the standard analysis of debt sustainability (e.g., Bohn, 1995; Hall, 2014) to the case of defaultable debt.

When all debt is in the form of short-term one-period bonds, we show that the bond price schedule is uniquely determined. However, this does not imply that the equilibrium path is unique because the revenue from debt issuance is not monotone in the amount of bonds issued—we call this revenue function a Laffer curve. Thus, for a given revenue that is needed by the government, there may be multiple bond prices that clear the market on different sides of the Laffer curve. With long-term debt we show the bond price schedule is no longer uniquely determined. Future bond prices now feed back into current bond prices. The coordination problem among investors takes an intertemporal dimension. As a result, the bad equilibrium cannot be prevented by coordinating current investors.

In our model, the self-fulfilling nature of crises is typically transitory. If the economy slips down the bad path long enough, eventually debt reaches a high enough level beyond which the good path is no longer possible. Although initially triggered by self-fulfilling pessimism, the crises itself eventually damages fundamentals. This suggests the importance of swiftly applying policies to escape or counteract a bad equilibrium, to avoid it from settling in.

Policy Rules and Debt Maturity. We use our model to investigate the role of the policy rule and of the debt maturity in making the economy vulnerable to self-fulfilling crises. We first show that when the fiscal rule is sufficiently active at reducing deficits when debt rises, the equilibrium is unique. The fiscal rule works to offset the negative feedback from interest rates. Higher interest rates induce a rise in debt, but if the borrower takes strong actions to repay when debt levels rises a self-fulfilling crisis is avoided. Note that we may never observe the need for such actions—all that matters is the expectation of such actions out-of-equilibrium. This may explain why some countries are more prone to such crises than others.

The usual requirement for stability without default is that the slope of the rule be greater than the interest rate (see, e.g., Hall, 2014). We show that the presence of default risk requires a more aggressive rule, not just because the interest rate is higher but because it responds endogenously to an increase in debt. These conditions are enough to guarantee local stability of a good equilibrium. However, our analysis also uncovers an inherent limitation in these local conditions, in that they cannot be met globally at all debt levels simply because the primary surplus is bounded above by finite tax capacity; Ghosh et al. (2011) refer to this problem as "fiscal fatigue." This creates the potential for the emergence of a bad equilibrium, even within regions where the fiscal rule is aggressive, in which debt is expected to growth to levels where the rule is insufficiently responsive.

We show that longer debt maturities contribute towards uniqueness. A short maturity requires constant refinancing, exposing the borrower to increases in interest rates and amplifying the feedback effects; in contrast, a longer maturity mutes these forces. This mechanism is distinct from the the one at play in rollover crises, where shorter maturities makes repayment more costly in the event of a rollover crisis, increasing the strategic motive for default.

Tipping Points. Our model captures the idea of a "tipping point" for debt dynamics, a notion which appears loosely in public debates on debt sustainability, but which acquires a precise form in our setting (see also Greenlaw et al., 2013). Even when the equilibrium is unique, there exists a threshold level of debt separating the good and bad paths for debt. Below this threshold, debt, interests rates and default probabilities fall over time; above it, they rise. By implication, the dynamics are very sensitive to debt and other variables near the tipping point. This captures a different source of volatility, with a equilibrium is unique outcomes cannot vary with non-fundamental shocks, yet they may be very susceptible to small changes in fundamentals.

Optimizing Government. Our focus on fiscal rules allows us to isolate the coordination problem amongst investors. However, our results and analysis do not depend on this modeling choice and we study version of our model with an optimizing borrower, maximizing expected utility under discretion.

With additive preferences, multiple equilibria arise from the non-negativity constraint on government consumption. If one ignores this constraint, the equilibrium is unique: there may be a tipping-point threshold, but it is uniquely pinned down by an arbitrage condition that equates the utility of the equilibrium path with falling and rising debt. However, this logic fails when one takes into account that spending cannot be negative: a government just above the threshold, facing high interest rates, may not be able to lower debt below the threshold, preventing the arbitrage condition. Importantly, the non-negativity constraint does not typically bind on the equilibrium path, but matters off the equilibrium path.

The more general point is that other relevant constraints lead to equilibrium multiplicity in a similar manner. Governments may face a host of restrictions, such as nonzero lower bounds on spending, upper bounds on taxes, or limits on the speed of change in spending and taxes. Such constraints may not bind along the equilibrium path, but they do condition deviations off the equilibrium path that set the stage for multiplicity.

Foundations for Timing and Interest Rate Ceilings. In most of the paper, we take the following approach: past debt obligations and the current surplus determine the government's financing needs, debt issuances and bond prices are then determined by the market. One may ask, why can the government not choose the quantity of bonds issued and select the equilibrium price? Our view is that this requires an implicit and strong commitment assumption: if the revenue of the bond issuance falls short, the government must adjust its surplus instantaneously and automatically. This seems implausible, as it requires an extremely responsive fiscal policy in the short run. Governments have limited room or desire for sharp adjustments in a such short time frame. Instead, it seems more reasonable to assume the the government will issue more bonds to make up for the lower price. This is the assumption we adopt.

At the same time, it is undeniable that in any given bond auction the government can enter with a pre-established offering and this seemingly provides the needed commitment. However, the government may quickly return to the market with another offering, undercutting its capacity to commitment in any given auction.

The two models in Section 5 provide a formalization of this argument. Namely, we formulate two explicit games with a government that issues bonds in repeated rounds. We assume the government can commit to the bond issuance for the current round, but not for future rounds. Importantly, preferences are not additively separable across rounds, so that lower spending today increases the desire for spending tomorrow. This assumption is especially natural given that each round is best interpreted as a short time interval, such as a day or a week.[3]

[3] Outlays on infrastructure investments have the desired property, e.g. lower spending on the bridge today requires more spending to complete the bridge tomorrow. The same is true for regular spending,

The first game assumes a potentially unlimited number of rounds. We show that the outcome is exactly as in the timing assumption adopted in most of the paper. The government loses all ability to commit to its bond issuance because it can always reverse or supplement issuances in a future round. The second game limits the number of rounds, but assumes the government is somewhat impatient to obtain funds in earlier rounds. Again, we obtain multiple equilibria of the same nature. Overall, these results indicate that lack of commitment may provide a game-theoretic foundation for the timing assumption we adopt.

Some authors, going back to Calvo (1988), have suggested a seemingly simple solution to rule out the bad equilibrium and select the good one: the government can announce a ceiling on interest rates, promising to abstain from bond issuances at higher interest rates. Our two explicit games clarify the inherent difficulties with this approach. In particular, an interest rate ceiling can only work if the government can back such an announcement with a credible commitment to slash spending, or ramp up taxes in the very short run, in the event of a shortfall in bond receipts. As we discussed above, such commitment powers seem implausible. Indeed, an interest rate ceiling violates the very spirit of the approach we pursue here. Unfortunately, in our view there appears to be no easy fix to the multiplicity problem, certainly not by way of interest rate ceilings.

Related literature. Aguiar and Amador (2013) provide a recent survey of the sovereign debt literature. The main precursor to our paper is Calvo (1988), who introduced the feedback between interest rates and the debt burden in a two-period model in which the government chooses its financing needs, while debt issuances and interest rates are jointly determined on the bond market. Our main goal is to reinvigorate this approach and expand it to a dynamic setting, more appropriate for the study of slow moving crises. A dynamic setting allows us to explore the conditions for multiple equilibria with respect to debt levels, debt maturity and fiscal policy rules. We also find that the nature of the multiplicity, coupled with stability refinements, is somewhat different in a more dynamic setting with long term bonds. A handful of recent papers also adopt elements of the Calvo (1988) approach. Corsetti and Dedola (2011) and Corsetti and Dedola (2013) investigate whether independent monetary policy is sufficient to insulate a government from confidence crises. Navarro et al. (2014) apply the same approach to a sovereign-default model with short-term debt similar to Arellano (2008), to investigate how likely are sovereign borrowers to enter the multiple equilibrium region.

such as on the government payroll and transfers, since temporary shortfalls in these outlays are possible by incurring an implicit and costly debt with government employees or pensioners.

Sovereign debt rollover crises were introduced by Giavazzi and Pagano (1989), Alesina et al. (1992) and Cole and Kehoe (1996).[4] Cole and Kehoe (2000) provides a workhorse model for the more recent literature, including Conesa and Kehoe (2012) and Aguiar et al. (2013). We see rollover crises and slow moving crises as complementary explanations for the turbulence in sovereign debt markets. Indeed, rollover crises are also possible in our model and slow moving crises are best thought of as less extreme versions of the same phenomena. For most of the paper we set rollover crises aside to focus on slow moving crises.

Our use of a fiscal rule in the baseline model follows the literature on debt sustainability (Bohn, 1995, 2005) and and the literature on the interaction of fiscal and monetary policy e.g. Leeper (1991). This literature asks what properties of fiscal policy rules ensures that holders of government debt are repaid with certainty at all future dates (Hall, 2014). Our paper extends this analysis to situations with positive default probability, asking what properties of fiscal policy rules avoid self-fulfilling debt crises.

2 Bond Price Schedules and Debt Dynamics

In this section, we show how to construct bond price functions and maximum debt revenue levels in a setup where the government is committed to follow a fixed fiscal rule and default occurs when the government is unable to raise enough debt revenue to cover its current financing needs.

Consider a discrete-time environments, with periods $t = 0, 1, 2, \ldots$ To simplify, we assume that all uncertainty is revealed at some finite date $T < \infty$. This assumption will be relaxed later, but it allows us to solve the model by backward induction, which is both revealing and simple, and ensures that multiplicity is not driven by an infinite horizon.

The government generates a sequence of primary fiscal surpluses $\{s_t\}$, representing total taxes collected minus total outlays on government purchases and transfers. A negative realization of s_t corresponds to a primary deficit.

The government issues non-contingent bonds in a competitive credit market to a continuum of risk-neutral investors with discount factor $\beta = 1/\left(1+r\right)$. Bonds have geometrically decreasing coupons: a bond issued at t promises to pay the sequence of coupons

$$\kappa, (1-\delta)\,\kappa, (1-\delta)^2\,\kappa, \ldots,$$

[4] Chamon (2007) argues that the coordination problem in rollover crises can be avoided by appropriately designing the way in which bonds are underwritten and offered for purchase to investors by investment banks.

where $\delta \in (0,1)$ and $\kappa > 0$. We normalize and set $\kappa = \delta + r$, so that the bond price equals 1 when the risk of default is zero at all future dates. This well-known formulation of long-term bonds is useful because it avoids having to carry the entire distribution of bonds of different maturities (see Hatchondo and Martinez, 2009). A bond issued at $t - j$ is equivalent to $(1-\delta)^j$ bonds issued at t, so the vector of outstanding bonds can be summarized by a single state variable b_t, which is equal to total debt in terms of equivalent newly issued bonds. The parameter δ controls the maturity of debt, with $\delta = 1$ corresponding to the case of a short-term bond and $\delta = 0$ corresponding to the case of a consol.

Absent default, the government budget constraint is

$$s_t + q_t(s^t) \cdot \left(b_{t+1}(s^t) - (1-\delta)\, b_t(s^{t-1})\right) = \kappa b_t(s^{t-1}), \tag{1}$$

where q_t is the price of a newly issued bond. Coupon payments on outstanding bonds are covered either by the primary surplus or by sales of newly issued bonds.

The fiscal policy rule and shocks to spending and taxes are all embedded in the stochastic process for the primary surplus, governed by

$$F(s_t \mid s^{t-1}, b_t),$$

a conditional cumulative distribution function, where $s^{t-1} = (s_1, s_2, \ldots, s_{t-1})$ denotes a history up to period $t-1$. We assume that s_t is bounded above: $s_t \leq \bar{s} < \infty$. At date T all uncertainty is resolved and the surplus is constant at s_T from then on. We allow the distribution of s_t to depend on current debt b_t to capture policy rules where the government responds to higher debt levels with fiscal efforts to cut spending or raise taxes. Fiscal rules of this kind are commonly adopted in the literature studying solvency (e.g. Bohn, 2005; Ghosh et al., 2011).

We assume that the government honors its debts whenever possible, so that default occurs only if the surplus and borrowing are insufficient to refinance outstanding debt. Let $\chi(s^t) = 1$ denote full repayment and $\chi(s^t) = 0$ denote default. We assume that after a default event debtors receive a recovery value $v_t\left(s^t\right) \geq 0$. Therefore, bond prices at date t satisfy

$$q\left(s^t\right) = \beta \mathbb{E}\left[\chi(s^{t+1})\left(\kappa + (1-\delta)\, q(s^{t+1})\right) + \left(1 - \chi(s^{t+1})\right) \frac{v\left(s^{t+1}\right)}{b_{t+1}\left(s^t\right)} \mid s^t\right]. \tag{2}$$

Our focus is on debt dynamics preceding default. Consequently, we characterize the equilibrium up to the first default episode and only derive the path for debt and prices

$b_{t+1}(s^t)$ and $q_t(s^t)$ along histories for which $\chi(s^\tau) = 1$ for all $\tau \leq t$.

