FEDERAL RESERVE BANK OF SAN FRANCISCO

WORKING PAPER SERIES

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December 2024

Working Paper 2020-35

https://doi.org/10.24148/wp2020-35

Suggested citation:

Christensen, Jens H. E., Glenn D. Rudebusch, and Patrick J. Shultz. 2024. "Accounting for Changes in Long-Term Interest Rates: Evidence from Canada." Federal Reserve Bank of San Francisco Working Paper 2020-35. https://doi.org/10.24148/wp2020-35

This paper was previously circulated under the title "Accounting for Low Long-Term Interest Rates: Evidence from Canada."

Accounting for Changes in Long-Term Interest Rates: Evidence from Canada

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Abstract

For several decades, long-term interest rates around the world trended down before reversing some of that decline in the past few years. We decompose these changes using a dynamic term structure model of Canadian nominal and real yields with adjustments for term, liquidity, and inflation risk premiums. Canada provides a novel perspective on this issue because of its established indexed debt market, modest distortions from monetary quantitative easing and tightening or the zero lower bound, and an absence of sovereign credit risk. From 1996 to 2021, we find that the steady-state real interest rate fell by more than 4 percentage points, long-term inflation expectations edged down modestly, and real bond and inflation risk premiums varied over time but showed little longer-run trend. The subsequent reversal in long-term interest rates is mostly driven by a sharp increase in the equilibrium real rate.

JEL Classification: C32, E43, E52, G12, G17

Keywords: affine arbitrage-free term structure model, liquidity risk, financial market frictions, r-star

We thank conference participants at the 2nd LTI/Bank of Italy Workshop on Long-Term Investors' Trends: Theory and Practice, the 37th International Conference of the French Finance Association (AFFI), the IFABS 2021 Oxford Conference, the INQUIRE Residential Seminar 2021, the RCEA Conference on Recent Developments in Economics, Econometrics, and Finance, the Midwest Finance Association 2022 Annual Meeting, and the IFABS 2022 Naples Conference for helpful comments and suggestions, including our discussants Guillaume Roussellet, Fan Dora Xia, and Juan Arismendi Zambrano. We also thank seminar participants at the Bank of Canada, the Federal Reserve Board, and the Norges Bank for helpful comments and suggestions. Furthermore, we thank Antonio Diez de los Rios for helpful comments on an early draft of the paper. Finally, we thank Gus Kmetz, Caroline Paulson, Dori Wilson, and Simon Zhu for outstanding research assistance. The views in this paper are solely the responsibility of the authors and should not be interpreted as reflecting the views of the Federal Reserve Bank of San Francisco or the Federal Reserve System.

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This version: December 16, 2024.

1 Introduction

The secular decline in global long-term nominal interest rates in the decades before the COVID-19 pandemic has been the subject of much discussion and research. Indeed, the possibility of a lower new normal for interest rates was at the center of key economic, financial, and policy debates about the transformation of the economy and bond market dynamics. Still, the source and permanence of that gradual decline in interest rates remain unresolved—especially given a general post-pandemic rise in global interest rates. Therefore, in this paper, we estimate a comprehensive dynamic term structure model on an under-utilized data set to provide a new perspective on the underlying nature of the changes in long-term interest rates.

Accounting for movements in long-term interest rates during the past few decades requires assessing the relative importance of nominal inflationary forces, real economic factors, and risk premiums. In the 1980s and 1990s, falling inflation expectations clearly played a substantial role in lowering long-term yields, but in the 2000s and 2010s, actual inflation as well as survey-based measures of longer-run inflation expectations remained relatively stable. Instead, the source of the general decline in interest rates in the first two decades of the century appears to reflect a variety of longer-run real-side factors—such as slower productivity growth and an aging population (e.g., Rachel and Smith, 2015, Christensen and Rudebusch, 2019, and Bauer and Rudebusch, 2020). More recently, the post-pandemic rise in interest rates may reflect various influences including elevated inflation, higher government debt levels, increased investment for the transition to a greener economy, and greater defense spending (e.g., Rogoff, 2023). These shifts in economic fundamentals can persistently affect nominal and real yield curves by changing the steady-state level of the safe short-term real interest rate—the so-called equilibrium or natural or neutral rate of interest. Many researchers have used macroeconomic models and data to try to pin down the equilibrium real rate. Inspired by Christensen and Rudebusch (2019, henceforth CR), we use a *financial* model that accounts for nominal and real factors.[1] Varying term, inflation risk, and liquidity risk premiums can also play a role in changing long-term yields, and one advantage of a financial modeling approach is that it allows for explicit risk premiums to account for changes in long-term interest rates (e.g., Bauer and Rudebusch, 2020). In particular, as price inflation fell and became better anchored at low levels in many countries before the pandemic, the inflation risk premium may have declined (e.g., Wright, 2011, and Bauer et al., 2014). By contrast, during the recent spell of more rapid price increases, the inflation risk premium may have reversed some of that decline. Our analysis assesses the relative contribution to bond yields of each of these various components.

[1] Our finance-based approach has several advantages relative to macro-based estimates. Most notably, our measure does not depend on obtaining a correct, complete, and stable specification of the macroeconomic dynamics of output and inflation, as described in CR. This latter requirement is particularly challenging during the COVID-19 pandemic with its unprecedented economic shocks.

For our analysis, we employ Canadian government bond prices, which provides a new—or at least, a relatively under-studied—perspective on persistent changes in long-term interest rates. Besides its novelty, a Canadian case study has several other advantages. First, Canada has deep and liquid markets for government debt. Also, the Bank of Canada has engaged much less than most other major central banks in unconventional monetary policies such as large-scale asset purchases (also known as quantitative easing or QE).[2] As a consequence, Canadian bond yields are less directly affected by such policies (although indirect spillover effects from QE in other countries is a possibility that we explore explicitly).[3] Arguably then, the Canadian government bond market offers a cleaner, less-managed setting for analyzing these questions than the euro area, Japan, the U.K., or the U.S., where such policies have been implemented for extended periods. Also, as the underlying factors affecting long-term interest rates are likely global in nature—such as worldwide demographic shifts or disinflationary and inflationary pressures—the Canadian government bond market may well be about as informative as any other major sovereign bond market. Furthermore, the Canadian government holds a AAA credit rating with a stable outlook from all major rating agencies, which also contrasts with some of its G7 peers. Therefore, there is no credit risk to account for in our Canadian bond price data, which is an additional advantage. Finally, the Canadian government has been issuing inflation-indexed debt since 1991, and Canada therefore offers a relatively long history of both nominal and real yields, only rivaled by U.K. and U.S. samples.[4]

We examine a sample of Canadian nominal government bond yields along with prices of Canadian government Real Return Bonds (RRBs), which have coupon and principal payments indexed to the Canadian Consumer Price Index (CPI). The indexed debt provides compensation to investors for the erosion of purchasing power due to price inflation with prices that can be expressed directly in terms of real yields. We assume that the embedded longer-term expectations in these asset prices reflect financial market participants' views about the steady state of the Canadian economy, including the natural rate of interest and the long-run level of inflation (i.e., the perceived Bank of Canada inflation target). Still, the use of RRBs for measuring the steady-state short-term real interest rate and inflation level does pose empirical challenges. One problem is that despite the long history and fairly large notional amount of outstanding RRBs, these securities potentially face appreciable liquidity risk as they tend to have smaller trading volumes and wider bid-ask spreads than Canadian nominal government

[2]See Azizova et al. (2024) for an evaluation of the effects of the Bank of Canada's government bond purchases during the pandemic. These purchases started in March 2020, ended in October 2021, and are expected to be fully unwound by 2025 entirely from not rolling over maturing bonds; see Kozicki (2024). At their peak, the Bank of Canada's bond holdings totaled slightly more than 20 percent of nominal GDP.

[3]See Kearns et al. (2020) for an analysis of monetary policy spillover effects from four major economies to 47 advanced and emerging economies, including Canada, covering the 1999-2019 period.

[4]A long sample allows for robust estimation of the models' objective factor dynamics, which are important for many of our conclusions. See Bauer et al. (2012) for a detailed discussion of the related finite-sample bias problem and its impact on yield curve model estimation.

securities. Presumably, investors require a premium for bearing the liquidity risk associated with holding RRBs, but the extent and time variation of this liquidity premium is unknown and apparently unresearched. This contrasts with numerous studies of the liquidity risk associated with U.S. Treasury Inflation-Protected Securities (TIPS) (e.g., Campbell et al., 2009, Pflueger and Viceira, 2016, and CR).

To estimate the natural rate of interest and long-term inflation expectations in the presence of liquidity and real and nominal term premiums, we use an arbitrage-free dynamic term structure model of nominal and real yields augmented with a liquidity risk factor. The identification of the liquidity risk factor comes from its unique loading for each individual RRB security as in Andreasen et al. (2021, henceforth ACR). Similar to ACR, our analysis combines a standard sample of nominal yields with the prices of individual RRBs. The underlying mechanism for identifying liquidity risk assumes that, over time, an increasing proportion of the outstanding inventory of each RRB is locked up in buy-and-hold investors' portfolios. Given forward-looking investor behavior, this lock-up effect means that a particular bond's sensitivity to the market-wide liquidity factor will vary depending on how seasoned the bond is and how close to maturity it is. In a careful study of nominal U.S. Treasuries, Fontaine and Garcia (2012) find a pervasive liquidity factor that affects all bond prices with loadings that vary with the maturity and age of each bond. By observing a cross section of RRB prices over time—each with a different time-since-issuance and time-to-maturity—we can identify the overall RRB liquidity factor and each bond's loading on that factor. This technique is particularly useful for analyzing inflation-indexed debt when only a limited sample of bonds may be available as in our case.[5,6]

As in ACR, the frictionless Canadian nominal and real yields follow the model of nominal and real yields introduced in Christensen et al. (2010, henceforth CLR), referred to throughout as the CLR model. We estimate CLR models and their liquidity-augmented extension, denoted CLR-L models. The theoretical arbitrage-free formulation of the CLR-L model also provides identification of a time-varying real term premium in the pricing of RRBs. Identifying the liquidity and term premiums allows us to estimate the underlying frictionless real rate term structure and the natural rate of interest, which we measure as the average expected real short rate over a five-year period starting five years ahead as in CR. Finally, to account

[5] Finlay and Wende (2012) examine prices from a limited number of Australian inflation-indexed bonds but do not account for their liquidity risk.

[6] We do not account for the liquidity risk in the Canadian nominal bond yields for two reasons. First, our focus is on the real yield aspect of our joint models of nominal and real yields, which is less sensitive to liquidity bias in nominal yields. Second and more importantly, Andreasen et al. (2019) find that pricing in the regular Canadian government bond market appears to be very efficient, which suggests that the liquidity premiums of standard fixed-coupon bonds are likely to be small. This may reflect the fact that the Bank of Canada in the past occasionally bought back seasoned nominal bond series and replaced them with new bond series, which helped maintain liquidity in the secondary market for these bonds. We note that the Bank of Canada stopped this practice a few years before the pandemic.

for the zero lower bound (ZLB) that may have constrained the behavior of the nominal yields in our Canadian data periodically since the Global Financial Crisis (GFC), we also estimate a shadow-rate version of the CLR-L model that enforces the ZLB for the nominal yields.[7]

Our preferred estimate of the natural rate of interest, r_t^*, is shown in Figure 1 along with measures of the ten-year nominal and real Canadian government bond yields. Both nominal and real long-term yields trended down together from the mid-1990s until early 2021 and sharply increased from late 2021 until the end of our sample, and this concurrence suggests little net change in inflation expectations or the inflation risk premium, which is supported by our results as evidenced by the estimated ten-year expected inflation also shown in Figure 1. Instead, the estimated equilibrium real rate is identified as the primary driver of the long-term yield declines since it fell from above 4 percent to below -1 percent in the 1996-2020 period. Accordingly, our results show that much of the 6-percentage-point decline in longer-term Canadian bond yields represents a reduction in the natural rate of interest, while ten-year inflation expectations are estimated to have declined only about 0.7 percentage point during that 25-year period before reversing about half of that modest decline during the last three years of our sample. Importantly, a sharp reversal in our r_t^* estimate accounts for the significant joint increase in nominal and real yields since 2021.

Furthermore, we find that the average liquidity premiums embedded in the yields of RRBs exhibit notable time variation reaching highs of close to 35 basis points around the peak of the GFC in the fall of 2008, and another sharp spike of almost similar magnitude around the peak of the COVID-19 pandemic in spring 2020. However, outside of those two periods of market stress, the average liquidity premium has been in a fairly narrow, slightly negative range. For the entire sample, the estimated RRB liquidity premiums average -4.8 basis points. This can be compared to the results of ACR, who report that the average liquidity premium in U.S. TIPS is estimated at 34 basis points for the 1997-2013 period. The difference in liquidity premium levels across the TIPS and the Canadian RRB markets is likely to be due to the much greater relative liquidity of nominal versus indexed U.S. Treasury securities compared to the more modest liquidity advantage of Canadian fixed-coupon government securities over Canadian RRBs.

In line with the existing literature on TIPS, we rely on a joint modeling of the nominal and real yield curves (e.g., CLR, Abrahams et al., 2016, and D'Amico et al., 2018). Although these joint specifications can also be used to estimate the steady-state real rate similar to our analysis, this earlier work has emphasized only the measurement of inflation expectations and risk. Importantly, our methodology requires a stable dynamic relationship between the pricing factors of both nominal and real bonds. This assumption is particularly demanding

[7]The ZLB constraint is ultimately tied to the fact that there is cash in circulation that investors can freely hold and therefore earn a minimum interest rate of zero; see Black (1995).

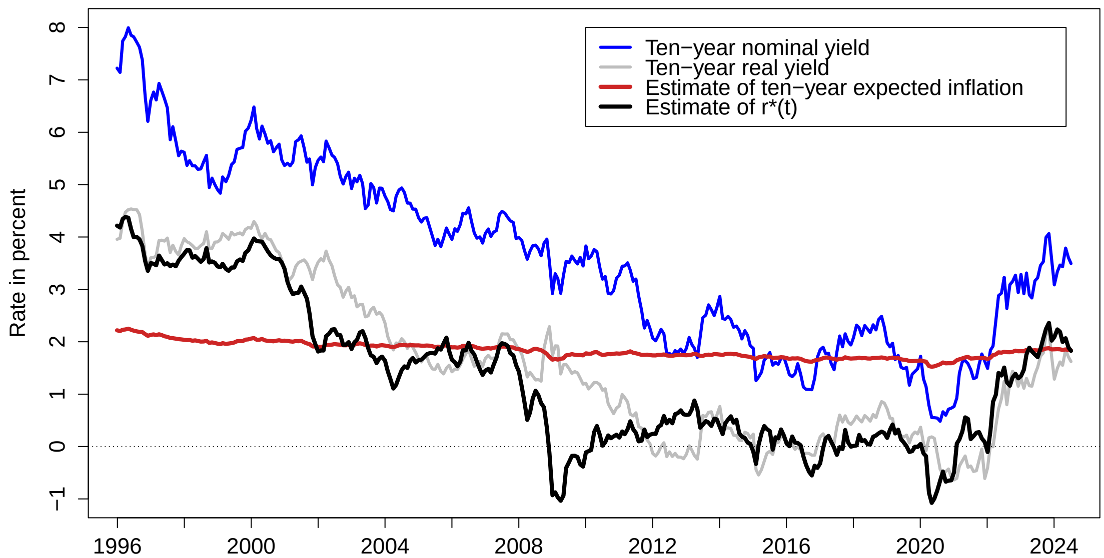


Figure 1: **Long-Term Nominal and Real Yields and an Estimate of Ten-Year Expected Inflation and r***

Illustration of (i) the ten-year nominal Canadian government bond yield from the Bolder et al. (2004) database maintained by staff at the Bank of Canada, (ii) the fitted ten-year real Canadian government bond yield from the CLR model with a diagonal specification of $K^{\mathbb{P}}$ and Σ, (iii) our preferred CLR-L model estimate of the ten-year expected inflation, and (iv) our preferred CLR-L model estimate of the equilibrium real short rate, r_t^*, i.e., the 5-to-10 year risk-neutral real rate.

during the period from 2009 to 2015 and again in the 2020-2021 period when the U.S. Federal Reserve kept the overnight federal funds rate at its effective ZLB, which drove Canadian short-term yields to historical lows as well. As a consequence, the dynamic interactions of Canadian short- and medium-term nominal yields were likely affected (see Swanson and Williams, 2014 and Christensen and Rudebusch, 2015, for discussions). However, our results for the shadow-rate version of the CLR-L model that respects the ZLB for nominal yields are very similar to the results from the unconstrained CLR-L model. This suggests that the dynamic properties of Canadian nominal yields are at most mildly affected by the nonnegative constraint on the overnight policy rate set by the Bank of Canada.

To demonstrate the applicability of the CLR-L model for economic analysis, we estimate it at daily frequency and use it to examine the market reaction to the decision by the Canadian Finance Department to terminate RRB issuance. This sudden policy change was announced on November 3, 2022, and came as a surprise to most investors and market observers. Our results indicate that the increase in RRB yields in response to the announcement does not reflect an increase in the RRB liquidity premium, rather this premium exhibited a declining pattern in the months following the announcement. Thus, it does not appear as if investors positioned themselves for somewhat slower RRB market trading going forward. Overall, though, the market reaction was tempered, and the RRB market has continued to function on par with the past through the end of our sample. Hence, for now, RRB trading remains active despite no new issuance has come to market since before November 2022. These findings also support our usage of the RRB data through the end of our sample, and presumably will support their usage well into the future given that the longest dated outstanding RRB can be expected to continue to trade until 2054.

As another policy-relevant application of our model, we examine international spillover effects from foreign unconventional monetary policies. Here, we follow Christensen and Rudebusch (2012) and analyze the responses of Canadian government bond yields to announcements by the U.S. Federal Reserve and the Bank of England of plans to buy longer-term government debt, commonly referred to as QE, in response to the GFC. Using our preferred shadow-rate B-CLR-L model estimated at daily frequency to account for the near-zero short-term nominal yields at the time, we decompose the nominal yield responses into changes in expectations about future monetary policy and changes in term premiums. We also decompose the responses of the difference between nominal and real yields of comparable maturity, known as breakeven inflation (BEI), into changes in expected inflation and changes in the inflation risk premium. We find that U.S. QE announcements mainly lowered Canadian bond yields through their impact on risk premiums with much smaller effects on Canadian policy and inflation expectations. These findings appear consistent with strong portfolio rebalancing spillover effects, while signaling effects play at best a secondary role. Although U.K. QE announcements produce much smaller reactions in the Canadian bond markets, it remains the case that the spillover effects seem to operate through the risk premiums. Thus, the portfolio balance channel seems to be relatively more important than the signalling channel when it comes to international spillover effects, at least for Canada during the specific period we consider. We stress that the GFC period was characterized by major market dislocations and associated elevated liquidity risk premiums. As a consequence, spillover effects may be different in more normal times. However, we leave it for future research to examine that conjecture further.

The analysis in this paper relates to several important literatures. Most directly, it speaks to the burgeoning literature on measurement of both the natural rate of interest and long-term inflation expectations. Second, our estimates of the real yield curve that would prevail without trading frictions have implications for asset pricing analysis on the true slope of the real yield curve. Third, our results relate to research on financial market liquidity. Indeed, the RRB liquidity premiums we estimate may serve as a benchmark for assessing liquidity premiums in other fixed-income markets in Canada and elsewhere. Furthermore, the paper is among the first to document what happens to trading and market dynamics when a government decides to terminate issuance of inflation-indexed debt. Finally, the paper contributes to the rapidly growing literature on the economic consequences of the COVID-19 pandemic.

The remainder of the paper is structured as follows. Section 2 contains the data description, while Section 3 provides a description of the no-arbitrage term structure models we use and presents the empirical results, including an analysis of the RRB liquidity premium and its empirical determinants. Section 4 examines the Canadian bond market response to the surprise cancellation of all future RRB bond issuance. Section 5 analyzes our market-based estimates of long-term inflation expectations and the natural rate, while Section 6 is dedicated to an analysis of spillover effects on Canadian interest rates from U.S. and U.K. unconventional monetary policies. Finally, Section 7 concludes the paper and offers directions for future research. An online appendix contains additional technical model details, estimation results, and robustness checks.

2 Canadian Government Bond Data

This section describes the Canadian government bond data used in our model estimation. As for the size of the Canadian government bond market, at the end of June 2024, the total outstanding notional amount of marketable bonds issued by the government of Canada was CAD1,076.5 billion of which CAD70.9 billion, or 6.6 percent, represented RRBs.[8] The Canadian government bond market is equivalent to about 37 percent of Canadian nominal GDP, and the Canadian government holds a AAA rating with a stable outlook by all major rating agencies.

2.1 Nominal Bonds

The Bank of Canada produces daily zero-coupon yield curves from a subset of the available universe of Canadian government fixed-coupon bonds using an "exponential spline model."[9] The database starts in January 1986 and is updated every Thursday with a two-week reporting lag. Note that, while the database contains the time series of maturities in quarterly increments from three months to thirty years, we limit our focus to a representative sample with the following eleven fixed maturities: 0.25, 0.5, 1, 2, 3, 5, 7, 10, 15, 20, and 30 years. Furthermore, we limit the sample to monthly data covering the period from January 31, 1991,

[8] This information is available at http://www.bankofcanada.ca/markets/government-securities-auctions/goc-t-bills-and-bonds-outstanding/

[9] See Bolder et al. (2004) for a description of these data, which can be accessed at http://www.bankofcanada.ca/rates/interest-rates/bond-yield-curves/. See Diez de los Rios (2015) for another empirical application using these data.

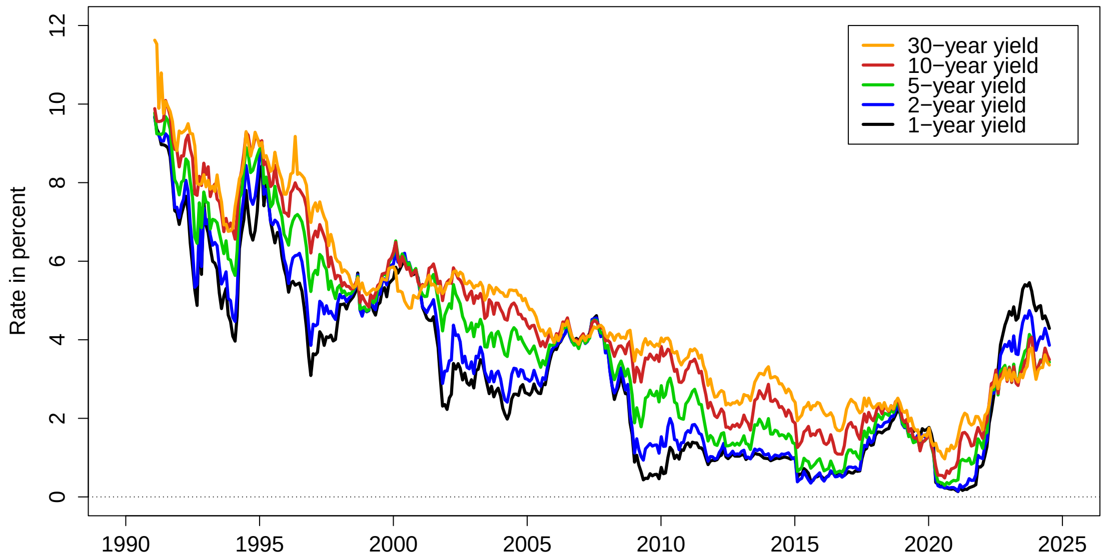


Figure 2: **Canadian Nominal Government Bond Yields**
Illustration of the Canadian government zero-coupon bond yields constructed by staff at the Bank of Canada with maturities of three months, two years, five years, ten years, and thirty years. The data series are monthly covering the period from January 31, 1991, to June 28, 2024.

to June 28, 2024, where the start date matches the launch of the RRB market in 1991 and also marks the initial introduction of the Bank of Canada's current inflation targeting regime.[10]

Figure 2 shows time series of the zero-coupon yields with maturities of three months, two years, five years, ten years, and thirty years. First, we note the downward trend of the general yield level between 1991 and 2021. The ten-year yield dropped from above 10 percent to below 1 percent during this thirty-year period. Also notable is the sharp, partial reversal since then that has brought interest rates in Canada back up close to 4 percent. Second, as in U.S. Treasury yield data, there is clear business cycle variation in the shape of the yield curve around these persistent trends. It is these characteristics that are the practical motivation behind our choice of using a three-factor model for the Canadian nominal government bond yield curve, adopting an approach similar to what is standard for U.S. and U.K. nominal yield data; see Christensen and Rudebusch (2012). This choice is also supported by a principal component analysis of our nominal bond yield sample, as described in online Appendix B.

Finally, regarding the important question of a lower bound, the Bank of Canada only tem-

[10]Although we rely on constructed synthetic nominal zero-coupon bond yields instead of bond prices, Andreasen et al. (2019) provide evidence that this conventional approach to term structure modeling delivers satisfactory estimates of investors' expectations and risk premiums, which is the focus of our paper.

Real return bond	No. obs.	Issuance		Total uplifted notional amount
		Date	amount	
(1) 4.25% 12/1/2021	299	12/10/1991	n.a.	8,508
(2) 4.25% 12/1/2026	343	12/7/1995	n.a.	9,599
(3) 4% 12/1/2031	295	3/8/1999	400	10,192
(4) 3% 12/1/2036	246	6/9/2003	400	9,121
(5) 2% 12/1/2041	200	6/4/2007	650	9,457
(6) 1.5% 12/1/2044	165	5/31/2010	700	10,696
(7) 1.25% 12/1/2047	126	12/2/2013	700	10,028
(8) 0.5% 12/1/2050	84	6/5/2017	700	9,390
(9) 0.25% 12/1/2054	37	6/4/2021	400	2,414

Table 1: **Sample of Canadian Government Real Return Bonds**
The table reports the characteristics, first issuance date and amount, and total uplifted notional amount outstanding either at maturity or as of June 30, 2024, in millions of Canadian dollars for the sample of Canadian government real return bonds. Also reported are the number of monthly observation dates for each bond during the sample period from December 29, 1995, to June 28, 2024.

porarily lowered its conventional policy rate to zero in response to the COVID-19 pandemic. As a consequence, the bond yields in the data have remained well above zero outside of this brief period. Thus, our benchmark model is an unconstrained Gaussian model. However, as a robustness check, we consider a shadow-rate specification of our benchmark model that respects the ZLB for the nominal yields. This allows us to assess the impact on our results from this constraint.

2.2 Real Bonds

Since 1991, the Canadian government has issued RRBs, which have semi-annual interest payments that are adjusted for inflation using the changes in the all-items Canadian CPI without seasonal adjustment.[11] These bonds have all been thirty-year bonds and are issued only once every 3-5 years. As a consequence, there is a limited universe of RRBs with a total of nine bonds having been issued by the end of our sample. Despite the limited number of RRBs in the early years and their correspondingly narrow maturity range, we start our sample of RRB prices in December 1995, when the second RRB was issued and started trading. Coincidentally, December 1995 also happens to be the time when the Bank of Canada's inflation target was lowered to its current 2 percent level; see Kozicki (2024). Crucially, this choice provides us with the longest possible perspective on persistent changes in Canadian real yields. Table 1 contains the contractual details of all nine RRBs as well as the number of monthly observations of each in our sample, while the time-varying maturity

[11] Coté et al. (1996) and Reid et al. (2004) are early studies comparing RRB and conventional Canadian nominal bond yields.

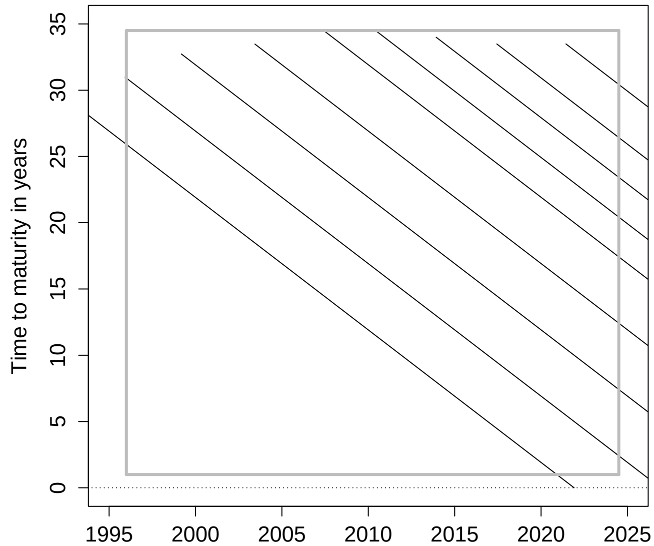


Figure 3: **Maturity Distribution of Canadian Government Real Return Bonds**
Illustration of the maturity distribution of the available universe of Canadian government real return bonds. The solid grey rectangle indicates the sample used in the empirical analysis, where the sample is restricted to start on December 29, 1995, and end on June 28, 2024, and limited to bond prices with more than one year to maturity.

distribution of the nine RRBs in our sample is illustrated in Figure 3, where each security is represented by a downward-sloping line showing its remaining years to maturity at each date.

Figure 4 shows the yields to maturity for all nine RRBs. The significant persistent fluctuations in real yields over this 28-year period are clearly visible. Canadian long-term real yields were close to 4 percent in the late 1990s and had dropped below zero by the time of the pandemic in spring 2020. Since then Canadian real yields have bounced back more than 2 percentage points. One empirical question is to what extent this persistent decline between 1996 and 2020 and the subsequent partial reversal represent changes in the natural real rate or are driven by other factors such as liquidity or other risk premiums.

