Supplement to BLP Estimation using Laplace Transformation and Overlapping Simulation Draws

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This supplement contains three sections. In the first section, we restate some useful definitions and theorems about functionals that were used in the proofs of our proposition 2. In the second section, we include the proofs of Theorems 4, and 7. In the third section, we give additional details about the relationship between our asymptotics and Freyberger (2015)'s asymptotics.

1 Definitions and Theorems about Functionals

Definition. Suppose $f : U \to Y$ is a mapping from an open subset $U \subset X$ of a Banach space to another Banach space Y. Then, f is **Fréchet Differentiable** at $u_0 \in U$ if there is a bounded linear map $Df(u_0) : X \to Y$ such that for every $\epsilon > 0$, there is a $\delta > 0$ such that whenever $0 < \|u - u_0\| < \delta$, we have

$$\frac{\|f(u) - f(u_0) - Df(u_0) \cdot (u - u_0)\|}{\|u - u_0\|} < \epsilon.$$

The **Fréchet Derivative** of f at u_0, $Df(u_0)$, is related to the directional derivative (sometimes called the Gateaux Derivative) of f at u_0 in the direction h:

$$Df(u_0) \cdot h = \lim_{t \to 0} \frac{f(u_0 + th) - f(u_0)}{t} \equiv f'_{u_0}(h).$$

The Mean Value Theorem can be extended to Fréchet differentiable functionals.

Theorem. *(Mean Value Theorem) Let $U \subset X$ be an open and convex subset of a Banach space X and let $f : U \to Y$ be a C^1 mapping from U to a Banach space Y. For $u, v \in U$,* *assume* $\{(1-t)u+tv | t\in[0,1]\}\subset U$. *Then,*

$$\begin{aligned} f(v)-f(u) &= \int_0^1 Df((1-t)u+tv)dt\cdot(v-u) \\ &= Df(u)\cdot(v-u)+\int_0^1 (Df((1-t)u+tv)-Df(u))dt\cdot(v-u). \end{aligned}$$

Corollary. *(Intermediate Value Theorem) Let* $U\subset X$ *be an open convex subset of a Banach space* X *and let* $f:U\to\mathbb{R}$ *be* C^1 *map. For all* $u,v\in U$ *, there exists a* $c=(1-t)u+tv$ *for some* $t\in[0,1]$ *such that* $f(v)-f(u)=Df(c)\cdot(v-u)$*.*

2 Additional Proofs of Theorems

2.1 Proof of Theorem 4

Proof. First we will show stochastic equicontinuity. Recall that the implicit function theorem applied to $s(\hat{\delta},X,\hat{F};\theta)=S$ implies that $\hat{\delta}(\theta)$ is a continuously differentiable function of θ. By the intermediate value theorem, there exists $\theta^*\in[\theta,\theta_0]$ such that $\hat{\delta}_t(\theta)-\hat{\delta}_t(\theta_0)=\frac{\partial\hat{\delta}_t(\theta^*)}{\partial\theta}(\theta-\theta_0)$. It follows that

$$\begin{aligned} &\|\hat{\gamma}(\theta)-\hat{\gamma}(\theta_0)-(\gamma(\theta)-\gamma(\theta_0))\| \\ &= \left\|\frac{1}{T}\sum_{t=1}^{T}\left\{Z_t\left(\hat{\delta}_t(\theta)-\hat{\delta}_t(\theta_0)-(X_t'\theta_1-X_t'\theta_{0,1})\right)\right\}-\lim_{T,R\to\infty}\frac{1}{T}\sum_{t=1}^{T}E\left[Z_t\left(\hat{\delta}_t(\theta)-\hat{\delta}_t(\theta_0)-(X_t'\theta_1-X_t'\theta_{0,1})\right)\right]\right\| \\ &\le \left\|\frac{1}{T}\sum_{t=1}^{T}Z_t\frac{\partial\hat{\delta}_t(\theta^*)}{\partial\theta_2}-\lim_{T,R\to\infty}\frac{1}{T}\sum_{t=1}^{T}E\left[Z_t\frac{\partial\hat{\delta}_t(\theta^*)}{\partial\theta_2}\right]\right\|_2\|\theta_2-\theta_{0,2}\|+\left\|\frac{1}{T}\sum_{t=1}^{T}Z_tX_t'-E[Z_tX_t']\right\|_2\|\theta_1-\theta_{0,1}\| \\ &\le \sup_{\theta\in\Theta}\left\|\frac{1}{T}\sum_{t=1}^{T}Z_t\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}-\lim_{T,R\to\infty}\frac{1}{T}\sum_{t=1}^{T}E\left[Z_t\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}\right]\right\|_2\|\theta_2-\theta_{0,2}\|+\left\|\frac{1}{T}\sum_{t=1}^{T}Z_tX_t'-E[Z_tX_t']\right\|_2\|\theta_1-\theta_{0,1}\|. \end{aligned}$$

Recall that $\frac{\partial\hat{\delta}(\theta)}{\partial\theta_1}=0$ and $\frac{\partial\hat{\delta}(\theta)}{\partial\theta_2}=-\left(\frac{1}{R}\sum_{r=1}^{R}G_\delta\left(\hat{\delta},X,v_r;\theta\right)\right)^{-1}\frac{1}{R}\sum_{r=1}^{R}G_{\theta_2}\left(\hat{\delta},X,v_r;\theta\right)$. If we can show that $E\left[\sup_{\theta\in\Theta}\left\|Z_t\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}\right\|_2\right]<\infty$, then since $\frac{\partial\hat{\delta}(\theta)}{\partial\theta_2}$ is continuous in θ and Θ is a compact set, we will have that the uniform law of large numbers holds (see e.g. Lemma 2.4 in Newey and McFadden (1994)):

$$\sup_{\theta\in\Theta}\left\|\frac{1}{T}\sum_{t=1}^{T}Z_t\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}-\lim_{T,R\to\infty}\frac{1}{T}\sum_{t=1}^{T}E\left[Z_t\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}\right]\right\|_2=o_p(1).$$

Recall that $Z_t\in\mathbb{R}^{L\times J}$ for finite L and that $E\left[\sup_{\theta\in\Theta}\left\|Z_t\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}\right\|_2\right]<\sqrt{L}E\left[\sup_{\theta\in\Theta}\left\|Z_t\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}\right\|_\infty\right]$.