Equilibrium. We derive the equilibrium conditions by backward induction. After period T the surplus is constant, so the government repays if and only if the present value exceeds the debt burden, or

$$s_T \geq r b_T.$$

Let the repayment function $X_T\left(b_T, s^T\right)$ equal 1 if this condition is met and 0 otherwise. Note that we have written the last period budget constraint as an inequality, instead of an equality, to allow larger surpluses than those needed to service the debt.[5]

We now have all the elements to compute the price of debt in period $T-1$ for every possible value of b_T,

$$Q_{T-1}(b_T, s^{T-1}) \equiv \beta \mathbb{E}\left[X_T(b_T, s^T)\,(\kappa + 1 - \delta) + \Big(1 - X_T(b_T, s^T)\Big)\frac{v\left(s^T\right)}{b_T} \mid s^{T-1}, b_T\right].$$

The maximal revenue from new debt issuances in period $T-1$ equals

$$m_{T-1}(b_{T-1}, s^{T-1}) \equiv \max_{b_T}\Big\{Q_{T-1}(b_T, s^{T-1}) \cdot (b_T - (1-\delta)b_{T-1})\Big\}.$$

Default can be avoided if and only if

$$\kappa b_{T-1} - s_{T-1} \leq m_{T-1}(b_{T-1}, s_{T-1}).$$

Let the repayment function $X_{T-1}(b_{T-1}, s^{T-1})$ equal 1 if this inequality is met and 0 otherwise. Whenever $X_{T-1}(b_{T-1}, s^{T-1}) = 1$ we select a value of b_T that solves

$$s_{T-1} + Q_{T-1}(b_T, s^{T-1}) \cdot (b_T - (1-\delta)b_{T-1}) = \kappa b_{T-1},$$

and denote the selected value as $B_T(b_{T-1}, s^{T-1})$.

The same procedure can be applied for earlier periods. Supposing we have $Q_{t+1}, m_{t+1}, X_{t+1}$ and B_{t+2}, we can construct Q_t, m_t, X_t and B_{t+1} as follows. In period t the price function

[5] One should hence interpret s_T as the potential maximal surplus, not the actual surplus. Of course, if $s_T > r b_T$, then any slack would be redirected towards lower taxes or increased spending and transfers. We abstract from describing this adjustment.

Q_t is given by

$$Q_t(b_{t+1}, s^t) \equiv \beta \mathbb{E}\Big[X_{t+1}(b_{t+1}, s^{t+1})(\kappa + (1-\delta)\, Q_{t+1}(B_{t+2}(b_{t+1}, s^{t+1}), s^{t+1})) \\ + (1 - X_{t+1}(b_{t+1}, s^{t+1}))\frac{v(s^{t+1})}{b_{t+1}} \Big| s^t, b_{t+1}\Big], \quad (3)$$

the maximal revenue function m_t is given by

$$m_t\left(b_t, s^t\right) \equiv \max_{b_{t+1}} \left\{Q_t(b_{t+1}, s^t) \cdot (b_{t+1} - (1-\delta)b_t)\right\}, \quad (4)$$

and let the repayment function $X_t(b_t, s^t) = 1$ if $s_t + m_t\left(b_t, s^t\right) \geq \kappa b_t$ and 0 otherwise. Whenever $X_t(b_t, s^t) = 1$ we selects a value of b_{t+1} solving

$$s_t + Q_t\left(b_{t+1}, s^t\right)(b_{t+1} - (1-\delta)b_t) = \kappa b_t. \quad (5)$$

and denote the selected value by $B_{t+1}(b_t, s^t)$.

Continuing in this way, we can solve iteratively for the functions m_t, Q_t, X_t, B_{t+1} in all earlier periods. In general, equation (5) admits multiple solutions so there may be multiple sequences of functions $\{m_t, Q_t, X_t, B_{t+1}\}$. For any given sequence, we can construct equilibrium outcome paths by iterating on $b_{t+1}(s^t) = B_{t+1}(b_t(s^t), s^t)$ starting with the given level of initial debt b_0; evaluating $q_t(s^t) = Q_t(b_t(s^t), s^t)$ then gives the sequence of bond prices.

Multiplicity, Selection and Timing Assumptions. If at each juncture in the backward induction one always selects the lowest b_{t+1} solving equation (5) the recursion pins down a unique sequence $\{m_t, Q_t, X_t, B_{t+1}\}$ and a unique outcome $b_t(s^t)$ and $q_t(s^t)$.

Proposition 1. *There is a unique equilibrium satisfying the requirement that $B_{t+1}(b_t, s^t)$ always equal the lowest possible solution to* (5).

If, instead, we do not impose a selection criterion, multiple equilibria arise whenever (5) admits multiple solutions for some realizations of s^t and b_t. This will be the approach we follow in the rest of the paper.

Let us conclude this section discussing briefly alternative approaches to selection and multiplicity. The sovereign-debt literature following Eaton and Gersovitz (1981)—and the subsequent quantitative literature started by Arellano (2008)—make the assumption that the government chooses the quantity of bonds it auctions each period and then investors bid and determine the bond price. This timing assumption acts as a selection criterion picking a unique value of b_{t+1} in each period, and thus leads to uniqueness.[6] Implicit in this timing is an assumption that the government has the ability to commit to a given bond issuance, both on and off the equilibrium path. However, committing to a given bond issuance requires a credible commitment to cut spending or raise taxes if bond prices turn out lower than expected. In practice, the credibility of an announcement of this kind seems questionable.[7]

Suppose a government is running a bond auction every week. On a given week, the government announces it will sell bonds with a face value of \$100bn, expecting a price of \$1, and will use the revenue to finance its current deficit needs of \$100bn. If the bond price turns out to be 90 cents, the government is short \$10bn. Should we expect the government to immediately implement tax increases or spending cuts for \$10bn, or should we expect it to increase its bond issuances in the following weeks? We see the second form of adjustment as more plausible in the short run.

In the model presented here, in which the government follows a fiscal rule, the primary surplus is fixed at the beginning of each period. In this context, we are naturally led to choose the timing assumption in which spending and taxes are fixed in the short run and, given bond prices, the government adjusts bond issuances to satisfy the budget constraint (5). All solutions to (5) are legitimate equilibrium outcomes.

A more general approach would be to allow for both margins of adjustment, and assume the fiscal rule adjusts the primary surplus *and* bond issuances to the current bond price. If it responds to the current price sufficiently, then this can rule out multiplicity. The idea is very similar to the logic that rules out multiplicity with a policy rule that reacts to the stock of accumulated debt, the main focus of this paper. However, we think this latter margin is more realistic.

We shall return to timing and commitment issues towards the end of the paper, in Section 5.

[6] In finite horizon versions of the models in Eaton and Gersovitz (1981) and Arellano (2008) the backward induction argument behind Proposition 1 can be easily extended to prove uniqueness. In the infinite horizon case things are more difficult, since a backward induction argument is unavailable. Auclert and Rognlie (2014) show uniqueness in the infinite horizon case under some conditions.

[7] Given a non-negativity constraints on spending and an upper bound on tax revenues, one also has to worry about the very feasibility of committing to a given level of bond issuances for any realization of the price q_t off the equilibrium path. Consider a setup with short-term debt, where the budget constraint is $q_t b_{t+1} \geq b_t - s_t$. Given a committed level b_{t+1}, if the price is low enough and the maximum achievable primary surplus is too low to fully repay b_t, then it is infeasible for the government to fulfill its commitment: either b_{t+1} has to be increased, or the government has to default. The timing in Cole and Kehoe (2000), by allowing for default triggered by low bond prices, has the virtue of ensuring that payoffs are well defined for all possible strategies.

3 Short and Long Debt

In this section we specialize the model laid out in the previous section. We first consider the case of short-term debt. We then consider a specific environment with long-term debt. In both setups we introduce a debt Laffer curve and investigate the relation between the shape of this curve and multiplicity.

3.1 Short-Term Debt

With short term debt, $\delta = 1$, the price functions are uniquely determined by backward induction.

Proposition 2. *With short-term debt $\delta = 1$ the functions m_t, Q_t, and X_t are uniquely defined and the function $m_t(b_t, s^t)$ does not depend on b_t.*

Laffer Curves and Multiplicity. Proposition 2 does not imply that the equilibrium path is unique. Multiple equilibria exist whenever

$$Q_t(b_{t+1}, s^t)b_{t+1} = (1+r)\, b_t - s_t \tag{6}$$

admits multiple solutions for b_{t+1}. The expression $Q_t(b_{t+1}, s^t)b_{t+1}$ defines a relation between debt issuance b_{t+1} and revenue. This Laffer curve, as we call it, is in general non-monotone because an increase in b_{t+1} reduces the probability of repayment at $t+1$, reducing the price that investors are willing to pay for bonds.

To see this, consider first the case with zero recovery, $v_t\left(s^t\right) = 0$ for all s^t. The bond price $Q_t(b_{t+1}, s^t)$ is then zero for b_{t+1} sufficiently large.[8] This ensures that the Laffer curve has at least two sides, implying the existence of multiple equilibrium paths.

Proposition 3 (Multiplicity)**.** *Suppose there is no recovery value, $v(s^t) = 0$. If $(1+r)b_0 - s_0 < m_0(b_0, s^0)$ then there are at least two equilibrium paths for debt and interest rates.*

The left panel of Figure 1 plots the Laffer curve. The amount the government must finance $(1+r)\, b_t - s_t$ is represented by the dashed horizontal line. There are two equilibrium values for b_{t+1}. The high-debt equilibrium is sustained by a higher interest rate that is self fulfilling: a lower bond price forces the government to sell more bonds to meet

[8] This follows because default at $t+1$ is certain if $m_t\left(b_{t+1}, s^{t+1}\right) + s_{t+1} < (1+r)\, b_{t+1}$ for b_{t+1} large enough, given a bounded support for s_{t+1}.

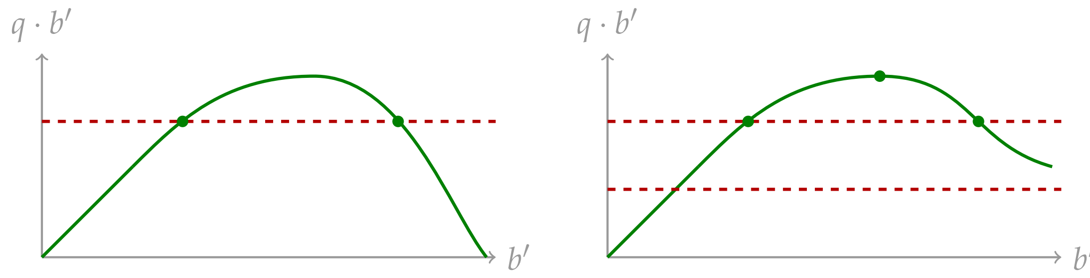


Figure 1: The Laffer curve: zero recovery (left) and positive recovery value (right).

its financial obligations; this higher debt leads to a higher probability of default in the future, lowering the price of the bond, which justifies the pessimistic outlook. This two-way feedback between high interest rates and high debt opens the door to multiple equilibria.

With a positive recovery value the Laffer curve converges to the recovery value for high debt. This implies a unique equilibrium for low enough financing needs. This case is illustrated in Figure 2.

Proposition 4. *Suppose the recovery value is bounded away from zero, $v(s^t) \geq \phi > 0$. Then for any history s^t, there is a unique b_{t+1} that solves (6) if $(1+r)\,b_t - s_t$ is low enough.*

Figure 1 illustrates that a sufficient reduction in the initial debt level b_t shifts the dashed red line downwards, eliminating the bad equilibrium.

What are the effects of inherited debt b_t? Along the good side of the Laffer curve an increase in b_t raises the current interest rate as well and the entire path of bond issuances and interest rates. Higher debt also increases the potential for multiple equilibria. This creates a feedback, from past equilibria selection, into the future. For example, if a bad equilibrium is selected at t, this raises b_{t+1} and the interest rate at $t+1$ even if the good equilibrium is expected to be played at $t+1$. It also raises the potential for equilibrium multiplicity. Notably, with short debt, there is no feedback running in the opposite direction, from the future to the present: as implied by Proposition 2 the Laffer curve $Q_t(b', s^t)b'$ is uniquely determined and independent of the equilibrium selection in future periods. Thus, expectations of bad equilibrium selection in the future have no effects in the current period. As we shall see in Section 3.2, this conclusion is no longer true with long-term debt.

Stable Equilibrium Selection. Equilibrium points where the Laffer curve is locally decreasing are "unstable" in the Walrasian sense that a small increase in the price of bonds reduces the supply by more than the demand, creating excess demand. These equilibria are also pathological on other grounds. First, they are also unlikely to be stable with

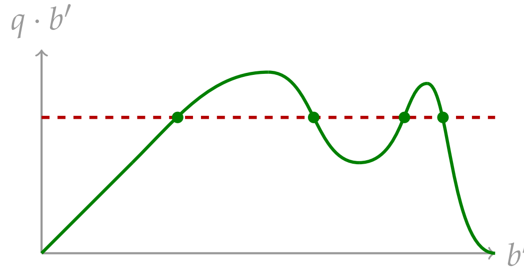


Figure 2: A case with 4 equilibria.

respect to most forms of learning dynamics. Second, Frankel, Morris and Pauzner (2003) show that global games do not select such equilibria. Finally, these equilibria lead to counterintuitive comparative statics: an increase in the current debt level b_t increases the bond price q_t, i.e. higher financing needs lower the equilibrium interest rate.