2.2.1 The Liquidity Risk of RRBs

Before turning to our models and their estimation, we examine RRB bid-ask spreads to provide support for the ACR approach to identify liquidity premiums. The spreads are constructed by converting the bid and ask prices into the corresponding yield to maturity and calculating the difference with all data downloaded from Bloomberg. Figure 5 shows two series of bid-ask spreads for Canadian RRBs, one represents the bid-ask spread of the

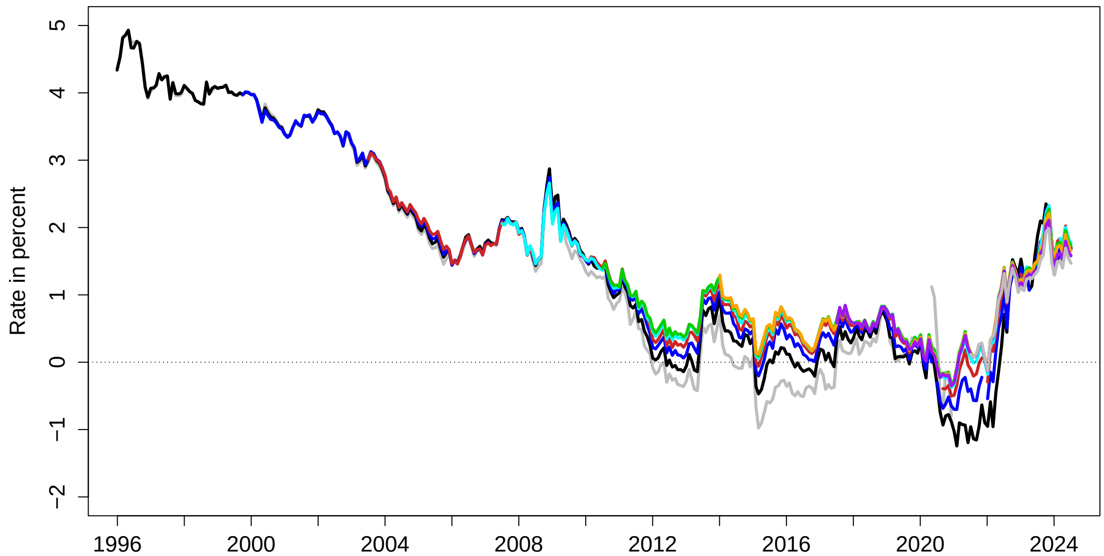


Figure 4: **Yield to Maturity of Canadian Government Real Return Bonds**
Illustration of the yield to maturity of the Canadian real return bonds considered in this paper covering the period from December 29, 1995, to June 28, 2024. Each bond yield series is shown with its own colored line.

first ever RRB issued in 1991, the other tracks the bid-ask spread of the most recently issued (on-the-run) RRB—a sequence of different underlying RRBs. Both series are smoothed four-week averages and measured in basis points.[12] Similar to what ACR document for U.S. TIPS, the RRB bid-ask spreads are wider for more seasoned securities than for recently issued securities. Rational, forward-looking investors are aware of these dynamics and the fact that future market liquidity of a given security is likely to be below its current market liquidity. This gives rise to liquidity premiums in the security price that serve as compensation for assuming the risk that it may be difficult to sell the security back to the market in the future at a satisfactory price and without incurring higher transaction costs. This pattern in observed measures of *current* market liquidity of RRBs is consistent with the factor loading of the liquidity risk factor in our approach that is intended to model the effects on current RRB prices of expected *future* market liquidity conditions. Although we mainly think of these premiums as liquidity discounts, we note that, given their high credit quality, these bonds may be viewed by investors as very safe assets and hence trade at a safety premium; see Christensen and Mirkov (2022). We stress that the model we use is flexible enough to

[12]The start date in the comparison is determined by data availability, while the end date is December 1, 2020, when the first RRB has one year remaining to maturity and is dropped from our sample.

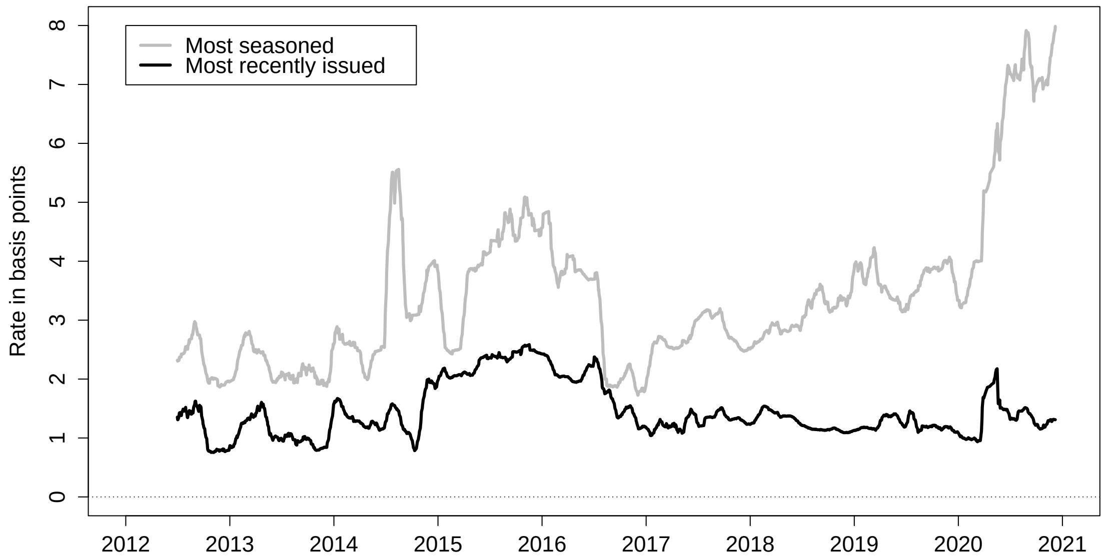


Figure 5: **Bid-Ask Spreads of Canadian Real Return Bonds**
Illustration of the four-week moving average of bid-ask spreads of Canadian RRBs constructed as explained in the main text. The series are daily covering the period from May 31, 2012, to December 1, 2020.

accommodate such outcomes.

3 Model Estimation and Results

In this section, we first detail the CLR-L model that serves as the benchmark in our analysis before we describe the restrictions imposed to achieve econometric identification of the model. We then compare its estimates to those from the CLR model without a liquidity adjustment and the shadow-rate B-CLR-L model that respects the ZLB for nominal yields. Finally, we describe the RRB liquidity premium implied by the estimated CLR-L model and examine its empirical determinants.

3.1 The CLR-L Model

To begin, let $X_t = (L_t^N, S_t, C_t, L_t^R, X_t^{liq})$ denote the state vector of the five-factor CLR-L model. Here, L_t^N and L_t^R denote the level factor unique to the nominal and real yield curve, respectively, while S_t and C_t represent slope and curvature factors common to both yield curves. Finally, X_t^{liq} represents the added liquidity risk factor unique to the RRB prices.

The instantaneous nominal and real risk-free rates are defined as

$$r_t^N = L_t^N + S_t, \tag{1}$$

$$r_t^R = L_t^R + \alpha^R S_t. \tag{2}$$

Note that the differential scaling of the real rates to the common slope factor is captured by the parameter α^R.[13]

The risk-neutral $\mathbb{Q}$-dynamics of the state variables used for pricing are given by

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \\ dX_t^{liq} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa_{liq}^{\mathbb{Q}} \end{pmatrix} \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \theta_{liq}^{\mathbb{Q}} \end{pmatrix} - \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \\ X_t^{liq} \end{pmatrix} \right] dt + \Sigma \begin{pmatrix} dW_t^{L^N,\mathbb{Q}} \\ dW_t^{S,\mathbb{Q}} \\ dW_t^{C,\mathbb{Q}} \\ dW_t^{L^R,\mathbb{Q}} \\ dW_t^{liq,\mathbb{Q}} \end{pmatrix},$$

where Σ is assumed to be a diagonal matrix as per Christensen et al. (2011).

Based on the $\mathbb{Q}$-dynamics above, nominal zero-coupon bond yields preserve a Nelson and Siegel (1987) factor loading structure

$$y_t^N(\tau) = L_t^N + \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau}\right) S_t + \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) C_t - \frac{A^N(\tau)}{\tau}, \tag{3}$$

where $A^N(\tau)$ is a convexity term that adjusts the functional form in Nelson and Siegel (1987) to ensure absence of arbitrage (see Christensen et al., 2011). The assumed Nelson and Siegel (1987) factor structure for the nominal yields is supported by a principal component analysis of our nominal yield data; see online Appendix B.

On the other hand, due to the lower liquidity in the market for real bonds, real yields are sensitive to liquidity pressures. As a consequence, the pricing of RRBs is not performed with the standard real discount function, but rather with a discount function that accounts for the liquidity risk:

$$\overline{r}^{R,i}(t, t_0^i) = r_t^R + \beta^i(1 - e^{-\lambda^{L,i}(t - t_0^i)})X_t^{liq} = L_t^R + \alpha^R S_t + \beta^i(1 - e^{-\lambda^{L,i}(t - t_0^i)})X_t^{liq},$$

where t_0^i denotes the date of issuance of the specific real bond and β^i is its sensitivity to the variation in the liquidity factor. Furthermore, the decay parameter $\lambda^{L,i}$ is assumed to vary across securities as well.

ACR show that the net present value of one consumption unit paid by real bond i at time

[13] In online Appendix B, we examine the behavior of the slope of the nominal and real yield curves and find them to be highly positively correlated similar to what ACR report for U.S. Treasury and TIPS yields. This high correlation is built explicitly into the CLR model structure.

$t+\tau$ has the following exponential-affine form

$$\begin{aligned} P_t(t_0^i,\tau) &= E_t^{\mathbb{Q}}\left[e^{-\int_t^{t+\tau^i}\overline{r}^{R,i}(s,t_0^i)ds}\right] \\ &= \exp\left(B_1(\tau)L_t^N + B_2(\tau)S_t + B_3(\tau)C_t + B_4(\tau)L_t^R + B_5(t,t_0^i,\tau)X_t^{liq} + A(t,t_0^i,\tau)\right), \end{aligned}$$

which implies that the model belongs to the class of Gaussian affine term structure models. Note also that, by fixing $\beta^i = 0$ for all i, we recover the CLR model.

Now, consider the whole value of the real bond i issued at time t_0^i with maturity at $t+\tau^i$ that pays an annual coupon C^R semi-annually. Its price is given by[14]

$$\begin{aligned} \overline{P}_t^{R,i}(t_0^i,\tau^i,C^R) &= C^R(t_1-t)E_t^{\mathbb{Q}}\left[e^{-\int_t^{t_1}\overline{r}^{R,i}(s,t_0^i)ds}\right] + \sum_{j=2}^{N}\frac{C^R}{2}E_t^{\mathbb{Q}}\left[e^{-\int_t^{t_j}\overline{r}^{R,i}(s,t_0^i)ds}\right] \\ &\quad + E_t^{\mathbb{Q}}\left[e^{-\int_t^{t+\tau^i}\overline{r}^{R,i}(s,t_0^i)ds}\right]. \end{aligned}$$

Unlike U.S. TIPS, Canadian RRBs have no embedded deflation protection option, which makes their pricing straightforward. The only minor omission in the bond price formula above is that we do not account for the lag in the inflation indexation of the real bond payoff. Grishchenko and Huang (2013) and D'Amico et al. (2018) find that this adjustment normally is within a few basis points for the implied yield on U.S. TIPS. It is likely to be very small for our Canadian data as well as our sample is dominated by very long-term bonds where this adjustment tends to be tiny.

Finally, within the CLR-L model described above, nominal bonds are assumed to have a minimum of liquidity risk, which is consistent with the empirical findings of Andreasen et al. (2019). Therefore, the nominal bonds are valued using the standard nominal zero-coupon yield described in equation (3).

So far, the description of the CLR-L model has relied solely on the dynamics of the state variables under the $\mathbb{Q}$-measure used for pricing. However, to complete the description of the model and to implement it empirically, we will need to specify the risk premiums that connect the factor dynamics under the $\mathbb{Q}$-measure to the dynamics under the real-world (or historical) $\mathbb{P}$-measure. It is important to note that there are no restrictions on the dynamic drift components under the empirical $\mathbb{P}$-measure beyond the requirement of constant volatility. To facilitate empirical implementation, we use the essentially affine risk premium specification introduced in Duffee (2002). In the Gaussian framework, this specification implies that the risk premiums Γ_t depend on the state variables; that is,

$$\Gamma_t = \gamma^0 + \gamma^1 X_t,$$

[14] This is the clean price that does not account for any accrued interest and maps to our observed RRB prices.

where $\gamma^0 \in \mathbf{R}^5$ and $\gamma^1 \in \mathbf{R}^{5\times 5}$ contain unrestricted parameters; see online Appendix A.

Thus, the resulting unrestricted five-factor CLR-L model has $\mathbb{P}$-dynamics given by

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \\ dX_t^{liq} \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{\mathbb{P}} & \kappa_{12}^{\mathbb{P}} & \kappa_{13}^{\mathbb{P}} & \kappa_{14}^{\mathbb{P}} & \kappa_{15}^{\mathbb{P}} \\ \kappa_{21}^{\mathbb{P}} & \kappa_{22}^{\mathbb{P}} & \kappa_{23}^{\mathbb{P}} & \kappa_{24}^{\mathbb{P}} & \kappa_{25}^{\mathbb{P}} \\ \kappa_{31}^{\mathbb{P}} & \kappa_{32}^{\mathbb{P}} & \kappa_{33}^{\mathbb{P}} & \kappa_{34}^{\mathbb{P}} & \kappa_{35}^{\mathbb{P}} \\ \kappa_{41}^{\mathbb{P}} & \kappa_{42}^{\mathbb{P}} & \kappa_{43}^{\mathbb{P}} & \kappa_{44}^{\mathbb{P}} & \kappa_{45}^{\mathbb{P}} \\ \kappa_{51}^{\mathbb{P}} & \kappa_{52}^{\mathbb{P}} & \kappa_{53}^{\mathbb{P}} & \kappa_{54}^{\mathbb{P}} & \kappa_{55}^{\mathbb{P}} \end{pmatrix} \left(\begin{pmatrix} \theta_1^{\mathbb{P}} \\ \theta_2^{\mathbb{P}} \\ \theta_3^{\mathbb{P}} \\ \theta_4^{\mathbb{P}} \\ \theta_5^{\mathbb{P}} \end{pmatrix} - \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \\ X_t^{liq} \end{pmatrix} \right) dt + \Sigma \begin{pmatrix} dW_t^{L^N,\mathbb{P}} \\ dW_t^{S,\mathbb{P}} \\ dW_t^{C,\mathbb{P}} \\ dW_t^{L^R,\mathbb{P}} \\ dW_t^{liq,\mathbb{P}} \end{pmatrix}.$$

This is the transition equation in the extended Kalman filter estimation of the CLR-L model.

3.1.1 The Shadow-Rate B-CLR-L Model

The CLR-L model relies on the standard affine specification for the nominal short rate, which does not enforce the ZLB. But, the policy rate in Canada was constrained by the ZLB in the first years following the COVID-19 pandemic, and possibly even occasionally in the earlier period from January 2009 to December 2015 when the U.S. monetary policy rate was constrained by the ZLB and Canadian nominal yields were low for that reason. To enforce this lower bound within our benchmark CLR-L model, we follow Black (1995) and replace r_t^N in equation (1) by $r_t^N = \max\{L_t^N + S_t, 0\}$ and solve for nominal yields using the approximation in Christensen and Rudebusch (2015), but the model is otherwise identical to the CLR-L model described above and denoted the B-CLR-L model.

3.2 Model Estimation and Econometric Identification

While nominal yields in the CLR-L model have a standard affine formulation given by equation (3), RRB prices are nonlinear functions of the state variables.[15] Due to this nonlinearity of the RRB pricing formulas, the model cannot be estimated with the standard Kalman filter. Instead, we use the extended Kalman filter as in Kim and Singleton (2012); see ACR for details. To make the fitted errors comparable across RRBs of various maturities, we follow ACR and scale each bond price by its duration. Thus, the measurement equation for the real bond prices takes the following form:

$$\frac{\overline{P}_t^R(t_0^i, \tau^i)}{D_t^R(\tau^i)} = \frac{\widehat{P}_t^R(t_0^i, \tau^i)}{D_t^R(\tau^i)} + \varepsilon_t^{R,i},$$

where $\widehat{P}_t^R(t_0^i, \tau^i)$ is the model-implied price of real bond i and $D_t^R(\tau^i)$ is its duration, which is fixed and calculated before estimation. See Andreasen et al. (2019) for evidence supporting this formulation of the measurement equation.

[15]Note that, in the B-CLR-L model, even the nominal yields are nonlinear functions of the state variables.

Since the liquidity risk factor is a latent factor that we do not observe, its level is not identified without additional restrictions. As a consequence, we let the first Canadian government real return bond issued, that is, the thirty-year real return bond with 4.25 percent coupon issued on December 10, 1991, with maturity on December 1, 2021, have a unit loading on the liquidity factor, that is, $\beta^i = 1$ for this security. This choice implies that the β^i sensitivity parameters measure liquidity sensitivity relative to that of the thirty-year 2021 RRB.

Furthermore, we note that the $\lambda^{L,i}$ parameters can be hard to identify if their values are too large or too small. As a consequence, we impose the restriction that they fall within the range from 0.0001 to 10, which is without practical consequences as demonstrated by ACR. Also, for numerical stability during the model optimization, we impose the restriction that the β^i parameters fall within the range from 0 to 250, which turns out not to be a binding constraint at the optimum.

Finally, we assume that all nominal yield measurement equations have *i.i.d.* fitted errors with zero mean and standard deviation σ_ε^N. Similarly, all RRB price measurement equations have fitted errors that are assumed to be *i.i.d.* with zero mean and standard deviation σ_ε^R.

3.3 Estimation Results

This section presents our benchmark estimation results. In the interest of simplicity, focus is in this section devoted to a version of the CLR-L model where $K^{\mathbb{P}}$ and Σ are diagonal matrices. As shown in ACR, these restrictions have hardly any effects on the estimated liquidity premium for each RRB, because it is identified from the model's $\mathbb{Q}$-dynamics, which is independent of $K^{\mathbb{P}}$ and only display a weak link to Σ through the small convexity adjustment in yields. Furthermore, we stress that we relax this assumption in Section 5 when we analyze estimates of long-term inflation expectations and r_t^*, which are indeed sensitive to the specification of the models' $\mathbb{P}$-dynamics.

Given that the CLR-L model includes standard nominal Canadian government zero-coupon bond yields, it seems natural to first explore how well it fits nominal yields. Table 2 documents that it provides a very satisfying fit to all nominal yields, where the overall root mean-squared error (RMSE) is just 12.95 basis points. The corresponding CLR model without a liquidity factor gives broadly a similar fit with an overall RMSE of 13.50 basis points, while the B-CLR-L model that respects the ZLB of nominal yields provides a slightly better fit with an overall RMSE of 12.66 basis points.[16] Thus, accounting for the liquidity risk of RRBs does not affect the ability of the CLR-L model to match nominal yields, which is consistent with the results in ACR. Still, we note some signs of poorer fit in the very long end of the nominal yield curve across the models.

[16]Unreported results further show that omitting RRB prices in the estimation gives basically the same satisfying fit of nominal yields.

Maturity in months	CLR		CLR-L		B-CLR-L	
	Mean	RMSE	Mean	RMSE	Mean	RMSE
3	-5.17	16.72	-4.97	14.53	-5.21	13.95
6	-2.05	7.06	-1.39	5.36	-1.26	5.33
12	3.51	14.37	4.61	14.09	4.93	13.91
24	4.11	13.85	4.89	12.32	5.04	11.73
36	3.59	11.28	3.44	9.71	3.28	9.07
60	-0.20	7.41	-1.95	7.09	-2.61	7.22
84	-3.21	8.99	-5.41	9.55	-6.30	10.11
120	-4.81	9.25	-6.16	9.81	-6.95	10.33
180	0.84	11.33	1.74	10.79	1.55	10.32
240	6.30	20.07	8.27	19.38	8.54	19.11
360	-4.01	19.90	-4.45	20.74	-4.18	19.80
All maturities	-0.10	13.50	-0.13	12.95	-0.29	12.66

Table 2: **Pricing Errors of Nominal Yields**
This table reports the mean pricing errors (Mean) and the root mean-squared pricing errors (RMSE) of nominal yields in the CLR, CLR-L, and B-CLR-L models estimated with a diagonal specification of $K^{\mathbb{P}}$ and Σ. All errors are reported in basis points.

RRB security	Pricing errors						Estimated parameters							
	CLR		CLR-L		B-CLR-L		CLR-L				B-CLR-L			
	Mean	RMSE	Mean	RMSE	Mean	RMSE	β^i	SE	$\lambda^{L,i}$	SE	β^i	SE	$\lambda^{L,i}$	SE
(1) 4.25% 12/1/2021	4.25	12.23	0.03	2.17	-0.02	2.20	1	n.a.	0.38	0.07	1	n.a.	9.16	70.53
(2) 4.25% 12/1/2026	-2.11	8.09	0.56	2.11	0.46	2.14	1.08	0.01	0.40	0.07	1.14	0.01	0.43	0.09
(3) 4% 12/1/2031	-1.67	4.54	-0.28	2.63	-0.10	2.67	0.84	0.02	10.00	0.08	0.99	0.02	10.00	0.11
(4) 3% 12/1/2036	-0.57	5.73	0.29	2.26	0.29	2.32	0.47	0.02	10.00	0.08	0.65	0.03	10.00	0.10
(5) 2% 12/1/2041	0.00	6.32	0.48	2.20	0.39	2.26	0.19	0.03	10.00	0.08	0.36	0.04	10.00	0.12
(6) 1.5% 12/1/2044	2.37	5.23	0.20	2.26	0.21	1.95	0.04	0.06	0.04	0.08	0.84	0.12	0.01	0.01
(7) 1.25% 12/1/2047	1.32	4.68	0.22	2.38	0.30	2.41	0.06	0.05	10.00	0.09	3.07	0.13	0.00	0.00
(8) 0.5% 12/1/2050	0.34	5.97	0.39	3.59	0.32	3.85	0.10	0.05	10.00	0.11	6.27	0.14	0.00	0.00
(9) 0.25% 12/1/2054	-3.16	10.84	-1.05	6.72	-1.18	6.91	0.12	0.06	10.00	0.16	0.11	0.08	5.98	0.19
All RRB yields	0.21	7.61	0.19	2.53	0.18	2.57	-	-	-	-	-	-	-	-
Max $\mathcal{L}^{EKF}$	30,652.57		31,953.26		32,049.10		-		-		-		-	

Table 3: **Pricing Errors of RRBs and Estimated Parameters for Liquidity Risk**
This table reports the mean pricing errors (Mean) and the root mean-squared pricing errors (RMSE) of RRBs in the CLR, CLR-L, and B-CLR-L models estimated with a diagonal specification of $K^{\mathbb{P}}$ and Σ. The errors are computed as the difference between the RRB market price expressed as yield to maturity and the corresponding model-implied yield. All errors are reported in basis points. Standard errors (SE) are not available (n.a.) for the normalized value of β^1.

The impact of accounting for liquidity risk is, however, more apparent in the RRB market. The first two columns in Table 3 show that the RRB pricing errors produced by the CLR model indicate a reasonable fit, with an overall RMSE of 7.61 basis points. The following two columns reveal a substantial improvement in the pricing errors when correcting for liquidity risk, as the CLR-L model has a very low overall RMSE of just 2.53 basis points, while the next two columns show that the B-CLR-L model provides an equally accurate fit with an

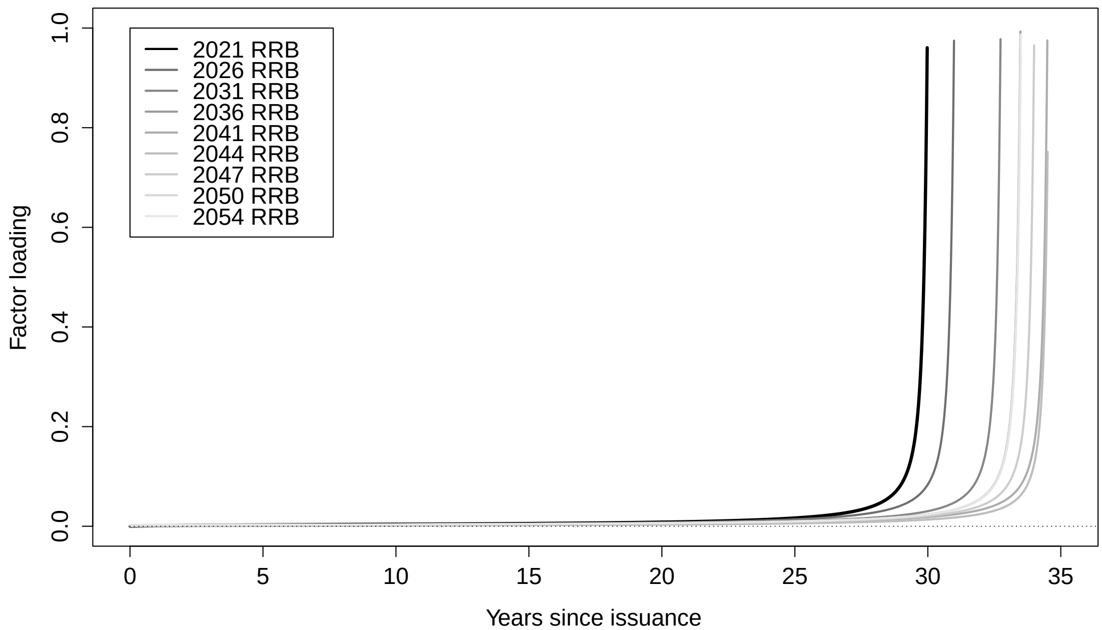
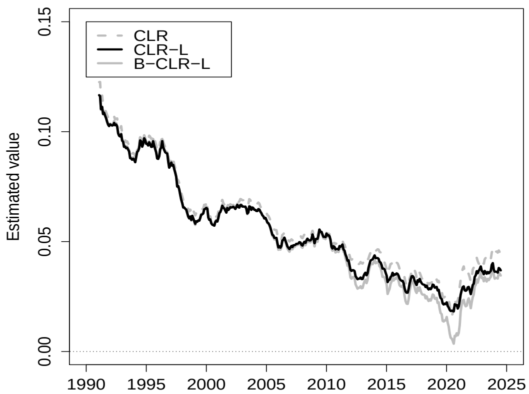


Figure 6: **The Term Structure of Liquidity Risk**
This figure shows the term structure of liquidity risk, where β^i is omitted to facilitate the comparison. That is, we report $\frac{\left(1-\exp\left\{-\kappa^{\mathbb{Q}}_{liq}(T-t)\right\}\right)}{\kappa^{\mathbb{Q}}_{liq}(T-t)} - \exp\left\{-\lambda^{L,i}\,(t-t_0)\right\}\frac{1-\exp\left\{-\left(\kappa^{\mathbb{Q}}_{liq}+\lambda^{L,i}\right)(T-t)\right\}}{\left(\kappa^{\mathbb{Q}}_{liq}+\lambda^{L,i}\right)(T-t)}$ for the yield related to the ith RRB as implied by the estimated version of the CLR-L model with a diagonal specification of $K^{\mathbb{P}}$ and Σ.

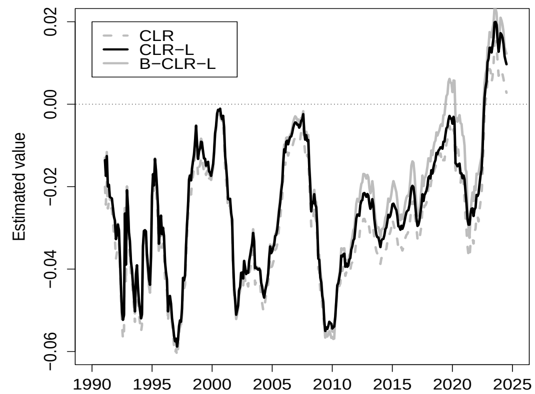
overall RMSE of 2.57 basis points. Hence, accounting for liquidity risk leads to a notable improvement in the ability of our benchmark model to explain RRB market prices.

The remaining columns of Table 3 report the estimates of the specific parameters attached to each RRB within the CLR-L and B-CLR-L models. We note that the 4 most recently issued RRBs in our sample are not exposed to liquidity risk, as their β^is are insignificantly different from zero at the conventional 5 percent level. In contrast, in the B-CLR-L model, it is only the most recently issued RRB that is insensitive to liquidity risk. Although the estimated values of $\lambda^{L,i}$ vary notably across securities, this does not imply major differences in the sensitivity of the RRBs to the liquidity risk factor as shown in Figure 6. The liquidity adjustment for all RRBs is increasing in t due to the strong mean-reversion in X_t^{liq} under the $\mathbb{Q}$-measure ($\kappa^{\mathbb{Q}}_{liq} = 12.05$ according to Table 4). Thus, liquidity risk operates as a traditional slope factor within the CLR-L model, although its steepness varies across the universe of RRBs.