It therefore suffices to show that $E\left[\underset{\theta\in\Theta}{sup}\left\|Z_t\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}\right\|_\infty\right]<\infty$. Note that

$$\begin{aligned}
&E\left[\underset{\theta\in\Theta}{sup}\left\|Z_t\frac{\partial\hat{\delta}_t\left(\theta\right)}{\partial\theta_2}\right\|_\infty\right]\\
&\leq E\left[\left\|Z_t\right\|_\infty\right]+E\left[\underset{\theta\in\Theta}{sup}\left\|\frac{\partial\hat{\delta}_t\left(\theta\right)}{\partial\theta_2}\right\|_\infty\right]\\
&\leq E\left[\left\|Z_t\right\|_\infty\right]+E\left[\underset{\theta\in\Theta}{sup}\left\|\left(\frac{1}{R}\sum_{r=1}^R\nabla_\delta g_t\left(\hat{\delta}_t,X_t,v_r;\theta\right)\right)^{-1}\right\|_\infty\underset{\theta\in\Theta}{sup}\left\|\frac{1}{R}\sum_{r=1}^R\nabla_{\theta_2}g_t\left(\hat{\delta}_t,X_t,v_r;\theta\right)\right\|_\infty\right].
\end{aligned}$$

We showed in lemma 1 that $\int\nabla_\delta g_t(\delta_t,X_t,v_r;\theta)dF(v_r)$ is strictly diagonally dominant for all θ, δ_t, X_t, and F, which implies $\frac{1}{R}\sum_{r=1}^R\nabla_\delta g_t\left(\delta_t,X_t,v_r;\theta\right)$ is strictly diagonally dominant for all θ, δ_t, and X_t. The Ahlberg-Nilson-Varah bound (Ahlberg and Nilson (1963); Varah (1975)) states that for all $t=1...T$,

$$\underset{\theta\in\Theta}{sup}\left\|\left(\frac{1}{R}\sum_{r=1}^R\nabla_\delta g_t\left(\hat{\delta}_t,X_t,v_r;\theta\right)\right)^{-1}\right\|_\infty\leq\underset{\theta\in\Theta}{sup}\frac{1}{\min_{1\leq i\leq JT}\left(|a_t^{ii}(\theta)|-\sum_{j\neq i}|a_t^{ij}(\theta)|\right)},$$

where $a_t^{ij}(\theta)$ is the i,jth element of $\frac{1}{R}\sum_{r=1}^R\nabla_\delta g_t\left(\hat{\delta}_t,X_t,v_r;\theta\right)$. Since $a_t^{ij}(\theta)\in(-1,0)\cup(0,1)$ for all θ, there exists a constant C such that $\underset{t=1...T}{max}\,\underset{\theta\in\Theta}{sup}\left\|\left(\frac{1}{R}\sum_{r=1}^R\nabla_\delta g_t\left(\hat{\delta}_t,X_t,v_r;\theta\right)\right)^{-1}\right\|_\infty<C$. Next we show $E\left[\underset{\theta\in\Theta}{sup}\left\|\frac{1}{R}\sum_{r=1}^R\nabla_{\theta_2}g_t\left(\hat{\delta}_t,X_t,v_r;\theta\right)\right\|_\infty\right]<\infty$ by showing that the vector $E\left[\underset{\theta\in\Theta}{sup}\frac{1}{R}\sum_{r=1}^R\left|\frac{\partial g_{jt}\left(\hat{\delta}_t,X_t,v_r;\theta\right)}{\partial\theta_2}\right|\right]<\infty$ for all $j=1...J$. Note that for all $t=1...T$, $j=1...J$, and $r=1...R$,

$$\begin{aligned}
&\underset{\theta\in\Theta}{sup}\left|\frac{\partial g_{jt}\left(\hat{\delta}_t,X_t,v_r;\theta\right)}{\partial\theta_2}\right|\\
&=\underset{\theta\in\Theta}{sup}\left|\frac{exp(\hat{\delta}_{jt}+\mu_{rjt})}{1+\sum_{k\in\mathcal{N}(t)}exp(\hat{\delta}_{kt}+\mu_{rkt})}\left(\left[1,x_{jt}'\right]'\circ v_r-\frac{\sum_{k\in\mathcal{N}(t)}exp(\hat{\delta}_{kt}+\mu_{rkt})x_{kt}\circ v_r}{1+\sum_{k\in\mathcal{N}(t)}exp(\hat{\delta}_{kt}+\mu_{rkt})}\right)\right|\\
&\leq\underset{k=1...J}{max}\left|[1,x_{kt}']'\circ v_r\right|\underset{\theta\in\Theta}{sup}\left|\left(\frac{exp(\hat{\delta}_{jt}+\mu_{rjt})}{1+\sum_{k\in\mathcal{N}(t)}exp(\hat{\delta}_{kt}+\mu_{rkt})}\right)\left(1+\frac{\sum_{k\in\mathcal{N}(t)}exp(\hat{\delta}_{kt}+\mu_{rkt})}{1+\sum_{k\in\mathcal{N}(t)}exp(\hat{\delta}_{kt}+\mu_{rkt})}\right)\right|\\
&\leq 2\underset{k=1...J}{max}\left|[1,x_{kt}']'\circ v_r\right|.
\end{aligned}$$