For all these reasons, we discard unstable equilibria and only consider stable equilibria, which in the present context requires the Laffer curve to be locally increasing.

With short term debt, multiple stable equilibria can only be obtained with primitives that generate a Laffer curve with multiple peaks. If the surplus process is fully exogenous (i.e., does not respond to b_t) and there is no recovery, one requires assumptions on the surplus distribution. Our next result shows that one needs a non-monotone hazard rate.

Proposition 5. *Suppose the recovery value is zero $v(s^t) = 0$ and the surplus process is independent of the debt level. If the hazard rate $\frac{f(s_t|s^{t-1})}{1-F(s_t|s^{t-1})}$ is monotone then the Laffer curve is single peaked and there is a unique equilibrium path that is interior and stable.*

One way to obtain multiple steady state equilibria is to specify distributions with non-monotone hazard rates. Navarro et al. (2014) show that a specification with a form of disaster risk can yield multi peaked Laffer curves and explore their implications. Another approach is to consider responsive policy rules. When debt affects the distribution for surplus this may generate multiple peaks, even if the conditional distribution satisfies the monotone hazard condition. However, as we shall see, once we introduce long term debt, multiple stable equilibria are possible even when the Laffer curve is single-peaked.

Rollover Crises. The same Walrasian tatonnement logic also leads us to consider the possibility of a different type of equilibrium, which is similar to a rollover crisis a la Cole and Kehoe (2000). Consider the left panel of Figure 1. Suppose we start immediately to the right of the bad, unstable equilibrium. At that point, there is an excess supply of bonds so bond prices should fall, pushing us further to the right. This continues until we reach a level of bond issuance associated with a zero probability of repayment and zero bond prices. This suggests the presence of a third, stable equilibrium, with a zero bond price and default in period t; a similar logic applies with a positive recovery value, for sufficiently high financing needs $(1+r)b_t - s_t$. To allow for this possibility, we must relax the assumption that default only happens when there is no issuance b_{t+1} that allows the government to finance $b_t - s_t$, and introduce the possibility of default triggered by zero bond prices today. This possibility can be interpreted as a rollover crisis, where pessimistic investors force immediate default and this default makes future repayments impossible, justifying investors' expectations. As discussed in the introduction, rollover crises are the focus of Cole and Kehoe (2000) and a large subsequent literature. For the remainder of this paper we set rollover crises aside to focus on slow moving crises. However, it is important to keep in mind that when one takes rollover crises into account, then multiplicity is relatively pervasive with short term debt, even when the Laffer curve is single peaked.

3.2 Long-Term Debt

We now turn to our benchmark case with long-term bonds. Long-term debt allows us to fully develop our idea of slow-moving crises, in which higher spreads lead to a gradual increase in debt servicing costs, as the government only replaces a fraction of maturing bonds with newly issued ones.

It is important for two other reasons. First, in most advanced economies the bulk of the debt is in the form of relatively long maturities, rather than three-month or one-year bonds (Arellano and Ramanarayanan, 2012). For example, the average maturity of sovereign debt for Greece, Spain, Portugal and Italy was 5-7 years over the 2000-2009 period. Second, it is widely believed that short-term borrowing exposes a sovereign borrower to debt crises and that longer maturities provides some protection. In the context of a rollover debt crises, Cole and Kehoe (1996) find support for this view. It is of interest to see if the same is true for slow moving debt crises.

A Special Case with Uncertainty at T. We first consider a case where all uncertainty is concentrated at some date $T > 0$. This special case is insightful because it can be mapped into the Laffer curve diagram, providing a bridge to the results with short term debt.

For $t \geq T$ the surplus is constant at $s_t = rS$, with S drawn from a continuous distribution $F(S)$. For $t < T$, the surplus is given by

$$s_t = (1-\lambda)\, s_{t-1} + \lambda\,(\alpha_0 + \alpha_1 b_t)\,. \tag{7}$$

Here $\alpha_0 + \alpha_1 b_t$ is a target surplus, which is increasing in the debt level if $\alpha_1 > 0$. The sur-

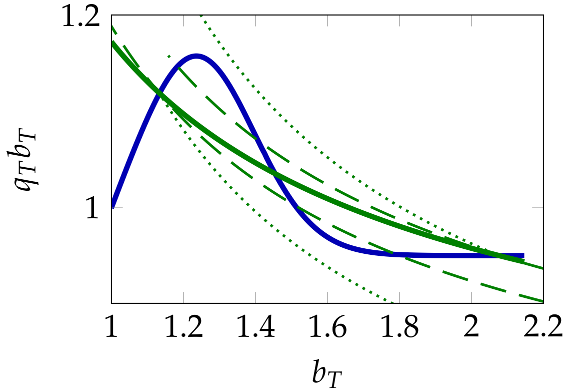


Figure 3: Solid green line shows $t = 0$, dashed green line $t = 14$, dotted green line $t = 35$.

plus adjusts to its target at a speed determined by λ. The parameter α_1 plays an important role:" larger values represent a more aggressive fiscal response to debt.

Given the concentration of news at T, default during the first stage can only occur at $t = 0$. We proceed by assuming that default does not occur during the first stage. If this construction does not yield an equilibrium, default is unavoidable and occurs at $t = 0$. Assuming no default during the first stage, for $t < T - 1$, the bond price satisfies the difference equation (using the normalization for κ)

$$q_t - 1 = \frac{1-\delta}{1+r}(q_{t+1} - 1), \tag{8}$$

In period $T - 1$ just before the resolution of uncertainty

$$q_{T-1} = 1 - F(b_T) + \beta \frac{\phi}{b_T} \int_0^{b_T} S\, dF(S). \tag{9}$$

The government's budget constraint for $t < T$ can be written as (using, once again, the normalization for κ)

$$s_t + (q_t - 1)\,(b_{t+1} - (1-\delta)b_t) + b_{t+1} = (1+r)b_t. \tag{10}$$

Equations (7), (8) and (10) describe the dynamics for s_t, q_t and b_t, respectively. The initial values for debt and surplus, b_0 and s_0, are given, while equation (9) provides a boundary condition for q_{T-1}.

To find an equilibrium, we proceed as follows. For each candidate value for the initial price $q_0 \in (0, \infty)$, we solve (7), (8) and (10) forward and find terminal values b_T and q_{T-1}. If these values satisfy (9) we have an equilibrium. Equation (9) is equivalent to

$$q_{T-1} b_T = (1 - F(b_T)) b_T + \beta\phi \int_0^{b_T} S \, dF(S),$$

This is a Laffer curve similar to the one analyzed in the case of short-term debt.

This construction is represented graphically in Figure 3 plotting b_T against $q_{T-1}b_T$. The single peaked curve is the Laffer curve. The downward sloping curve plots the values of b_T and $q_{T-1}b_T$ obtained by solving (7), (8) and (10) forward for all values of $q_0 \in (0, \infty)$. Equilibria are represented by intersections of these two curves. The following parameters were chosen to capture some elements of the Italian case

$$T = 120, \quad \delta = \frac{1}{7} \cdot \Delta, \quad \beta = 1.02^{-\Delta}, \quad \phi = 0.7, \quad \log S \sim N\left(0.3, 0.1^2\right),$$

Here Δ represents the length of a time period, measured in annual terms. We set $\Delta = \frac{1}{12}$ to represent a month. In our example, all uncertainty is resolved in 10 years and average debt maturity is 7 years. The risk-free interest rate is 2% and the recovery rate in case of default is 70%. The distribution of the present value of surplus, after uncertainty is resolved has mean 1.357 and standard deviation 0.136. It is useful to interpret all these numbers as being relative to a country's GDP. The initial conditions are set to

$$s_0 = -0.1 \cdot \Delta, \quad b_0 = 1,$$

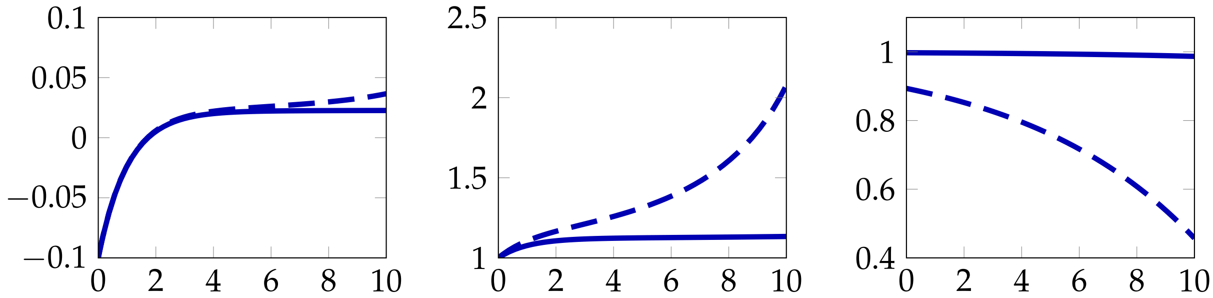
and the fiscal policy parameters are

$$\lambda = \Delta, \quad \alpha_0 = 0, \quad \alpha_1 = 0.02,$$

so that a unit increase in the debt stock implies a fiscal effort to increase the surplus by 0.02 in annual terms; note that this would exactly stabilize the debt with a risk free interest of 2%.

Figure 3 shows the presence of three equilibria as the intersection of the two solid lines (the dashed lines will be described below). Importantly, both the first and third equilibrium are stable, in the sense discussed earlier. Thus, in the presence of long-term debt it is possible to obtain multiple equilibria that are interior and stable even with a Laffer curve that is singled peaked. Figure 4 shows the dynamics for the primary surplus, debt and bond prices for the two stable equilibria, which we term "good" (solid lines) and "bad" (dashed lines).

This stylized example is able to capture various features of recent episodes of sovereign market turbulence. Sovereign bond yields experience a sudden and unexpected jump,



(a) Primary Surplus (b) Debt (c) Bond Price

Figure 4: Dynamics over 10 years (120 month period) good (solid) and bad (dashed line) equilibrium.

when shifting unexpectedly from the good to the bad equilibrium path. The debt-to-GDP ratio increases slowly but steadily. Auctions for new debt issues continue without any signs of illiquidity, but interest rates climb along with the level of debt. Large differences in debt dynamics appears gradually, as bond prices diverge and a larger fraction of debt is issued at crisis prices.

Eventual Uniqueness. A characteristic feature of our model is that multiplicity only plays out in the early phase of a crisis. Indeed, along both equilibrium paths, multiplicity eventually vanishes. Figure 3 illustrates this point. Each dashed and dotted line corresponds to a different time horizon and initial debt condition. In particular, we plot them for $t = 14$, and $t = 35$ and use as initial conditions the values of s_t and b_t reached under the good and the bad equilibrium paths from Figure 4. At $t = 14$ the two dashed lines show that multiplicity is still present. For example, the top dashed line indicates that it would be possible to snap out of a crisis after enduring the bad path for 14 months and switch to a good path.[9] However, after the crises has gone on for 35 months a switch is no longer possible, the dotted line representing $t = 35$ features only one intersection.

There are two reasons why multiplicity disappears as we approach T. First, there is a direct or mechanical effect, since the remaining time for debt to accumulate or decumulate shortens, weakening the feedback loop that gives rise to multiplicity. Second, along the bad path debt may grow to a high enough level making the bad path the only possible outcome; conversely, along the good path debt may fall to a low enough level making the good path the only possible outcome. In Section 4.1 we consider a stationary model where this first effect is not present, but the second effect still implies that the intermediate levels of debt where multiplicity is possible is transitory.

[9]Clearly, the switch needs to be unexpected for prices to be in equilibrium between $t = 0$ and $t = 14$.

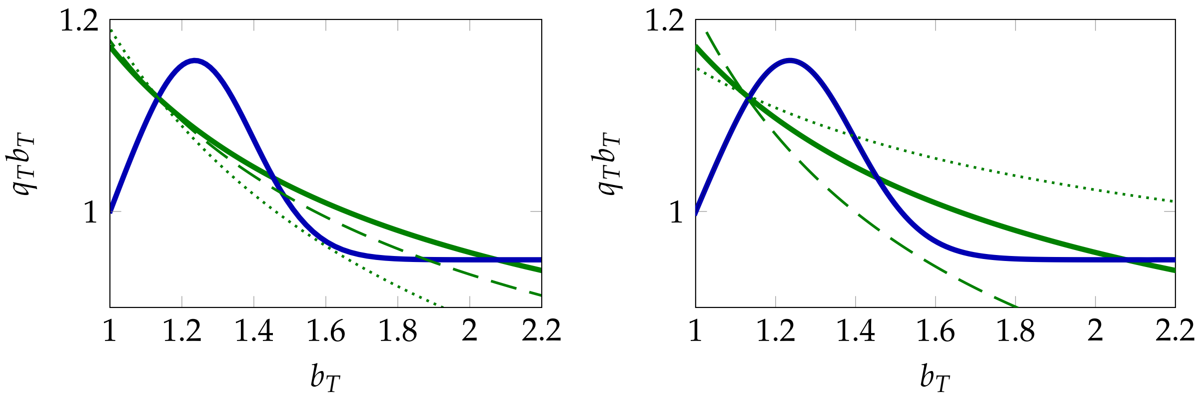


Figure 5: Left Panel: solid green line $\alpha_1 = 0.02 \cdot \Delta$, dashed green line $\alpha_1 = 0.03 \cdot \Delta$, dotted green line $\alpha_1 = 0.05 \cdot \Delta$. Right panel: solid green line $\delta = \frac{1}{7}\Delta$, dashed green line $\delta = \frac{1}{10}\Delta$, dotted green line $\delta = \frac{1}{5}\Delta$.