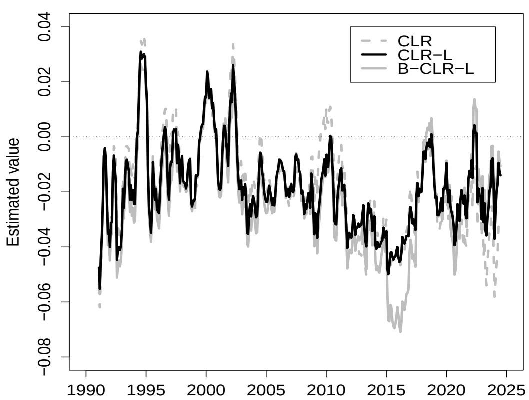
The remaining estimated model parameters are provided in Table 4, which shows that the dynamics of the four frictionless factors, (L_t^N, S_t, C_t, L_t^R), are very similar across the CLR, CLR-L, and B-ClR-L models, both under the $\mathbb{P}$- and the $\mathbb{Q}$-measure. We draw the



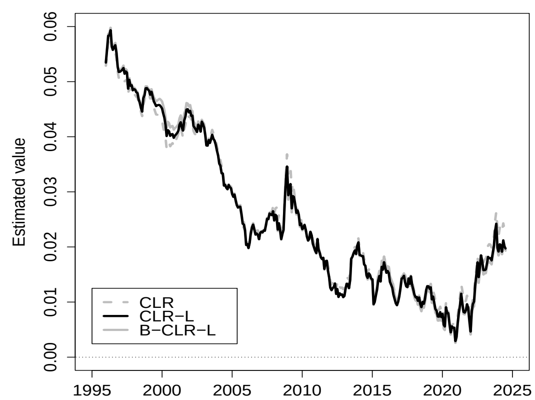
(a) L_t^N



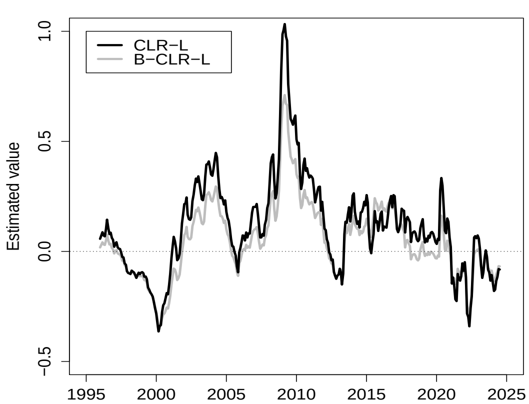
(b) S_t



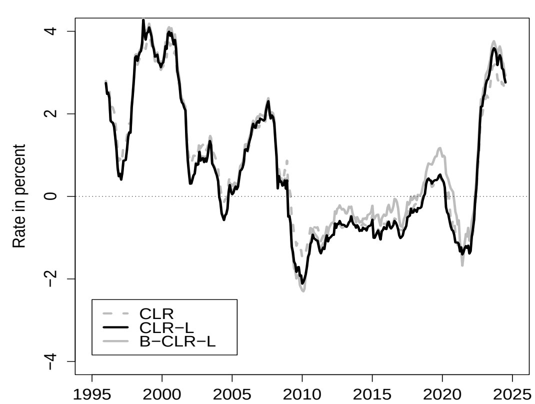
(c) C_t



(d) L_t^R



(e) X_t^{liq}



(f) The instantaneous real rate

Figure 7: **Estimated State Variables**

Illustration of the estimated state variables and instantaneous real rate from the CLR, CLR-L, and B-CLR-L models estimated with a diagonal specification of $K^{\mathbb{P}}$ and Σ.

Parameter	CLR		CLR-L		B-CLR-L	
	Est.	SE	Est.	SE	Est.	SE
$\kappa_{11}^{\mathbb{P}}$	0.0127	0.0387	0.0089	0.0336	0.0048	0.0171
$\kappa_{22}^{\mathbb{P}}$	0.2390	0.1455	0.2110	0.0737	0.2238	0.0962
$\kappa_{33}^{\mathbb{P}}$	0.7119	0.2054	0.9954	0.0715	0.6084	0.0972
$\kappa_{44}^{\mathbb{P}}$	0.0130	0.0435	0.0107	0.0296	0.0107	0.0320
$\kappa_{55}^{\mathbb{P}}$	-	-	0.4601	0.0795	0.3658	0.0935
σ_{11}	0.0058	0.0001	0.0051	0.0001	0.0059	0.0001
σ_{22}	0.0116	0.0003	0.0117	0.0003	0.0140	0.0004
σ_{33}	0.0225	0.0008	0.0226	0.0008	0.0216	0.0007
σ_{44}	0.0042	0.0001	0.0040	0.0001	0.0040	0.0001
σ_{55}	-	-	0.2224	0.0123	0.1482	0.0096
$\theta_1^{\mathbb{P}}$	0.0780	0.0383	0.0733	0.0361	0.1042	0.0998
$\theta_2^{\mathbb{P}}$	-0.0244	0.0110	-0.0197	0.0116	-0.0287	0.0132
$\theta_3^{\mathbb{P}}$	-0.0194	0.0055	-0.0185	0.0041	-0.0215	0.0057
$\theta_4^{\mathbb{P}}$	0.0528	0.0781	0.0534	0.0456	0.0534	0.0494
$\theta_5^{\mathbb{P}}$	-	-	0.0982	0.0648	0.0395	0.0788
λ	0.3106	0.0035	0.4030	0.0049	0.4431	0.0049
α^R	0.6957	0.0079	0.8150	0.0174	0.8115	0.0210
$\kappa_{liq}^{\mathbb{Q}}$	-	-	12.0480	0.0788	7.9287	0.1177
$\theta_{liq}^{\mathbb{Q}}$	-	-	-0.0013	0.0001	-0.0015	0.0001
σ_y	0.0015	6.74×10^{-6}	0.0014	5.60×10^{-6}	0.0014	5.60×10^{-6}
σ_{TIPS}	0.0010	6.47×10^{-6}	0.0004	4.95×10^{-6}	0.0004	4.95×10^{-6}

Table 4: **Estimated Dynamic Parameters**
The table shows the estimated dynamic parameters for the CLR, CLR-L, and B-CLR-L models estimated with a diagonal specification of $K^{\mathbb{P}}$ and Σ.

same conclusion from Figure 7, which plots the estimated factors in all three models. The differences tend to be short-lived and barely noticeable. For that reason it is also the case that the frictionless instantaneous real rates $r_t^R = L_t^R + \alpha^R S_t$ are almost indistinguishable across the three models as documented in panel (f) of Figure 7. Finally, panel (e) shows the estimated liquidity factors, X_t^{liq}, which are unique to the CLR-L and B-CLR-L models and also very similar.

3.4 The Estimated RRB Liquidity Premium

We now use the estimated CLR-L model to extract the liquidity premium in the RRB market. To compute this premium, we first use the estimated parameters and the filtered states $\left\{X_{t|t}\right\}_{t=1}^T$ to calculate the fitted RRB prices $\left\{\hat{P}_t^{RRB,i}\right\}_{t=1}^T$ for all outstanding securities in our sample. These bond prices are then converted into yields to maturity $\left\{\hat{y}_t^{c,i}\right\}_{t=1}^T$ by solving the fixed-point problem

$$\hat{P}_t^{RRB,i} = C(t_1 - t)\exp\left\{-(t_1 - t)\hat{y}_t^{c,i}\right\} + \sum_{k=2}^{n}\frac{C}{2}\exp\left\{-(t_k - t)\hat{y}_t^{c,i}\right\} + \exp\left\{-(T - t)\hat{y}_t^{c,i}\right\}, \tag{4}$$

for $i = 1, 2, ..., n_{RRB}$, meaning that $\left\{\hat{y}_t^{c,i}\right\}_{t=1}^{T}$ is approximately the real rate of return on the ith RRB if held until maturity (see Sack and Elsasser 2004). To obtain the corresponding yields with correction for liquidity risk, a new set of model-implied bond prices are computed from the estimated CLR-L and B-CLR-L models but using only their frictionless part, i.e., using the constraints that $X_{t|t}^{liq} = 0$ for all t as well as $\sigma_{55} = 0$ and $\theta_{liq}^{Q} = 0$. These prices are denoted $\left\{\tilde{P}_t^{RRB,i}\right\}_{t=1}^{T}$ and converted into yields to maturity $\tilde{y}_t^{c,i}$ using (4). They represent estimates of the prices that would prevail in a world without any financial frictions. The liquidity premium for the ith RRB is then defined as

$$\Psi_t^i \equiv \hat{y}_t^{c,i} - \tilde{y}_t^{c,i}. \tag{5}$$

Figure 8 shows the average RRB liquidity premium $\bar{\Psi}_t$ across the outstanding RRB at a given point in time. The average estimated RRB liquidity premium clearly varies notably over time with a maximum of 34 basis points achieved at the peak of the financial crisis and a low of -31 basis points in the spring of 2000. Note also the sharp short-lived spike at the onset of the COVID-19 pandemic in spring 2020. Importantly for our analysis, they are stationary near zero with an average for the entire period equal to -4.84 basis points and a standard deviation of 10.79 basis points.

In Figure 8, we also show the average RRB liquidity premium implied by the B-CLR-L model that accounts for the asymmetric behavior of nominal yields near the ZLB. We note that accounting for this asymmetry matters little for the estimated RRB liquidity premiums, although the estimated liquidity premiums implied by the shadow-rate B-CLR-L model tend to be slightly below those implied by the unconstrained Gaussian CLR-L model.[17]

In summary, these results allow us to conclude with great confidence that changes in RRB liquidity premiums are not a factor behind the persistent decline in Canadian long-term interest rates observed in Figure 1.[18] Still, thanks to their significant time variation, it is important to account for the liquidity premiums in our analysis as we demonstrate later on.

[17] ACR report a similar pattern across these two models for their estimated U.S. TIPS liquidity premiums.

[18] The robustness of the estimated RRB liquidity premium series is further documented in online Appendix C.

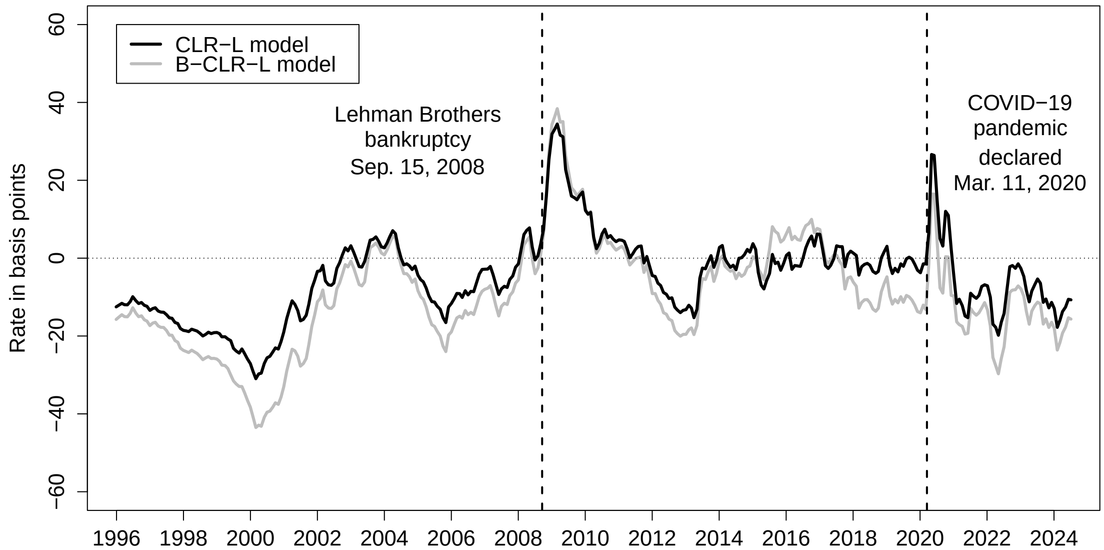


Figure 8: **Average Estimated RRB Liquidity Premium**
Illustration of the average estimated real return bond liquidity premium for each observation date implied by the CLR-L and B-CLR-L models. The real return bond liquidity premiums are measured as the estimated yield difference between the fitted yield-to-maturity of individual real return bonds and the corresponding frictionless yield-to-maturity with the liquidity risk factor turned off. The data cover the period from December 29, 1995, to June 28, 2024.

3.4.1 Determinants of the RRB Liquidity Premium

In this section, we aim to show that the RRB liquidity premium is related to observable proxies for liquidity risk as well as factors that are fundamental determinants of the prices of RRBs. For ease of exposition, we limit our focus to the estimated RRB liquidity risk premium series implied by our benchmark CLR-L model.

The first variable we consider is the VIX options-implied volatility index, which represents near-term uncertainty in the Standard & Poor's 500 stock market index. This is a widely used gauge of investor fear and risk aversion. The second variable is the yield difference between the seasoned (off-the-run) ten-year U.S. Treasury as provided by Gürkaynak et al. (2007) and the most recently issued (on-the-run) U.S. Treasury of the same maturity from the H.15 series at the Federal Reserve Board of Governors. The on-the-run security is typically the most traded security and therefore penalized the least in terms of liquidity premiums, which explains the mostly positive spread. For our analysis, the important thing to note is that, if there is a wide yield spread between liquid on-the-run and comparable seasoned U.S. Treasuries, we would expect liquidity premiums in the Canadian bond market to also

Explanatory variables	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
VIX (%)	0.308*								0.188*
	(0.185)								(1.68)
OTR premium (bps)		0.076							-0.010
		(0.179)							(-0.09)
TED spread (bps)			0.016						-0.050***
			(0.042)						(-3.29)
MOVE Index (bps)				0.056				0.167***	0.161***
				(0.048)				(6.27)	(5.50)
10yr US Treasury yield (%)					-3.497***			-3.794***	-3.310***
					(0.571)			(-6.84)	(-5.57)
Canadian CPI inflation (%)						-1.928**		-2.330***	-2.321***
						(0.773)		(-4.10)	(-4.48)
Debt-to-GDP ratio (%)							-0.588***	-0.361***	-0.442***
							(-6.37)	(-5.02)	(-6.03)
Constant	-11.12***	-5.718***	-5.475***	-9.919**	7.839***	-0.713	26.62***	18.01***	19.40***
	(3.300)	(1.654)	(1.553)	(4.041)	(2.440)	(2.273)	(5.17)	(4.69)	(4.61)
N	343	343	343	343	343	342	343	342	342
Adjusted R^2	0.047	0.003	-0.000	0.020	0.252	0.054	0.267	0.573	0.589

Table 5: **Regression Results for Average Estimated RRB Liquidity Premium**
The table reports the results of regressions with the average estimated RRB liquidity premium as the dependent variable and seven explanatory variables. Standard errors computed by the Newey-West estimator (with 3 lags) are reported in parentheses. Asterisks *, ** and *** indicate significance at the 10 percent, 5 percent and 1 percent levels, respectively.

be elevated. The third variable is the U.S. TED spread, which is calculated as the difference between the three-month U.S. LIBOR and the three-month U.S. T-bill interest rate. This spread represents a measure of the perceived general credit risk in global financial markets. To control for factors related to the uncertainty about the interest rate environment, we include the MOVE index. The next variable is the ten-year U.S. Treasury yield from the Federal Reserve's H.15 database, which is included to control for reach-for-yield effects. This may be particularly relevant for our sample during the period between December 2008 and December 2015 and again in the 2020-2021 period when U.S. short-term interest rates were constrained by the ZLB. Our final two variables represent domestic factors that are fundamental to the prices of RRBs, namely Canadian CPI inflation and the Canadian government debt-to-GDP ratio.[19]

We now run standard linear regressions to more formally assess the relative importance of each of these seven variables. First, we run regressions with each explanatory variable in isolation with the results reported in columns (1)-(7) of Table 5. The VIX, the U.S.

[19]The Canadian government debt-to-GDP ratio comes from Statistics Canada and is defined as "Federal general government gross debt to gross domestic product." Moreover, the data has been interpolated from a quarterly frequency to a monthly frequency. Source: Statistics Canada. Table 38-10-0237-01 Financial indicators of general government sector, national balance sheet accounts, https://www150.statcan.gc.ca/t1/tbl1/en/tv.action?pid=3810023701

Treasury on-the-run premium, the TED spread, and the MOVE index all have estimated positive coefficients on a stand-alone basis, but none of them are statistically significant at the conventional 5 percent level. In contrast, the ten-year U.S. Treasury yield, the Canadian CPI inflation, and the Canadian government debt-to-GDP ratio all have highly statistically significant negative coefficients in these stand-alone regressions.

A one percentage point increase in the ten-year U.S. Treasury yield is associated with a decline in the average RRB liquidity premium of slightly more than 3 basis points. During most of our sample, U.S. long-term interest rates were declining thanks to slowing growth and low inflation. In that environment, inflation-indexed bonds like RRBs do not represent particularly attractive investment opportunities, which may explain the tendency for the RRB liquidity premiums to be slightly higher under those circumstances.

A one percentage point increase in the Canadian CPI inflation tends to coincide with RRB liquidity premiums that are almost 2 basis points lower. Given that higher domestic CPI inflation directly increases the cash flows of RRBs, we interpret the decline in the RRB liquidity premium as a sign that RRBs are more convenient assets to hold under those conditions.

Lastly, a one percentage point increase in the Canadian government debt-to-GDP ratio is associated with a decline in the average RRB liquidity premium of 0.6 basis point. Since greater government debt likely requires increased issuance of RRBs, we interpret the negative coefficient as a positive supply effect whereby increased RRB issuance leads to more trading and lower financial frictions in the market for these bonds.

These encouraging preliminary results made us look for a preferred regression model the results for which are reported in column 8 of Table 5. In addition to the three significant variables discussed above, the preferred model includes the MOVE index that now has a highly statistically significant positive coefficient. This implies that increased uncertainty about the near-term interest rate environment in the U.S. tends to coincide with higher liquidity premiums in the RRB market. Moreover, this preferred regression produces an adjusted R^2 of 0.57.

For completeness, the final column of Table 5 reports the results of regressing the average RRB liquidity premium from the CLR-L model on all seven explanatory variables jointly. We note that this only produces a slightly higher adjusted R^2 of 0.59. Moreover, the four explanatory variables from our preferred regression model preserve the sign, size, and high statistical significance of their coefficients in this joint regression. We interpret this as a sign of the robustness of our preferred regression model, even though we note that the model also leaves a notable part of the variation in the RRB liquidity premium series unexplained.

4 The Termination of the RRB Program

In this section, we examine the market response to the Canadian government's decision to terminate the RRB program announced on November 3, 2022.

4.1 The Announcement

The decision to cease issuance of RRBs was made by the Canadian Department of Finance and communicated to the public through its 2022 Fall Economic Statement released on November 3, 2022. In Annex 2, page 69, a brief section on the RRB program contains the following notable paragraph:[20]

"*The government has decided to cease issuance of Real Return Bonds (RRBs) effective immediately. This decision reflects low demand for this product and will allow the government to promote liquidity by consolidating funding within our core funding sectors.*"

The announcement caught investors and market observers by surprise as there had been no prior indication of any changes to the RRB program. Moreover, conversations with staff at the Bank of Canada, who act as the debt manager for the Department of Finance, indicate that the Bank was not consulted in this matter.

In response, one asset manager, SLC Management, issued a note on November 4, 2022, stating:[21]

"*Real return bond markets froze after the announcement. We expect that over the next few days institutional investors, investment managers and index providers will have discussions on a path forward for their existing products and allocations. We suspect that total return investors may no longer wish to hold on to their RRBs as the asset class becomes increasingly illiquid, and this could result in some additional supply in the short to medium term. However, investors that hold real return bonds to back inflation-linked liabilities, such as pension plans and insurance companies, will likely hold their existing positions to maturity and may look to absorb any additional supply to satisfy any future planned RRB purchases.*"

According to the same note, the missing issuance in the fiscal year that would end on March 31, 2023, would equal about 0.5 percent of all government bonds, or CAD1.4 billion. Given that RRBs are only issued with long maturities, this would represent about 6 percent

[20] The full report is available at https://www.budget.canada.ca/fes-eea/2022/report-rapport/FES-EEA-2022-en.pdf.

[21] The full note is available at https://www.slcmanagement.com/ca/en/insights/all-insights/SLC-Management-Update-real-return-bonds-cessation-announcement/

RRB	4.25% 12/1-26	4% 12/1-31	3% 12/1-36	2% 12/1-41	1.5% 12/1-44	1.25% 12/1-47	0.5% 12/1-50	0.25% 12/1-54
Mat. (yrs)	4.08	9.08	14.08	19.08	22.08	25.08	28.08	32.08
11/2-2022	145.8	136.4	139.0	138.2	139.7	136.2	132.6	127.8
11/3-2022	156.7	143.3	144.8	148.2	145.2	142.9	136.1	131.5
11/4-2022	156.6	144.4	150.2	149.5	149.3	147.6	141.1	130.5
1-day change	10.9	6.9	5.8	10.0	5.5	6.7	3.5	3.7
2-day change	10.8	8.0	11.2	11.3	9.6	11.4	8.5	2.7

Table 6: **Response of Canadian Real Government Bond Yields to RRB Termination Announcement**
The table reports the one- and two-day responses of the outstanding Canadian RRB yields to the announcement to terminate all future RRB issuance on November 3, 2022. All numbers are measured in basis points. The data are mid-market quoted yields to maturity at market close downloaded from Bloomberg.

of the thirty-year government bond issuance in that fiscal year.

Regarding any market implications of this announcement, we note that, on one hand, it seems investors feared selling pressures and associated market illiquidity. However, at the same time, there is reason for cautious optimism that some current long-term holders of RRBs, such as pension funds, would remain willing buyers knowing that no further supply would come to market. The purpose of this section is to assess both the immediate market reaction to this surprise announcement as well as any longer-term impact on the functioning of the RRB market.

4.2 The Market Reaction

In the following, we assess the impact of the termination of all future RRB issuance on both RRB yields directly and on our estimated RRB liquidity risk premiums.

To begin, we examine the immediate market reaction to the surprise announcement. Table 6 reports the one- and two-day yield changes for the RRBs outstanding at the time of the announcement. The results indicate a notable market reaction with a two-day change in the observed RRB yields of about 10 basis points in the 4- to 28-year maturity range.

Since the responses in Table 6 reflect changes in yields to maturity, they are sensitive to both the bond coupon sizes and the shape of the underlying real yield curve and therefore hard to interpret and compare across bonds. For a cleaner read, Table 7 reports the response of fitted real zero-coupon yields. We focus on the important five- to ten-year maturity range that is commonly used in the construction of BEI measures. Note the initial one-day reaction between 6 and 8 basis points, which increases to a material response between 10 and 23 basis points with a two-day event window.

Although this appears like a strong market reaction, this assessment is tempered by the fact that nominal yields increased by a similar magnitude as documented in Table 8. This is

Maturity	5-year	6-year	7-year	8-year	9-year	10-year
11/2-2022	131.96	129.39	129.75	131.19	132.90	134.53
11/3-2022	140.09	136.01	135.71	137.00	138.83	140.73
11/4-2022	154.90	150.13	146.74	144.94	144.42	144.78
1-day change	8.13	6.62	5.96	5.81	5.93	6.19
2-day change	22.93	20.73	16.99	13.75	11.52	10.24

Table 7: **Response of Canadian Real Zero-Coupon Yields to RRB Termination Announcement**
The table reports the two-day response of Canadian real government zero-coupon bond yields to the announcement by the Canadian Finance Department to permanently terminate its issuance of RRBs on November 3, 2022. All numbers are measured in basis points. See online Appendix A for details on the construction of the real zero-coupon yields.

Maturity	1-year	2-year	3-year	5-year	7-year	10-year
11/2-2022	437.34	390.17	370.39	339.26	327.41	331.48
11/3-2022	445.11	401.04	381.87	350.42	337.41	339.35
11/4-2022	454.85	407.91	388.57	358.46	346.96	349.80
1-day change	7.78	10.87	11.48	11.16	10.00	7.86
2-day change	17.52	17.74	18.18	19.20	19.55	18.32

Table 8: **Response of Canadian Nominal Zero-Coupon Yields to RRB Termination Announcement**
The table reports the two-day response of Canadian nominal government zero-coupon bond yields to the announcement by the Canadian Finance Department to permanently terminate its issuance of RRBs on November 3, 2022. All numbers are measured in basis points.

suggestive of Canadian bond markets reacting for other reasons, including the U.S. Federal Open Market Committee (FOMC) meeting on November 2, 2022, when the FOMC decided to raise the target for the overnight federal funds rate by 0.75 percent.

Still, at face value, the market reaction suggests that investors wanted to get out of holding RRBs following the announcement. Alternatively, they may have become concerned about the trading conditions in the RRB market going forward and demanded a higher liquidity premium as a result.

4.3 Impact on RRB Market Conditions

In this section, we explore the latter conjecture above by examining data on trading conditions for evidence of any dislocations to the functioning of the RRB market.

First, we examine bid-ask spreads of the outstanding set of RRBs. We think of these spreads as representative measures of the current trading conditions in the market for these

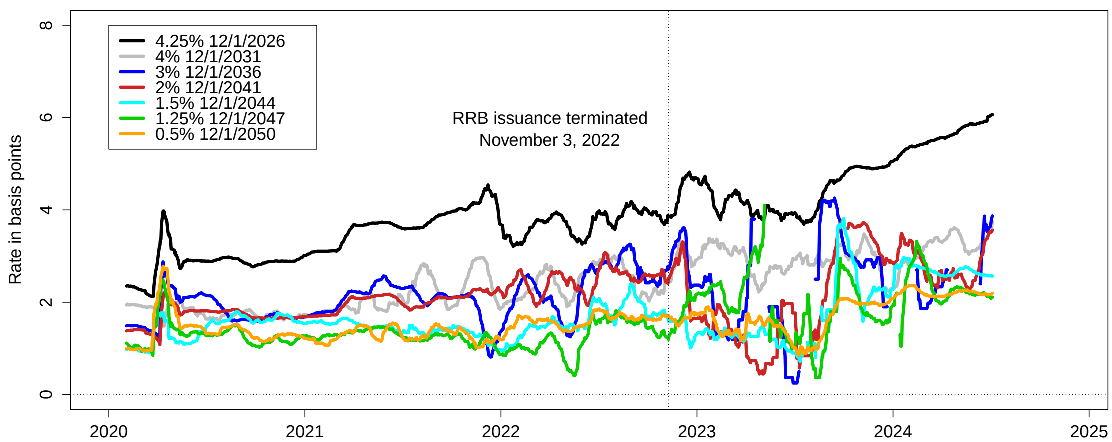


Figure 9: **Bid-Ask Spreads of Canadian Real Return Bonds: 2020-2024**
Illustration of the four-week moving average of bid-ask spreads of Canadian RRBs constructed as described in Section 2.2. The series are daily covering the period from January 2, 2020, to June 28, 2024.

bonds. Figure 9 shows four-week moving averages of the bid-ask spreads for each RRB from January 2020 through the end of our sample. We first note the general upward trend in the bid-ask spread series that is also evident in Figure 5 and caused by the fact that the bonds become more seasoned and less liquid as time passes. In addition, we do see some tendency for greater volatility of the bid-ask spreads in 2023, but this change seems to have started already in late 2021, well before the November 2022 announcement. We speculate that this increase in volatility may reflect the monetary policy tightening cycle on-going at the time as dealers and intermediaries manage their bond holdings more actively to avoid losing money on long-term bond inventories in a rising interest rate environment. Importantly, though, there is no major change in the general bid-ask spread levels in the period following the November 3, 2022, announcement. Thus, the RRB trading conditions do not seem to have fundamentally changed.

Second, we assess whether there seems to have been any impact on the performance of the CLR-L model and its ability to fit the RRB bond prices. To that end, we estimate the CLR-L model using daily data instead of the monthly frequency considered so far. Figure 10 shows the resulting average estimated RRB liquidity premium series since the start of 2020 through the end of June 2024. First, there is hardly any reaction in the estimated RRB liquidity premium in the days following the announcement to cease all future RRB issuance. This contrasts with the market movements at the peak of the pandemic in spring 2020 when the RRB liquidity premium series spiked up more than 30 basis points. Thus, based on our

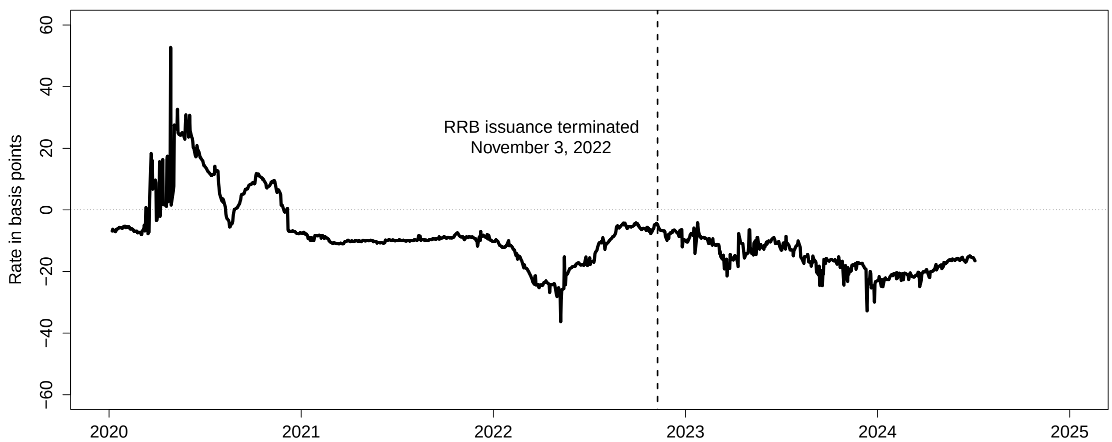


Figure 10: **Average Daily Estimated Real Bond Liquidity Premium since 2020**
Illustration of the average estimated real return bond liquidity premium for each observation date implied by the CLR-L model estimated at daily frequency with a diagonal specification of $K^{\mathbb{P}}$ and Σ. The real return bond liquidity premiums are measured as the estimated yield difference between the fitted yield-to-maturity of individual real return bonds and the corresponding frictionless yield-to-maturity with the liquidity risk factor turned off. The shown data cover the period from January 2, 2020, to June 28, 2024.

model results, investors did not seem to worry much about the future liquidity in the RRB market despite no new supply being issued.