Since $E\left[\underset{j=1\ldots J}{max}\left|\left[1,x_{jt}'\right]'\circ v_r\right|\right]<\infty$ by assumption, $E\left[\underset{\theta\in\Theta}{sup}\frac{1}{R}\sum_{r=1}^{R}\left|\frac{\partial g_{jt}\left(\hat{\delta}_t,X_t,v_r;\theta\right)}{\partial\theta_2}\right|\right]\leq$ $E\left[\frac{1}{R}\sum_{r=1}^{R}\underset{j=1\ldots J}{max}\left|\left[1,x_{jt}'\right]'\circ v_r\right|\right]<\infty$ and $E\left[\underset{\theta\in\Theta}{sup}\left\|\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}\right\|_\infty\right]\leq$ $CE\left[\underset{\theta\in\Theta}{sup}\left\|\frac{1}{R}\sum_{r=1}^{R}\nabla_{\theta_2}g_t\left(\hat{\delta}_t,X_t,v_r;\theta\right)\right\|_\infty\right]<\infty$. This combined with $E\left[\|Z_t\|_\infty\right]<\infty$ implies that $E\left[\underset{\theta\in\Theta}{sup}\left\|Z_t\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}\right\|_\infty\right]<\infty$ which implies that $E\left[\underset{\theta\in\Theta}{sup}\left\|Z_t\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}\right\|_2\right]<\infty$. It follows that

$$\underset{\theta\in\Theta}{sup}\left\|\frac{1}{T}\sum_{t=1}^{T}Z_t\frac{\partial\hat{\delta}_t\left(\theta\right)}{\partial\theta_2}-\underset{T,R\to\infty}{lim}\frac{1}{T}\sum_{t=1}^{T}E\left[Z_t\frac{\partial\hat{\delta}_t\left(\theta\right)}{\partial\theta_2}\right]\right\|_2=o_p(1).$$

Additionally, since $E\left[\|Z_tX_t'\|_2\right]<\infty$, the weak law of large numbers implies that

$$\left\|\frac{1}{T}\sum_{t=1}^{T}Z_tX_t'-E\left[Z_tX_t'\right]\right\|_2=o_p(1).$$

Therefore, stochastic equicontinuity holds:

$$\begin{aligned}&\underset{\|\theta-\theta_0\|\leq\kappa_m}{sup}\sqrt{m}\left\|\hat{\gamma}\left(\theta\right)-\hat{\gamma}\left(\theta_0\right)-\left(\gamma\left(\theta\right)-\gamma\left(\theta_0\right)\right)\right\|/\left(1+\sqrt{m}\left\|\theta-\theta_0\right\|\right)\\ \leq&\underset{\|\theta-\theta_0\|\leq\kappa_m}{sup}\left\|\hat{\gamma}\left(\theta\right)-\hat{\gamma}\left(\theta_0\right)-\left(\gamma\left(\theta\right)-\gamma\left(\theta_0\right)\right)\right\|/\left\|\theta-\theta_0\right\|\\ \leq&\underset{\theta\in\Theta}{sup}\left\|\frac{1}{T}\sum_{t=1}^{T}Z_t\frac{\partial\hat{\delta}_t\left(\theta\right)}{\partial\theta_2}-\underset{T,R\to\infty}{lim}\frac{1}{T}\sum_{t=1}^{T}E\left[Z_t\frac{\partial\hat{\delta}_t\left(\theta\right)}{\partial\theta_2}\right]\right\|_2+\left\|\frac{1}{T}\sum_{t=1}^{T}Z_tX_t'-E\left[Z_tX_t'\right]\right\|_2\\ =&o_p(1).\end{aligned}$$

Using similar arguments, we can show that for all $\theta',\theta''\in\Theta$,

$$\begin{aligned}\left\|\hat{\gamma}\left(\theta'\right)-\hat{\gamma}\left(\theta''\right)\right\|&\leq\underset{\theta\in\Theta}{sup}\left\|\frac{1}{T}\sum_{t=1}^{T}Z_t\frac{\partial\hat{\delta}_t\left(\theta\right)}{\partial\theta_2}\right\|_2\left\|\theta_2'-\theta_2''\right\|+\left\|\frac{1}{T}\sum_{t=1}^{T}Z_tX_t'\right\|_2\left\|\theta_1'-\theta_1''\right\|\\&\leq B_T\left\|\theta'-\theta''\right\|\end{aligned}$$

for $B_T=\underset{\theta\in\Theta}{sup}\left\|\frac{1}{T}\sum_{t=1}^{T}Z_t\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}\right\|_2+\left\|\frac{1}{T}\sum_{t=1}^{T}Z_tX_t'\right\|_2\leq\frac{1}{T}\sum_{t=1}^{T}\left(\underset{\theta\in\Theta}{sup}\left\|Z_t\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}\right\|_2+\|Z_tX_t'\|_2\right)=$ $O_p(1)$ since $E\left[\underset{\theta\in\Theta}{sup}\left\|Z_t\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}\right\|_2\right]<\infty$ and $E\left[\|Z_tX_t'\|_2\right]<\infty$.
Since $\gamma\left(\theta\right)=\underset{T,R\to\infty}{lim}\frac{1}{T}\sum_{t=1}^{T}E\left[Z_t\left(\hat{\delta}_t\left(\theta\right)-X_t'\theta_1\right)\right]$ is continuous in θ, Θ is a compact set,

and $\|\hat{\gamma}(\theta) - \gamma(\theta)\| \xrightarrow{p} 0$ for each θ, Lemma 2.9 in Newey and McFadden (1994) implies that

$$\sup_{\theta\in\Theta} \|\hat{\gamma}(\theta) - \gamma(\theta)\| \xrightarrow{p} 0.$$