Note that even if we switch from the bad path to the good path, unexpectedly, the economy inherits a higher debt level and higher interest rates with it. This can be seen in the figure: the new leftmost intersection with the upper dashed line is further to the right, up the good side of the Laffer curve, relative to the intersection of the two solid lines, representing the outcome if the economy had never ventured off the good equilibrium path.

Fiscal Rules. How does the fiscal policy rule affect the equilibrium or the existence of multiple equilibria? Figure 5 shows the effects of increasing α_1, while adjusting α_0 to keep the good equilibrium unchanged. As shown, a high enough value for α_1 rules out the bad equilibrium. As investors contemplate the effect of lower bond prices and increased debt issuances that entails, they also realize that the government will make a greater fiscal adjustment in the future. This helps counter the feedback loop between interest rates and debt.

Of course, an extremely responsive fiscal rule might, in principle, ensure that debt is stabilized for any sequence of interest rates. Or, even more extreme, prevent default altogether. It is important to emphasize that nothing as drastic as this is required to obtain a unique equilibrium. In our parameterization, lower bond prices do lead to higher debt accumulation; higher values of α_1 mitigate the rise in debt, but never offset it completely.

Initial Debt. What are the effects of initial debt? Figure 6 shows the parameter space (α_1, b_0) and divide it into four regions. We now make no adjustment to α_0. In the red region there is a single equilibrium, in the bottom portion debt is low and on the good side of the Laffer curve, while in the upper portion (above pink region) the unique equilibrium lies on the bad side of the Laffer curve. There are three equilibria in the pink region, just as in our calibrated example. In the yellow region no equilibrium with debt exists, implying immediate default at $t = 0$.

Consider for example, the case $\alpha_1 = 0.01$ in the graph, in which four cases are possible. For low levels of b_0, we get a unique equilibrium on the increasing portion of the Laffer curve (lower portion of the red region). For higher levels of b_0, we have three equilibria, as depicted in Figure 6 (pink region). For even higher levels of b_0, we have a unique equilibrium again, but this time on the bad side of the Laffer curve. Finally, for very high values of b_0, there is no equilibrium without default.

Debt Maturity. Consider next the impact of debt maturity, captured by δ. Figure 5 shows the effects of varying δ around our benchmark value, while adjusting α_0 to keep the good equilibrium unchanged. A longer maturity, with a low enough value for δ, leads to a unique equilibrium. Intuitively, shorter maturities require greater refinancing, increasing the exposure to self-fulfilling high interest rates. The debt burden of longer maturities, in contrast, is less sensitive to the interest rate, mitigating the feedback loop that leads to equilibrium multiplicity.

The right panel in Figure 6 is similar to the left panel, but over the parameter space (δ, b_0) instead of (α, b_0). Again, we divide the figure into four regions. There are three equilibria in the pink region, just as in our calibrated example. In the red region there is a single equilibrium. In the bottom portion of the red region the equilibrium lies on the good side of the Laffer curve, while in the upper portion (above the pink region) it is on the bad side of the Laffer curve. In the yellow region no equilibrium exists, implying immediate default at $t = 0$.

For given δ, we see the same effects with respect the level of initial debt. Turning to the effects of δ, shorter maturities, higher values for δ, place the economy in a "danger zone" (pink region) with 3 equilibrium values for the interest rate. Still higher values for δ may lead to a unique bad equilibrium (upper red region) or to non-existence prompting immediate default (upper right, yellow region). These last two conclusions depend on the fact that α_0 was not adjusted here, unlike in Figure 5.

4 Tipping Points and Optimizing Governments

This section explores a stationary setting where all uncertainty is resolved after the arrival of a Poisson shock. For convenience, we now work in continuous time, which allows us to study the model's dynamics on a phase diagram. We first consider fiscal policy described

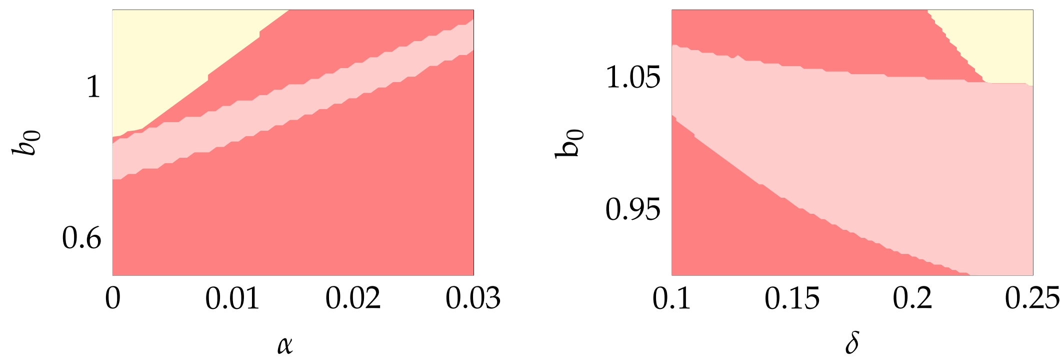


Figure 6: Regions with unique equilibrium (red), three equilibria (pink) or immediate default (yellow).

by a fiscal rule and then consider an optimizing government with additive preferences.

4.1 Fiscal Rules

Time is continuous and runs forever. Investors are risk neutral discounting at rate r. Bonds issued at time t pay a coupon $\kappa e^{-\delta(\tau-t)}$ for all $\tau > t$, the continuous-time analog of the bonds used in previous sections. We adopt the normalization $\kappa = r + \delta$ again, so that the bond price equals 1 in the absence of default risk.

The primary surplus evolves in two stages. In the first stage, it is a deterministic function of the stock of outstanding debt b,

$$s = h(b),$$

where h is a weakly increasing function with $h(b) = \bar{s} > 0$ for $b \geq \bar{b}$. For simplicity, this fiscal rule imposes a direct relationship between b and s in levels, rather than of a relation between b and the rate of change of s as in Section 3.2.

At a Poisson arrival rate λ we reach the second stage, where all uncertainty is resolved. The present value of future surpluses, S, is drawn from a continuous distribution $F(S)$ over $[\underline{S}, \overline{S}]$, with $\underline{S} \geq 0$. If $S \geq b$ default is avoided and the bond price equals 1. If $S < b$ bond holders obtain a recovery value ϕS, with $\phi < 1$. Thus, the bond price upon entering the second stage but immediately before the resolution of uncertainty is

$$q = \Psi(b) \equiv 1 - F(b) + \frac{\phi}{b} \int_{\underline{S}}^{b} S \, dF(S).$$

We now derive the dynamics for q and b during the first stage. The assumption that

$\bar{s} > 0$ ensure that default never occurs within the first stage, as we discuss further below. Thus, in the first stage the bond price solves

$$rq = \kappa - \delta q + \dot{q} + \lambda \left(\Psi \left(b \right) - q \right), \tag{11}$$

Bonds decay at rate δ, pay coupon κ, earn a capital gain $\dot{q}$ before uncertainty is revealed, and have an expected capital gain $\Psi \left(b \right) - q$ with Poisson arrival λ.

The government budget constraint is

$$h \left(b \right) + q \left(\dot{b} + \delta b \right) = \kappa b, \tag{12}$$

The dynamics of q and b solve the ODEs (11)–(12). We develop boundary conditions below.

Steady States. Steady states are found at the intersection of the loci $\dot{q} = 0$ and $\dot{b} = 0$. The locus $\dot{q} = 0$ is given by

$$q = \frac{\kappa + \lambda \Psi \left(b \right)}{r + \delta + \lambda} \tag{13}$$

and is downward sloping since $\Psi' \left(b \right) < 0$. A larger stock of bonds implies a smaller probability of repayment after the resolution of uncertainty, lowering bond prices.

The locus $\dot{b} = 0$ is given by

$$q = \frac{\kappa b - h \left(b \right)}{\delta b} \tag{14}$$

and is decreasing if and only if

$$h' \left(b \right) > \kappa - \delta q. \tag{15}$$

We assume that there is a range of b for which (15) is satisfied, so the $\dot{b} = 0$ locus is decreasing on that range. As we shall see, this assumption is necessary to ensure the existence of a stable steady state. On the other hand, our assumption that h is bounded above at $\bar{s}$ implies that for high enough levels of debt (15) is violated and the locus $\dot{b} = 0$ is increasing.

In Figure 7 we show two examples. In each graph, the blue line represents the $\dot{q} = 0$ locus and the green line the $\dot{b} = 0$ locus. Both examples display two steady states. A low-debt steady state, in which bond prices are high, debt issuances $\delta q b$ cover a large fraction of the coupon payments κb and the government runs a low primary surplus, consistent with its policy rule $h(b)$. And a high-debt steady state, in which bond prices are low and debt issuances cover a smaller portion of coupon payments, so the government needs to run a larger surplus. We will return shortly to the dynamics of these two examples.

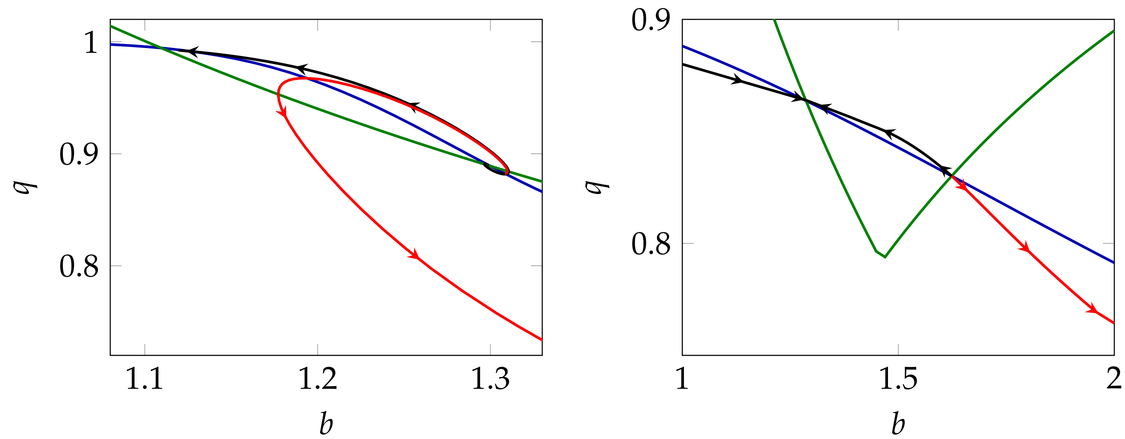


Figure 7: Phase diagrams with fiscal rule.

Boundary Conditions. The ODEs must be complemented by boundary conditions. We consider two types of boundary conditions: either the economy converges to a steady state with constant b and q, or it converges to a path with ever growing debt and ever decreasing bond prices.

The path with ever growing debt can be characterized analytically using the fact that $h(b)$ and S are bounded above and that these bounds become binding for large enough b. Given these properties, it is easy to show that there exists a cutoff $\hat{b}$ such that for any initial debt level $b(0) \geq \hat{b}$, there is a path for b and q that satisfies (11)–(12), $b \to \infty$ and $q \to 0$.[10] Along this path the primary surplus is constant at $\bar{s} > 0$, lenders anticipate certain default as soon as stage 2 is reached, and the total value of debt is constant and equal to

$$bq = \hat{v} \equiv \frac{\bar{s} + \lambda \Psi\left(\overline{S}\right) \overline{S}}{r + \lambda}.$$

Effectively, an investor that always buys new issuances gets the expected value of the surplus in stage 1 plus the expected recovery value in stage 2. Therefore, we use as boundary condition the point $(\hat{q}, \hat{b})$, where $\hat{q} \equiv \hat{v}/\hat{b}$.

It is simple to impose alternative boundary conditions. For example, we could assume that whenever debt reaches some high arbitrary threshold $\hat{b}$ this triggers a renegotiation between investors and the borrower, perhaps intermediated by the IMF or some other organization. As long as we can predict the outcome of such a renegotiation, this pins down the value of debt at $\hat{b}$, providing a boundary condition to solve the ODE system.

[10]The cutoff is

$$\hat{b} = \max\left\{\bar{s}, \overline{S}, \left[\bar{s} + \delta \hat{v}\right] / \kappa\right\},$$

with $\hat{v}$ defined below in the text.

Stable Steady States. For a steady state to serve as a boundary condition it must be locally stable. Our first result provides a necessary and sufficient condition for local (saddle-path) stability.