Second, in the time since the announcement, the estimated RRB liquidity premium has declined and turned even more negative. This suggests that RRBs have become scarce and now trade at a notable convenience premium. Crucially, this decline underscores the advantage of our model approach that allows us to account for the variation in these bond-specific premiums in individual RRB prices.

As for the model fit specifically, Figure 11 shows the daily fitted error series for all 9 RRBs in our sample going back to December 1995. We note some tendency for larger and more volatile fitted errors since late 2021, but it is not clear that it is tied to the November 3, 2022, announcement.

To examine this in greater detail, Table 9 reports the summary statistics of the fitted errors for each RRB broken down into the pre- and post-announcement period with the full sample statistics reported for completeness. We note somewhat higher fitted errors for five of the RRBs in the period since the termination of RRB issuance, but this is partially offset by improved fit for the remaining three RRBs. As a result, the overall fit has only deteriorated modestly. Hence, the CLR-L model clearly remains able to fit and explain the variation in the entire RRB yield curve.

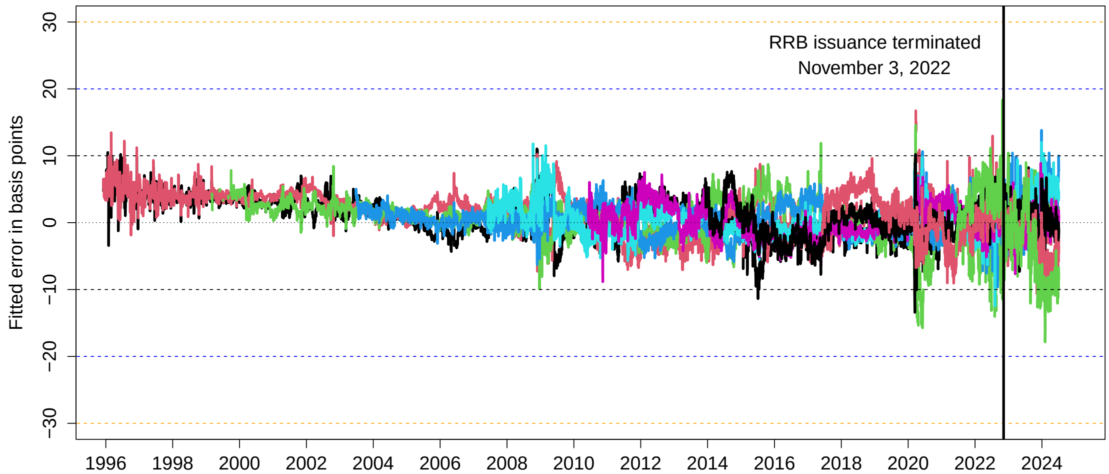


Figure 11: **Fitted Errors of RRB**
Illustration of the fitted errors of RRB yields to maturity implied by the CLR-L model estimated at daily frequency for the period from December 7, 1995, to June 28, 2024.

RRB security	Full sample		Before 11/3-2022		After 11/3-2022	
	Mean	RMSE	Mean	RMSE	Mean	RMSE
(1) 4.25% 12/1/2021	0.93	3.05	0.93	3.05	n.a.	n.a.
(2) 4.25% 12/1/2026	1.32	3.14	1.43	3.21	-0.65	1.69
(3) 4% 12/1/2031	0.43	2.93	0.37	2.90	1.34	3.33
(4) 3% 12/1/2036	0.55	2.25	0.40	2.02	2.75	4.39
(5) 2% 12/1/2041	0.33	2.40	0.07	2.12	3.25	4.42
(6) 1.5% 12/1/2044	0.07	2.25	0.07	2.27	0.11	2.05
(7) 1.25% 12/1/2047	0.04	2.55	0.00	2.58	0.29	2.35
(8) 0.5% 12/1/2050	0.28	3.78	1.22	3.56	-3.11	4.47
(9) 0.25% 12/1/2054	-0.76	5.96	3.66	4.71	-4.83	6.92
All RRB yields	0.60	2.91	0.67	2.79	-0.20	4.04

Table 9: **Pricing Errors of RRBs**
This table reports the mean pricing errors (Mean) and the root mean-squared pricing errors (RMSE) of RRBs in the CLR-L model estimated with a diagonal specification of $K^{\mathbb{P}}$ and Σ using daily data covering the period from December 7, 1995, to June 28, 2024. The first two columns contain the results for the full sample, while columns 3 and 4 contain the results for the period up to November 2, 2022. Finally, the last two columns report the results for the period from November 3, 2022, to June 28, 2024.

4.4 Summary

To summarize, despite a negative initial reaction with RRB prices declining, or equivalently, RRB yields rising, the decision by the Canadian Finance Department to cease all future RRB issuance does not seem to have impacted RRB market conditions much, if at all.

With a mildly negative RRB liquidity premium from November 2022 through the end of our sample, there is little that points to disadvantageous trading conditions of RRBs in the secondary market. Data for bid-ask spreads of individual RRBs support this interpretation. Moreover, the CLR-L model remains able to provide a very close fit to the RRB bond price data. As a consequence, it remains valid to continue to use the RRB bond price data despite the decision by the Canadian government to terminate all future RRB issuance.

Furthermore, based on our results, RRB liquidity premiums actually declined in the year after the announcement. Thus, the liquidity discount has become a scarcity convenience premium. This suggests that, rather than terminating the RRB program, the Canadian government could benefit from resuming and increasing RRB issuance.

With remaining maturities up to thirty years, it is feasible for the existing set of RRBs to continue to trade for several decades, even without any new supply coming to market. Thus, it seems recommendable to continue to use the RRB data for both market monitoring and policy analysis.

Overall, this decision appears to be short sighted. The Canadian government spent thirty years on establishing this market and incurred the high real rates early on visible in Figure 4, only to shut down the program at the first sight of a meaningful spike in inflation. This comes across as a less than well contemplated decision. Furthermore, both fiscal and monetary policy benefit from access to directly observable measures of real yields. As we document in this paper, we can measure and decompose BEI and track the underlying inflation expectations among investors in the Canadian government bond market. Moreover, the RRB market offers direct measures of real yields that can be decomposed and used to obtain estimates of the natural real rate that can be used to assess the stance of monetary policy—as we demonstrate later on—and the sustainability of public finances. Finally, this analysis provides a concrete example of how a model like ours can leverage the RRB pricing data to examine the implications of important policy decisions like the one made by the Finance Department when it ceased all future RRB issuance.

5 A New Normal for Canadian Interest Rates?

After first defining breakeven inflation (BEI) and the natural real rate, we select a preferred specification of the CLR-L model's objective $\mathbb{P}$-dynamics consistent with the data. We use this preferred CLR-L model to account for liquidity and term premiums in RRB prices and obtain estimates of investors' long-term inflation expectations and an associated measure of the equilibrium real rate, which we compare to other market- and macro-based estimates from the literature. We also leverage our natural real rate estimate to measure the stance of monetary policy in Canada. Finally, we end the section using our preferred model to make ten-year projections for both long-term expected inflation and the equilibrium real rate as of June 2024.

5.1 Definition of BEI and the Natural Rate

To begin the analysis in this section, we note that BEI is defined as

$$BEI_t(\tau) \equiv y_t^N(\tau) - y_t^R(\tau) = \pi_t^e(\tau) + \phi_t(\tau), \tag{6}$$

that is, the difference between nominal and real yields of the same maturity, which can then be decomposed into the sum of expected inflation $\pi_t^e(\tau)$ and the inflation risk premium $\phi_t(\tau)$.

CLR show that the market-implied average rate of inflation expected at time t for the period from t to $t+\tau$ is

$$\pi_t^e(\tau) = -\frac{1}{\tau} \ln E_t^{\mathbb{P}}\left[\frac{\Pi_t}{\Pi_{t+\tau}}\right] = -\frac{1}{\tau} \ln E_t^{\mathbb{P}}\left[e^{-\int_t^{t+\tau}(r_s^N - r_s^R)ds}\right]$$

and the associated inflation risk premium for the same time period is

$$\phi_t(\tau) = -\frac{1}{\tau} \ln\left(1 + \frac{cov_t^{\mathbb{P}}\left[\frac{M_{t+\tau}^R}{M_t^R}, \frac{\Pi_t}{\Pi_{t+\tau}}\right]}{E_t^{\mathbb{P}}\left[\frac{M_{t+\tau}^R}{M_t^R}\right] \times E_t^{\mathbb{P}}\left[\frac{\Pi_t}{\Pi_{t+\tau}}\right]}\right),$$

where Π_t is the price level and M_t^R is the real stochastic discount factor.

This last equation highlights that the inflation risk premium can be positive or negative. It is positive if and only if

$$cov_t^{\mathbb{P}}\left[\frac{M_{t+\tau}^R}{M_t^R}, \frac{\Pi_t}{\Pi_{t+\tau}}\right] < 0.$$

That is, the riskiness of nominal bonds relative to real bonds depends on the covariance between the real stochastic discount factor and inflation, and is ultimately determined by investor preferences.

Furthermore, following CR, our definition of the equilibrium real rate of interest r_t^* is

$$r_t^* = \frac{1}{5}\int_{t+5}^{t+10} E_t^{\mathbb{P}}[r_{t+s}^R]ds, \tag{7}$$

that is, the average expected real short rate over a five-year period starting five years ahead where the expectation is with respect to the objective $\mathbb{P}$-probability measure. As noted in the introduction, this 5yr5yr forward average expected real short rate should be little affected

Alternative specifications	Goodness of fit statistics			
	$\log L$	k	p-value	BIC
(1) Unrestricted $K^{\mathbb{P}}$	31,995.08	58	n.a.	-63,642.37
(2) $\kappa_{24}^{\mathbb{P}} = 0$	31,995.07	57	0.89	-63,648.34
(3) $\kappa_{24}^{\mathbb{P}} = \kappa_{13}^{\mathbb{P}} = 0$	31,994.99	56	0.69	-63,654.18
(4) $\kappa_{24}^{\mathbb{P}} = \kappa_{13}^{\mathbb{P}} = \kappa_{12}^{\mathbb{P}} = 0$	31,994.75	55	0.49	-63,659.70
(5) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{15}^{\mathbb{P}} = 0$	31,994.46	54	0.45	-63,665.11
(6) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{43}^{\mathbb{P}} = 0$	31,994.01	53	0.34	-63,670.21
(7) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{35}^{\mathbb{P}} = 0$	31,993.56	52	0.34	-63,675.30
(8) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{53}^{\mathbb{P}} = 0$	31,993.56	51	1.00	-63,681.30
(9) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{45}^{\mathbb{P}} = 0$	31,992.57	50	0.16	-63,685.32
(10) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{32}^{\mathbb{P}} = 0$	31,991.52	49	0.15	-63,689.21
(11) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{34}^{\mathbb{P}} = 0$	31,991.36	48	0.57	-63,694.89
(12) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{31}^{\mathbb{P}} = 0$	31,991.08	47	0.45	-63,700.33
(13) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{21}^{\mathbb{P}} = 0$	31,986.16	46	< 0.01	-63,696.48
(14) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{52}^{\mathbb{P}} = 0$	31,985.82	45	0.41	-63,701.80
(15) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{23}^{\mathbb{P}} = 0$	31,985.57	44	0.48	-63,707.30
(16) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{51}^{\mathbb{P}} = 0$	31,985.53	43	0.78	-63,713.21
(17) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{54}^{\mathbb{P}} = 0$	31,984.82	42	0.23	**-63,717.79**
(18) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{25}^{\mathbb{P}} = 0$	31,976.90	41	< 0.01	-63,707.95
(19) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{42}^{\mathbb{P}} = 0$	31,966.61	40	< 0.01	-63,693.36
(20) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{41}^{\mathbb{P}} = 0$	31,965.21	39	0.09	-63,696.56
(21) $\kappa_{24}^{\mathbb{P}} = \ldots = \kappa_{14}^{\mathbb{P}} = 0$	31,953.27	38	< 0.01	-63,678.67

Table 10: **Evaluation of Alternative Specifications of the CLR-L Model**
There are twenty-one alternative estimated specifications of the CLR-L model. Each specification is listed with its maximum log likelihood ($\log L$), number of parameters (k), the p-value from a likelihood ratio test of the hypothesis that it differs from the specification above with one more free parameter, and the Bayesian information criterion (BIC). The period analyzed covers monthly data from January 31, 1991, to June 28, 2024.

by short-term transitory shocks.[22] Alternatively, r_t^* could be defined as the expected real short rate at an infinite horizon as discussed in Bauer and Rudebusch (2020). However, this quantity will depend crucially on whether the factor dynamics exhibit a unit root. As is well known, the typical spans of time series data that are available do not distinguish strongly between highly persistent stationary processes and non-stationary ones. Our model follows the finance literature and adopts the former structure, so strictly speaking, our infinite-horizon steady state expected real rate is constant. However, our data sample likely has insufficient information in the ten-year to infinite horizon to definitively pin down that steady state.

$K^{\mathbb{P}}$	$K^{\mathbb{P}}_{\cdot,1}$	$K^{\mathbb{P}}_{\cdot,2}$	$K^{\mathbb{P}}_{\cdot,3}$	$K^{\mathbb{P}}_{\cdot,4}$	$K^{\mathbb{P}}_{\cdot,5}$	$\theta^{\mathbb{P}}$		Σ
$K^{\mathbb{P}}_{1,\cdot}$	0.6105	0	0	-0.6655	0	0.0845	σ_{11}	0.0051
	(0.0471)			(0.0537)		(0.0031)		(0.0001)
$K^{\mathbb{P}}_{2,\cdot}$	0	0.4425	0	0	0.0377	-0.0151	σ_{22}	0.0114
		(0.0630)			(0.0135)	(0.0045)		(0.0002)
$K^{\mathbb{P}}_{3,\cdot}$	0	0	0.9984	0	0	-0.0187	σ_{33}	0.0227
			(0.0577)			(0.0040)		(0.0009)
$K^{\mathbb{P}}_{4,\cdot}$	-0.7243	-0.2207	0	0.8126	0	0.0596	σ_{44}	0.0040
	(0.0505)	(0.0326)		(0.0563)		(0.0032)		(0.0001)
$K^{\mathbb{P}}_{5,\cdot}$	0	0	0	0	0.5511	0.0955	σ_{55}	0.2745
					(0.0568)	(0.0541)		(0.0151)

Table 11: **Estimated Dynamic Parameters of the Preferred CLR-L Model**
The table shows the estimated parameters of the $K^{\mathbb{P}}$ matrix, $\theta^{\mathbb{P}}$ vector, and diagonal Σ matrix for the preferred CLR-L model according to the BIC. The estimated value of λ is 0.4013 (0.0050), while $\alpha^R =$ 0.8108 (0.0170), $\kappa^{\mathbb{Q}}_{liq} =$ 14.72 (0.0659), and $\theta^{\mathbb{Q}}_{liq} =$ -0.0013 (0.0001). The maximum log likelihood value is 31,984.82. The numbers in parentheses are the estimated parameter standard deviations.

5.2 Model Selection

For decompositions of BEI and estimation of the natural real rate and associated inflation, nominal, and real risk premiums, the specification of the mean-reversion matrix $K^{\mathbb{P}}$ is critical as noted earlier. To select the best fitting specification of the model's real-world dynamics, we use a general-to-specific modeling strategy in which the least significant off-diagonal parameter of $K^{\mathbb{P}}$ is restricted to zero and the model is re-estimated. This strategy of eliminating the least significant coefficient is carried out down to the most parsimonious specification, which has a diagonal $K^{\mathbb{P}}$ matrix. The final specification choice is based on the value of the Bayesian information criterion (BIC) as in Christensen et al. (2014).[23]

The summary statistics of the model selection process are reported in Table 10. The BIC is minimized by specification (17), which has a $K^{\mathbb{P}}$ matrix given by

$$K^{\mathbb{P}}_{BIC} = \begin{pmatrix} \kappa^{\mathbb{P}}_{11} & 0 & 0 & \kappa^{\mathbb{P}}_{14} & 0 \\ 0 & \kappa^{\mathbb{P}}_{22} & 0 & 0 & \kappa^{\mathbb{P}}_{25} \\ 0 & 0 & \kappa^{\mathbb{P}}_{33} & 0 & 0 \\ \kappa^{\mathbb{P}}_{41} & \kappa^{\mathbb{P}}_{42} & 0 & \kappa^{\mathbb{P}}_{44} & 0 \\ 0 & 0 & 0 & 0 & \kappa^{\mathbb{P}}_{55} \end{pmatrix}.$$

The estimated parameters of the preferred specification are reported in Table 11. The estimated $\mathbb{Q}$-dynamics used for pricing and determined by (Σ, λ, α^R, $\kappa^{\mathbb{Q}}_{liq}$, $\theta^{\mathbb{Q}}_{liq}$) are close to

[22]Later on, we show that the reported results are robust to using alternative definitions of r^*_t.

[23]The Bayesian information criterion is defined as $\text{BIC} = -2\log L + k\log T$, where k is the number of model parameters and T is the number of data observations. We have 402 nominal yield and 343 real yield monthly observations. We follow CLR and interpret T as referring to the longest data series and fix it at 402.

those reported in Table 4 for the CLR-L model with diagonal $K^{\mathbb{P}}$. This implies that both model fit and the estimated RRB liquidity premiums from the preferred CLR-L model are similar to those already reported and therefore not shown here, but can be found in online Appendix C. Furthermore, the estimated objective $\mathbb{P}$-dynamics in terms of $\theta^{\mathbb{P}}$ and Σ are also qualitatively similar to those reported in Table 4. Finally, we note that the liquidity factor matters for the expected excess return of nominal bonds through $\kappa_{25}^{\mathbb{P}}$ in addition to its effect on RRB pricing, while the slope factor is important for the expected return of both nominal bonds and RRBs. This comes about both through its own direct effect on their pricing and through its dynamic interactions with the other state variables. In contrast, the third factor, i.e. the curvature factor, is the only independent factor without any dynamic interactions with any of the other state variables.

5.3 Empirical BEI Decomposition

In this section, we describe the decomposition of the ten-year BEI implied by our estimation results.

The starting point for the decomposition is the fitted ten-year BEI rate from the CLR model, which offers the cleanest and most straightforward fit of the raw bond data without any adjustments. This measure of ten-year BEI is shown with a solid black line in Figure 12. Now, the estimated ten-year frictionless BEI from the CLR-L model, which does not contain any RRB liquidity risk premiums, is shown with a solid grey line in Figure 12. It fluctuates above and below the ten-year fitted BEI as the estimated RRB liquidity premiums switch sign. Hence, the difference between these two series represent an alternative measure of the RRB liquidity premiums, which is different from the estimate in Figure 8 as the former has a constant maturity.

As explained in Section 5.1, the CLR-L model also provides a decomposition of the estimated ten-year frictionless BEI into an expected CPI inflation component and the associated inflation risk premium shown in Figure 12 with a solid red and green line, respectively.

The ten-year inflation risk premium is variable and was mostly positive until mid-2015. Since then the Canadian inflation risk premium seems to have switched sign and has remained mostly mildly negative.[24] Notably, it dropped back into negative territory the last year of our sample despite the recent spell of high inflation also on vivid display in Figure 12. This premium was also significantly negative in the late 1990s when Canadian CPI inflation barely averaged 1 percent. Moreover, the inflation risk premium experienced a temporary softening around the peak of the financial crisis in late 2008, when CPI inflation in Canada and elsewhere started to fall; see Christensen et al. (2012) for a U.S. analysis of this episode. It also turned

[24]Note that, due to the model's Gaussian dynamics, the conditional variance of expected inflation is constant. As a result, changes in the inflation risk premium reflect changes in the risk premiums within the model.

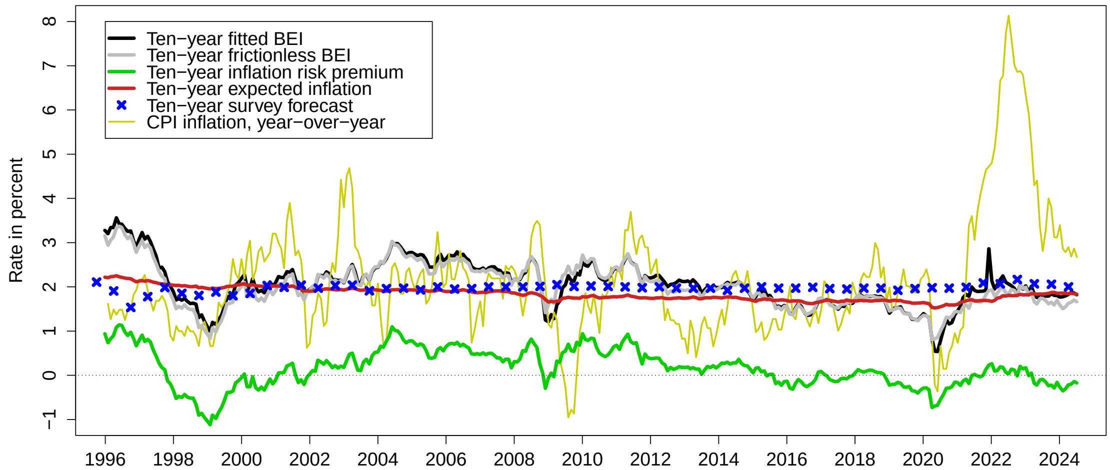


Figure 12: **Ten-Year BEI Decomposition**
Illustration of the ten-year fitted BEI implied by the CLR model, i.e. the model without adjustment for the RRB liquidity premium, and its decomposition into (1) the fitted frictionless BEI, (2) the ten-year expected inflation, and (3) the residual ten-year inflation risk premium based on the preferred CLR-L model. Also shown is the ten-year inflation forecasts from the Consensus Forecasts and Canadian CPI inflation calculated as the 12-month change in the CPI.

negative briefly in late 2015 and early 2016 when global energy and commodity prices fell sharply. In addition, it fell further into negative territory around the peak of the COVID-19 pandemic in spring 2020 when price inflation also feel sharply. Finally, we note that it did temporarily turn positive during the post-pandemic economic reopening when inflation in Canada and elsewhere spiked up to levels not seen in more than 40 years. Based on these observations we consider the estimated inflation risk premium from the CLR-L model to be reasonable in terms of both its level and time-series variation.

Most existing studies have found inflation risk premiums to be positive on average and relatively stable; see Ang et al. (2008), Buraschi and Jiltsov (2005), and Hördahl and Tristani (2014), among many others. Thus, the tilt towards a negative inflation risk premium the past decade seems to be a more recent and novel development that merits monitoring going forward. At face value, it entails that investors in the Canadian bond markets fear future recessions happening in the context of disinflationary price dynamics similar to the GFC and the pandemic. In that regard, the recent spell of high inflation is not very informative in shedding light on the crucial question of the sign of the inflation risk premium because it happened in the context of a booming economy in both Canada and the United States.

In comparison with the inflation risk premium, the estimated ten-year inflation expectations are much less variable, although still characterized by a very persistent and gradual decline since the mid-1990s when bond investors' long-term inflation expectations were about a quarter point above the Bank of Canada's two percent inflation target. By the time of the pandemic, these expectations had fallen to a low of 1.52 percent, or about a 0.7 percentage point drop, before partially reversing that decline during the last three years of our sample to a level of 1.82 percent by June 2024. This contrasts with the responses to the Consensus Forecasts survey of professional forecasters, who twice a year are asked about their expectations for inflation over the following ten years. The mean responses in each survey since October 1995 are shown with blue crosses in Figure 12 and have remained very close to two percent since the late 1990s. As a consequence, the forecasters view the gyrations in Canadian long-term BEI rates to be entirely driven by changes in the inflation risk premiums.

In support of the decline in our preferred CLR-L model's estimated ten-year expected inflation, we note that Canadian 12-month CPI inflation averaged 2.04 percent in the 12 and a half years from January 1996 through the end of June 2008. In contrast, it only averaged 1.58 percent in the 12 and a half years from July 2008 through the end of December 2020. Thus, the decline in the model-implied ten-year expected inflation series is of a magnitude similar to the decline in actual inflation outcomes during this 25-year period.

We take the modest increase in the ten-year expected inflation in the 2021-2024 period against the backdrop of the outsized increase in realized inflation as a sign that investors' inflation expectations in Canada were, and continue to remain, well anchored close to the Bank of Canada's 2 percent inflation target.

5.4 Estimates of the Natural Rate

Our market-based measure of the natural rate is the average expected real short rate over a five-year period starting five years ahead. This 5yr5yr forward average expected real short rate should capture the persistent trends in the natural real rate.

Figure 13 shows the preferred CLR-L model decomposition of the 5yr5yr forward frictionless real yield based on the standard definition of the real term premium

$$TP_t^R(\tau) = y_t^R(\tau) - \frac{1}{\tau}\int_t^{t+\tau} E_t^{\mathbb{P}}[r_s^R]ds,$$

where $y_t^R(\tau)$ is the fitted frictionless real zero-coupon yield with maturity in τ years; see online Appendix A for details. The solid grey line is the 5yr5yr forward real term premium, which has fluctuated around a fairly stable level since 1996 with spikes in September 2001, during the GFC in the 2008-2009 period, and also during the peak of the COVID-19 pandemic in spring 2020. Hence, as suggested by theory, the term premium is countercyclical and elevated during economic recessions. In contrast, the estimate of the natural real rate of interest implied by

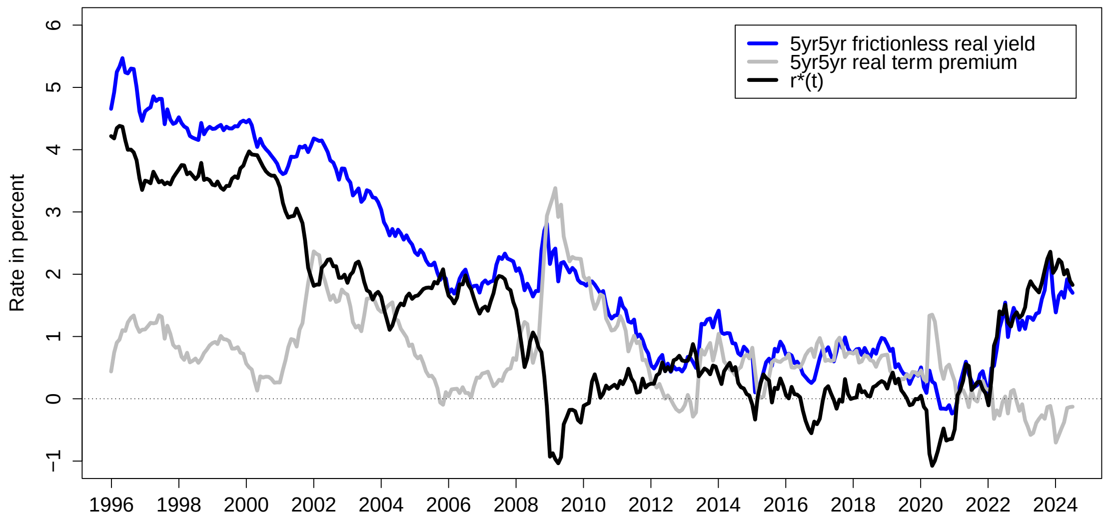


Figure 13: **CLR-L Model 5yr5yr Real Yield Decomposition**

the CLR-L model—the black line in Figure 13—shows a very gradual and persistent decline from above 4 percent in late 1995 to below -1 percent by spring 2020 followed by a sharp partial reversal during the last three years of the sample that leaves it at 1.83 percent by June 2024. Thus, much of the downward trend in the 5yr5yr forward real yield in the 1995-2020 period is driven by declines in this measure of r_t^*. Similarly, the increase in the 5yr5yr forward real yield since 2021 is fully accounted for by a matching increase in r_t^*, while the real term premium has declined slightly on net during this most recent three-year period.

To examine the sensitivity of our r_t^* estimate to our choice to define r_t^* as the average expected real short rate over a five-year period starting five years ahead, we consider three alternative definitions that all embed a longer view about the time it takes for the economy to reach steady state. The first assumes that this takes seven years and then measures the neutral real rate as the average expected real short over the following three years. It is referred to as the 7yr3yr r_t^* estimate. The second alternative takes an even longer view and assumes that it takes nine years to reach steady state and then measures the neutral real rate as the average expected real short rate over a short one-year period. It is referred to as the 9yr1yr r_t^* estimate. Finally, the third alternative takes the longest view and assumes that it takes ten years for the economy to reach steady state and then measures r_t^* as the average expected real short rate over a five-year horizon as in our benchmark definition. It is referred to as the 10yr5yr r_t^* estimate. Figure 14 shows all four r_t^* estimates, which are very close to each other thanks to the high estimated persistence of the state variables within our preferred CLR-L

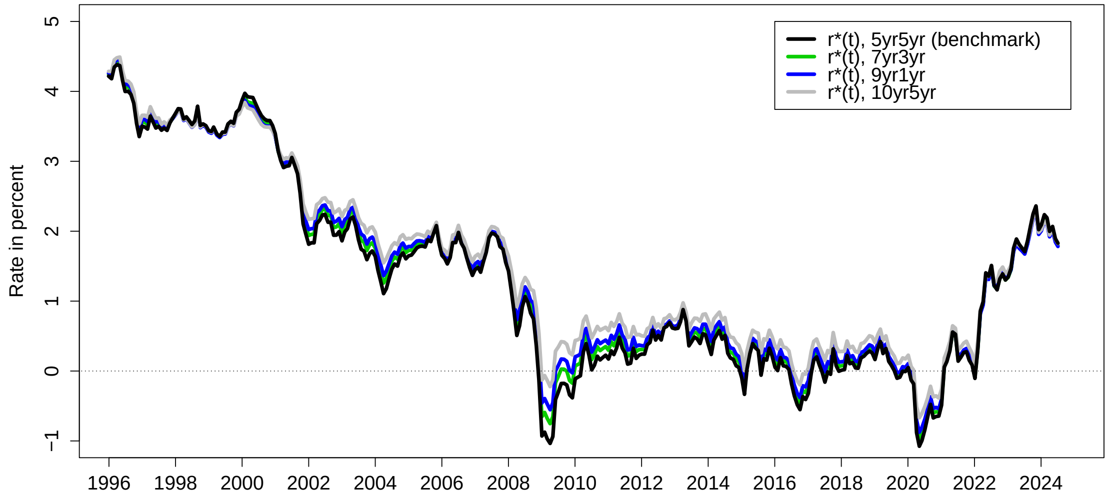


Figure 14: **The Sensitivity of r* Estimate to Alternative Definitions**

model. Hence, our r_t^* estimate of the neutral real rate has very little sensitivity to the specific definition of r_t^* used. Thus, our reported results are very robust from that perspective.