■

2.2 Proof of Theorem 7

PROOF. Recall that

$$\frac{\partial\hat{\delta}(\theta)}{\partial\theta_2} = -\left(\frac{1}{R}\sum_{r=1}^{R} G_\delta\left(\hat{\delta}, X, v_r; \theta\right)\right)^{-1} \frac{1}{R}\sum_{r=1}^{R} G_{\theta_2}\left(\hat{\delta}, X, v_r; \theta\right).$$

We showed in theorem 4 that $E\left[\sup_{\theta\in\Theta}\left\|Z_t\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}\right\|_2\right] < \infty$. Since $\hat{\theta} \xrightarrow{p} \theta_0$ and $\frac{\partial\hat{\delta}(\theta)}{\partial\theta_2}$ is continuous in θ, by Lemma 4.3 of Newey and McFadden (1994) and the weak law of large numbers,

$$\hat{\Gamma} = \left[\; -\tfrac{1}{T}\textstyle\sum_{t=1}^{T} Z_t X_t', \quad \tfrac{1}{T}\sum_{t=1}^{T} Z_t \left.\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}\right|_{\hat{\theta}} \;\right] \xrightarrow{p} \left[\; -E\left[Z_t X_t'\right], \quad \lim_{T,R\to\infty} \tfrac{1}{T}\textstyle\sum_{t=1}^{T} E\left[Z_t \left.\frac{\partial\hat{\delta}_t(\theta)}{\partial\theta_2}\right|_{\theta_0}\right] \;\right].$$

Since we also assumed $E\|Z_t X_t'\|_2 < \infty$ and $Z_t\left(\hat{\delta}_t(\theta) - X_t'\theta_1\right)$ is integrable for θ in a neighborhood of θ_0, we can interchange differentiation and expectation so that $plim\hat{\Gamma} = \Gamma$. Furthermore, $W_T = \hat{\Sigma}^{-1} \xrightarrow{p} W = \Sigma^{-1}$. Therefore, $\hat{\Gamma}'W_T\hat{\Gamma} \xrightarrow{p} \Gamma'W\Gamma$.
To show that $\hat{\Omega} \xrightarrow{p} \Omega$, note that since $\hat{\delta}\left(\hat{\theta}_2\right) \xrightarrow{p} \delta_0$, $\hat{\theta} \xrightarrow{p} \theta_0$, and there exists $\kappa_m \downarrow 0$ such that $E\left[\sup_{\|\theta-\theta_0\|\le\kappa_m}\left\|Z_t\left(\hat{\delta}_t(\theta) - X_t'\theta_1\right)\right\|\right] < \infty$, by Lemma 4.3 of Newey and McFadden (1994),

$$\frac{1}{T}\sum_{t=1}^{T}\left(Z_t\left(\hat{\delta}_t\left(\hat{\theta}_2\right) - X_t'\hat{\theta}_1\right)\right)\left(Z_t\left(\hat{\delta}_t\left(\hat{\theta}_2\right) - X_t'\hat{\theta}_1\right)\right)' - E\left[\left(Z_t\left(\delta_{0t} - X_t'\theta_{0,1}\right)\right)\left(Z_t\left(\delta_{0t} - X_t'\theta_{0,1}\right)\right)'\right] \xrightarrow{p} 0.$$

To show that $\hat{\Sigma}_h \xrightarrow{p} \Sigma_h$, we first show that $\max_{r=1\ldots R}\left\|\hat{h}\left(v_r;\hat{\theta}\right) - \tilde{h}\left(v_r;\theta_0\right)\right\|_\infty \xrightarrow{p} 0$, where

$$\hat{h}\left(v_r;\hat{\theta}\right) = -\frac{1}{T}\sum_{t=1}^{T} Z_t\left(\frac{1}{R}\sum_{r'=1}^{R}\nabla_\delta g_t(\hat{\delta}_t, X_t, v_{r'};\hat{\theta})\right)^{-1}\left(g_t(\hat{\delta}_t, X_t, v_r;\hat{\theta}) - \frac{1}{R}\sum_{r=1}^{R} g_t(\hat{\delta}_t, X_t, v_r;\hat{\theta})\right)$$
$$\tilde{h}\left(v_r;\theta_0\right) = -\frac{1}{T}\sum_{t=1}^{T} Z_t\left(\int\nabla_\delta g_t(\delta_{0t}, X_t, v;\theta_0)dF_0(v)\right)^{-1}\left(g_t(\delta_{0t}, X_t, v_r;\theta_0) - E_v[g_t(\delta_{0t}, X_t, v;\theta_0)]\right).$$