Lemma 1. *A steady state with positive debt is locally stable if and only if the $\dot{b} = 0$ locus is downward sloping and steeper than the $\dot{q} = 0$ locus, or equivalently,*

$$h'(b) > \kappa - \delta q - \frac{\delta\lambda}{r+\delta+\lambda}\Psi'(b)\, b. \tag{16}$$

In a steady state with no default risk $q = 1$ and $\lambda\Psi'(b) = 0$, so condition (16) reduces to $h'(b) > r$. This is the standard stability condition used for a fiscal rule with no default risk. In the literature, conditions analogous to $h'(b) > r$ are used to describe a "Ricardian regime" or a "passive" fiscal policy in the terminology of Leeper (1991).

When default risk is positive, local stability requires a stronger condition. Condition (16) is stronger than $h'(b) > r$ for two reasons. First, the average cost of debt servicing (coupon payments net of receipts from replacing depreciated bonds) is higher with default risk, since $\kappa - \delta q = r + \delta(1-q) > r$. Second, the marginal cost of debt servicing is higher than the average cost because q is endogenous and decreasing in b. This effect is captured by the last term in (16), since $\Psi'(b) < 0$.

Notice that (16) is stronger than condition (15). The latter condition ensures that the $\dot{b} = 0$ locus is decreasing, which the former ensures that the $\dot{b} = 0$ locus is steeper than the $\dot{q} = 0$ locus.

When it comes to ruling out multiple equilibria, stability is a helpful property. It produces virtuous dynamics that favor a good equilibrium path. However, given the model non-linearities, the existence of a stable steady state is not enough to rule out other steady states or multiple equilibria. Indeed, multiple steady states are always present whenever one of the steady state is stable.

Proposition 6. *If there exists a stable steady state with positive debt, then there exists an unstable steady state with higher debt.*

The result follows from the fact that surplus is bounded above by $\bar{s} > 0$, implying that the fiscal policy rule cannot be too responsive at high debt levels. Indeed, for high enough debt the locus for $\dot{b} = 0$ is increasing and must intersect the decreasing locus for $\dot{q} = 0$.

Multiple Equilibria. We now examine the possibility of multiple equilibria. The next proposition establishes the existence of a Markov equilibrium and provides a sufficient condition for multiplicity. For simplicity, we focus on the case in which there is a single stable steady state.

Proposition 7. *Suppose there is a unique stable steady state* (b^s, q^s) *and let* (b^u, q^u) *be the unstable steady state, with* $b^s < b^u$*. Then there are two functions* $Q^- : [b^-, \infty) \to \mathbb{R}_+$ *and* $Q^+ : (-\infty, b^+] \to \mathbb{R}_+$ *with* $b^- \leq b^u \leq b^+$ *with* $Q^-(b) < Q^+(b)$ *for* $b \in [b^-, b^+]$*. For any threshold* $\hat{b} \in [b^-, b^+]$ *there is a Markov equilibrium with*

$$Q(b) = \begin{cases} Q^+(b) & \text{for } b \leq \hat{b}, \\ Q^-(b) & \text{for } b > \hat{b}. \end{cases}$$

Debt dynamics satisfy $\dot{b} < 0$ *for* $b < \hat{b}$ *and* $\dot{b} > 0$ *for* $b > \hat{b}$*.*

Suppose that at the unstable steady state

$$4\delta\lambda\Psi'(b^u)\, q^u b^u + \left(\delta q^u + h'(b^u) + \lambda\Psi(b^u)\right)^2 < 0,$$

then $b^- < b^+$*, so there are multiple Markov equilibria.*

The condition for multiplicity in the proposition is sufficient but not necessary. Multiple equilibria can also arise when both eigenvalues are real and positive if the nonlinearity of the system generates an overlapping interval for both paths.

The proposition shows that two possible outcomes are possible. When $b^- < b^+$ the domains of the two functions Q^- and Q^+ overlap and there are multiple Markov equilibria because any intermediate threshold can serve to switch between Q^- and Q^+. When $b^- = b^+$ the equilibrium is unique. Figure 7 illustrate these two cases. In both panels, the low-debt steady state is saddle-path stable and the high-debt steady state is not, as can be inspected from the slopes of the $\dot{b} = 0$ and $\dot{q} = 0$ loci. We show two paths that satisfy the ODE system: a black path that converges to the low-debt steady state boundary and a red path that converges to the upper-debt boundary.

The left panel displays an example where the high-debt steady state has local spiral-like dynamics and the two paths unwind outwards from the high-debt steady state (the linearized system's eigenvalues are complex). For the path converging to the low-debt steady state corresponds to Q^+; while the path converging to the upper debt boundary corresponds to Q^-. The two paths overlap over an interval that contains the high-debt steady state, b^u. Any threshold $\hat{b}$ within this interval defines a price function $Q(b)$ that switches between Q^+ and Q^- at $\hat{b}$.

Eventual Uniqueness. Note that along the bad path debt eventually exits the interval of multiplicity given by $[b^-, b^+]$. For some time the good path may remain available, but there is a point of no return. A self-fulfilling crisis driven by bad expectations eventually turns into an insolvency crisis with bad fundamentals. This result is reminiscent of a result obtained in the non-stationary model of Section 3.2. There were two forces at work there: a shrinking time horizon and the level of debt. In the current stationary setting only the latter force is at work.

Tipping Points with Uniqueness. Multiple steady states do not imply multiple equilibria. The right panel of Figure 7 shows a case with two steady states and a unique equilibrium, with $b^- = b^+$. At the high debt unstable steady state b^u the two eigenvalues are real and positive. As a result the dynamics do not spiral, so the good and bad path to not overlap.

Although the equilibrium is unique, long-run dynamics are very sensitive to initial conditions. Just below the high steady state, b^u, debt converge to the low steady state $b(t) \to b^s$; just above the high steady state, debt grows. This formalizes the notion that debt dynamics may display a "tipping point" where debt-sustainability concerns drastically alter debt dynamics.

An Example Based on Italy. We now adapt the model and choose parameters to capture the dynamics of bond prices and government debt for Italy during the summer of 2012.

We first adapt the model slightly by introducing growth. It is easy to reinterpret our model in terms of debt-to-GDP and primary-surplus-to-GDP ratios. The only equation that needs to be modified is the government budget constraint which becomes

$$q\left(\dot{b} + (\delta + g)\, b\right) = \kappa b - h\,(b)\,,$$

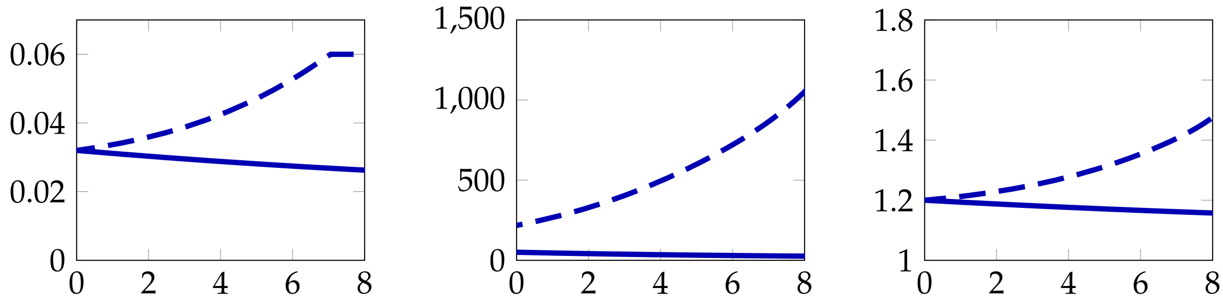
where g is the growth rate of GDP. We also allow the long run interest rate to be a random variable, drawn after the realization of uncertainty. This only affects the calculation of the Ψ function which becomes

$$\Psi\,(b) \equiv \int_{S \geq b} \frac{\kappa}{\tilde{r} + \delta} dF\,(\tilde{r}, S) + \phi \int_{S < b} \frac{S}{b} dF\,(\tilde{r}, S)\,,$$

where F is the joint distribution of $\tilde{r}$ and S.

The fiscal rule is chosen to fit the observed relation between the debt-to-GDP ratio and the primary surplus in the period 1988–2012. A linear regression of the primary surplus on the debt-to-GDP ratio yields $\alpha_0 = -0.13$ and $\alpha_1 = 0.135$. We also assume the fiscal surplus is bounded above at $\bar{s} = 6\%$ and use the rule

$$s = \min\left\{\alpha_0 + \alpha_1 b, \bar{s}\right\}.$$



(a) Surplus. (b) Interest Rate Spread. (c) Debt.

Figure 8: Two equilibrium paths starting at $b = 1.2$.

We choose $\delta = 1/7$ to match the average maturity of Italian government debt, which is about 7 years. We choose $r = 3\%$ to match the 10 year nominal bond yield in Germany in 2011 and $g = 2\%$ to capture nominal growth equal to the ECB inflation target of 2% plus zero real growth. We assume that the interest rate after the resolution of uncertainty is $\tilde{r} = 5\%$. We choose $\phi = 0.5$ and assume that S is normally distributed. The mean and variance of S are chosen so that in the low-debt steady state debt is essentially safe (with a spread of 10 basis points) and so that the high-debt steady state is $b = 1.3$.

We start the economy at $b = 1.2$, which corresponds to Italy's debt-to-GDP ratio in 2011. Given the parameters above, the model features multiple equilibria and $b = 1.2$ is in the multiplicity region. The dynamics of bond spreads, of the primary surplus and of debt-to-GDP are plotted in Figure 8. We can then imagine Italy following a path of slow debt reduction, as in the purple-line equilibrium. At date 0, if investors' sentiment shift to the bad equilibrium path, spreads jump from 50bp to 220bp, which is roughly the order of magnitude of the increase in spreads in the summer of 2011.[11] Therefore, this simple model is able to account for a crisis in which spreads increase suddenly, but not to the point of shutting off Italy from financial markets, and in which debt dynamics only slowly incorporate the effect of the higher spreads.

4.2 An Optimizing Government

In this section, we consider a model analogous to the stationary model from the previous subsection, but we derive fiscal policy endogenously from an optimizing government with additively separable preferences lacking commitment. The goal is to show that the a similar multiplicity of equilibrium is present here.

[11] Yields on 10-year Italian bonds went from 4.75 in May 2011 to 7.6 in November of the same year. Similar magnitudes of "sentiment" shocks can be read off the estimate of the effects of OMT announcements in Krishnamurthy et al. (2013).

The government now has additively separable preferences

$$\int_0^\infty e^{-\rho t} u(c(t)) dt$$

where c is government spending, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ and $\sigma, \rho > 0$.

As in Section 4.1, we work in continuous time with an infinite horizon. Lenders are risk neutral with discount rate r, the country issues long-term bonds with coupon κ that decays at rate δ, and we set $\kappa = r + \delta$ so the bond price is 1 if there is no default risk.

Uncertainty is fully resolved at some random date with Poisson arrival rate λ, at which point we enter stage 2 and the government receives a constant stream of tax revenue $\tilde{y}$ drawn from the continuous distribution $H(\tilde{y})$. The government then decides to repay or default. In the latter case tax revenue is reduced to $\eta\tilde{y}$ forever, with $\eta < 1$. We assume the country is not excluded from financial markets upon default.[12] The country repays if and only if the present value of tax revenue net of repayment is greater than that after default,

$$\tilde{y} - rb \geq \eta\tilde{y}. \tag{17}$$

The value to the government immediately before uncertainty is resolved is

$$W(b) = \frac{1}{\hat{\rho}} \int_0^\infty u\left(\frac{\hat{\rho}}{r} \max\left\langle \tilde{y} - rb, \eta\tilde{y} \right\rangle\right) dH(\tilde{y}),$$

for $\hat{\rho} = \rho + \left(\frac{1}{\sigma} - 1\right)(\rho - r)$.[13]

If default occurs, investors recover a proportion of the tax revenue $\zeta\tilde{y}$ and we assume $\eta + \zeta < 1$. Define

$$S \equiv \frac{1-\eta}{r}\tilde{y} \qquad \text{and} \qquad F(S) = H\left(\frac{r}{1-\eta}S\right),$$

where F is the c.d.f. for S. Condition (17) is then equivalent to $S \geq b$ and investors recover in present value ϕS, where $\phi \equiv \frac{\zeta}{1-\eta} < 1$. The pricing condition before the Poisson event is then identical to equation (11) from Section 4.1, which we rewrite here for convenience:

$$\dot{q} = (r + \delta + \lambda) q - \kappa - \lambda \Psi(b), \tag{18}$$

where, just as before, $\Psi(b) \equiv 1 - F(b) + \frac{\phi}{b} \int_{\underline{S}}^{b} S dF(S)$.

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[12] The exact same results obtain if they are, provided $r = \rho$.

[13] In general $\hat{\rho} \neq \rho$ because the government does not consume a constant path, unless $\rho = r$. We also have $\hat{\rho} = \rho$ in the logarithmic case $\sigma = 1$.

During stage 1, before the resolution of uncertainty the government receives a constant y facing the budget constraint

$$y - c + q\left(\dot{b} + \delta b\right) = \kappa b.$$

This is identical to equation (12) except that the primary surplus $y - c$ is now chosen by the government.