5.4.1 Comparison of Estimates of the Natural Rate

In this section, we compare other existing estimates of the equilibrium or natural interest rate in the literature to our Canadian estimate. To start, we compare the Canadian r_t^* estimate from our preferred CLR-L model to the U.S. finance-based estimate generated by CR using solely the prices of U.S. TIPS. These two market-based estimates of the natural rate are shown in Figure 15. Their high positive correlation and similar downward trend for the overlapping period are both evident. Moreover, they share the feature that much of their common declines happened outside of the GFC. This suggests that the factors depressing both Canadian and U.S. interest rates during the overlapping period are likely to be global in nature and do not just reflect the immediate financial market dislocations during the crisis.

Now, we turn to the crucial comparison of our finance-based estimate of r_t^* with estimates based on macroeconomic data. Figure 16 shows the r_t^* estimate from our preferred CLR-L model, along with the macro-based estimate of r_t^* from Holston et al. (2017, henceforth HLW), which is the filtered estimate generated by applying the approach described in Laubach and Williams (2003) to Canadian macroeconomic series. The r_t^* estimate from HLW starts in 1961 at a level above 5 percent. Since then it has followed a steady declining trend that leaves it close to 1.5 percent by the end of 2023 and hence close to the finance-based estimate.

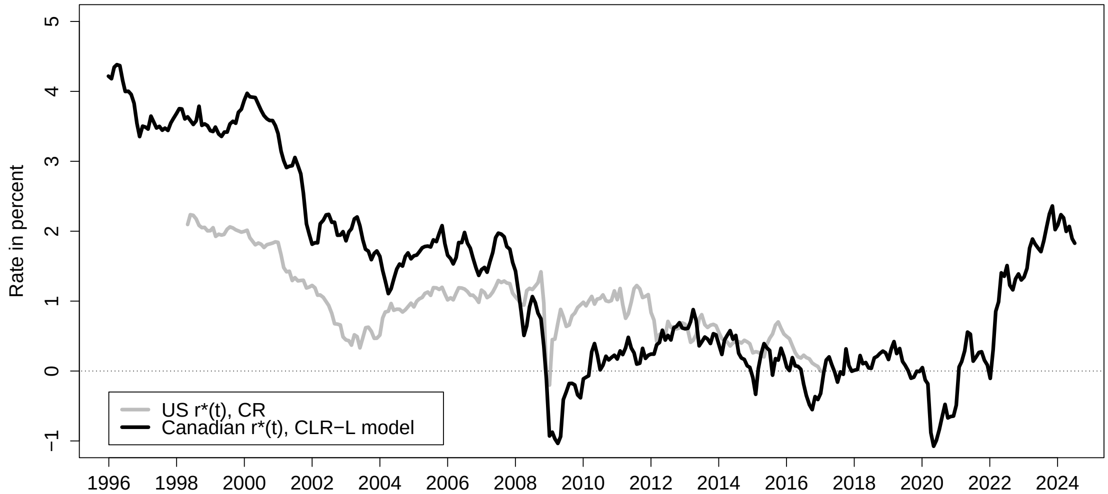


Figure 15: **Comparison with a U.S. Market-Based Estimate of r***

The second macro-based estimate of r_t^* is taken from Del Negro et al. (2019, henceforth DGGT). They estimate a flexible vector autoregression model with common trends for a sample of data from 7 advanced economies, including Canada, covering the period from 1870 to 2016 and here extended through the end of 2020.[25] They use annual data on short- and long-term government bond yields, consumer prices, and real consumption per capita in addition to Moody's Baa corporate bond yields. In their analysis, it is an assumption of no arbitrage in the long run that implies a factor structure for the trend of real interest rates across countries. They find that real interest rates across these 7 countries share a global common trend that has been particularly pronounced since the 1970s. Moreover, using regression analysis, they find that declining consumption growth and increasing convenience yields from the safety and liquidity offered by government bonds from these countries are the main drivers of declining real rates the past 40 years. Their r_t^* estimate for Canada is shown with a solid blue line in Figure 16. Note that their r_t^* estimate was increasing back in the 1960s and 1970s before starting a pronounced secular decline in the early 1980s. The trend lower continues through the end of the shown sample and leaves it close to zero by 2020 and very close to our finance-based r_t^* estimate at the time.

The third macro-based estimate of r_t^* is taken from Ferreira and Shousha (2023, henceforth FS) and shown with a solid green line in Figure 16. They consider a panel of 11 advanced economies and estimate the longer-run neutral real interest rates while accounting for changes

[25]We thank Marco Del Negro for sharing the updated data.

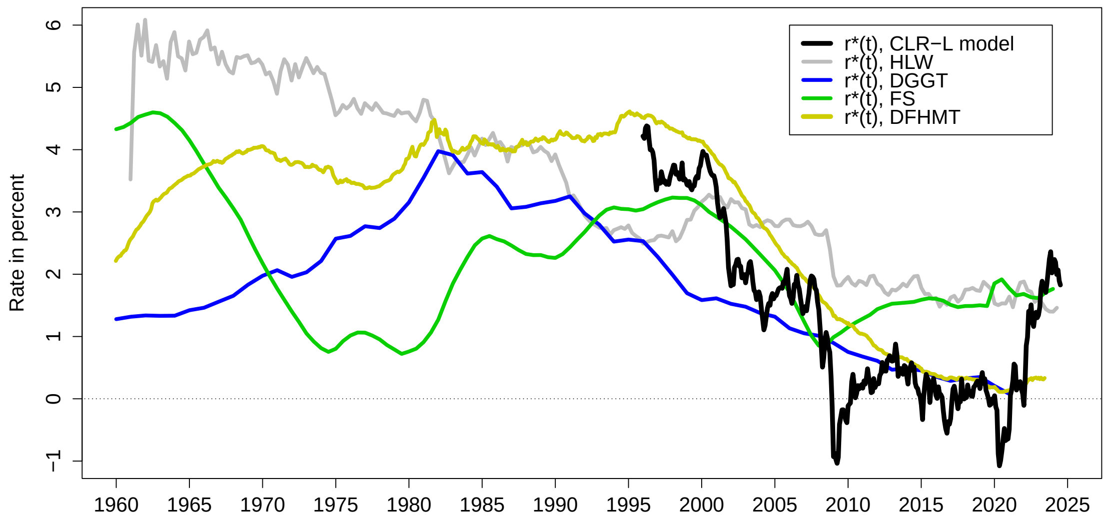


Figure 16: **Comparison with Macro-Based Estimates of r***

in the global supply of safe assets and their convenience yields in addition to productivity and demographics and global spillovers from their developments. Their r_t^* estimate for Canada starts in 1960 and fell steadily until the mid-1970s. It reversed some of the decline in the early 1980s and remained fairly stable until the late 1990s. It then steadily declined until 2008 when it reached a low of 0.85 percent. Since then it has gradually trended higher and stood at 1.76 percent by the end of 2023. This upward trend the past 15 years with a net increase of about 1 percentage point sets it apart from the other macro-based estimates, which mostly trend lower during this period.

The fourth and final macro-based estimate of r_t^* comes from Davis et al. (2024, henceforth DFHMT). They introduce a unified no-arbitrage macro-finance model with two trend factors used to estimate the natural rate r_t^* for 10 advanced economies, including Canada. Using a multitude of data sources on trend growth and inflation in addition to risk premium series, DFHMT also underscore the need for a coherent model approach like ours. Importantly, the interpretation of their natural rate r_t^* is consistent with Laubach and Williams (2003) of representing a medium-run real rate anchor for monetary policy. Our finance-based definition taken from CR is intended to capture the same concept. Hence, the DFHMT r_t^* estimate should be comparable to ours. One notable difference, though, is that, by relying on a single average yield equation, their estimation is not fully exploiting all the information in the yield curve unlike our approach. Their r_t^* series is shown with a solid yellow line in Figure 16. While increasing from the 1960s and into the 1990s, the DFHMT r_t^* estimate peaked in the mid-

1990s. In the subsequent more than 25 years, r_t^* fell more than 4 percentage points according to their estimate and ends the sample slightly above zero. For the overlapping period this entails a close similarity between their r_t^* estimate and our finance-based r_t^* estimate. Overall, this pattern aligns well with the observed RRB yields shown in Figure 4. Moreover, using panel regressions, DFHMT relate their r_t^* estimates to economic growth and demographic variables and find that slowing growth and population aging have been significant factors in driving down the natural rate r_t^* globally, and in Canada as well. Given the similarity of our finance-based r_t^* to their estimate, we speculate that our estimate is likely influenced by those same factors.

Finally, we note that the Bank of Canada publishes an annual analysis of the level of the neutral rate in Canada. The most recent release of this analysis described in Adjalala et al. (2024) indicates an estimate between 2.25 and 3.25 percent for the neutral rate using five different approaches. Assuming inflation at 2 percent in the long run consistent with the Bank of Canada's inflation target, this would translate into a range between 0.25 and 1.25 percent for the natural real rate. This puts our r_t^* estimate of 1.85 percent at the end of our sample a notch above the range considered relevant by staff at the Bank of Canada. However, in light of the involved estimation uncertainty and the disagreement across macro-based models on full display in Figure 16, we consider our r_t^* estimate to be reasonable and broadly consistent with those implied by standard macroeconomic models.

5.4.2 The Stance of Canadian Monetary Policy

In this section, as an additional application of our market-based estimate of r_t^*, we follow Christensen and Mouabbi (2024) and use it to construct a measure of the stance of the Bank of Canada's monetary policy.

As described in Christensen and Mouabbi (2024), the stance of monetary policy would in theory be given by the difference between the current real instantaneous short rate and its neutral level as reflected in r_t^*, i.e., it would be defined as

$$\zeta_t = r_t - r_t^*.$$

The intuition behind this definition is straightforward. When the current real short rate is above its neutral level, interest rates of all kinds are likely to be above their steady-state level and will provide some headwind for new economic activity through higher borrowing costs and help slowdown the economy. And vice versa, when the current real short rate is below its neutral level, the general interest rate level is likely to be below what is needed to maintain trend growth, and businesses and households may take advantage of that by making investments in new projects or housing at cheap financing rates, which will help boost economic activity.

Unfortunately, the instantaneous real short rate is not directly observable because we do not have a continuous measure of the very short end of the RRB yield curve, given that individual RRBs reach maturity infrequently as noted in Figure 3. Furthermore, as explained earlier, RRBs, like other inflation-indexed bonds, tend to have rather erratic prices close to maturity thanks to both low liquidity and the unpredictability of the final inflation adjustments to be earned—the sudden and very sharp spike in Canadian CPI inflation in 2021-2022 period is very illustrative in this regard.[26] Thus, to make the definition above operational, we consider instead two proxies that we think of as reasonable substitutes for r_t. The first proxy is the one-year fitted real RRB yield from an estimation of the CLR model without censoring any bond price information, that is, RRB prices remain in the sample until they mature. This provides the best possible coverage around the one-year maturity point but comes at the cost of adding noise from the prices of RRBs close to maturity. Still, one can argue that this yield measures the full actual real yields observed in financial markets—including noise and frictions—and hence represents the most realistic real-world equivalent to the textbook short-term real rate embedded in the definition of ζ_t. The second proxy is the one-year frictionless real yield implied by our preferred CLR-L model. This is a cleaner and more stable measure of the one-year real yield as it adjusts for the noise from the bond-specific risk premiums in the RRB prices. However, in doing so, it may be different from the textbook concept of the real short rate r_t appearing in the original definition of ζ_t. Moreover, as RRB prices with less than one year to maturity are censored in the estimation of our preferred CLR-L model, it may capture the short end of the RRB real yield curve less accurately.

The resulting two empirical measures of the stance of Canadian monetary policy are shown in Figure 17. In general, the two measures are very similar and highly positively correlated (99 percent), although we note greater volatility in the measure based on the fitted real yield in 2021. Hence, they paint a very similar picture for the stance of monetary policy in Canada the past 28 years. First, monetary policy in Canada reached a restrictive stance in the years ahead of the mild recession in 2000 caused by the burst of the IT stock price bubble at the time. Second, monetary policy in Canada was tight going into the GFC in 2007 and remained above neutral into 2009 before finally reaching an accommodative level that remained in place until 2018 when the U.S. Federal Reserve had also raised its overnight policy rate for some time. Third, at the peak of the COVID-19 pandemic in spring 2020, Canadian monetary policy briefly reached a very restrictive stance before becoming very accommodative relatively quickly. Finally and similar to the United States, the response of the Bank of Canada to the spike in inflation following the global economic reopening after

[26]For comparison, a standard fixed-coupon bond pays a principal of 1 and fixed coupons C. Thus, there is no uncertainty about its final cash flow in the months leading up to its maturity date, which helps maintain the liquidity of these securities.

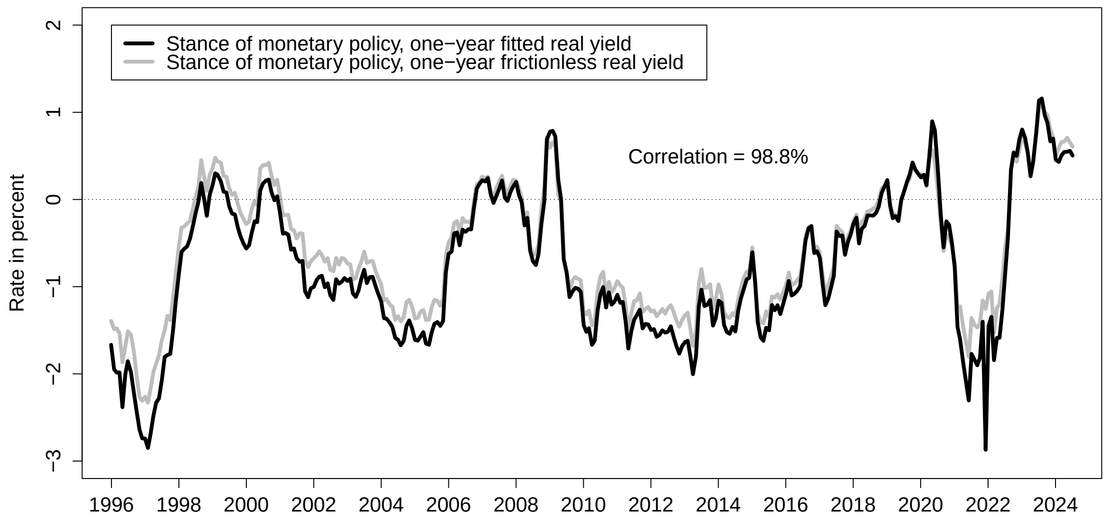


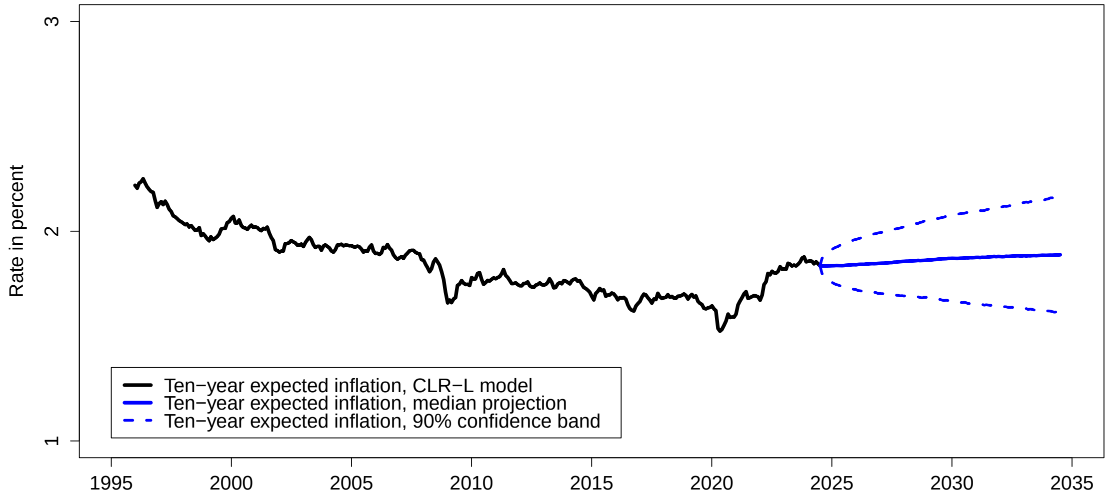
Figure 17: **Market-Based Measures of the Stance of Canadian Monetary Policy**

the pandemic was delayed, which had the implication that monetary policy remained very accommodative for an extended period of time and did not reach a tightening posture until spring 2022 according to our measures. This may have contributed to prolonging the spell of high inflation in Canada during this period, but it falls outside the focus of this paper to make any determinations to that effect, so we leave it for future research to explore that question further.

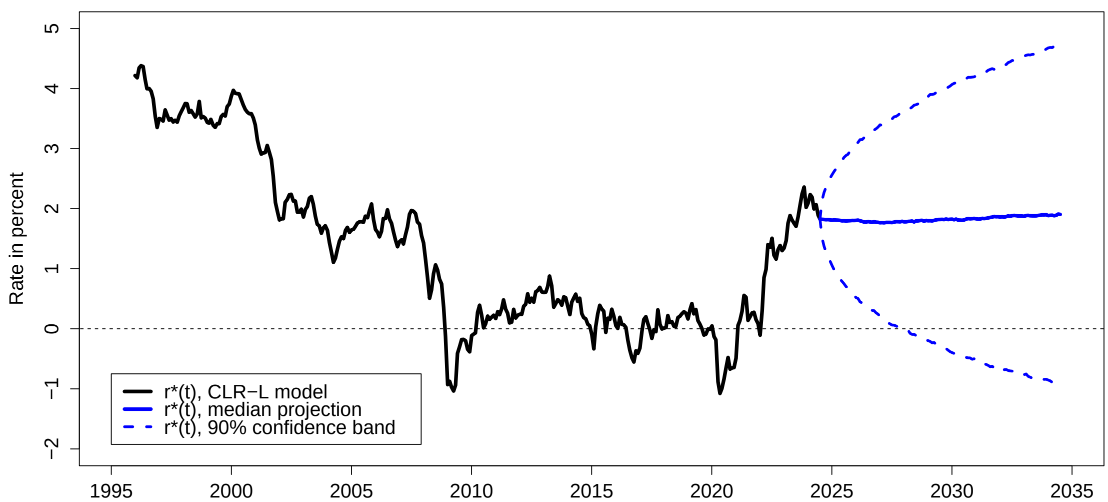
Based on these observations we think of our empirical market-based measures of the stance of Canadian monetary policy as realistic. Moreover, they can be estimated at daily frequency and used for truly real-time policy analysis. This represents a major advantage relative to existing macro-based estimates, which are only available with a lag and may be subject to significant data revisions.

5.5 Projections of Expected Inflation and the Natural Rate

Given the debate among researchers, investors, and policymakers about the persistence of the recent rise in interest rates, we analyze the outlook for long-term inflation expectations and the natural rate based on our preferred CLR-L model. We follow the approach of Christensen et al. (2015) and simulate 10,000 factor paths over a ten-year horizon conditioned on the shapes of the nominal and real yield curves and investors' embedded forward-looking expectations as of the end of our sample (that is, using estimated state variables and factor dynamics as of June 28, 2024). The simulated factor paths are then converted into forecasts of ten-



(a) Ten-year expected inflation



(b) r_t^*

Figure 18: **Ten-Year Projections**

year expected inflation and r_t^*. Figure 18 shows the median projection and the 5th and 95th percentile values for the simulated ten-year expected inflation and natural rate over a ten-year forecast horizon.[27]

The median projections of both ten-year expected inflation and r_t^* show essentially no

[27]Note that the lines do not represent paths from a single simulation run over the forecast horizon; instead, they delineate the distribution of all simulation outcomes at a given point in time.

change in either measure over the coming decade. We note that for the projected ten-year expected inflation the 90 percent confidence band is fairly narrow, even at the ten-year projection horizon, consistent with the stable in-sample ten-year expected inflation series. For the r_t^* estimate the projected values are notably more uncertain. The upper 95th percentile continues to rise rapidly reaching the high levels last seen in the mid-1990s towards the end of the projection period, while the lower 5th percentile represents outcomes with the natural rate returning to the near-zero values prevailing in the decade before the pandemic.

We take the stable projected long-term expected inflation as additional strong evidence that long-term inflation expectations in Canada continue to remain anchored close to the central bank target despite the recent spell of highly elevated inflation. Furthermore, the fact that the 90% confidence band of both ten-year expected inflation and r_t^* are on par with the historical variation in each respective series since 1996 suggests that the model's estimated factor dynamics imply a realistic amount of uncertainty about investors' long-term inflation expectations and the natural real rate.

This exercise also demonstrates that our preferred CLR-L model is able to simultaneously imply modest uncertainty about long-term inflation expectations, while allowing for quite significant uncertainty about the medium-term steady state real rate as reflected in our definition r_t^*. Mapping to a Taylor-based rule for monetary policy, these findings imply little uncertainty about long-run inflation, or π^*, in Canada, but much uncertainty about the neutral level of the overnight policy rate thanks to the persistent and significant trends in r_t^*.

Finally, our market-based estimate of r_t^* remains relevant to the debate about the source of the persistent trends in the equilibrium real rate. In particular, although our measure of the real rate fluctuated a bit at the start of the GFC, our average r_t^* estimate appears to be dominated by deeper and more structural persistent trends. This suggests that flight-to-safety and safety premium explanations of the trends in the equilibrium real rate, in particular following the GFC, are unlikely to be key drivers of the downtrend in global interest rates in the decades prior to the pandemic (as proposed by Hall, 2016, among others). Instead, our estimate appears more broadly consistent with many of the explanations that attribute the decline in the natural rate to real-side fundamentals such as changing demographics (e.g., Carvalho et al., 2016, Favero et al., 2016, and Gagnon et al., 2016). Lastly, the notable uptick in 2022 along with the sharp increases in global long-term interest rates suggests that, even with inflation brought back under control, real interest rates in Canada may not return to their pre-pandemic lows, which contrasts with the view of Blanchard (2023).

6 International Spillover of Unconventional Monetary Policy

To further demonstrate the potential usefulness of our model for policymakers and investors, we examine the spillover to Canadian bond markets from foreign unconventional monetary policies enacted in response to the GFC of 2008-2009.[28] Given the absence of any domestic Canadian unconventional monetary policies during that period, Canada is a useful laboratory for studying such international spillover effects.[29]

To the best of our knowledge, we are the first to examine the impact of foreign QE announcements on domestic BEI and underlying inflation expectations and risk premiums in a major advanced economy. In addition, we quantify their effects on Canadian nominal yields and their embedded risk premiums as others have done; see Bauer and Neely (2014, henceforth BN) for an example.

As this exercise is intended to be more illustrative than comprehensive, we limit our focus to the QE events considered in Christensen and Rudebusch (2012, henceforth CR). Their study represents a useful benchmark as it offers an international perspective by studying the domestic bond market reactions to U.S. Federal Reserve and Bank of England announcements regarding their respective QE programs launched as part of the early response to the financial crisis.[30] Moreover, that study relies on dynamic term structure models of nominal yields estimated at a daily frequency for its assessment. Hence, we provide an extension by estimating our preferred shadow-rate B-CLR-L model of nominal and real yields using daily data. We note that, although the policy rate in Canada never went below 0.5 percent during the period examined here, we rely on the shadow-rate specification of our preferred CLR-L model for this exercise to account for the possibility that investors may have expected Canadian short-term nominal interest rates to be constrained by the ZLB similar to their U.S counterparts.[31]

We begin by analyzing spillover effects from the Fed's first QE program followed by a similar analysis of spillovers from QE announcements made by the Bank of England.

6.1 U.S. QE

In late 2008, the Federal Reserve lowered its target policy rate—the overnight federal funds rate—effectively to its ZLB. Given a deteriorating outlook for economic growth and a perceived threat of price deflation, the Fed began to purchase longer-term securities to push down bond yields and provide additional monetary policy stimulus to the economy.

[28]We thank an anonymous referee for encouraging this analysis.

[29]Christensen and Zhang (2024) examine international spillover effects in Sweden during the 2015-2019 period when both the domestic Swedish Riksbank and the neighboring European Central Bank were operating simultaneous QE programs.

[30]Glick and Leduc (2012) analyze international effects of U.S. and U.K. QE announcements, but limit their focus to exchange rates and energy prices.

[31]See online Appendix F for details of our preferred shadow-rate B-CLR-L model.

To shed some light on the various channels through which these foreign bond purchases may have affected Canadian bond prices, we examine the responses of Canadian bond yields using an event study methodology as in CR. Specifically, we quantify the effects as the changes in Canadian interest rates over two-day intervals around announcements of future bond purchases.[32] Although widely used, this is an imperfect technique. We use a two-day window to allow investors in the Canadian bond market to fully digest the news reflected in each announcement and determine its implications for the Canadian economy and monetary policy. The drawback of this choice is that other news may have been released that significantly affected Canadian interest rates and obscured the effects we are trying to assess.[33]

6.1.1 Nominal Yield Decompositions

We estimate the effect of the Fed's QE announcements on expected short-term interest rates and inflation as well as associated risk premiums with our preferred B-CLR-L model estimated using daily data through the end of June 2024. We focus on the model decomposition of the reaction of the ten-year nominal yield and BEI rate over two-day periods.[34]

Recall that nominal term premiums are defined as the difference in expected nominal return between a buy-and-hold strategy for a τ-year nominal bond and an instantaneous rollover strategy at the risk-free nominal short rate r_t^N

$$TP_t^N(\tau) = y_t^N(\tau) - \frac{1}{\tau}\int_t^{t+\tau} E_t^{\mathbb{P}}[r_s^N]ds.$$

Based on this formula, our preferred B-CLR-L model is used to decompose the Canadian ten-year nominal zero-coupon yield into three components:

(i) the estimated average expected nominal short rate until maturity;

(ii) the nominal term premium defined as the difference between the model fitted nominal yield and the average expected nominal short rate; and

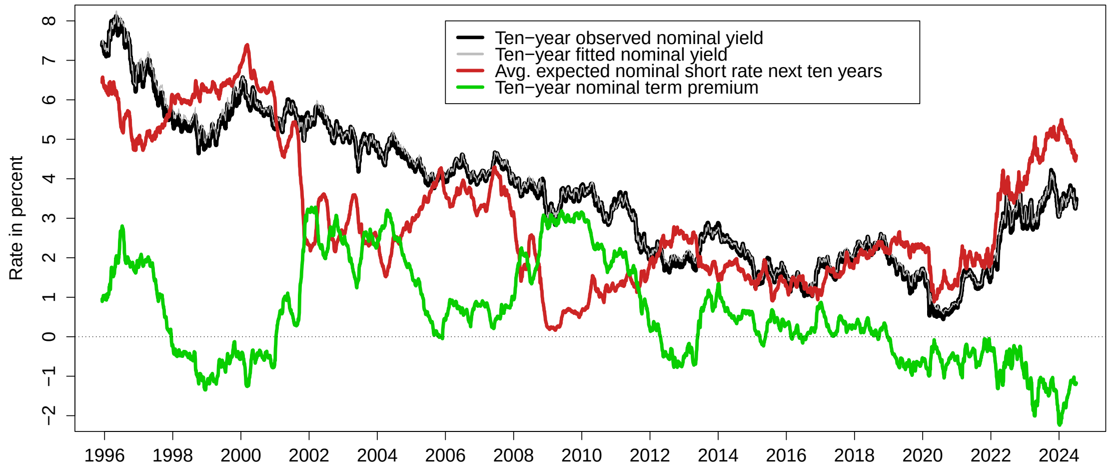
(iii) a residual that reflects variation not accounted for by the model.

This breakdown of the ten-year nominal yield for the full sample is shown in Figure 19(a).

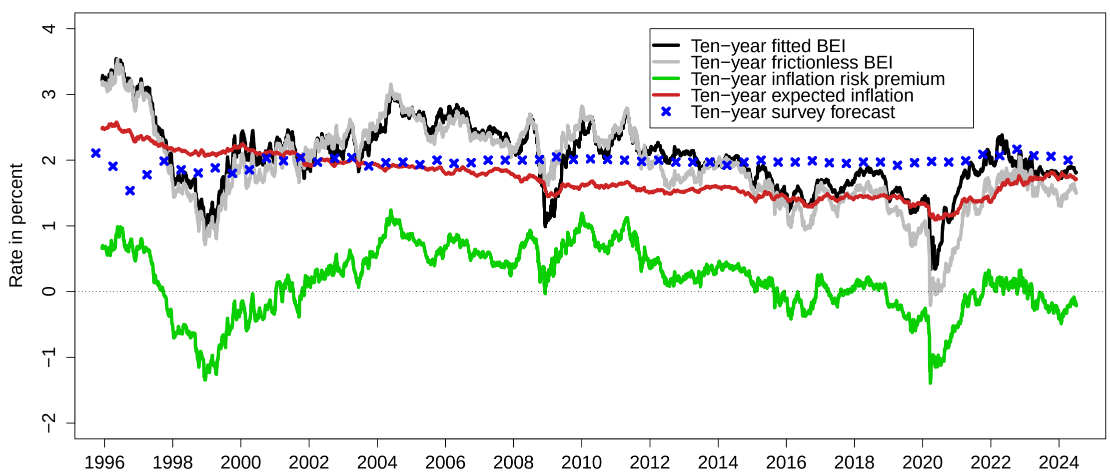
[32]The list of U.S. and U.K. announcements analyzed can be found in online Appendix D. It also reports the reaction of a representative set of nominal yields and BEI rates during the event windows.