Note that for all $t = 1...T$, $\nabla_\delta g_t(\delta_t, X_t, v_r; \theta)$ is continuous in δ_t and θ, and $\nabla_\delta g_t(\delta_t, X_t, v_r; \theta) \in (-1, 0) \cup (0, 1)$ for all δ_t, X_t, v_r, and θ. Since $\hat{\theta} \xrightarrow{p} \theta_0$ and $\hat{\delta}\left(\hat{\theta}_2\right) \xrightarrow{p} \delta_0$, by Lemma 4.3 of Newey and McFadden (1994),

$$\max_{t=1...T} \left\| \frac{1}{R}\sum_{r=1}^{R} \nabla_\delta g_t(\hat{\delta}_t, X_t, v_r; \hat{\theta}) - \int \nabla_\delta g_t(\delta_{0t}, X_t, v_r; \theta_0) dF_0(v_r) \right\|_\infty \xrightarrow{p} 0.$$

By the Continuous Mapping Theorem,

$$\max_{t=1...T} \left\| \left(\frac{1}{R}\sum_{r=1}^{R} \nabla_\delta g_t(\hat{\delta}_t, X_t, v_r; \hat{\theta}) \right)^{-1} - \left(\int \nabla_\delta g_t(\delta_{0t}, X_t, v; \theta_0) dF_0(v) \right)^{-1} \right\|_\infty \xrightarrow{p} 0.$$

Similarly, note that for all $t = 1...T$, $g_t(\delta_t, X_t, v_r; \theta)$ is continuous in δ_t and θ, and $g_t(\delta_t, X_t, v_r; \theta) \in (0, 1)$ for all δ_t, X_t, v_r, and θ. Since $\hat{\theta} \xrightarrow{p} \theta_0$ and $\hat{\delta}\left(\hat{\theta}_2\right) \xrightarrow{p} \delta_0$, by Lemma 4.3 of Newey and McFadden (1994),

$$\max_{t=1...T} \left\| \frac{1}{R}\sum_{r=1}^{R} g_t\left(\hat{\delta}_t, X_t, v_r; \hat{\theta}\right) - E_v\left[g_t(\delta_{0t}, X_t, v; \theta_0)\right] \right\|_\infty \xrightarrow{p} 0.$$

Note that the Ahlberg-Nilson-Varah (Ahlberg and Nilson (1963); Varah (1975)) bound on the strictly diagonally dominant matrices $\int \nabla_\delta g_t(\delta_{0t}, X_t, v_r; \theta_0)\, dF(v_r)$ implies that there exists a constant C such that

$$\max_{t=1...T} \left\| \left(\int \nabla_\delta g_t(\delta_{0t}, X_t, v_r; \theta_0)\, dF_0(v_r) \right)^{-1} \right\|_\infty < C.$$

Also, since $g_t(\delta_t, X_t, v_r; \theta) \in (0, 1)$ for all δ_t, X_t, v_r, and θ, there exists B such that

$$\max_{t=1...T} \max_{r=1...R} \left\| g_t\left(\hat{\delta}_t, X_t, v_r; \hat{\theta}\right) - \frac{1}{R}\sum_{r'=1}^{R} g_t\left(\hat{\delta}_t, X_t, v_{r'}; \hat{\theta}\right) \right\|_\infty < B.$$

We assumed in theorem 4 that $E\left\|Z_t\right\|_\infty < \infty$, which implies that $\left\|Z_t\right\|_\infty = O_p(1)$. Furthermore, we assumed

$$\max_{r=1...R} \max_{t=1...T} \left\| g_t\left(\hat{\delta}_t, X_t, v_r; \hat{\theta}\right) - g_t(\delta_{0t}, X_t, v_r; \theta_0) \right\|_\infty \xrightarrow{p} 0.$$