Given the presence of a positive recovery rate, we must introduce some limit on debt issuance to ensure the borrower's problem is well defined. We do so by assuming that if b reaches some upper bound $\bar{b}$ before the resolution of uncertainty, renegotiation takes place between the borrower and its creditors. Following renegotiation, the government agrees to receive a net transfer τ (possibly negative) in all future periods before the resolution of uncertainty and not to issue any additional debt, so the creditors will receive the expected value $\Psi\left(\bar{b}\right)\bar{b}$ when uncertainty is resolved.

Markov Equilibria. A Markov equilibrium is a price function $Q\left(b\right)$ and a government consumption function $C\left(b\right)$ such that: (i) government behavior is optimal taking the price function as given; (ii) the price function provides a fair price to investors given government behavior. We also require Q to be piecewise differentiable. Just as in Section 4.1 the function Q may have a point of discontinuity at a threshold that divides a path with falling and rising debt.

Let $V\left(b\right)$ denote the value function before the resolution of uncertainty. The Hamilton-Jacobi-Bellman equation associated to the government's optimization problem is

$$0 = \max_{c \geq 0}\left\{u(c) + V'(b)\left(\frac{\kappa b - y + c}{Q(b)} - \delta b\right) + \lambda(W(b) - V(b)) - \rho V(b)\right\}, \qquad (19)$$

with first-order condition for an interior solution

$$Q(b)u'(c) = -V'(b).$$

The bond price must satisfy (18) along the path induced by government policy $C(b)$. Differentiating $q(t) = Q(b(t))$ along an equilibrium path gives $\dot{q} = Q'\left(b\right)\dot{b}$ where $Q'(b)$ can be interpreted as the left derivative if $\dot{b} < 0$ and the right derivative if $\dot{b} > 0$. Rewriting condition (18) in terms of Q gives

$$Q'(b)\left(\frac{\kappa b - y + c}{Q(b)} - \delta b\right) = \left(r + \delta + \lambda\right)Q\left(b\right) - \kappa - \lambda\Psi(b). \qquad (20)$$

ODEs and Boundary Conditions. Equations (19) and (20) provide a system of ordinary differential equations (ODEs) for the pair of functions $V(b)$ and $Q(b)$. There are two alternative boundary conditions. The first is at $b = \underline{S}$, the lowest value in the support of S. This represents the safe level of debt where default is avoided. We assume $r = \rho$. The second boundary condition is obtained by assuming that there exists a high enough level of debt $\bar{b}$ where renegotiation is triggered. We assume this delivers some given values to the borrower and investors, pinning down $V(\bar{b})$ and $Q(\bar{b})$.

An Example with Multiple Equilibria. We construct a numerical example displaying multiple equilibria. The example is meant as an illustrative, not as a calibration. The parameters are set to

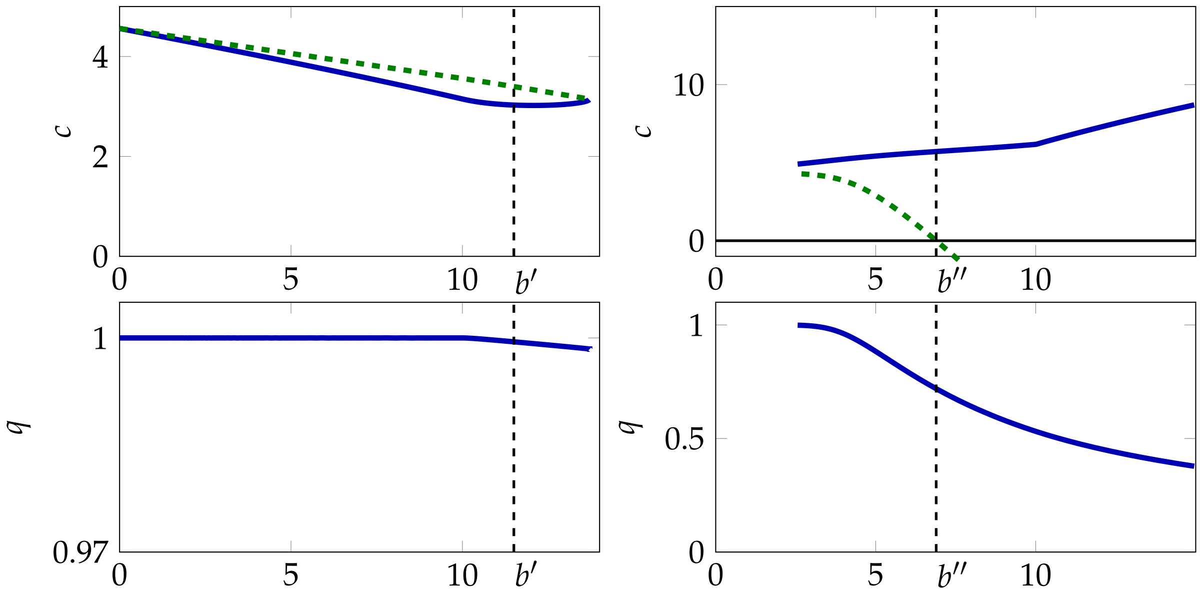
$$r = \rho = 0.1, \quad \delta = 2, \quad \lambda = 0.1, \quad \eta = 0.5, \quad \zeta = 0.1, \quad \sigma = 4.$$

The distribution of $\tilde{y}$ is chosen such that S is uniformly distributed on $[10, 100]$. The value of y is set so that the model admits a steady state at $b = 0$ with no default and $q = 1$. The upper bound for debt is $\bar{b} = 30$ and we assume that upon reaching this level, investors provide a transfer to the borrower equal to $\tau = \frac{1}{2}\lambda\Psi(\bar{b})\bar{b}$. In other words, they split the residual value of debt equally.

Figure 9a plots consumption and bond price functions $C^{-}(b)$ and $Q^{-}(b)$ consistent with a path with $\dot{b} < 0$ converging to $b = 0$. We solve the HBJ equation (19) and the pricing equation (20) moving upwards from the steady state at $b = 0$. Also shown, in the top panel, is a dashed green line representing the consumption level that would yield $\dot{b} = 0$, which is above $C^{-}(b)$ since $\dot{b} < 0$.

Figure 9b is analogous, but for the path with $\dot{b} > 0$ converging to the renegotiation boundary $\hat{b}$, solved using the ODEs starting at $\hat{b}$ and moving downwards. Importantly, for debt above b'' setting $\dot{b} = 0$ requires negative consumption. In other words, given the bond price in the lower panel, feasibility requires $\dot{b} > 0$ for all $b > b''$. In that region, bond prices are so low that the revenue from replacing old bonds $q\delta b$, plus current income y, are not sufficient to cover the coupon payment κb. The borrower needs to go into further debt to finance the difference even it made the maximal effort of setting $c = 0$.

In Figure 10, we plot the value function $V(b)$ both paths. Whenever the borrower can select which path to be on, it chooses the one with the highest value: $\dot{b} < 0$ to the left of b' and $\dot{b} > 0$ to the right of b'. Indeed, this constitutes a Markov equilibrium, with a cutoff of b'. If it were not for the non-negativity constraint on consumption, this would be the unique equilibrium.



(a) Equilibrium with $\dot{b} < 0$.

(b) Equilibrium path with $\dot{b} > 0$.

Figure 9: Consumption and bond price functions.

However, the borrower may lack the power to select equilibria in the region (b'', b'): if investors price bonds according to Q^+ the borrower must choose a path with $\dot{b} > 0$ leading to $\hat{b}$. But conditional on doing so, the borrower chooses the optimal path leading to $\hat{b}$, validating investors' expectations. This implies that we can choose any cutoff $\hat{b} \in (b'', b']$ and construct a Markov equilibrium, as we did in Section 4.1.

Proposition 8. *For any $\hat{b} \in (b'', b']$, there is a Markov equilibrium with*

$$Q(b) = \begin{cases} Q^-(b) & b \leq \hat{b} \\ Q^+(b) & b > \hat{b} \end{cases} \qquad \textit{and} \qquad C(b) = \begin{cases} C^-(b) & b \leq \hat{b} \\ C^+(b) & b > \hat{b} \end{cases},$$

Equilibria are Pareto ranked by the threshold $\hat{b}$, with the best equilibrium $\hat{b} = b'$.[14]

[14] Within the region $[0, b'']$ the equilibrium is unique. To understand why, suppose the we have a threshold equilibrium with $b' < b''$. Then for a borrower just above b' there is a discrete drop in utility. This is not consistent with optimal behavior: a borrower would willingly sacrifice lower consumption for a small amount of time to set $\dot{b} < 0$ and reach b' in order to experience a discrete improvement in utility. This argument breaks down, however, in the region (b'', b'), precisely because $\dot{b} < 0$ is no longer feasible.

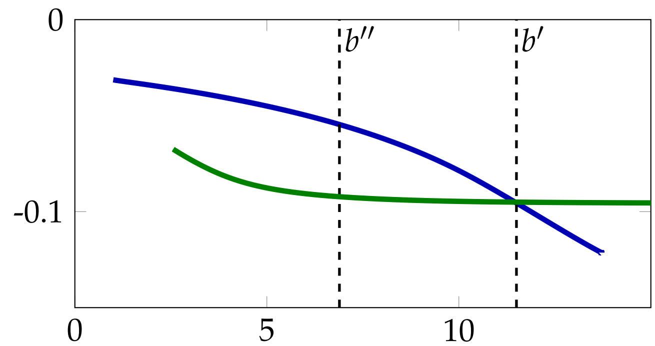


Figure 10: Value functions: $\dot{b} < 0$ (blue line) and $\dot{b} > 0$ (red line).

5 Commitment and Multiplicity

In previous sections, we showed that the government budget constraint may be satisfied at various bond prices, creating vulnerability to self-fulfilling crises. We assumed that the government cannot select the best bond price by, for example, committing to the quantity of bonds issued. Commitment of this kind can be interpreted as a timing convention where the government issues a certain about of debt and then investors bid on these bonds determining the price, as in Eaton and Gersovitz (1981). Instead, we have assumed that the borrower takes the bond price as given, which may be interpreted as the reverse timing assumption or, equivalently, as a situation where the government fixes the amount it borrows, the funds it needs today, but does so at a variable interest rate determined by the market, as in Calvo (1988). In this section, we study simple game-theoretic models that allow the government to commit to bond issuances in the very short run, but provide microfoundations for our timing assumption. In a sense, we adopt the Eaton and Gersovitz (1981) timing, but obtain the Calvo (1988) outcome.

We endow the government with partial commitment powers: it can commit to sell a fixed amount of bonds in the present auction, but cannot make binding commitments regarding future auctions. The key idea is that if a bond auction delivers a lower price the government is expected to make further bond issuances to make up for the shortfall. Lack of commitment across auctions renders short-run commitment, within an auction, ineffective.

We present two models. The first model features a potentially unbounded number of bond auctions within each period; the government cares about total spending within the period. This delivers a very sharp result: equilibrium outcomes coincide exactly with those studied in previous sections. The idea is reminiscent of the Coase theorem, whereby a durable goods (in our case, long term bonds) monopolist competes intertemporally with itself and is reduced to competitive behavior. The second model has a finite number of rounds, but assumes impatience, a preference for earlier rounds. Depending on parameters, multiple equilibria of the same nature arise.

Overall, our results show that the opportunity to raise funds in the future may send a borrower to the wrong side of the Laffer curve. An intertemporal commitment problem leads to an intertemporal coordination failure.

5.1 A Game with No Commitment

The idea of the first model is to split each period into rounds, similar to the sequential banking model in Bizer and DeMarzo (1992).[15] Formally, we allow for a potentially infinite number of rounds within a period, but the important point is that the government can always return to the market. The government values total spending within the period, across all rounds. It can commit to the quantity of bonds it issues in the current round, but not to future rounds. A period may be thought of as a month, a quarter or a year, over which the government's funding needs are determined by fiscal policy decisions that adjust slowly. Rounds are best thought of as days over which auctions of Treasury bonds take place.

The government's objective function is

$$u\left(c\right)+\beta V\left(b\right),$$

where c is spending in the first period and b is the stock of bonds issued in the first period that are to be repaid in the second period. Both u and V are decreasing, differentiable and concave functions. We can interpret u as the payoff resulting from a full specification of the benefits of public expenditure and the costs of taxation and V as the expectation of a value function in an optimizing model with an infinite horizon.

The government receives a given tax revenue y and has a stock of bonds b_- inherited from the past that it needs to repay at the end of the first period. Thus, in the first period it must borrow to finance $c-y+b_-$.

There is a continuum of risk neutral atomistic investors with discount factor β. Because of risk neutrality and because all bond holders are treated equally, only the expected payment by the government in the second period is required to determine price bonds. If the total debt owed to investors is b, the expected repayment per bond is given by the

[15] The main differences with their setup are that (a) they studied a particular investment model with moral hazard; (b) they allow debt to have seniority clauses; as a result of which (c) the equilibrium in their model is unique. Instead, we focus on a general borrowing problem, without seniority clauses (not employed in sovereign debt contexts) and focus on the potential for multiple equilibria.

non-increasing function $\Psi(b)$. This function encapsulates all the relevant considerations regarding repayment, including the probability of default as well as the recovery value in the case of default and how these vary with the level of indebtedness. Note that this framework could capture strategic default or moral hazard by the government, as all these consideration can be embedded in V and Ψ.