[33]We did an extensive search of major Canadian news releases and policy announcements and failed to find any that overlapped with our foreign event windows.

[34]Ten-year yields are commonly used as the benchmark long-term yield in most government bond markets, including Canada. They are also key long-term rates of interest for monetary policy, and have served as the most popular maturity for studies of financial market reactions to unconventional monetary policies. For examples, see Gagnon et al. (2011), Christensen and Rudebusch (2012), Christensen and Krogstrup (2019), and Christensen and Spiegel (2022), among numerous others.



(a) Ten-year nominal yield decomposition



(b) Ten-year BEI decomposition

Figure 19: **Ten-Year Nominal Yield and BEI Decompositions**

Table 12 reports the result of these decompositions for our eight U.S. QE announcements. Overall, these indicate small changes in the nominal short-rate expectations component—accounting, at most, for about a third of the total reduction. Consequently, signaling effects appear secondary in the declines in Canadian nominal bond yields around these announcements. This contrasts with the findings of CR and Bauer and Rudebusch (2014), both of which find significant signaling effects in the U.S. Treasury bond market reaction to these events. More importantly, these results also contrast with those reported by BN, who consider a slightly different set of event windows and use term structure models of Canadian

Event		Avg. short rate next ten years	Ten-year term premium	Residual	Ten-year yield
I	Nov. 24, 2008	65	305	1	371
	Nov. 26, 2008	60	302	-6	357
	Change	**-5**	**-3**	**-7**	**-15**
II	Nov. 28, 2008	55	301	-4	353
	Dec. 2, 2008	49	293	-5	337
	Change	**-6**	**-8**	**-1**	**-16**
III	Dec. 15, 2008	34	298	-3	329
	Dec. 17, 2008	29	288	-7	310
	Change	**-5**	**-10**	**-4**	**-19**
IV	Jan. 27, 2009	21	300	-7	315
	Jan. 29, 2009	24	307	1	332
	Change	**2**	**7**	**8**	**17**
V	Mar. 17, 2009	18	294	-7	305
	Mar. 19, 2009	16	285	-18	283
	Change	**-2**	**-9**	**-11**	**-22**
VI	Aug. 11, 2009	70	302	-5	368
	Aug. 13, 2009	70	301	-2	368
	Change	**-1**	**-1**	**2**	**1**
VII	Sep. 22, 2009	65	299	-2	361
	Sep. 24, 2009	64	298	-4	358
	Change	**-1**	**-1**	**-1**	**-3**
VIII	Nov. 3, 2009	58	309	-4	363
	Nov. 5, 2009	59	313	1	373
	Change	**2**	**4**	**4**	**10**
Total net change		**-15**	**-21**	**-10**	**-46**

Table 12: **Decomposition of Two-Day Responses of Canadian Ten-Year Nominal Yield to U.S. QE Announcements**
The decomposition of two-day responses of the Canadian ten-year nominal government bond yield on eight U.S. QE announcement dates into changes in (i) the average expected nominal short rate over the next ten years, (ii) the ten-year nominal term premium, and (iii) the unexplained residual based on the preferred CLR-L model estimated with daily data. All numbers are measured in basis points.

nominal yields to decompose the changes in the ten-year nominal yield into changes in the expected nominal short rate and the nominal term premium. They interpret their results to suggest significant signaling spillover effects onto the Canadian government bond market from the U.S. QE events. Instead, our results suggest declines in the nominal term premiums to be the primary source behind the observed strong market reaction. Finally, the unexplained residuals account for another 20 percent of the market reaction, underscoring that the outcomes of the decompositions are associated with some uncertainty. We take this evidence to imply that Canadian bond markets were affected by these U.S. policy actions mainly through a portfolio re-balance channel; see Christensen and Krogstrup (2019) for a discussion. This is also consistent with the results of Dahlhaus et al. (2018), who use a FAVAR model to examine the international spillover effects on the broader Canadian economy from the U.S. QE programs and find that they operated mainly through a financing channel. Furthermore, our results are in line with the analysis in Neely (2015), which suggests that simple portfolio balance effects may have played an important role as an international spillover transmission mechanism of unconventional monetary policy to Canada and other advanced economies.

Another approach in the literature uses vector autoregressions (VAR) to estimate the yield impulse responses to changes in the quantity of asset purchases by the Federal Reserve. Kabaca and Tuzcuoglu (2022) use a VAR to show that a 1 percent asset purchase announcement by the U.S. Federal Reserve (as a share of U.S. GDP) is followed by a (median) ten basis point decline in the ten-year Canadian yield and a two basis point decline in the one-year Canadian yield. Similarly, Fratzscher et al. (2018) include Canada in a basket of advanced economies and show that QE in the U.S. significantly lowered yields in this set of advanced economies.

6.1.2 BEI Decompositions

Building on equation (6), the B-CLR-L model allows us to make a decomposition of the τ-year BEI as follows:

$$BEI_t = \pi_t^e(\tau) + \psi_t(\tau) - \Psi_t(\tau),$$

where the τ-year constant maturity RRB liquidity premium, $\Psi_t(\tau)$, is given by the difference between the fitted BEI and the corresponding frictionless BEI implied by our preferred B-CLR-L model.

While the result of this decomposition of BEI at the ten-year maturity for the full sample is shown in Figure 19(b), the results of the ten-year BEI decomposition on the eight U.S. announcement dates are reported in Table 13. Here, we see only very small reactions in the estimated inflation expectations. Instead, the market reactions are driven by a mix of the inflation and RRB liquidity risk premiums. On net, the inflation risk premium increases a bit, while the negative of the RRB liquidity risk premium declines, i.e., the liquidity risk premiums increase as well. This suggests that investors grew somewhat more fearful about upside risks to inflation, but clearly not to the point that they felt compelled to raise their expectations for inflation. Furthermore, the results indicate that this happened in the context of somewhat deteriorating financial market conditions as reflected in the uptick in the liquidity premiums they demand for holding RRBs.

Event		Ten-year exp. inflation	Ten-year IRP	neg. RRB liq. premium	Ten-year BEI
I	Nov. 24, 2008	162	7	-44	125
	Nov. 26, 2008	160	2	-48	114
	Change	**-2**	**-5**	**-4**	**-11**
II	Nov. 28, 2008	159	0	-50	108
	Dec. 2, 2008	156	-3	-53	100
	Change	**-3**	**-3**	**-3**	**-8**
III	Dec. 15, 2008	152	9	-56	105
	Dec. 17, 2008	149	19	-61	107
	Change	**-2**	**9**	**-5**	**2**
IV	Jan. 27, 2009	148	31	-62	117
	Jan. 29, 2009	150	33	-60	123
	Change	**1**	**2**	**2**	**6**
V	Mar. 17, 2009	147	44	-61	129
	Mar. 19, 2009	145	54	-64	135
	Change	**-1**	**10**	**-3**	**6**
VI	Aug. 11, 2009	163	83	-27	219
	Aug. 13, 2009	163	82	-27	218
	Change	**0**	**0**	**0**	**0**
VII	Sep. 22, 2009	160	77	-26	211
	Sep. 24, 2009	160	78	-27	211
	Change	**0**	**2**	**-1**	**1**
VIII	Nov. 3, 2009	158	95	-23	229
	Nov. 5, 2009	158	90	-22	226
	Change	**1**	**-5**	**2**	**-3**
Total net change		**-6**	**10**	**-12**	**-9**

Table 13: **Decomposition of Two-Day Responses of Canadian Ten-Year BEI to U.S. QE Announcements**

The decomposition of one-day responses of the Canadian ten-year BEI rate on eight U.S. QE announcement dates into changes in (i) the average expected inflation over the next ten years, (ii) the ten-year inflation risk premium, and (iii) the negative of the RRB liquidity premium based on the preferred CLR-L model estimated with daily data. All numbers are measured in basis points.

6.2 U.K. QE

The Bank of England first announced its asset purchases in February 2009 to provide additional stimulus during the GFC. By March 2009, the facilities were operational, and the Bank of England had set the policy rate to 0.5 percent, close to the effective ZLB. Despite substantial stimulus coming from fiscal policy and the main interest rate being cut close to zero, the Bank of England decided that QE would help stimulate nominal spending and stabilize

Event		Avg. short rate next ten years	Ten-year term premium	Residual	Ten-year yield
I	Feb. 10, 2009	22	305	-2	325
	Feb. 11, 2009	22	303	-4	321
	Change	**0**	**-2**	**-2**	**-4**
II	Mar. 4, 2009	21	292	-1	312
	Mar. 5, 2009	20	289	-6	303
	Change	**-1**	**-3**	**-5**	**-10**
III	May 6, 2009	25	307	-7	325
	May 7, 2009	26	309	-2	333
	Change	**1**	**2**	**5**	**8**
IV	Aug. 5, 2009	69	308	-1	376
	Aug. 6, 2009	70	306	-4	372
	Change	**0**	**-2**	**-2**	**-4**
V	Nov. 4, 2009	59	311	-1	368
	Nov. 5, 2009	59	313	1	373
	Change	**1**	**2**	**2**	**5**
VI	Feb. 3, 2010	69	295	3	367
	Feb. 4, 2010	70	293	-2	361
	Change	**0**	**-2**	**-4**	**-6**
VII	Oct. 5, 2011	163	60	1	225
	Oct. 6, 2011	163	65	5	234
	Change	**1**	**4**	**4**	**9**
Total net change		**2**	**0**	**-4**	**-2**

Table 14: **Decomposition of One-Day Responses of Canadian Ten-Year Nominal Yield to U.K. QE Announcements**
The decomposition of one-day responses of the Canadian ten-year nominal government bond yield on seven U.K. QE announcement dates into changes in (i) the average expected nominal short rate over the next ten years, (ii) the ten-year nominal term premium, and (iii) the unexplained residual based on the preferred CLR-L model estimated with daily data. All numbers are measured in basis points.

inflation relative to its target.[35]

6.2.1 Nominal Yield Decompositions

To further illuminate the effects of U.K. QE announcements on Canadian yields, we again use our preferred B-CLR-L model to decompose the ten-year nominal yield changes into the estimated average expected nominal short rate until maturity, the nominal term premium, and a residual.

Table 14 presents the results for these decompositions for our seven U.K. announcement

[35]For more details regarding the initial implementation of QE in the U.K.; see Benford et al. (2009).

days. We find that for all days besides the May 2009 announcement, the change in the average expected nominal short rate is close to zero. This result is similar to what we find for the U.S., indicating that the signaling effect plays a minor role in the changes in Canadian nominal bond yields around these announcement days. While the net effect on the nominal term premium is smaller than the expectations component, the change within each event tends to be larger for the term premium than the expectations component. Taken together with the evidence from the U.S. data, it appears that foreign announcements regarding QE impact domestic yields through a term premium channel rather than a signaling channel, a result that contrasts with findings of U.S. Federal Reserve announcements on the U.S. Treasury bond market, but is consistent with the U.K. findings reported by CR. Notably, the residual component has the most considerable net change indicating that the results for the U.K. come with more uncertainty than observed for the U.S. case.

6.2.2 BEI Decompositions

Finally, we examine the B-CLR-L model-implied decompositions of the response of the Canadian ten-year BEI to the U.K. QE announcements. These results are reported in Table 15. On net, the change in the ten-year BEI is four basis points indicating that U.K. QE announcements had a minimal effect on inflation expectations in Canada. Of these four basis points, most are ascribed to changes in the ten-year inflation risk premium, and only one comes from changes in the ten-year expected inflation. Across all announcements, changes in inflation expectations and inflation risk components for the ten-year BEI tend to be zero or close to it, except for an increase in the inflation risk premium of three basis points around the March and May 2009 Bank of England QE announcements. Overall, these small responses indicate little, if any, spillover effects from the QE announcements in the U.K. to Canadian inflation expectations. Together with the U.S. Federal Reserve QE announcement effects on Canadian yields, these results are the first to document that asset purchase announcements appear to have limited effects on foreign inflation expectations and inflation risk premiums despite the significant spillover effects on nominal yields.

Our results are thus comparable to other findings in the literature on the effects of U.S. QE announcements on Canadian yields. However, relative to the earlier literature, we consider movements in breakeven inflation and inflation risk premiums on QE announcement days. We find that QE spillovers to these variables are smaller than for the nominal yields. Additionally, we provide the first analysis of the effects of U.K. QE announcements on Canadian yields and find U.S. announcements tend to have a relatively larger effect on Canadian yields.

Event		Ten-year exp. inflation	Ten-year IRP	neg. RRB liq. premium	Ten-year BEI
I	Feb. 10, 2009	149	48	-61	136
	Feb. 11, 2009	149	49	-61	137
	Change	**0**	**1**	**0**	**1**
II	Mar. 4, 2009	147	28	-59	117
	Mar. 5, 2009	146	31	-61	117
	Change	**-1**	**3**	**-2**	**0**
III	May 6, 2009	151	50	-50	151
	May 7, 2009	152	54	-51	155
	Change	**1**	**3**	**0**	**4**
IV	Aug. 5, 2009	163	84	-27	221
	Aug. 6, 2009	163	85	-27	221
	Change	**0**	**0**	**0**	**0**
V	Nov. 4, 2009	158	92	-22	228
	Nov. 5, 2009	158	90	-22	226
	Change	**0**	**-2**	**0**	**-1**
VI	Feb. 3, 2010	162	103	-21	243
	Feb. 4, 2010	162	102	-22	242
	Change	**0**	**0**	**-1**	**-1**
VII	Oct. 5, 2011	155	41	6	202
	Oct. 6, 2011	155	40	8	203
	Change	**0**	**-1**	**2**	**1**
Total net change		**1**	**4**	**0**	**4**

Table 15: **Decomposition of One-Day Responses of Canadian Ten-Year BEI to U.K. QE Announcements**
The decomposition of one-day responses of the Canadian ten-year BEI rate on seven U.K. QE announcement dates into changes in (i) the average expected inflation over the next ten years, (ii) the ten-year inflation risk premium, and (iii) the negative of the RRB liquidity premium based on the preferred CLR-L model estimated with daily data. All numbers are measured in basis points.

7 Conclusion

Given the historic persistent trends in yields in recent decades, many researchers have investigated the factors affecting interest rates. Much of this work has focused on the steady-state level of the safe short-term real interest rate based on *macroeconomic* models and U.S. data. Instead, we consider a finance-based decomposition of interest rates from empirical dynamic term structure models estimated on a sample of standard Canadian nominal bond yields combined with the prices of Canadian inflation-indexed bonds known as RRBs. By adjusting for both RRB liquidity premiums and nominal and real term premiums, we uncover investors' expectations for the underlying frictionless real short rate for the five-year period starting five years ahead. This measure of the natural rate of interest exhibits a persistent gradual decline over the 1996-2021 period that accounts for about two thirds of the general decline in Canadian yields. It also accounts for much of the sharp spike in Canadian interest rates following the economic reopening after the COVID-19 pandemic. Furthermore, model projections that exploit the estimated factor dynamics suggest that this measure of the natural rate is likely to remain near its current higher level for the foreseeable future.

Equally important, our joint model of nominal and real yields also produces estimates of investors' inflation expectations and associated inflation risk premiums. Here, we find that a modest decline in investors' long-term inflation expectations of about 0.7 percentage point in the 1996-2021 period contributed further to the decline in Canadian long-term nominal bond yields prior to the pandemic. Moreover, about half of that decline has been retraced in the years since then in response to the transitory spell of high inflation in the 2022-2023 period. Crucially, our model-based projections as of the end of June 2024 indicate that investors' long-term inflation expectations are likely to experience a further modest increase in the coming years, which will make them approach the Bank of Canada's 2 percent inflation target. We interpret this as a sign that Canadian inflation expectations are very well anchored near the central banks's desired level.

For monetary policy analysis, the proposed finance-based approach to estimate both inflation expectations and the natural real rate also offers notable advantages as they are available in real-time and not subject to data revision unlike estimates based on macroeconomic data. Since our measures are based on the forward-looking information priced into the active RRB market and can be updated at daily frequency, they could serve as an important input for real-time monetary policy analysis. As one envisioned application, we follow the literature and calculate a measure of the stance of monetary policy that suggests that monetary policy in Canada remains restrictive at the end of our sample. We demonstrate two further applications by first examining the market reaction to the surprise announcement by the Canadian Finance Department to permanently halt RRB issuance made public on November 3, 2022. We document that the RRB market has continued to function on par with the historical experience prior to the announcement. This supports our continued usage of the RRB data. We then proceed to an examination of the international spillovers to Canadian interest rates from the first round of U.S. and U.K. QE programs launched around the peak of the GFC. Using high-frequency daily analysis, we document significant effects from these programs on bond risk premiums, which points to a portfolio rebalancing effect as an international QE transmission mechanism.

For future research, our methods can be further expanded along an international dimension. With a significant degree of capital mobility, the natural rate will depend on global saving and investment, so the joint modeling of inflation-indexed bonds in several countries could be informative similar to HLW. Finally, our measure could be incorporated into an expanded joint macroeconomic and finance analysis—particularly with an eye towards further understanding the determinants of the persistent trends in interest rates. In this regard, Bauer and Rudebusch (2020) show that accounting for fluctuations in the natural rate substantially improves long-range interest rate forecasts and helps predict excess bond returns.

References

Abrahams, Michael, Tobias Adrian, Richard K. Crump, Emanuel Moench, and Rui Yu, 2016, "Decomposing Real and Nominal Yield Curves," *Journal of Monetary Economics*, Vol. 84, 182-200.

Adjalala, Frida, Felipe Alves, H'el'ene Desgagnés, Wei Dong, Dmitry Matveev, and Laure Simon, 2024, "Assessing the U.S. and Canadian Neutral Rates: 2024 Update," Bank of Canada Staff Analytical Note No. 2024-09.

Andreasen, Martin M., Jens H. E. Christensen, and Simon Riddell, 2021, "The TIPS Liquidity Premium," *Review of Finance*, Vol. 25, No. 6, 1639-1675.

Andreasen, Martin M., Jens H. E. Christensen, and Glenn D. Rudebusch, 2019, "Term Structure Analysis with Big Data: One-Step Estimation Using Bond Prices," *Journal of Econometrics*, Vol. 212, 26-46.

Andreasen, Martin M. and Andrew Meldrum, 2019, "A Shadow Rate or a Quadratic Policy Rule? The Best Way to Enforce the Zero Lower Bound in the United States," *Journal of Financial and Quantitative Analysis*, Vol. 54, No. 5, 2261-2292.

Ang, Andrew, Geert Bekaert, and Min Wei, 2008, "The Term Structure of Real Rates and Expected Inflation," *Journal of Finance*, Vol. 63, No. 2, 797-849.

Azizova, Chinara, Jonathan Witmer, and Xu Zhang, 2024, "Assessing the Impact of the Bank of Canada's Government Bond Purchases," Bank of Canada Staff Discussion Paper No. 2024-5.

Bauer, Michael D. and Christopher J. Neely, 2014, "International Channels of the Fed's Unconventional Monetary Policy," *Journal of International Money and Finance*, Vol. 44, 24-46.

Bauer, Michael D. and Glenn D. Rudebusch, 2020, "Interest Rates Under Falling Stars," *American Economic Review*, Vol. 110, No. 5, 1316-1354.

Bauer, Michael D., Glenn D. Rudebusch, and Jing (Cynthia) Wu, 2012, "Correcting Estimation Bias in Dynamic Term Structure Models," *Journal of Business and Economic Statistics*, Vol. 30, No. 3, 454-467.

Bauer, Michael D., Glenn D. Rudebusch, and Jing (Cynthia) Wu, 2014, "Term Premia and Inflation Uncertainty: Empirical Evidence from an International Panel Dataset: Comment," *American Economic Review*, Vol. 104, No. 1, 323-337.

Benford, James, Stuart Berry, Kalin Nikolov, and Chris Young, 2009, "Quantitative Easing," *Bank of England Quarterly Bulletin*, 2009 Q2, 90-100.

Black, Fisher, 1995, "Interest Rates as Options," *Journal of Finance*, Vol. 50, No. 7, 1371-1376.

Blanchard, Olivier, 2023, *Fiscal Policy under Low Interest Rates*. MIT Press: Cambridge, Massachusetts.

Bolder, David J., Grahame Johnson, and Adam Metzler, 2004, "An Empirical Analysis of the Canadian Term Structure of Zero-Coupon Interest Rates," Bank of Canada Working Paper No. 2004-48.

Buraschi, Andrea and A. Jiltsov, 2005, "Inflation Risk Premia and the Expectations Hypothesis," *Journal of Financial Economics*, Vol. 75, 429-490.

Campbell, John Y., Robert J. Shiller, and Luis M. Viceira, 2009, "Understanding Inflation-Indexed Bond Markets," *Brookings Papers on Economic Activity*, Spring, 79-120.

Carvalho, Carlos, Andrea Ferrero, and Fernanda Nechio, 2016, "Demographics and Real Interest Rates: Inspecting the Mechanism," *European Economic Review*, Vol. 88, 208-226.

Christensen, Jens H. E., Francis X. Diebold, and Glenn D. Rudebusch, 2011, "The Affine Arbitrage-Free Class of Nelson-Siegel Term Structure Models," *Journal of Econometrics*, Vol. 164, No. 1, 4-20.

Christensen, Jens H. E. and Signe Krogstrup, 2019, "Transmission of Quantitative Easing: The Role of Central Bank Reserves," *Economic Journal*, Vol. 129, 249-272.

Christensen, Jens H. E., Jose A. Lopez, and Glenn D. Rudebusch, 2010, "Inflation Expectations and Risk Premiums in an Arbitrage-Free Model of Nominal and Real Bond Yields," *Journal of Money, Credit and Banking*, Supplement to Vol. 42, No. 6, 143-178.

Christensen, Jens H. E., Jose A. Lopez, and Glenn D. Rudebusch, 2012, "Extracting Deflation Probability Forecasts from Treasury Yields," *International Journal of Central Banking*, Vol. 8, No. 4, 21-60.

Christensen, Jens H. E., Jose A. Lopez, and Glenn D. Rudebusch, 2014, "Do Central Bank Liquidity Facilities Affect Interbank Lending Rates?," *Journal of Business and Economic Statistics*, Vol. 32, No. 1, 136-151.

Christensen, Jens H. E., Jose A. Lopez, and Glenn D. Rudebusch, 2015, "A Probability-Based Stress Test of Federal Reserve Assets and Income," *Journal of Monetary Economics*, Vol. 73, 26-43.

Christensen, Jens H. E. and Nikola Mirkov, 2022, "The Safety Premium of Safe Assets," Working Paper 2019-28, Federal Reserve Bank of San Francisco.

Christensen, Jens H. E. and Sarah Mouabbi, 2024, "The Natural Rate of Interest in the Euro Area: Evidence from Inflation-Indexed Bonds," Working Paper 2024-08, Federal Reserve Bank of San Francisco.

Christensen, Jens H. E. and Glenn D. Rudebusch, 2012, "The Response of Interest Rates to U.S. and U.K. Quantitative Easing," *Economic Journal*, Vol. 122, F385-F414.

Christensen, Jens H. E. and Glenn D. Rudebusch, 2015, "Estimating Shadow-Rate Term Structure Models with Near-Zero Yields," *Journal of Financial Econometrics*, Vol. 13, No. 2, 226-259.

Christensen, Jens H. E. and Glenn D. Rudebusch, 2019, "A New Normal for Interest Rates? Evidence from Inflation-Indexed Debt," *Review of Economics and Statistics*, Vol. 101, No. 5, 933-949.

Christensen, Jens H. E. and Mark M. Spiegel, 2022, "Monetary Reforms and Inflation Expectations in Japan: Evidence from Inflation-Indexed Bonds," *Journal of Econometrics*, Vol. 231, No. 2, 410-431.

Christensen, Jens H. E. and Xin Zhang, 2024, "Quantitative Easing, Bond Risk Premia and the Exchange Rate in a Small Open Economy," Working Paper 2024-13, Federal Reserve Bank of San Francisco.

Coté, Agathe, Jocelyn Jacob, John Nelmes, and Miles Whittingham, 1996, "Inflation Expectations and Real Return Bonds," *Bank of Canada Review* (Summer), 40-53.

Dahlhaus, Tatjana, Kristina Hess, and Abeer Reza, 2018, "International Transmission Channels of U.S. Quantitative Easing: Evidence from Canada," *Journal of Money, Credit and Banking*, Vol. 50, No. 2-3, 545-563.

D'Amico, Stefania, Don H. Kim, and Min Wei, 2018, "Tips from TIPS: The Informational Content of Treasury Inflation-Protected Security Prices," *Journal of Financial and Quantitative Analysis*, Vol. 53, No. 1, 243-268.

Davis, Josh, Cristian Feunzalida, Leon Huetsch, Benjamin Mills, and Alan M. Taylor, 2024, "Global Natural Rates in the Long Run: Postwar Macro Trends and the Market-Implied r^* in 10 Advanced Economies," *Journal of International Economics*, Vol. 149, 103919.

Del Negro, Marco, Domenico Giannone, Marc P. Giannoni, and Andrea Tambalotti, 2019, "Global Trends in Interest Rates," *Journal of International Economics*, Vol. 118, 248-262.

Diez de los Rios, Antonio, 2015, "A New Linear Estimator for Gaussian Dynamic Term Sructure Models," *Journal of Business and Economic Statistics*, Vol. 33, No. 1, 282-295.

Duffee, Gregory R., 2002, "Term Premia and Interest Rate Forecasts in Affine Models," *Journal of Finance*, Vol. 57, No. 1, 405-443.

Favero, Carlo A., Arie E. Gozluklu, and Haoxi Yang, 2016, "Demographics and the Behavior of Interest Rates " *IMF Economic Review* Vol. 64, No. 4, 732-776.

Ferreira, Thiago R. T. and Samer Shousha, 2023, "Determinants of Global Neutral Interest Rates," *Journal of International Economics*, Vol. 145, 103833.

Finlay, Richard and Sebastian Wende, 2012, "Estimating Inflation Expectations with a Limited Number of Inflation-Indexed Bonds," *International Journal of Central Banking*, Vol. 8, No. 2, 111-142.

Fontaine, Jean-Sébastien and René Garcia, 2012, "Bond Liquidity Premia," *Review of Financial Studies*, Vol. 25, No. 4, 1207-1254.

Fratzscher Marcel, Marco Lo Duca, and Roland Straub, 2018, "On the International Spillovers of US Quantitative Easing," *Economic Journal*, Vol. 128, No. 608, 330-377.

Gagnon, Etienne, Benjamin K. Johannsen, and David Lopez-Salido, 2016, "Understanding the New Normal: The Role of Demographics," Finance and Economics Discussion Series 2016-080. Washington: Board of Governors of the Federal Reserve System, http://dx.doi.org/10.17016/FEDS.2016.080.

Gagnon, Joseph, Matthew Raskin, Julie Remache, and Brian Sack, 2011, "The Financial Market Effects of the Federal Reserve's Large-Scale Asset Purchases," *International Journal of Central Banking*, Vol. 7, No. 1, 3-43.

Glick, Reuven and Sylvain Leduc, 2012, "Central Bank Announcements of Asset Purchases and the Impact on Global Financial and Commodity Markets," *Journal of International Money and Finance*, Vol. 31, No. 8, 2078-2101.

Grishchenko, Olesya V. and Jing-Zhi Huang, 2013, "Inflation Risk Premium: Evidence from the TIPS Market," *Journal of Fixed Income*, Vol. 22, No. 4, 5-30.

Gürkaynak, Refet S., Brian Sack, and Jonathan H. Wright, 2007, "The U.S. Treasury Yield Curve: 1961 to the Present," *Journal of Monetary Economics*, Vol. 54, No. 8, 2291-2304.

Gürkaynak, Refet S., Brian Sack, and Jonathan H. Wright, 2010, "The TIPS Yield Curve and Inflation Compensation," *American Economic Journal: Macroeconomics*, Vol. 2, No. 1, 70-92.

Hall, Robert E., 2016, "The Role of the Growth of Risk-Averse Wealth in the Decline of the Safe Real Interest Rate," Manuscript, Stanford University.

Holston, Kathryn, Thomas Laubach, and John C. Williams, 2017, "Measuring the Natural Rate of Interest: International Trends and Determinants," *Journal of International Economics*, Vol. 108, 559-575.

Hördahl, Peter and Oreste Tristani, 2014, "Inflation Risk Premia in the Euro Area and the United States," *International Journal of Central Banking*, Vol. 10, No. 3, 1-47.

Kabaca, Serdar and Kerem Tuzcuoglu, 2022, "International Transmission of Quantitative Easing Policies: Evidence from Canada," *Bank of Canada Staff Working Paper* 2022-30.

Kearns, Jonathan, Andreas Schrimpf, and Fan Dora Xia, 2020, "Explaining Monetary Spillovers: The Matrix Reloaded," CEPR Discussion Paper No. 15006.

Kim, Don H. and Kenneth J. Singleton, 2012, "Term Structure Models and the Zero Bound: An Empirical Investigation of Japanese Yields," *Journal of Econometrics*, Vol. 170, No. 1, 32-49.

Kozicki, Sharon, 2024, "Exceptional Policies for an Exceptional Time: From Quantitative Easing to Quantitative Tightening," Speech at the Canadian Association for Business Economics, Ottawa, Ontario, June 13, 2024.