Therefore,

$$
\begin{aligned}
&\max_{r=1\ldots R}\left\|\hat{h}\left(v_r;\hat{\theta}\right)-\tilde{h}\left(v_r;\theta_0\right)\right\|_\infty \\
\leq&\max_{r=1\ldots R}\left\|\frac{1}{T}\sum_{t=1}^{T}Z_t\left(\left(\frac{1}{R}\sum_{r'=1}^{R}\nabla_\delta g_t\left(\hat{\delta}_t,X_t,v_{r'};\hat{\theta}\right)\right)^{-1}-\left(\int\nabla_\delta g_t\left(\delta_{0t},X_t,v_{r'};\theta_0\right)dF_0\left(v_{r'}\right)\right)^{-1}\right)\right.\\
&\qquad\left.\left(g_t\left(\hat{\delta}_t,X_t,v_r;\hat{\theta}\right)-\frac{1}{R}\sum_{r'=1}^{R}g_t\left(\hat{\delta}_t,X_t,v_{r'};\hat{\theta}\right)\right)\right\|_\infty \\
&+\max_{r=1\ldots R}\left\|\frac{1}{T}\sum_{t=1}^{T}Z_t\left(\int\nabla_\delta g_t\left(\delta_{0t},X_t,v_{r'};\theta_0\right)dF_0\left(v_{r'}\right)\right)^{-1}\left(g_t\left(\hat{\delta}_t,X_t,v_r;\hat{\theta}\right)-g_t\left(\delta_{0t},X_t,v_r;\theta_0\right)\right)\right\|_\infty \\
&+\left\|\frac{1}{T}\sum_{t=1}^{T}Z_t\left(\int\nabla_\delta g_t\left(\delta_{0t},X_t,v_{r'};\theta_0\right)dF_0\left(v_{r'}\right)\right)^{-1}\left(\frac{1}{R}\sum_{r'=1}^{R}g_t\left(\hat{\delta}_t,X_t,v_{r'};\hat{\theta}\right)-E_v\left[g_t\left(\delta_{0t},X_t,v;\theta_0\right)\right]\right)\right\|_\infty \\
\leq&\max_{r=1\ldots R}\max_{t=1\ldots T}\left\|Z_t\left(\left(\frac{1}{R}\sum_{r'=1}^{R}\nabla_\delta g_t\left(\hat{\delta}_t,X_t,v_{r'};\hat{\theta}\right)\right)^{-1}-\left(\int\nabla_\delta g_t\left(\delta_{0t},X_t,v_{r'};\theta_0\right)dF_0\left(v_{r'}\right)\right)^{-1}\right)\right.\\
&\qquad\left.\left(g_t\left(\hat{\delta}_t,X_t,v_r;\hat{\theta}\right)-\frac{1}{R}\sum_{r'=1}^{R}g_t\left(\hat{\delta}_t,X_t,v_{r'};\hat{\theta}\right)\right)\right\|_\infty \\
&+\max_{r=1\ldots R}\max_{t=1\ldots T}\left\|Z_t\left(\int\nabla_\delta g_t\left(\delta_{0t},X_t,v_{r'};\theta_0\right)dF_0\left(v_{r'}\right)\right)^{-1}\left(g_t\left(\hat{\delta}_t,X_t,v_r;\hat{\theta}\right)-g_t\left(\delta_{0t},X_t,v_r;\theta_0\right)\right)\right\|_\infty \\
&+\max_{t=1\ldots T}\left\|Z_t\left(\int\nabla_\delta g_t\left(\delta_{0t},X_t,v_{r'};\theta_0\right)dF_0\left(v_{r'}\right)\right)^{-1}\left(\frac{1}{R}\sum_{r'=1}^{R}g_t\left(\hat{\delta}_t,X_t,v_{r'};\hat{\theta}\right)-E_v\left[g_t\left(\delta_{0t},X_t,v;\theta_0\right)\right]\right)\right\|_\infty \\
\leq&\max_{t=1\ldots T}\left\|Z_t\right\|_\infty\max_{t=1\ldots T}\left\|\left(\frac{1}{R}\sum_{r'=1}^{R}\nabla_\delta g_t\left(\hat{\delta}_t,X_t,v_{r'};\hat{\theta}\right)\right)^{-1}-\left(\int\nabla_\delta g_t\left(\delta_{0t},X_t,v_{r'};\theta_0\right)dF_0\left(v_{r'}\right)\right)^{-1}\right\|_\infty \\
&\max_{r=1\ldots R}\max_{t=1\ldots T}\left\|g_t\left(\hat{\delta}_t,X_t,v_r;\hat{\theta}\right)-\frac{1}{R}\sum_{r'=1}^{R}g_t\left(\hat{\delta}_t,X_t,v_{r'};\hat{\theta}\right)\right\|_\infty \\
&+\max_{t=1\ldots T}\left\|Z_t\right\|_\infty\max_{t=1\ldots T}\left\|\left(\int\nabla_\delta g_t\left(\delta_{0t},X_t,v_{r'};\theta_0\right)dF_0\left(v_{r'}\right)\right)^{-1}\right\|_\infty \\
&\left\{\max_{r=1\ldots R}\max_{t=1\ldots T}\left\|g_t\left(\hat{\delta}_t,X_t,v_r;\hat{\theta}\right)-g_t\left(\delta_{0t},X_t,v_r;\theta_0\right)\right\|_\infty\right.\\
&\left.+\max_{t=1\ldots T}\left\|\frac{1}{R}\sum_{r'=1}^{R}g_t\left(\hat{\delta}_t,X_t,v_{r'};\hat{\theta}\right)-E_v\left[g_t\left(\delta_{0t},X_t,v;\theta_0\right)\right]\right\|_\infty\right\} \\
=&o_p\left(1\right).
\end{aligned}
$$

Then it follows that

$$\begin{aligned}
\hat{\Sigma}_h &= \frac{1}{R}\sum_{r=1}^{R}\hat{h}\left(v_r;\hat{\theta}\right)\hat{h}\left(v_r;\hat{\theta}\right)' \\
&= \frac{1}{R}\sum_{r=1}^{R}\left(\tilde{h}\left(v_r;\theta_0\right)+o_p\left(1\right)\right)\left(\tilde{h}\left(v_r;\theta_0\right)+o_p\left(1\right)\right)' \\
&= \frac{1}{R}\sum_{r=1}^{R}\tilde{h}\left(v_r;\theta_0\right)\tilde{h}\left(v_r;\theta_0\right)'+o_p\left(1\right).
\end{aligned}$$

Note that $\frac{1}{R}\sum_{r=1}^{R}\tilde{h}\left(v_r;\theta_0\right)\tilde{h}\left(v_r;\theta_0\right)'$ is a two-sample V-statistic, which is known to satisfy a weak law of large numbers (see e.g. van der Vaart (1998)). Therefore, for $k\left(\delta_{0t},X_t,v_r;\theta_0\right)=-Z_t\left(\int\nabla_\delta g_t(\delta_{0t},X_t,v;\theta_0)dF_0(v)\right)^{-1}\left(g_t(\delta_{0t},X_t,v_r;\theta_0)-E_v[g_t(\delta_{0t},X_t,v;\theta_0)]\right)$,

$$\frac{1}{R}\sum_{r=1}^{R}\tilde{h}\left(v_r;\theta_0\right)\tilde{h}\left(v_r;\theta_0\right)'\xrightarrow{p}E\left[k\left(\delta_{0t},X_t,v_r;\theta_0\right)k\left(\delta_{0t},X_t,v_r;\theta_0\right)'\right]=\Sigma_h.$$