The first period is divided into infinite rounds $i = 1, 2, \ldots$ and the government can run an auction in each round. Think of auctions taking place in real time at $t = 0$, $1/2$, $2/3$, $3/4$, ... At $t = 1$, the government collects the revenue from all these auctions, repays b_-, buys c, and the payoff $u(c)$ is realized. Finally, at $t = 2$ the payoff $V(b)$ is realized. Letting d_i denote bond issuances in round i, total bonds issued in period 1 are then

$$b = \sum_{i=0}^{\infty} d_i.$$

At each round i the investors bid price q_i for the issuance d_i.

The crucial assumption we make is that in each auction the government cannot commit to the size of debt issuances in future auctions. Strategies are described by functions $d_i = D_i(d^{i-1}, q^{i-1})$ and $q_i = Q_i(d^i, q^{i-1})$, where superscripts denote sequences up to round i. A subgame perfect equilibrium requires that:

i. In round i, after any history (d^{i-1}, q^{i-1}), the government strategy D_j for the remaining rounds $j = i, i+1, \ldots$ is optimal, given that future prices satisfy $q_j = Q(d^j, q^{j-1})$ at $j = i, i+1, \ldots$

ii. The price in round i after history (d^i, q^{i-1}) satisfies $Q(d^i, q^{i-1}) = \Psi(\sum_{i=0}^{\infty} d_i)$ where $\{d_i\}$ is computed using the government strategy D_j for $j = i, i+1, \ldots$ and future bond prices Q_j for $j = i+1, \ldots$

For an equilibrium to be well defined, the sequence $\{d_i\}$ must be summable in equilibrium and after any possible deviation. Moreover, since investors are atomistic, the only restriction on prices is that they be consistent with expected repayment, which in turn is determined by total debt issued. Observe that along an equilibrium path the bond price is constant across rounds $q_i = q^*$. We can then denote by (c^*, b^*, q^*) an equilibrium outcome of the game in terms of government spending, total debt issued and bond price. The main result of this section is a tight characterization of all possible equilibrium outcomes.

Proposition 9. *A triplet (c^*, b^*, q^*) is the outcome of a subgame perfect, pure strategy equilibrium if and only if*

$$(c^*, b^*) \in \arg\max_{c,b} u(c) + \beta V(b) \qquad s.t. \qquad c + b_- = y + q^* b$$

and

$$q^* = \beta \Psi (b^*).$$

The assumption that a further round is always available delivers equilibria with outcomes that are equivalent to that of a price-taking government. The government solving the maximization problem in Proposition 9 is the polar opposite of a government that can fully commit to issuances and solves

$$\max_{c,b} u(c) + \beta V(b) \qquad \text{s.t.} \qquad c + b_- = y + \beta \Psi(b) b.$$

This is the assumption typically adopted in the literature following Eaton and Gersovitz (1981). Instead, we assume that the government can commit to bond issuances in each round, but found that the outcome is *as if* it lacked any such commitment, as in Calvo (1988).

5.2 A Game with Limited Commitment

We now look at an intermediate cases where so commitment is available. We develop a simple three-period model, with one auction per period.

The Game. There are three periods, $t = 0, 1, 2$. The government wishes to finance consumption in all three periods, but has income available only in the last period. It issues a long-term bonds that pay in period 2. To relate the current setting with the previous one it may be useful to think of $t = 0, 1$ as two rounds within the "first period" and $t = 2$ as the "second period".

In period 0, the government decides how many bonds b_1 to sell. Next, an auction takes place and risk neutral investors bid q_0 for these bonds,[16] the government receives $q_0 b_1$ from investors and uses it to finance spending

$$c_0 = q_0 b_1.$$

In period 1, the government chooses b_2, the investors bid q_1, and the government raises

[16] The particular auction protocol is not important, but for concreteness we can assume investors play a second price auction.

$q_1(b_2 - b_1)$ financing spending

$$c_1 = q_1(b_2 - b_1).$$

Finally, in period 2 the random income y is drawn from a cumulative distribution function $F(y)$ on $[0, \infty)$. The government then decides to repay or default on debt. In the event of default all income y is lost: there is no recovery value for investors and government consumption is zero. Thus, the government repays if $y \geq b_2$ and defaults otherwise.

Investors are risk neutral and do not discount future payoffs. The government objective is to maximize expected utility

$$\mathbb{E}U(c_0, c_1, c_2).$$

Key to our results will be that U is not additively separable between $t = 0$ and $t = 1$.

Strategies and Equilibrium. The government's strategy is given by a choice for b_1 and a function $B_2(b_1, q_0)$ that gives b_2 for each past history (b_1, q_0). The investors' strategy is summarized by two pricing functions $Q_0(b_1)$ and $Q_1(b_1, q_0, b_2)$.

We study subgame perfect equilibria. In period 1, investors bid

$$Q_1(b_1, q_0, b_2) = 1 - F(b_2),$$

so the price is independent of b_1 and q_0. In period 1, given b_1 and q_0, the government then solves

$$\max_{b_2} \mathbb{E}U(q_0 b_1, (1 - F(b_2))(b_2 - b_1), \max\{y - b_2, 0\}). \tag{21}$$

Assume, for simplicity, that this problem yields a unique best response $B_2(b_1, q_0)$. In period 0, investors bid

$$q_0 = 1 - F(B_2(b_1, q_0)). \tag{22}$$

The fact that q_0 appears on both sides of this equation is crucial in generating multiplicity. A necessary condition for multiplicity is that $B_2(b_1, q_0)$ be decreasing in q_0 for some value of b_1. Since q_0 enters the maximization problem (21) by determining the value of c_0, this requires non-separability of U between c_0 and c_1. This provides a channel by which a high realized bond price q_0, by allowing the government to finance a high level of spending in period 0, reduces the incentive to finance high-spending in period 1. Below, we work out a specific example in which this effect is present.

Depending on the value of b_1, there may be multiple values of q_0 that solve (22). Let $\mathcal{Q}(b_1)$ be a function that selects a solution to (22) for each b_1. Let $\mathcal{B}_2(b_1) = B_2(b_1, \mathcal{Q}(b_1))$

denote the associated value for b_2.[17] The choice of b_1 at $t = 0$ must maximize

$$\mathbb{E}U(\mathcal{Q}(b_1)\,b_1, \mathcal{Q}(b_1)\cdot(\mathcal{B}_2(b_1) - b_1), \max\{y - \mathcal{B}_2(b_1), 0\}). \tag{23}$$

Summing up, an equilibrium is characterized by a value b_1 and by functions $B_2(b_1, q_0)$, $\mathcal{Q}(b_1)$ and $\mathcal{B}_2(b_1)$such that: b_1 solves the maximization problem (23), $B_2(b_1, q_0)$ solves the maximization problem (21), $\mathcal{Q}(b_1)$ solves equation (22) and $\mathcal{B}_2(b_1) = B_2(b_1, \mathcal{Q}(b_1))$.

An Example with Multiplicity. For concreteness, we now present an example that allow us to solve the model analytically and provide conditions for equilibrium multiplicity. We let income have an exponential distribution $F(y) = 1 - e^{-\lambda y}$ with $\lambda > 0$. We adopt the utility function

$$U(c_0, c_1, c_2) = \alpha \min\{c_0, \bar{c}\} + \theta \min\{c_0 + c_1, \bar{c}\} + c_2.$$

This utility function captures the idea that in the first two periods the government has a target level of spending $\bar{c}$ and has a preference for early spending. The parameter $\theta > 1$ determines the loss from not meeting the target $\bar{c}$, while the parameter $\alpha > 0$ measures the desire for early spending.[18]

Before proceeding we make a number of parametric assumptions. We assume $\bar{c}$ is below the peak of the Laffer curve $\bar{c} < \max_b e^{-\lambda b} b = (\lambda e)^{-1}$; denote the two solutions to $\bar{c} = e^{-\lambda b} b$ by $\underline{b} < \overline{b}$. We also assume that $\theta(1 - \lambda \underline{b}) > 1$, to ensure that at the good equilibrium the government has a sufficiently strong incentive to spend in periods 0 and 1. Our next result shows the possibility of multiple equilibria.

Proposition 10. *There is always a good equilibrium with strategy $\mathcal{B}_2(b_1) = \underline{b}$ and outcome $b_1 = b_2 = \underline{b}$. If*

$$\alpha\left[1 - 1/\theta - \lambda\left(\overline{b} - \underline{b}\right)\right]\overline{b} > \overline{b} - \underline{b},$$

there is also a bad equilibrium with outcome $b_1 = b_2 = \overline{b}$. In both equilibria $c_0 = \bar{c}$.

Total spending $c_0 + c_1 = \bar{c}$ is identical in both the good and bad equilibrium, but the bad equilibrium lies on the wrong side of a Laffer curve. In both equilibria all bond issuances take place in the first period. However, the possibility of further bond issuances in period 1 is crucial, otherwise only the good outcome is an equilibrium.

[17] We could easily extend the analysis to introduce a sunspot and allow a stochastic selection of equilibria.

[18] For example, investment spending on infrastructure requires some total outlay over an extended time horizon, but with a preference for early completion. As another example consider the payment of government wages. The payment may be delayed, if needed, at a cost, since workers are impatient and demand compensation.

At $t = 1$, the government can commit to the number of bonds it issues, so it chooses a level of b_2 on the increasing side of the Laffer curve for new issuances, given by

$$(1 - F(b_2))(b_2 - b_1).$$

Given our functional forms, this implies $1 - \lambda(b_2 - b_1) \geq 0$. If the government could commit to total debt $b_2 = b_1 = b$ at time $t = 0$, the *commitment Laffer curve* for total debt would be

$$(1 - F(b))\, b.$$

The slope of this Laffer curve is $1 - \lambda b$, which can be negative at $b = \overline{b}$ even if $1 - \lambda(b_2 - b_1) = 1 > 0$. Thus, the bad equilibrium has total debt on the wrong side of the commitment Laffer curve. Moreover, at date 0 the government cannot improve matters by issuing slightly less debt $b_1 < \overline{b}$, because investors expects the government to make up for this by issuing the remainder $\overline{b} - b_1$ at $t = 1$ to reach $b_2 = \overline{b}$. Thus, the pricing function $\mathcal{Q}_0(b_1)$ is flat for b_1 near $\overline{b}$, so the Laffer curve at $t = 0$, which is equal to

$$\mathcal{Q}_0(b_1)\, b_1 = (1 - F(\mathcal{B}_2(b_1)))\, b_1,$$

is locally increasing. Getting to a lower price, that lies on the good side of the commitment Laffer curve, is possible, but requires a discrete reduction in debt b_1 below a cutoff $\hat{b}_1 < \overline{b}$, which is derived in the appendix. However, if α is large enough the effort of reducing b_1 below $\hat{b}_1$ becomes too costly in terms of delayed spending, sustaining the bad equilibrium.

6 Conclusions

Our formal analysis supports the idea that countries may become entrapped in self-fulfilling crises. During such a slow-moving crisis, interest rates rise, increasing the path for debt and raising future default probabilities. Fortunately, our results indicate that such slow moving crises are far from inevitable. Our analysis shows that aggressive fiscal policy rules and long debt maturities help prevent the occurrence of slow moving debt crises.

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7 Appendix (For Online Publication)

7.1 Proof of Proposition 2

The argument is by backward induction. The functions X_T, Q_{T-1} and m_{T-1} are uniquely defined. The first step at which multiple equilibria can arise is in the selection of b_T when constructing the bond issuance function $B_T(b_{T-1}, s^T)$. However, when $\delta = 1$, the bond issuance function B_T does not affect the construction of the repayment function X_{T-1} and of the pricing function Q_{T-2}, as repayment only depends on the maximum of the function $Q_{T-1}\left(b_T, s^{T-1}\right) b_T$ and the term $(1-\delta)\, Q_{T-1}\left(B_T(b_{T-1}, s^{T-1}), s^{T-1}\right)$ in (3) disappears when $\delta = 1$. The same argument applies in all previous periods.

7.2 Proof of Proposition 5

In the case considered, the Laffer curve takes the form $[1 - F\left((1+r)\, b - m\right)]\, b$ (omitting time subscripts and dependence on s^t to simplify notation). The slope of the Laffer curve is

$$1 - F\left((1+r)\, b - m\right) - (1+r)\, f\left((1+r)\, b - m\right) b$$

which has the same sign of

$$1 - (1+r)\, \frac{f\left((1+r)\, b - m\right)}{1 - F\left((1+r)\, b - m\right)} b.$$

So if $f / (1 - F)$ is monotone non-decreasing, the derivative can only change sign once.