Laubach, Thomas and John C. Williams, 2003, "Measuring the Natural Rate of Interest," *Review of Economics and Statistics*, Vol. 85, No. 4, 1063-1070.

Laubach, Thomas and John C. Williams, 2016, "Measuring the Natural Rate of Interest Redux," *Business Economics*, Vol. 51, No. 2, 57-67.

Neely, Christopher J., 2015, "Unconventional Monetary Policy had Large International Effects," *Journal of Banking and Finance*, Vol. 52, 101-111.

Nelson, Charles R. and Andrew F. Siegel, 1987, "Parsimonious Modeling of Yield Curves," *Journal of Business*, Vol. 60, No. 4, 473-489.

Pflueger, Carolin E. and Luis M. Viceira, 2016, "Return Predictability in the Treasury Market: Real Rates, Inflation, and Liquidity," in Handbook of Fixed-Income Securities, edited by P. Veronesi, Wiley, Hoboken, N.J., pp. 191-209 (Chapter 10).

Rachel, Lukasz, and Thomas D Smith, 2015, "Secular Drivers of the Global Real Interest Rate," Staff working paper 571, Bank of England.

Reid, Christopher, Frédéric Dion, and Ian Christensen, 2004, "Real Return Bonds: Monetary Policy Credibility and Short-Term Inflation Forecasting," *Bank of Canada Review* (Autumn), 15-26.

Rogoff, Kenneth, 2023, "Higher Interest rates Are Here to Stay," Project Syndicate, December 5, 2023, link: https://www.project-syndicate.org/commentary/era-of-low-interest-rates-has-come-to-an-end-by-kenneth-rogoff-2023-12

Sack, Brian and Robert Elsasser, 2004, "Treasury Inflation-Indexed Debt: A Review of the U.S. Experience," *Federal Reserve Bank of New York Economic Policy Review*, Vol. 10, No. 1, 47-63.

Swanson, Eric T. and John C. Williams, 2014, "Measuring the Effect of the Zero Lower Bound on Medium- and Longer-Term Interest Rates," *American Economic Review*, Vol. 104, No. 10, 3154-3185.

Wright, Jonathan H., 2011, "Term Premia and Inflation Uncertainty: Empirical Evidence from an International Panel Dataset," *American Economic Review*, Vol. 101, No. 4, 1514-1534.

Online Appendix

Accounting for Changes in Long-Term Interest Rates: Evidence from Canada

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This version: December 16, 2024.

Contents

A Identifying the Natural Rate of Interest with Bonds

In this appendix, we first describe how nominal and real bond yields can be decomposed into underlying nominal and real short-rate expectations components and residual nominal and real term premiums in a world without any frictions to trading. We then introduce our benchmark frictionless model of nominal and real yields. This model of frictionless dynamics is fundamental to our empirical analysis in the paper. We end the section by relating our frictionless model to the Gaussian canonical class of dynamic term structure models introduced in Dai and Singleton (2000).

A.1 Decomposing Nominal and Real Yields with Frictionless Models

We begin our analysis assuming a world in which there are no frictions to the trading of financial claims, i.e., there are no bid-ask spreads and any financial claim can be traded in arbitrarily small or large amounts without affecting its price. As a consequence, financial market prices contain no liquidity premiums as there is no liquidity risk to be rewarded. Under such ideal conditions, nominal and real yields vary either because fundamental factors in the economy have changed or because investors have altered their perceptions of, or aversions to, the risks that those economic fundamentals represent.

An arbitrage-free term structure model of nominal and real yields can be used to decompose nominal and real yields into the sum of the corresponding short-rate expectations and associated term premiums. We follow Merton (1974) and assume a continuum of nominal and real zero-coupon bonds exists with no frictions to their continuous trading. To begin the model description, define the nominal and real stochastic discount factors, denoted M_t^N and M_t^R, respectively. The no-arbitrage condition enforces a consistency of pricing for any security over time. Specifically, the price of a nominal bond that pays one dollar in τ years and the price of a real bond that pays one consumption unit in τ years must satisfy the conditions that

$$P_t^N(\tau) = E_t^{\mathbb{P}}\left[\frac{M_{t+\tau}^N}{M_t^N}\right] \quad \text{and} \quad P_t^R(\tau) = E_t^{\mathbb{P}}\left[\frac{M_{t+\tau}^R}{M_t^R}\right],$$

where $P_t^N(\tau)$ and $P_t^R(\tau)$ are the observed prices of the zero-coupon, nominal and real bonds for maturity τ at time t and $E_t^{\mathbb{P}}[.]$ is the conditional expectations operator under the real-world (or $\mathbb{P}$-) probability measure.

Our working definition of the equilibrium real rate of interest r_t^* is

$$r_t^* = \frac{1}{5}\int_{t+5}^{t+10} E_t^{\mathbb{P}}[r_{t+s}^R]ds, \tag{1}$$

that is, the average expected real short rate over a five-year period starting five years ahead where the expectation is with respect to the objective $\mathbb{P}$-probability measure. As noted in the paper, this 5yr5yr forward average expected real short rate should be little affected by short-term transitory shocks.

In the empirical analysis, we rely on the market prices of nominal and real bonds to construct this market-based measure of the natural rate. In doing so, it is important to acknowledge that financial market prices do not reflect objective $\mathbb{P}$-expectations as in equation (1). Instead, they reflect expectations adjusted with the premiums investors demand for being exposed to the underlying risks. We follow the usual empirical finance approach that models bond prices with latent factors, here denoted as X_t, and the assumption of no residual arbitrage opportunities.[1] We assume that X_t follows an affine Gaussian process with constant volatility, with dynamics in continuous time given by the solution to the following stochastic differential equation (SDE):

$$dX_t = K^{\mathbb{P}}(\theta^{\mathbb{P}} - X_t) + \Sigma dW_t^{\mathbb{P}},$$

where $K^{\mathbb{P}}$ is an $n \times n$ mean-reversion matrix, $\theta^{\mathbb{P}}$ is an $n \times 1$ vector of mean levels, Σ is an $n \times n$ volatility matrix, and $W_t^{\mathbb{P}}$ is an n-dimensional Brownian motion. The dynamics of the nominal and real stochastic discount factors are given by

$$\begin{aligned} dM_t^N/M_t^N &= -r_t^N dt - \Gamma_t' dW_t^{\mathbb{P}}, \\ dM_t^R/M_t^R &= -r_t^R dt - \Gamma_t' dW_t^{\mathbb{P}}, \end{aligned}$$

and the instantaneous, risk-free nominal and real short rates, r_t^N and r_t^R, are assumed affine in the state variables

$$\begin{aligned} r_t^N &= \delta_0^N + \delta_1^N X_t, \\ r_t^R &= \delta_0^R + \delta_1^R X_t, \end{aligned}$$

[1] Ultimately, of course, the behavior of the stochastic discount factor is determined by the preferences of the agents in the economy, as in, for example, Rudebusch and Swanson (2011).

where $\delta_0^N \in \mathbf{R}$, $\delta_0^R \in \mathbf{R}$, $\delta_1^N \in \mathbf{R}^n$ and $\delta_1^R \in \mathbf{R}^n$. The risk premiums, Γ_t, are also affine

$$\Gamma_t = \gamma_0 + \gamma_1 X_t,$$

where $\gamma_0 \in \mathbf{R}^n$ and $\gamma_1 \in \mathbf{R}^{n\times n}$.

Duffie and Kan (1996) show that these assumptions imply that zero-coupon nominal and real yields are also affine in X_t:

$$\begin{aligned} y_t^N(\tau) &= -\frac{1}{\tau}A^N(\tau) - \frac{1}{\tau}B^N(\tau)' X_t, \\ y_t^R(\tau) &= -\frac{1}{\tau}A^R(\tau) - \frac{1}{\tau}B^R(\tau)' X_t, \end{aligned}$$

where $A^N(\tau)$, $A^R(\tau)$, $B^N(\tau)$, and $B^R(\tau)$ are given as solutions to the following system of ordinary differential equations

$$\begin{aligned} \frac{dB^N(\tau)}{d\tau} &= -\delta_1^N - (K^{\mathbb{P}} + \Sigma\gamma_1)' B^N(\tau), \quad B^N(0) = 0, \\ \frac{dA^N(\tau)}{d\tau} &= -\delta_0^N + B^N(\tau)'(K^{\mathbb{P}}\theta^{\mathbb{P}} - \Sigma\gamma_0) + \frac{1}{2}\sum_{j=1}^{n}\left(\Sigma' B^N(\tau) B^N(\tau)'\Sigma\right)_{j,j}, \quad A^N(0) = 0, \\ \frac{dB^R(\tau)}{d\tau} &= -\delta_1^R - (K^{\mathbb{P}} + \Sigma\gamma_1)' B^R(\tau), \quad B^R(0) = 0, \\ \frac{dA^R(\tau)}{d\tau} &= -\delta_0^R + B^N(\tau)'(K^{\mathbb{P}}\theta^{\mathbb{P}} - \Sigma\gamma_0) + \frac{1}{2}\sum_{j=1}^{n}\left(\Sigma' B^R(\tau) B^R(\tau)'\Sigma\right)_{j,j}, \quad A^R(0) = 0. \end{aligned}$$

Thus, the $A^N(\tau)$, $A^R(\tau)$, $B^N(\tau)$, and $B^R(\tau)$ functions are calculated *as if* the dynamics of the state variables had a constant drift term equal to $K^{\mathbb{P}}\theta^{\mathbb{P}} - \Sigma\gamma_0$ instead of the actual $K^{\mathbb{P}}\theta^{\mathbb{P}}$ and a mean-reversion matrix equal to $K^{\mathbb{P}} + \Sigma\gamma_1$ as opposed to the actual $K^{\mathbb{P}}$.[2] The difference is determined by the risk premium Γ_t and reflects investors' aversion to the risks embodied in X_t.

Finally, we define the nominal and real term premiums as

$$TP_t^N(\tau) = y_t^N(\tau) - \frac{1}{\tau}\int_t^{t+\tau} E_t^{\mathbb{P}}[r_s^N]ds, \tag{2}$$

$$TP_t^R(\tau) = y_t^R(\tau) - \frac{1}{\tau}\int_t^{t+\tau} E_t^{\mathbb{P}}[r_s^R]ds. \tag{3}$$

[2]The probability measure with these alternative dynamics is frequently referred to as the risk-neutral, or $\mathbb{Q}$, probability measure since the expected return on any asset under this measure is equal to the risk-free real rate r_t that a risk-neutral investor would demand.

That is, the nominal term premium is the difference in expected nominal return between a buy and hold strategy for a τ-year nominal bond and an instantaneous rollover strategy at the risk-free nominal rate r_t^N. The real term premium has a similar interpretation, but in real terms. This model thus allows us to decompose nominal and real yields into their respective term premium and short-rate expectations components.

A.2 A Frictionless Arbitrage-Free Model of Nominal and Real Yields

Building on the insights from the general theoretical discussion in the previous section, we need an accurate model of the instantaneous nominal and real rate, r_t^N and r_t^R, in order to precisely measure nominal and real term premiums. With that goal in mind we choose to focus on the tractable affine dynamic term structure model of nominal and real yields introduced in Christensen et al. (2010, henceforth CLR) and briefly summarized below. We emphasize that even though the model is not formulated using the canonical form of affine term structure models introduced by Dai and Singleton (2000), it can be viewed as a restricted version of the canonical Gaussian model as we show in the next section.

The CLR model of nominal and real yields represents an extension of the three-factor, arbitrage-free Nelson-Siegel (AFNS) model developed by Christensen et al. (2011, henceforth CDR) for nominal yields. In the CLR model, the state vector is denoted by $X_t = (L_t^N, S_t, C_t, L_t^R)$, where L_t^N is the level factor for nominal yields, S_t and C_t represent slope and curvature factors common to both nominal and real yields, and L_t^R is the level factor for real yields.[3] The instantaneous nominal and real risk-free rates are defined as

$$r_t^N = L_t^N + S_t, \tag{4}$$

$$r_t^R = L_t^R + \alpha^R S_t. \tag{5}$$

Note that the differential scaling of the real rates to the common slope factor is captured by the parameter α^R. To preserve the Nelson and Siegel (1987) factor loading structure in the yield functions, the risk-neutral (or $\mathbb{Q}$-) dynamics of the state variables are given by the

[3] Chernov and Mueller (2012) provide evidence of a hidden factor in the nominal yield curve that is observable from real yields and inflation expectations. The CLR model accommodates this stylized fact via the L_t^R factor.

stochastic differential equations:[4]

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \end{pmatrix} dt + \Sigma \begin{pmatrix} dW_t^{L^N,\mathbb{Q}} \\ dW_t^{S,\mathbb{Q}} \\ dW_t^{C,\mathbb{Q}} \\ dW_t^{L^R,\mathbb{Q}} \end{pmatrix}, \tag{6}$$

where Σ is the constant covariance (or volatility) matrix.[5] Based on this specification of the $\mathbb{Q}$-dynamics, nominal zero-coupon bond yields preserve the Nelson-Siegel factor loading structure as

$$y_t^N(\tau) = L_t^N + \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) S_t + \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) C_t - \frac{A^N(\tau)}{\tau}, \tag{7}$$

where the nominal yield-adjustment term is given by

$$\begin{aligned} \frac{A^N(\tau)}{\tau} &= \frac{\sigma_{11}^2}{6}\tau^2 + \sigma_{22}^2\Big[\frac{1}{2\lambda^2} - \frac{1}{\lambda^3}\frac{1-e^{-\lambda\tau}}{\tau} + \frac{1}{4\lambda^3}\frac{1-e^{-2\lambda\tau}}{\tau}\Big] \\ &\quad + \sigma_{33}^2\Big[\frac{1}{2\lambda^2} + \frac{1}{\lambda^2}e^{-\lambda\tau} - \frac{1}{4\lambda}\tau e^{-2\lambda\tau} - \frac{3}{4\lambda^2}e^{-2\lambda\tau} + \frac{5}{8\lambda^3}\frac{1-e^{-2\lambda\tau}}{\tau} - \frac{2}{\lambda^3}\frac{1-e^{-\lambda\tau}}{\tau}\Big]. \end{aligned}$$

Similarly, real zero-coupon bond yields have a Nelson-Siegel factor loading structure expressed as

$$y_t^R(\tau) = L_t^R + \alpha^R\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) S_t + \alpha^R\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) C_t - \frac{A^R(\tau)}{\tau}, \tag{8}$$

where the real yield-adjustment term is given by

$$\begin{aligned} \frac{A^R(\tau)}{\tau} &= \frac{\sigma_{44}^2}{6}\tau^2 + \sigma_{22}^2(\alpha_S^R)^2\Big[\frac{1}{2\lambda^2} - \frac{1}{\lambda^3}\frac{1-e^{-\lambda\tau}}{\tau} + \frac{1}{4\lambda^3}\frac{1-e^{-2\lambda\tau}}{\tau}\Big] \\ &\quad + \sigma_{33}^2(\alpha_S^R)^2\Big[\frac{1}{2\lambda^2} + \frac{1}{\lambda^2}e^{-\lambda\tau} - \frac{1}{4\lambda}\tau e^{-2\lambda\tau} - \frac{3}{4\lambda^2}e^{-2\lambda\tau} + \frac{5}{8\lambda^3}\frac{1-e^{-2\lambda\tau}}{\tau} - \frac{2}{\lambda^3}\frac{1-e^{-\lambda\tau}}{\tau}\Big]. \end{aligned}$$

To complete the description of the model and to implement it empirically, we will need to specify the risk premiums that connect these factor dynamics under the $\mathbb{Q}$-measure to the dynamics under the real-world (or physical) $\mathbb{P}$-measure. It is important to note that there are no restrictions on the dynamic drift components under the empirical $\mathbb{P}$-measure beyond

[4] As discussed in CDR, with unit roots in the two level factors, the model is not arbitrage-free with an unbounded horizon; therefore, as is often done in theoretical discussions, we impose an arbitrary maximum horizon.

[5] As per CDR, Σ is a diagonal matrix, and $\theta^{\mathbb{Q}}$ is set to zero without loss of generality.

the requirement of constant volatility. To facilitate empirical implementation, we use the essentially affine risk premium specification introduced in Duffee (2002). In the Gaussian framework, this specification implies that the risk premiums Γ_t depend on the state variables; that is,

$$\Gamma_t = \gamma^0 + \gamma^1 X_t,$$

where $\gamma^0 \in \mathbf{R}^4$ and $\gamma^1 \in \mathbf{R}^{4\times 4}$ contain unrestricted parameters.

Thus, the resulting unrestricted four-factor CLR model has $\mathbb{P}$-dynamics given by

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ L_t^R \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{\mathbb{P}} & \kappa_{12}^{\mathbb{P}} & \kappa_{13}^{\mathbb{P}} & \kappa_{14}^{\mathbb{P}} \\ \kappa_{21}^{\mathbb{P}} & \kappa_{22}^{\mathbb{P}} & \kappa_{23}^{\mathbb{P}} & \kappa_{24}^{\mathbb{P}} \\ \kappa_{31}^{\mathbb{P}} & \kappa_{32}^{\mathbb{P}} & \kappa_{33}^{\mathbb{P}} & \kappa_{34}^{\mathbb{P}} \\ \kappa_{41}^{\mathbb{P}} & \kappa_{42}^{\mathbb{P}} & \kappa_{43}^{\mathbb{P}} & \kappa_{44}^{\mathbb{P}} \end{pmatrix} \left(\begin{pmatrix} \theta_1^{\mathbb{P}} \\ \theta_2^{\mathbb{P}} \\ \theta_3^{\mathbb{P}} \\ \theta_4^{\mathbb{P}} \end{pmatrix} - \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \end{pmatrix} \right) dt + \Sigma \begin{pmatrix} dW_t^{L^N,\mathbb{P}} \\ dW_t^{S,\mathbb{P}} \\ dW_t^{C,\mathbb{P}} \\ dW_t^{L^R,\mathbb{P}} \end{pmatrix}.$$

This is the transition equation in the Kalman filter estimation of the CLR model.

A.3 Mapping from the Canonical $A_0(4)$ Model to the CLR Model

In this appendix, we build on the classification of ATSMs introduced in Dai and Singleton (2000) and detail the connection between the canonical $A_0(4)$ model of nominal and real yields and its counterpart CLR model. It should be noted that implicit in the description is the usage of the extended affine risk premium specification of Cheridito et al. (2007), which for Gaussian $A_0(N)$ models is equivalent to the essentially affine risk premium specification introduced in Duffee (2002). By limiting the focus to affine risk premium specifications, the models preserve affine factor dynamics under both probability measures and the invariant affine transformations of Dai and Singleton (2000) apply.

Derivation of the restrictions imposed on the canonical representation of the $A_0(4)$ model needed to get to a specification that has a distribution identical to the CLR model considered in the paper starts from a general affine diffusion process represented by

$$dY_t = K_Y^{\mathbb{Q}}[\theta_Y^{\mathbb{Q}} - Y_t]dt + \Sigma_Y dW_t^{\mathbb{Q}}.$$

Now, consider the affine transformation $\mathcal{T}_A : AY_t + \eta$, where A is a nonsingular square matrix of the same dimension as Y_t and η is a vector of constants of the same dimension as Y_t. Denote the transformed process by $X_t = AY_t + \eta$. By Ito's lemma, it follows that

$$\begin{aligned} dX_t &= AdY_t = [AK_Y^{\mathbb{Q}}\theta_Y^{\mathbb{Q}} - AK_Y^{\mathbb{Q}}Y_t]dt + A\Sigma_Y dW_t^{\mathbb{Q}} \\ &= AK_Y^{\mathbb{Q}}A^{-1}[A\theta_Y^{\mathbb{Q}} - AY_t - \eta + \eta]dt + A\Sigma_Y dW_t^{\mathbb{Q}} \\ &= AK_Y^{\mathbb{Q}}A^{-1}[A\theta_Y^{\mathbb{Q}} + \eta - X_t]dt + A\Sigma_Y dW_t^{\mathbb{Q}} = K_X^{\mathbb{Q}}[\theta_X^{\mathbb{Q}} - X_t]dt + \Sigma_X dW_t^{\mathbb{Q}}. \end{aligned}$$

Thus, X_t is itself an affine diffusion process with parameter specification:

$$K_X^{\mathbb{Q}} = AK_Y^{\mathbb{Q}}A^{-1}, \quad \theta_X^{\mathbb{Q}} = A\theta_Y^{\mathbb{Q}} + \eta, \quad \text{and} \quad \Sigma_X = A\Sigma_Y.$$

A similar result holds for the dynamics under the $\mathbb{P}$-measure.

As for the short rate process, there exists the following relationship:

$$\begin{aligned} r_t &= \delta_0^Y + (\delta_1^Y)'Y_t = \delta_0^Y + (\delta_1^Y)'A^{-1}AY_t = \delta_0^Y + (\delta_1^Y)'A^{-1}[AY_t + \eta - \eta] \\ &= \delta_0^Y - (\delta_1^Y)'A^{-1}\eta + (\delta_1^Y)'A^{-1}X_t. \end{aligned}$$

Thus, defining $\delta_0^X = \delta_0^Y - (\delta_1^Y)'A^{-1}\eta$ and $\delta_1^X = (\delta_1^Y)'A^{-1}$, the short rate process is left unchanged and may be represented in either way

$$r_t = \delta_0^Y + (\delta_1^Y)'Y_t = \delta_0^X + (\delta_1^X)'X_t.$$

Because both Y_t and X_t are affine latent factor processes that deliver the same distribution for the short rate process r_t, they are equivalent representations of the same fundamental model; hence, $\mathcal{T}_A$ is called an affine invariant transformation.

In the canonical representation of the subset of $A_0(4)$ affine term structure models considered here, the $\mathbb{Q}$-dynamics are[6]

$$\begin{pmatrix} dY_t^1 \\ dY_t^2 \\ dY_t^3 \\ dY_t^4 \end{pmatrix} = - \begin{pmatrix} \kappa_{11}^{Y,\mathbb{Q}} & \kappa_{12}^{Y,\mathbb{Q}} & \kappa_{13}^{Y,\mathbb{Q}} & \kappa_{14}^{Y,\mathbb{Q}} \\ 0 & \kappa_{22}^{Y,\mathbb{Q}} & \kappa_{23}^{Y,\mathbb{Q}} & \kappa_{24}^{Y,\mathbb{Q}} \\ 0 & 0 & \kappa_{33}^{Y,\mathbb{Q}} & \kappa_{34}^{Y,\mathbb{Q}} \\ 0 & 0 & 0 & \kappa_{44}^{Y,\mathbb{Q}} \end{pmatrix} \begin{pmatrix} Y_t^1 \\ Y_t^2 \\ Y_t^3 \\ Y_t^4 \end{pmatrix} dt + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dW_t^{1,\mathbb{Q}} \\ dW_t^{2,\mathbb{Q}} \\ dW_t^{3,\mathbb{Q}} \\ dW_t^{4,\mathbb{Q}} \end{pmatrix},$$

[6]Note that we follow Singleton (2006) and impose the identifying restrictions on the $\mathbb{Q}$-dynamics, which contrasts with the approach of Dai and Singleton (2000) where they are imposed on the $\mathbb{P}$-dynamics.

and the $\mathbb{P}$-dynamics are left unconstrained

$$\begin{pmatrix} dY_t^1 \\ dY_t^2 \\ dY_t^3 \\ dY_t^4 \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{Y,\mathbb{P}} & \kappa_{12}^{Y,\mathbb{P}} & \kappa_{13}^{Y,\mathbb{P}} & \kappa_{14}^{Y,\mathbb{P}} \\ \kappa_{21}^{Y,\mathbb{P}} & \kappa_{22}^{Y,\mathbb{P}} & \kappa_{23}^{Y,\mathbb{P}} & \kappa_{24}^{Y,\mathbb{P}} \\ \kappa_{31}^{Y,\mathbb{P}} & \kappa_{32}^{Y,\mathbb{P}} & \kappa_{33}^{Y,\mathbb{P}} & \kappa_{34}^{Y,\mathbb{P}} \\ \kappa_{41}^{Y,\mathbb{P}} & \kappa_{42}^{Y,\mathbb{P}} & \kappa_{43}^{Y,\mathbb{P}} & \kappa_{44}^{Y,\mathbb{P}} \end{pmatrix} \left[\begin{pmatrix} \theta_1^{Y,\mathbb{P}} \\ \theta_2^{Y,\mathbb{P}} \\ \theta_3^{Y,\mathbb{P}} \\ \theta_4^{Y,\mathbb{P}} \end{pmatrix} - \begin{pmatrix} Y_t^1 \\ Y_t^2 \\ Y_t^3 \\ Y_t^4 \end{pmatrix} \right] dt + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dW_t^{1,\mathbb{P}} \\ dW_t^{2,\mathbb{P}} \\ dW_t^{3,\mathbb{P}} \\ dW_t^{4,\mathbb{P}} \end{pmatrix}.$$

Finally, the instantaneous risk-free rate is

$$r_t = \delta_0^Y + \delta_{1,1}^Y Y_t^1 + \delta_{1,2}^Y Y_t^2 + \delta_{1,3}^Y Y_t^3 + \delta_{1,4}^Y Y_t^4.$$

This equation shows that we will be considering two cases jointly: (1) The case of nominal yields discounted with the nominal risk-free rate r_t^N and (2) the case of real yields discounted with the real risk-free rate r_t^R. Thus, the short rate processes in the canonical model are

$$\begin{aligned} r_t^N &= \delta_0^{N,Y} + \delta_{1,1}^{N,Y} Y_t^1 + \delta_{1,2}^{N,Y} Y_t^2 + \delta_{1,3}^{N,Y} Y_t^3 + \delta_{1,4}^{N,Y} Y_t^4, \\ r_t^R &= \delta_0^{R,Y} + \delta_{1,1}^{R,Y} Y_t^1 + \delta_{1,2}^{R,Y} Y_t^2 + \delta_{1,3}^{R,Y} Y_t^3 + \delta_{1,4}^{R,Y} Y_t^4. \end{aligned}$$

There are 35 parameters in this maximally flexible canonical representation of the $A_0(4)$ class of models for nominal and real yields separately. Once we join the information set and model r_t^N and r_t^R simultaneously, there are 40 parameters in the canonical $A_0(4)$ model and we now present the parameter restrictions needed to arrive at the CLR model of nominal and real yields with diagonal Σ matrix analyzed in the paper.

To begin, let the state vector be denoted by $X_t = (L_t^N, S_t, C_t, L_t^R)$, where L_t^N is the level factor for nominal yields, S_t is the common slope factor, C_t is the common curvature factor, and L_t^R is the level factor for real yields.

The maximally flexible specification of the CLR model is

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{\mathbb{P}} & \kappa_{12}^{\mathbb{P}} & \kappa_{13}^{\mathbb{P}} & \kappa_{14}^{\mathbb{P}} \\ \kappa_{21}^{\mathbb{P}} & \kappa_{22}^{\mathbb{P}} & \kappa_{23}^{\mathbb{P}} & \kappa_{24}^{\mathbb{P}} \\ \kappa_{31}^{\mathbb{P}} & \kappa_{32}^{\mathbb{P}} & \kappa_{33}^{\mathbb{P}} & \kappa_{34}^{\mathbb{P}} \\ \kappa_{41}^{\mathbb{P}} & \kappa_{42}^{\mathbb{P}} & \kappa_{43}^{\mathbb{P}} & \kappa_{44}^{\mathbb{P}} \end{pmatrix} \left(\begin{pmatrix} \theta_1^{\mathbb{P}} \\ \theta_2^{\mathbb{P}} \\ \theta_3^{\mathbb{P}} \\ \theta_4^{\mathbb{P}} \end{pmatrix} - \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \end{pmatrix} \right) dt + \Sigma \begin{pmatrix} dW_t^{L^N,\mathbb{P}} \\ dW_t^{S,\mathbb{P}} \\ dW_t^{C,\mathbb{P}} \\ dW_t^{L^R,\mathbb{P}} \end{pmatrix},$$

while its $\mathbb{Q}$-dynamics are given by

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \end{pmatrix} dt + \Sigma \begin{pmatrix} dW_t^{L^N,\mathbb{Q}} \\ dW_t^{S,\mathbb{Q}} \\ dW_t^{C,\mathbb{Q}} \\ dW_t^{L^R,\mathbb{Q}} \end{pmatrix},$$

where Σ is a diagonal matrix

$$\Sigma = \begin{pmatrix} \sigma_{L^N} & 0 & 0 & 0 \\ 0 & \sigma_S & 0 & 0 \\ 0 & 0 & \sigma_C & 0 \\ 0 & 0 & 0 & \sigma_{L^R} \end{pmatrix}.$$

The instantaneous nominal and real risk-free rates are defined as:

$$\begin{aligned} r_t^N &= L_t^N + S_t, \\ r_t^R &= L_t^R + \alpha^R S_t. \end{aligned}$$

This model has a total of 26 parameters; thus, 14 parameter restrictions need to be imposed on the canonical $A_0(4)$ model with a joint representation of nominal and real yields.