We have shown that $\hat{\Sigma}=min\left(1,\frac{R}{T}\right)\hat{\Omega}+min\left(1,\frac{T}{R}\right)\hat{\Sigma}_h\xrightarrow{p}\Sigma=(1\wedge k)\Omega+(1\wedge 1/k)\Sigma_h$. Therefore,

$$\widehat{AsyVar}\left[\hat{\theta}\right]=\left(\hat{\Gamma}'W_T\hat{\Gamma}\right)^{-1}\hat{\Gamma}'W_T\hat{\Sigma}W_T\hat{\Gamma}\left(\hat{\Gamma}'W_T\hat{\Gamma}\right)^{-1}\xrightarrow{p}\left(\Gamma'W\Gamma\right)^{-1}\Gamma'W\Sigma W\Gamma\left(\Gamma'W\Gamma\right)^{-1}.$$

■

3 Comparison with Freyberger (2015)

We reproduce the formulas for Φ_1 and Φ_2 in Freyberger (2015) and compare them to our Ω and Σ_h. In the following expressions, we reproduce Freyberger (2015)'s notation under the assumption of overlapping simulation draws: $v_{rt}=v_r$ for all t. First note that Freyberger (2015) uses ν_{jt} when defining the market shares while we use g_{jt}.

$$\begin{aligned}
\nu_{jt}\left(\theta,x_t,\xi_t,v_r\right) &= \frac{exp\left(X_{jt}'\theta_1+\xi_{jt}+\mu_{rjt}\right)}{1+\sum_{k\in\mathcal{N}(t)}exp\left(X_{kt}'\theta_1+\xi_{jt}+\mu_{rkt}\right)}=g_{jt}\left(\delta_t,X_t,v_r;\theta\right) \\
\sigma_t\left(\theta,x_t,\xi_t,P_t\right) &= \int\nu_t\left(\theta,x_t,\xi_t,v\right)dP_t\left(v\right)=\int g_t\left(\delta_t,X_t,v;\theta\right)dF\left(v\right).
\end{aligned}$$

Freyberger (2015) defines H_{0t} as the Jacobian matrix of the true market shares with respect to ξ_t, which is the same as our Jacobian matrix of the true market shares with respect to $\delta_t = X_t'\theta_1 + \xi_t$.

$$H_{0t} = \frac{\partial \sigma_t\left(\theta_0, \xi_t\left(\theta_0, P_{0t}\right), P_{0t}\right)}{\partial \xi} = \int \nabla_\delta g_t\left(\delta_{0t}, X_t, v_r; \theta_0\right) dF_0\left(v_r\right).$$

Freyberger (2015) also defines

$$\epsilon_{0rt} = \nu_t\left(\theta_0, x_t, \xi_t\left(\theta_0, P_{0t}, s_t\right), v_r\right) - \int \nu_t\left(\theta_0, x_t, \xi_t\left(\theta_0, P_{0t}, s_t\right), v\right) dP_{0t}\left(v\right).$$

Freyberger (2015) uses $z_t \in \mathbb{R}^{J\times p}$ to denote the matrix of instruments for market t while we use $Z_t \in \mathbb{R}^{p\times J}$. His expression for Φ_1 coincides with our Ω under our assumption that the data are i.i.d. across markets.

$$\begin{aligned}
\Phi_1 &= \underset{T\to\infty}{lim} E\left[\frac{1}{T}\sum_{t=1}^{T} z_t'\xi_t\left(\theta_0, P_{0t}, s_t\right)\xi_t\left(\theta_0, P_{0t}, s_t\right)' z_t\right] \\
&= \underset{T\to\infty}{lim} E\left[\frac{1}{T}\sum_{t=1}^{T} Z_t\left(\delta_{0t} - X_t'\theta_{01}\right)\left(\delta_{0t} - X_t'\theta_{01}\right)' Z_t'\right] \\
&= E\left[Z_t\left(\delta_{0t} - X_t'\theta_{01}\right)\left(\delta_{0t} - X_t'\theta_{01}\right)' Z_t'\right] = \Omega.
\end{aligned}$$

Freyberger (2015)'s Φ_2 is not exactly the same as our Σ_h, but only differs by a $o(1)$ term. To see this, note that $\Phi_2 = \underset{T\to\infty}{lim} \frac{1}{T}\sum_{t=1}^{T} Var\left[z_t' H_{0t}^{-1}\epsilon_{0rt}\right]$ can be rewritten using the fact that v_r are i.i.d. as $\underset{T\to\infty}{lim} Var\left[\frac{1}{TR}\sum_{t=1}^{T}\sum_{r=1}^{R} q\left(Z_t, X_t, v_r; \theta_0, \delta_{0t}\right)\right]$, where

$$q\left(Z_t, X_t, v_r; \theta_0, \delta_{0t}\right) = -Z_t\left(\int \nabla_\delta g_t\left(\delta_{0t}, X_t, v; \theta_0\right) dF_0\left(v\right)\right)^{-1}\left(g_t\left(\delta_{0t}, X_t, v_r; \theta_0\right) - \int g_t\left(\delta_{0t}, X_t, v_r; \theta_0\right) dF_0\left(v_r\right)\right).$$