7.3 Proof of Lemma 1

Using steady-state conditions, the Jacobian can be written as

$$J = \left[\begin{array}{cc} \frac{\kappa - h'(b)}{q} - \delta & -\frac{\delta b}{q} \\ -\lambda \Psi'(b) & r + \delta + \lambda \end{array} \right].$$

A necessary and sufficient condition for a saddle is a negative determinant of J, i.e., $J_{11}J_{22} < J_{12}J_{21}$. Since $J_{12} < 0$ and $J_{22} > 0$, this is equivalent to $-J_{11}/J_{12} < -J_{21}/J_{22}$, which means that the $\dot{b} = 0$ locus is downward sloping and steeper than the $\dot{q} = 0$ locus. Condition (16) then follows.

7.4 Proof of Proposition 6

Consider the functions on the right-hand sides of (13) and (14), which are both continuous for $b > 0$. If there is a saddle-path stable steady state at b', the second function is steeper, from Lemma 1, and so is below the first function at $b' + \epsilon$ for some $\epsilon > 0$. Taking limits for $b \to \infty$ the the second function yields $q \to \kappa/\delta$ and the first yields

$$q \to \frac{\kappa + \lambda \Psi\left(\overline{S}\right)}{r + \delta + \lambda} < \frac{\kappa}{\delta},$$

where the inequality can be proved using $\Psi\left(\overline{S}\right) < 1$ and $\kappa = r + \delta$. Therefore, the second function is above the first for some b'' large enough. The intermediate value theorem implies that a second steady state exists in $(b' + \epsilon, b'')$.

7.5 Proof of Proposition 7

Consider the path that solves our ODE system going backwards in time, starting on the saddle path converging to the low-debt steady state, at some value of $b = b' + \epsilon$. Given a small enough $\epsilon > 0$ the saddle path must lie above the $\dot{q} = 0$ locus. Moreover, between b' and b'' the $\dot{q} = 0$ locus lies strictly above the $\dot{b} = 0$ locus. Therefore, the path can never cross the $\dot{q} = 0$ locus because along the path $\dot{b} < 0$ and $\dot{q} > 0$. Therefore, it is possible to solve the ODE backwards until b approaches b'' from below. This implies that for all $b(0) < b''$ we can select a path with $\dot{b} < 0$ and $b \to b'$. Consider next the path that solves the ODE going backwards starting at $\left(\hat{b}, \hat{q}\right)$. By construction the point $\left(\hat{b}, \hat{q}\right)$ must lie in the region of the phase diagram below both the $\dot{b} = 0$ locus and the $\dot{q} = 0$ locus (to see this notice that at the definition of $\hat{b}$ implies that $\dot{b} > 0$ at $\left(\hat{b}, \hat{q}\right)$ and the constancy of qb implies $\dot{q} < 0$). If $\hat{b} < b''$ the path with $qb = \hat{v}$ is an equilibrium for all initial conditions in $[\hat{b}, \infty)$, so the interesting case is $\hat{b} > b''$. In this case, we can solve backward the ODE. As long as $b > b''$ the $\dot{b} = 0$ locus lies strictly above the $\dot{q} = 0$ locus. Therefore, the path can never cross the $\dot{q} = 0$ locus, because along the path $\dot{b} > 0$ and $\dot{q} < 0$. Therefore, it is possible to solve the ODE backwards until b approaches b'' from above. This implies that for all $b(0) > b''$ we can select a path with $\dot{b} > 0$ and $b \to \infty$.

Turning to multiplicity, consider the first path constructed above. As we approach b'' two possibilities arise. Either q remains bounded away from its steady state value q'' or q converges to q''. In the first case, $\dot{b}$ is bounded above by a negative value, so we must cross b'' and can extend the solution in some interval $[b'', b'' + \epsilon)$. In this case, we have multiple equilibria because for some $b > b''$ we can select both an equilibrium path with $\dot{b} < 0$ and an equilibrium path with $\dot{b} > 0$. In the second case, the path converges to the steady state (b'', q'') along a monotone path with $\dot{b} < 0$. However, if the local dynamics near (b'', q'') are characterized by a spiral, we reach a contradiction (since the path must cross the arms of the spiral and then convergence can no longer be monotone).

7.6 Proof of Proposition 9

We start with the sufficiency part, by constructing an equilibrium which implements the desired outcome. The equilibrium pricing function sets the price $Q\left(d^i, q^{i-1}\right) = q^*$ for any history (d^i, q^{i-1}) with $q^{i-1} = \{q^*, ..., q^*\}$. The strategy of the government is to issue $b^* - \sum_{j=0}^{i} d_j$ after any history with $q^{i-1} = \{q^*, ..., q^*\}$.[19] The resulting equilibrium play is that the government issues b^* in the first auction and no further auction takes place. Since at each round the price is independent of the amount of bonds issued, the government cannot gain by changing its bonds issuances. Investors, on the other hand, expect that if the government deviates and offers anything other than $b^* - \sum_{j=0}^{i} d_j$ in the current round, it will adjust its issuances in the next round so as to reach the debt level b^*. This justifies their bid being independent of the amount of bonds issued in the current round.

Turning to the necessity part, suppose we have an equilibrium with outcome (s, b) and define $q = (b_- - c)/b$. We want to prove that $q = MRS \equiv V'(b)/u'(s)$. Suppose, towards a contradiction, that we have a proposed equilibrium where instead $q \neq MRS$. For concreteness suppose $q > MRS$. The other case is symmetric.

In equilibrium the borrower is supposed to exit with (b, c) at some round. Suppose that instead, upon reaching this round, the government considers a deviation, does not exit and instead issues a small extra $\varepsilon > 0$ amount of debt in the next round, for a current total of

$$\tilde{b} = b + \varepsilon.$$

The market must then respond with a price $\tilde{q}$ for this round. The current price yields a current revised consumption

$$\tilde{c} = c + \tilde{q}\varepsilon.$$

In the equilibrium of the ensuing sub-game, the price in all future rounds must be constant and given by $\tilde{q} = \tilde{q}^* = G(\tilde{b}^*)$ where $\tilde{b}^*$ is the end outcome for debt following this sub-game. The associated end outcome for consumption is then $\tilde{c}^* = c + \tilde{q}(\tilde{b}^* - b)$.

[19] It is not difficult to complete the description of the equilibrium constructing continuation strategies after histories with $q_i \neq q^*$. However, given the atomistic nature of investors, these off-equilibrium paths are irrelevant for the borrower's maximization problem.

The following inequalities hold

$$u(c)+V(b)\geq u(\tilde{c}^{*})+V(\tilde{b}^{*})\geq u(\tilde{c})+V(\tilde{b}). \tag{24}$$

The first inequality follows because (b,c) is an equilibrium outcome. The second because otherwise in the sub-game the borrower would prefer to stop after the initial deviation. We can now prove that the end outcome of the sub-game cannot have more total debt than the initial deviation, that is, $\tilde{b}^{*}\leq\tilde{b}$. Suppose, by contradiction, that $\tilde{b}^{*}>\tilde{b}$. Let

$$\lambda=\frac{\tilde{b}-b}{\tilde{b}^{*}-b}\in(0,1),$$

and notice that

$$(\tilde{b},\tilde{c})=(1-\lambda)(b,c)+\lambda(\tilde{b}^{*},\tilde{c}^{*}).$$

Strict concavity and the first inequality in (24) then imply $u(\tilde{c}^{*})+V(\tilde{b}^{*})<u(\tilde{c})+V(\tilde{b})$, which contradicts the second inequality in (24).

Since the function $G(b)/b$ is non-increasing in b, $\tilde{b}^{*}\leq\tilde{b}$ implies

$$\tilde{q}=\frac{G(\tilde{b}^{*})}{\tilde{b}^{*}}\geq\frac{G(\tilde{b})}{\tilde{b}}.$$

By choosing an initial deviation with $\varepsilon>0$ small enough, the borrower can ensure that the lower bound on $\tilde{q}$ is arbitrarily close to q, since $G(\tilde{b})/\tilde{b}\to q$ as $\tilde{b}\to b$. But then, since $q>MRS$, this implies that along this deviation the borrower can sell bonds at a price $\tilde{q}>MRS$, which implies $u(\tilde{c})+V(\tilde{b})>u(c)+V(b)$. Therefore, if the proposed equilibrium satisfies $q>MRS$, there is a profitable deviation by the borrower, a contradiction.

7.7 Proof of Proposition 10

We start by characterizing the government's optimal choice for debt $B_2(b_1,q_0)$ at $t=1$, given values of b_1 and q_0.

Lemma 2. *Given q_0 and b_1, the optimal choice of b_2 must satisfy either*

$$\begin{aligned} q_0 b_1+(1-F(b_2))(b_2-b_1)&<\bar{c}\\ \theta(1-\lambda(b_2-b_1))&=1\end{aligned}$$

or

$$q_0 b_1 + (1 - F(b_2))(b_2 - b_1) = \bar{c}$$
$$\theta(1 - \lambda(b_2 - b_1)) \geq 1.$$

Proof. In equilibrium we always have $q_0 b_1 \leq \bar{c}$. Therefore, the marginal benefit of increasing b_2 is

$$\theta(1 - F(b_2) - f(b_2)(b_2 - b_1)) - (1 - F(b_2)) = (1 - F(b_2))[\theta(1 - \lambda(b_2 - b_1)) - 1]$$

if $c_0 + (1 - F(b_2))(b_2 - b_1) < \bar{c}$ and 0 otherwise. The statement follows immediately. □

Define the cutoff

$$\hat{b}_1 = \bar{b} - \frac{1}{\lambda}\left(1 - \frac{1}{\theta}\right) \in \left(0, \bar{b}\right). \tag{25}$$

We can now characterize the continuation equilibria that arise after the choice of b_1 by the government at date 0, that is, we look for candidates for the equilibrium selections $\mathcal{Q}_0(b_1)$ and $\mathcal{B}_2(b_1)$. We first consider the case in which b_1 is below the cutoff $\hat{b}_1$.

Lemma 3. *If $b_1 < \hat{b}_1$ there is a unique continuation equilibrium, with $b_2 = \underline{b}$. If $b_1 \geq \hat{b}_1$ there are two continuation equilibria, one with $b_2 = \underline{b}$ and one with $b_2 = \bar{b}$.*

Proof. The equilibrium with $b_2 = \underline{b}$ exists because $(1 - F(b_2)) b_2 = \bar{c}$ at $b_2 = \underline{b}$ and the assumption that $\theta(1 - \lambda\underline{b}) > 1$ implies $\theta(1 - \lambda(b_2 - b_1)) > 1$ for any $b_1 \geq 0$. To prove uniqueness notice that we cannot have $b_2 \in (\underline{b}, \bar{b})$ in equilibrium, otherwise $e^{-\lambda b_2} b_2 > \bar{c}$, we cannot have $b_2 \geq \bar{b}$, otherwise $\theta(1 - \lambda(b_2 - b_1)) < 1$, and we cannot have $b_2 < \underline{b}$, otherwise $e^{-\lambda b_2} b_2 < \bar{c}$ and $\theta(1 - \lambda(b_2 - b_1)) > 1$ (always using Lemma 2).

The second equilibrium exists because $b_1 \geq \hat{b}_1$ is equivalent to

$$\theta\left(1 - \lambda\left(\bar{b} - b_1\right)\right) \geq 1.$$

□

The lemma implies that a possible selection for continuation equilibria is

$$\mathcal{B}_2(b_1) = \begin{cases} \underline{b} & \text{if } b_1 \leq \hat{b}_1 \\ \bar{b} & \text{if } b_1 > \hat{b}_1 \end{cases}.$$

We go back to period 0 and study the government's optimization problem with this selection. The government chooses b_1 to maximize

$$\alpha e^{-\lambda \mathcal{B}_2(b_1)} b_1 + \theta \min \left\{ e^{-\lambda \mathcal{B}_2(b_1)} \mathcal{B}_2\left(b_1\right), \bar{c} \right\} + \frac{1}{\lambda} e^{-\lambda \mathcal{B}_2(b_1)}.$$

The government faces a trade-off here. If it chooses $b_1 \leq \hat{b}_1$ it ensures that in the continuation game investors will expect low issuance of bonds in period 1 and so only $\underline{b}$ bonds will be eventually issued, keeping the government on the good side of the Laffer curve. However, to keep b_1 low the government foregoes the benefits from early spending α. In particular, choosing $0 \leq b_1 \leq \hat{b}_1$ we have

$$\alpha e^{-\lambda \underline{b}} b_1 + \theta \bar{c} + \frac{1}{\lambda} e^{-\lambda \underline{b}}.$$

While choosing $\hat{b}_1 < b_1 \leq \bar{b}$ we have

$$\alpha e^{-\lambda \bar{b}} b_1 + \theta \bar{c} + \frac{1}{\lambda} e^{-\lambda \bar{b}}.$$

Clearly, the only possible optimal choices are $b_1 = \hat{b}_1$ and $b_1 = \bar{b}$. It is optimal to choose $b_1 = \bar{b}$ if

$$\alpha e^{-\lambda \bar{b}} \bar{b} + \frac{1}{\lambda} e^{-\lambda \bar{b}} > \alpha e^{-\lambda \underline{b}} \hat{b}_1 + \frac{1}{\lambda} e^{-\lambda \underline{b}}.$$

Using (25) to substitute for $\hat{b}_1$ in this inequality completes the proof.