Recall that $\frac{1}{TR}\sum_{t=1}^{T}\sum_{r=1}^{R} q\left(Z_t, X_t, v_r; \theta_0, \delta_{0t}\right)$ is a two-sample U-statistic whose decomposition is

$$\frac{1}{TR}\sum_{t=1}^{T}\sum_{r=1}^{R} q\left(Z_t, X_t, v_r; \theta_0, \delta_{0t}\right) = \frac{1}{R}\sum_{r=1}^{R} h\left(v_r; \theta_0\right) + o_p\left(1\right).$$

$$h\left(v_r; \theta_0\right) = -\int\left\{Z_t\left(\int \nabla_\delta g_t\left(\delta_{0t}, X_t, v; \theta_0\right) dF_0\left(v\right)\right)^{-1}\left(g_t\left(\delta_{0t}, X_t, v_r; \theta_0\right) - \int g_t\left(\delta_{0t}, X_t, v_r; \theta_0\right) dF_0\left(v_r\right)\right)\right\} dP\left(Z_t, X_t, \xi_t\right).$$

It follows that $\Phi_2 = \underset{T\to\infty}{lim} Var\left[h\left(v_r;\theta_0\right)+o_p\left(1\right)\right] = \Sigma_h + o\left(1\right)$. Next we explain why $\hat{\Phi}_1 = \hat{\Omega}$ and $\hat{\Phi}_2 = \hat{\Sigma}_h$. Note that

$$\begin{aligned}
\hat{H}_t = \frac{\partial \sigma_t\left(\hat{\theta}, x_t, \hat{\xi}_t, P_{rt}\right)}{\partial \xi} &= \frac{1}{R}\sum_{r=1}^{R} diag\left(\nu_t\left(\hat{\theta}, x_t, \hat{\xi}_t, v_r\right)\right) - \nu_t\left(\hat{\theta}, x_t, \hat{\xi}_t, v_r\right)\nu_t\left(\hat{\theta}, x_t, \hat{\xi}_t, v_r\right)' \\
&= \frac{1}{R}\sum_{r'=1}^{R} \nabla_\delta g_t\left(\hat{\delta}_t, X_t, v_{r'};\hat{\theta}\right).
\end{aligned}$$

$$\begin{aligned}
\bar{\nu}_t\left(\hat{\theta}, x_t, \hat{\xi}_t, v_r\right) &= \nu_t\left(\hat{\theta}, x_t, \hat{\xi}_t, v_r\right) - \sigma_t\left(\hat{\theta}, x_t, \hat{\xi}_t, P_{rt}\right) \\
&= g_t\left(\hat{\delta}_t, X_t, v_r;\hat{\theta}\right) - \frac{1}{R}\sum_{r'=1}^{R} g_t\left(\hat{\delta}_t, X_t, v_{r'};\hat{\theta}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\hat{\Phi}_1 &= \frac{1}{T}\sum_{t=1}^{T} z_t'\hat{\xi}_t\hat{\xi}_t' z_t = \frac{1}{T}\sum_{t=1}^{T} Z_t\left(\hat{\delta}_t - X_t'\hat{\theta}_1\right)\left(\hat{\delta}_t - X_t'\hat{\theta}_1\right)' Z_t' = \hat{\Omega}. \\
\hat{\Phi}_2 &= \frac{1}{RT}\sum_{r=1}^{R}\sum_{t=1}^{T} z_t'\hat{H}_t^{-1}\bar{\nu}_t\left(\hat{\theta}, x_t, \hat{\xi}_t, v_r\right)\bar{\nu}_t\left(\hat{\theta}, x_t, \hat{\xi}_t, v_r\right)'\left(\hat{H}_t^{-1}\right)' z_t \\
&= \frac{1}{R}\sum_{r=1}^{R}\hat{h}\left(v_r;\hat{\theta}\right)\hat{h}\left(v_r;\hat{\theta}\right)' = \hat{\Sigma}_h.
\end{aligned}$$

$$\hat{h}\left(v_r;\hat{\theta}\right) = -\frac{1}{T}\sum_{t=1}^{T} Z_t\left(\frac{1}{R}\sum_{r'=1}^{R}\nabla_\delta g_t\left(\hat{\delta}_t, X_t, v_{r'};\hat{\theta}\right)\right)^{-1}\left(g_t\left(\hat{\delta}_t, X_t, v_r;\hat{\theta}\right) - \frac{1}{R}\sum_{r'=1}^{R} g_t\left(\hat{\delta}_t, X_t, v_{r'};\hat{\theta}\right)\right).$$

We can also show that our standard errors are the same as Freyberger (2015)'s standard errors. Our estimate of the finite sample variance of $\hat{\theta}$ is $\frac{1}{m}\left(\hat{\Gamma}'W_T\hat{\Gamma}\right)^{-1}\hat{\Gamma}'W_T\hat{\Sigma}W_T\hat{\Gamma}\left(\hat{\Gamma}'W_T\hat{\Gamma}\right)^{-1}$ where

$$\begin{aligned}
\frac{1}{m}\hat{\Sigma} &= \frac{1}{m}min\left(1, \frac{R}{T}\right)\hat{\Omega} + \frac{1}{m}min\left(1, \frac{T}{R}\right)\hat{\Sigma}_h \\
&= \frac{1}{T}\hat{\Omega} + \frac{1}{R}\hat{\Sigma}_h \\
&= \frac{1}{T}\left(\hat{\Phi}_1 + \frac{T}{R}\hat{\Phi}_2\right).
\end{aligned}$$

$\frac{1}{T}\left(\hat{\Gamma}'W_T\hat{\Gamma}\right)^{-1}\hat{\Gamma}'W_T\left(\hat{\Phi}_1+\frac{T}{R}\hat{\Phi}_2\right)W_T\hat{\Gamma}\left(\hat{\Gamma}'W_T\hat{\Gamma}\right)^{-1}$ is Freyberger (2015)'s estimate of the finite sample variance of $\hat{\theta}$.

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