It is easy to verify that the affine invariant transformation $\mathcal{T}_A(Y_t) = AY_t + \eta$ with

$$A = \begin{pmatrix} \sigma_{L^N} & 0 & 0 & 0 \\ 0 & \sigma_S & 0 & 0 \\ 0 & 0 & \sigma_C & 0 \\ 0 & 0 & 0 & \sigma_{L^R} \end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

will convert the canonical representation into the CLR model as described above. For the mean-reversion matrices, the relationship between the two representations is

$$\begin{aligned} K_X^{\mathbb{P}} = AK_Y^{\mathbb{P}}A^{-1} \quad &\Longleftrightarrow \quad K_Y^{\mathbb{P}} = A^{-1}K_X^{\mathbb{P}}A, \\ K_X^{\mathbb{Q}} = AK_Y^{\mathbb{Q}}A^{-1} \quad &\Longleftrightarrow \quad K_Y^{\mathbb{Q}} = A^{-1}K_X^{\mathbb{Q}}A. \end{aligned}$$

The equivalent mean-reversion matrix under the $\mathbb{Q}$-measure is then

$$K_Y^{\mathbb{Q}} = \begin{pmatrix} \frac{1}{\sigma_{L^N}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sigma_S} & 0 & 0 \\ 0 & 0 & \frac{1}{\sigma_C} & 0 \\ 0 & 0 & 0 & \frac{1}{\sigma_{L^N}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_{L^N} & 0 & 0 & 0 \\ 0 & \sigma_S & 0 & 0 \\ 0 & 0 & \sigma_C & 0 \\ 0 & 0 & 0 & \sigma_{L^R} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda\frac{\sigma_C}{\sigma_S} & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, eight restrictions need to be imposed on the upper triangular mean-reversion matrix $K_Y^{\mathbb{Q}}$:

$$K_{11}^{Y,\mathbb{Q}} = K_{12}^{Y,\mathbb{Q}} = K_{13}^{Y,\mathbb{Q}} = K_{14}^{Y,\mathbb{Q}} = K_{24}^{Y,\mathbb{Q}} = K_{34}^{Y,\mathbb{Q}} = K_{44}^{Y,\mathbb{Q}} = 0 \quad \text{and} \quad K_{33}^{Y,\mathbb{Q}} = K_{22}^{Y,\mathbb{Q}}.$$

Furthermore, notice that $K_{23}^{Y,\mathbb{Q}}$ will always have the opposite sign of $K_{22}^{Y,\mathbb{Q}}$ and $K_{33}^{Y,\mathbb{Q}}$, but its absolute size can vary independently of these two parameters. Because $K_X^{\mathbb{P}}$ is an unconstrained 4×4 matrix, there are no restrictions on $K_Y^{\mathbb{P}}$.

Finally, we can study the factor loadings in the affine function for the short rate processes.

In the CLR model, the nominal risk-free rate is $r_t^N = L_t^N + S_t$, which is equivalent to fixing

$$\delta_0^{N,X} = 0, \quad \delta_1^{N,X} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

From the relation $(\delta_1^{N,X})' = (\delta_1^{N,Y})' A^{-1}$ it follows that

$$(\delta_1^{N,Y})' = (\delta_1^{N,X})' A = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_{L^N} & 0 & 0 & 0 \\ 0 & \sigma_S & 0 & 0 \\ 0 & 0 & \sigma_C & 0 \\ 0 & 0 & 0 & \sigma_{L^R} \end{pmatrix} = \begin{pmatrix} \sigma_{L^N} & \sigma_S & 0 & 0 \end{pmatrix}.$$

For the constant term it holds that

$$\delta_0^{N,X} = \delta_0^{N,Y} - (\delta_1^{N,Y})' A^{-1} \eta \iff \delta_0^{N,Y} = \delta_0^{N,X} = 0.$$

Thus, we have obtained three additional parameter restrictions

$$\delta_0^{N,Y} = 0 \quad \text{and} \quad \delta_{1,3}^{N,Y} = \delta_{1,4}^{N,Y} = 0.$$

In the CLR model, the real risk-free rate is $r_t^R = L_t^R + \alpha_R S_t$, which is equivalent to fixing

$$\delta_0^{R,X} = 0, \quad \delta_1^{R,X} = \begin{pmatrix} 0 \\ \alpha_R \\ 0 \\ 1 \end{pmatrix}.$$

From the relation $(\delta_1^{R,X})' = (\delta_1^{R,Y})' A^{-1}$ it follows that

$$(\delta_1^{R,Y})' = (\delta_1^{R,X})' A = \begin{pmatrix} 0 & \alpha_R & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{L^N} & 0 & 0 & 0 \\ 0 & \sigma_S & 0 & 0 \\ 0 & 0 & \sigma_C & 0 \\ 0 & 0 & 0 & \sigma_{L^R} \end{pmatrix} = \begin{pmatrix} 0 & \alpha_R \sigma_S & 0 & \sigma_{L^R} \end{pmatrix}.$$

For the constant term it holds that

$$\delta_0^{R,X} = \delta_0^{R,Y} - (\delta_1^{R,Y})' A^{-1} \eta \iff \delta_0^{R,Y} = \delta_0^{R,X} = 0.$$

Thus, we have obtained another three additional parameter restrictions

$$\delta_0^{R,Y} = 0 \quad \text{and} \quad \delta_{1,1}^{R,Y} = \delta_{1,3}^{R,Y} = 0,$$

which brings the total to 14 parameter restrictions as required.

B Characteristics of Canadian Nominal and Real Yields

In this appendix, we first provide support for our choice to focus on the arbitrage-free Nelson-Siegel (AFNS) class of yield curve models for our sample of Canadian nominal yields. We then provide support for the common slope factor structure embedded in the CLR model we rely on for the joint modeling of nominal and real yields.

B.1 Properties of Nominal Yields

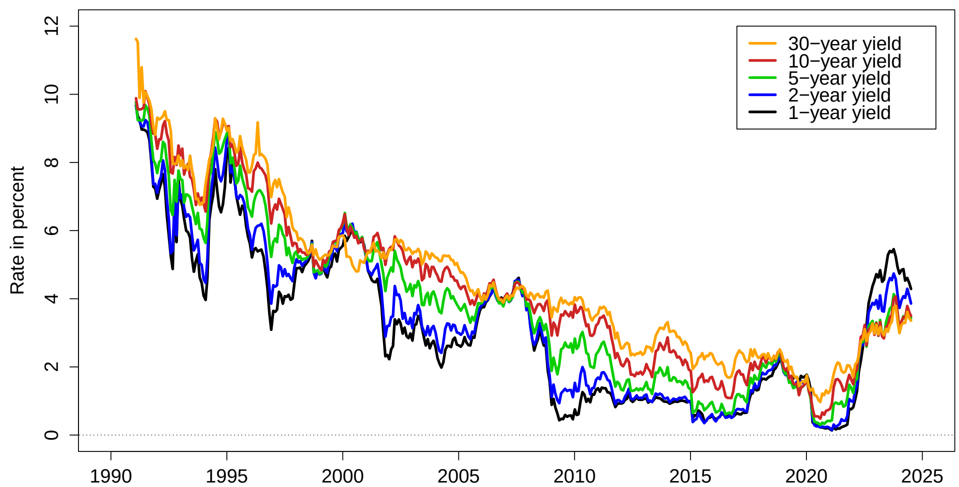


Figure 1: **Canadian Nominal Government Bond Yields**
Illustration of the Canadian government zero-coupon bond yields constructed by staff at the Bank of Canada with maturities of three months, two years, five years, ten years, and thirty years. The data series are monthly covering the period from January 31, 1991, to June 28, 2024.

Figure 1 shows time series of the zero-coupon yields with maturities of three months, two years, five years, ten years, and thirty years. First, we note the downward trend of the general yield level between 1991 and 2021. The ten-year yield dropped from above 10 percent to below 1 percent during this thirty-year period. Also notable is the sharp, partial reversal since then that has brought interest rates in Canada back up close to 4 percent. Second, as in U.S. Treasury yield data, there is clear business cycle variation in the shape of the yield curve around these persistent trends.

It is these characteristics that are the practical motivation behind our choice of using a three-factor model for the Canadian nominal government bond yield curve, adopting an

Maturity in months	First P.C.	Second P.C.	Third P.C.
3	0.28	0.44	-0.50
6	0.28	0.43	-0.25
12	0.29	0.38	0.08
24	0.30	0.21	0.35
36	0.31	0.09	0.41
60	0.31	-0.06	0.34
84	0.31	-0.15	0.22
120	0.31	-0.23	0.05
180	0.31	-0.32	-0.14
240	0.31	-0.36	-0.24
360	0.30	-0.33	-0.38
% explained	94.53	4.95	0.34

Table 1: **Factor Loadings of Canadian Nominal Government Bond Yields**
The top rows show the eigenvectors corresponding to the first three principal components (PC). Put differently, they show how bond yields at various maturities load on the first three principal components. In the final row the proportion of all bond yield variability explained by each principal component is shown. The data are end-of-month Canadian nominal zero-coupon government bond yields from January 31, 1991, to June 28, 2024, a total of 402 observations for each yield series.

approach similar to what is standard for U.S. and U.K. nominal yield data; see Christensen and Rudebusch (2012).

To support that choice more formally, we note that researchers have typically found that three factors are sufficient to model the time variation in the cross section of U.S. Treasury yields (e.g., Litterman and Scheinkman 1991). We perform a similar analysis based on our sample of Canadian nominal government bond yields with 11 constant maturities: 0.25, 0.5, 1, 2, 3, 5, 7, 10, 15, 20, and 30 years. The result of a principal component analysis of the yield panel is reported in Table 1. The top panel reports the eigenvectors that correspond to the first three principal components. The first principal component accounts for 94.5 percent of the variation in the bond yields, and its loading across maturities is uniformly positive. Thus, similar to a level factor, a shock to this component changes all yields in the same direction irrespective of maturity. The second principal component accounts for 5.0 percent of the variation in these data and has sizable positive loadings for the shorter maturities and sizable negative loadings for the long maturities. Thus, similar to a slope factor, a shock to this component steepens or flattens the yield curve. Finally, the third component, which accounts for 0.3 percent of the variation, has a hump shaped factor loading as a function of maturity, which is naturally interpreted as a curvature factor. These three factors combined account for 99.8 percent of the total variation. This motivates our choice to focus on the Nelson and Siegel (1987) model with its level, slope, and curvature factors for modeling this sample of Canadian bond yields. However, for theoretical consistency, we use the arbitrage-free version of this class of models derived in Christensen et al. (2011). Lastly, we stress that the estimated nominal state variables in our models are *not* identical to the principal component factors discussed here, but estimated through Kalman filtering.[7]

B.2 Properties of Real Yields

Due to the limited number of real return bonds (RRB) in our sample, the principal component exercise above cannot meaningfully be replicated for our sample of Canadian real yields.

Instead, to get a sense of the dynamic factor structure embedded in our sample of RRB prices, we construct synthetic real zero-coupon bond yields by fitting the flexible Svensson (1995) yield curve to the set of RRB prices observed for each observation date during the most recent four-year period from January 2, 2020, to June 28, 2024, when we are close to observing the entire RRB yield curve.[8]

We then calculate the slope of the real yield curve measured as the difference between the ten-year real yield and the five-year real yield. This series is shown with a solid grey line in Figure 2. For comparison, we show the slope of the nominal yield curve also calculated as the difference between the ten-year nominal yield and the five-year nominal yield. This series is shown with a solid black line in Figure 2.

We note the high positive correlation of 71.4 percent. More importantly, the general similarity is striking. The two yield curve slopes operate at very similar levels. Moreover, they are inverted at the same time, and they are also steeply upward sloping during the same time periods. This high positive correlation between the slope of the nominal yield curve and the slope of the real yield curve is built directly into the structure of the CLR model through its assumption that the slope factor S_t is common to both yield curves. The only minor caveat about this analysis is that it only provides this supportive evidence for the most recent 4 and a half years of data. However, it looks similar to the evidence reported by Andreasen et al. (2021) for the connection between the slopes of the U.S. Treasury and TIPS yield curves based on data back to the late 1990s. Thus, we are comfortable imposing this cross-curve

[7] A number of papers use principal components as state variables. Joslin et al. (2011) is an early example.

[8] Technically, we proceed as described in Andreasen et al. (2019).

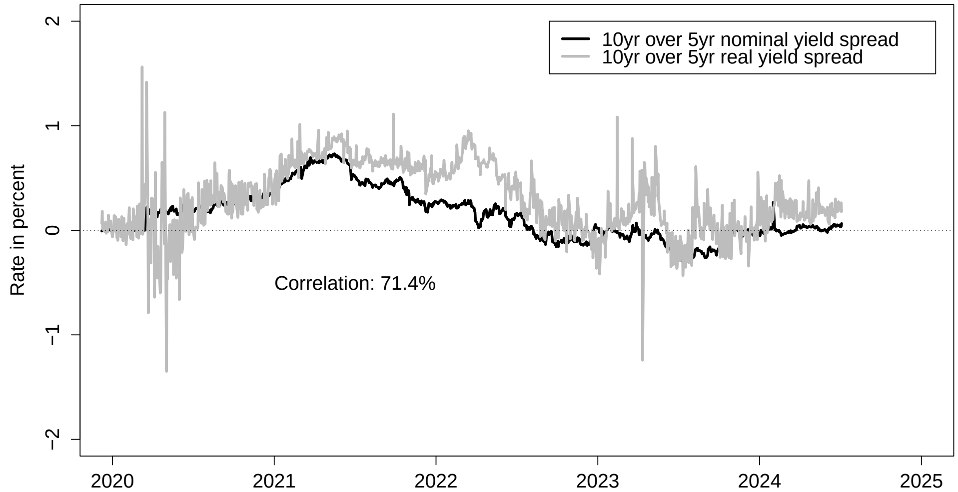


Figure 2: **Slopes of Canadian Nominal and Real Government Bond Yield Curves**
Illustration of the difference between ten-year and five-year real zero-coupon government bond yields constructed as described in the text. Also shown is the difference between ten-year and five-year nominal zero-coupon government bond yields constructed by staff at the Bank of Canada. In both cases, the data series are daily covering the period from January 2, 2020, to June 28, 2024.

restriction for our entire sample of RRB prices going back to December 1995.

C Robustness of the Estimated RRB Liquidity Premium

In this appendix, we analyze the real return bond (RRB) liquidity premium implied by the estimated CLR-L model and calculated as described in Section 3.4 in the main text. First, we examine its sensitivity to both data frequency and the CLR-L model specification. We then explore its robustness to assuming alternative factor dynamics. We end by comparing it to an estimate of the liquidity premium in the U.S. Treasury Inflation-Protected Securities (TIPS) market.

C.1 Sensitivity to Data Frequency

To begin, we assess whether the data frequency plays any role for the estimated RRB liquidity premium. To do so, we estimate the CLR-L model using daily and monthly data, where we focus on the most parsimonious CLR-L model with diagonal $K^{\mathbb{P}}$ and Σ matrices, which is sufficient as explained in Section 3.3 in the main text. Figure 3 shows the estimated average

Figure 3: **Sensitivity of the RRB Liquidity Premium to Data Frequency**
Illustration of the average estimated real return bond liquidity premium for each observation date implied by the CLR-L model when estimated using monthly and daily data. In both cases, the real return bond liquidity premiums are measured as the estimated yield difference between the fitted yield to maturity of individual real return bonds and the corresponding frictionless yield to maturity with the liquidity risk factor turned off. The monthly data cover the period from December 29, 1995, to June 28, 2024, while the daily data cover the period from December 7, 1995, to June 28, 2024.

RRB liquidity premium series from both estimations. Note that the two series are barely distinguishable. Thus, we conclude that data frequency matters little for our results. Clearly, at the higher daily frequency, there are a few isolated spikes that are absent in the monthly series, but they are too few to have an impact on the estimation results.

C.2 Sensitivity to Model Specifications

Next, we study the sensitivity of the estimated RRB liquidity premium to the choice of dynamic specification within the CLR-L model. To do so, we compare the RRB liquidity premium series implied by our preferred CLR-L model described in Section 5.2 in the main text to the corresponding estimates implied by the CLR-L model with unrestricted $K^{\mathbb{P}}$ matrix and diagonal $K^{\mathbb{P}}$ matrix, respectively.

The three different liquidity premium series are shown in Figure 4 with the one generated by the most parsimonious CLR-L model with diagonal $K^{\mathbb{P}}$ and Σ matrices shown with a solid blue line, while the one generated by the preferred CLR-L model is shown with a solid black line. Finally, the series implied by CLR-L model with unrestricted $K^{\mathbb{P}}$ matrix is shown with

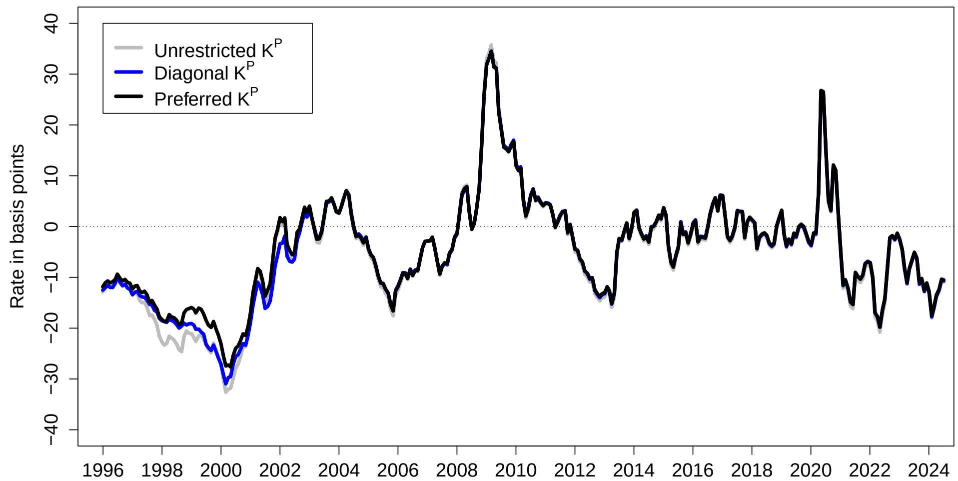


Figure 4: **Sensitivity of the RRB Liquidity Premium to Model Specification**
Illustration of the average estimated real return bond liquidity premium for each observation date implied by the CLR-L model estimated with three different specifications of $K^{\mathbb{P}}$. Note that Σ has a diagonal specification in all three estimations. The real return bond liquidity premiums are measured as the estimated yield difference between the fitted yield to maturity of individual real return bonds and the corresponding frictionless yield to maturity with the liquidity risk factor turned off. The data cover the period from December 29, 1995, to June 28, 2024.

a solid gray line.

We note some dispersion across specifications early in our sample period when we observe prices for only a small set of RRB with very long maturities. This suggests that the liquidity risk factor may not be fully identified in this period. However, since 2002 the estimated RRB liquidity premium series are very close to each other across specifications. Overall, these results confirm the findings of Andreasen et al. (2021, henceforth ACR), namely that liquidity premiums estimated with their approach are extracted primarily from the cross-sectional price information on each observation date with relatively little sensitivity to the specification of the time series dynamics. As a consequence, we limit the regression analysis in Section 3.4.1 in the main text to the average RRB liquidity premium estimated by the most parsimonious specification of the CLR-L model with diagonal $K^{\mathbb{P}}$ and Σ matrices as it is representative of the liquidity premiums one would estimate with other more flexible specifications.

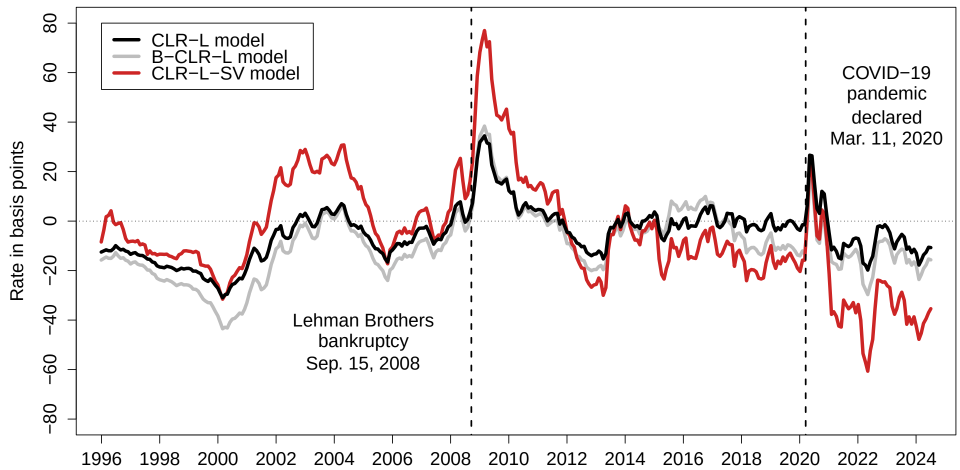


Figure 5: **Sensitivity of the RRB Liquidity Premium to Alternative Model Dynamics**

Illustration of the average estimated real return bond liquidity premium for each observation date implied by the CLR-L model and compared with the corresponding series implied by the shadow-rate B-CLR-L model and the CLR-SV-L model with stochastic volatility. In all three cases, the real return bond liquidity premiums are measured as the estimated yield difference between the fitted yield to maturity of individual real return bonds and the corresponding frictionless yield to maturity with the liquidity risk factor turned off. The data cover the period from December 29, 1995, to June 28, 2024.

C.3 Sensitivity to Alternative Model Dynamics

Our benchmark CLR-L model has a standard affine specification for the nominal short rate, which does not enforce the zero lower bound (ZLB). However, a large fraction of our sample (2009-2017 & 2020-2021) is near the ZLB, and we therefore briefly explore whether our estimated liquidity premium is robust to accounting for the ZLB through a shadow-rate extension of the CLR-L model. We adopt an approach inspired by Black (1995) and replace r_t^N in equation (4) by $r_t^N = \max\left(L_t^N + S_t, 0\right)$ and solve for nominal yields using the approximation in Christensen and Rudebusch (2015), but this B-CLR-L model is otherwise identical to the CLR-L model.

As another alternative modification, we also consider the CLR model extension offered by Christensen et al. (2016), who incorporate stochastic volatility into the nominal and real level factors in the CLR model. We augment their model with a liquidity factor as before and refer to it as the CLR-L-SV model.

Figure 5 shows that the estimated liquidity premiums from these two model alternatives

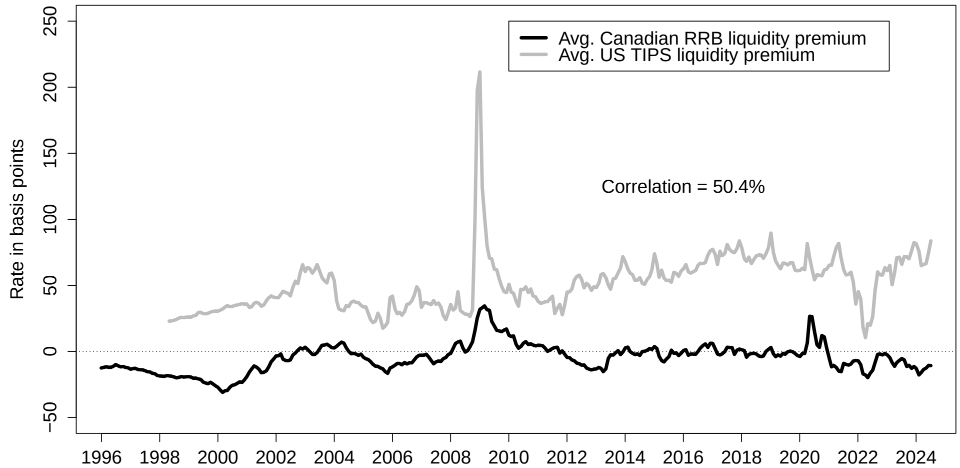


Figure 6: **Comparison of Canadian RRB and U.S. TIPS Liquidity Premiums**
Illustration of the average estimated real return bond liquidity premium implied by the CLR-L model and a comparison with the U.S. TIPS liquidity premium series implied by the model described in Christensen and Rudebusch (2019). In both cases, the liquidity premiums are measured as the estimated yield difference between the fitted yield to maturity of individual real return bonds and the corresponding frictionless yield to maturity with the liquidity risk factor turned off. The Canadian RRB data cover the period from December 29, 1995, to June 28, 2024, while the U.S. TIPS data cover the period from April 30, 1998, to June 28, 2024.

are qualitatively similar to our benchmark estimate from the CLR-L model. Thus, neither the presence of the ZLB nor allowing for stochastic volatility seem to materially affect the estimated RRB liquidity premiums similar to what ACR report in the context of U.S. Treasuries and TIPS. Crucially, there are no trends in any of the three liquidity premium series. Thus, these results support the conclusion that RRB liquidity premiums cannot account for the persistent changes in Canadian long-term real yields since 1995.

C.4 Comparison with U.S. TIPS Liquidity Premiums

Given the geographical proximity of Canada to the United States, we next compare our average estimated RRB liquidity premium series with the average estimated U.S. TIPS liquidity premium from an update of the analysis in Christensen and Rudebusch (2019). The resulting average TIPS liquidity premium is shown with a solid grey line in Figure 6. We note that the two series share a number of broad trends. First, they both reach a peak in late 2002 followed by a steady decline with a trough in 2005. Both series spike at the peak of the financial crisis in the fall of 2008 followed by a gradual reversal in the immediate years after the crisis. Finally, they are both close to their respective historical averages at the end of the sample. This broadly similar underlying trend implies a high positive correlation (50%).

D U.S. and U.K. QE Announcements

In this appendix, we briefly describe key QE announcements by the U.S. Federal Reserve and the Bank of England and their spillover effects on the Canadian bond markets, which Section 6 of the paper analyzes in detail.

No.	Date	Event	Description
I	Nov. 25, 2008	Initial LSAP announcement	Fed announces purchases of $100 billion in GSE debt and up to $500 billion in MBS.
II	Dec. 1, 2008	Bernanke speech	Chairman Bernanke indicates that the Fed could purchase long-term Treasury securities.
III	Dec. 16, 2008	FOMC statement	The first FOMC statement that mentions possible purchases of long-term Treasuries.
IV	Jan. 28, 2009	FOMC statement	FOMC states that it is ready to expand agency debt and MBS purchases and to purchase long-term Treasuries.
V	Mar. 18, 2009	FOMC statement	Fed will purchase an additional $750 billion in agency MBS and $100 billion in agency debt. Also, it will purchase $300 billion in long-term Treasury securities.
VI	Aug. 12, 2009	FOMC statement	Fed is set to slow the pace of the LSAP. The final purchases of Treasury securities will be in the end of October instead of mid-September.
VII	Sep. 23, 2009	FOMC statement	Fed's purchases of agency debt and MBS will end in the first quarter of 2010, while its Treasury purchases will end as planned in October.
VIII	Nov. 4, 2009	FOMC statement	Amount of agency debt capped at $175 billion instead of the $200 billion previously announced.

Table 2: **Key U.S. Federal Reserve QE Announcements**

Given that the analysis is intended to be illustrative rather than comprehensive, we only consider the announcements studied in Christensen and Rudebusch (2012, henceforth CR). These include eight key announcements regarding the Federal Reserve's first QE program, which are listed in Table 2. In addition, there are seven key announcements made by the Bank of England's Monetary Policy Committee (MPC) regarding its QE program, which are

No.	Date	Event	Description
I	Feb. 11, 2009	February Inflation Report	Press conference and Inflation Report indicated that asset purchases were likely.
II	Mar. 5, 2009	MPC statement	The MPC announced that it would purchase £75 billion of assets over three months. Gilt purchases would be restricted to the 5-25 year maturity range.
III	May 7, 2009	MPC statement	The MPC announced that the amount of asset purchases would be extended by a further £50 billion to a total of £125 billion.
IV	Aug. 6, 2009	MPC statement	The MPC announced that the amount of asset purchases would be extended to £175 billion and that the buying range would be extended to include gilts with residual maturity greater than three years.
V	Nov. 5, 2009	MPC statement	The MPC announced that the asset purchases would be extended to £200 billion.
VI	Feb. 4, 2010	MPC statement	The MPC announced that the amount of asset purchases would be maintained at £200 billion.
VII	Oct 6, 2011	MPC statement	The MPC announced that the asset purchases would be extended to £275 billion.

Table 3: **Key Bank of England QE Announcements**

listed in Table 3.

We start our analysis with a model-free inspection of the changes in the Canadian yield data around our U.S. QE event windows. Table 4 shows the changes on these dates in seven of the eleven nominal yield maturities we use in the model estimation in the main text. The largest changes are in the five- to ten-year maturity range, although both shorter- and longer-term nominal yields exhibited notable reactions to the announcements as well. Furthermore, with few exceptions, the entire nominal yield curve moves in the same direction in response to these eight announcements. Importantly, the net change for the Canadian one-year nominal yield is similar to the one-day response of U.S. Treasury yields reported in CR. However, for the nominal yields with maturities from two years and up, the net response is only about half as large as the one-day response of U.S. Treasury yields reported in CR. On balance, we take this as evidence of strong spillover effects from the U.S. QE announcements onto Canadian nominal bond yields, so the forceful U.S. central bank actions also helped lower Canadian interest rates significantly during this challenging period for the global economy.

Turning to breakeven inflation (BEI), Table 5 shows the changes on the eight U.S. an-

Event		Nominal yields						
		1-year	2-year	5-year	7-year	10-year	20-year	30-year
I	Nov. 24, 2008	170.2	187.1	276.8	317.8	371.3	448.8	402.8
	Nov. 26, 2008	163.1	178.9	264.8	303.3	356.5	436.8	390.2
	Change	**-7.1**	**-8.2**	**-12.1**	**-14.6**	**-14.8**	**-12.0**	**-12.6**
II	Nov. 28, 2008	158.9	175.8	260.1	299.8	352.8	432.8	386.4
	Dec. 2, 2008	148.8	162.9	242.9	283.3	337.3	422.3	373.9
	Change	**-10.0**	**-12.9**	**-17.2**	**-16.5**	**-15.5**	**-10.6**	**-12.6**
III	Dec. 15, 2008	122.6	148.8	229.2	270.4	329.1	416.5	374.1
	Dec. 17, 2008	102.3	129.9	208.1	249.9	310.2	396.4	353.1
	Change	**-20.3**	**-18.9**	**-21.1**	**-20.5**	**-18.9**	**-20.1**	**-21.0**
IV	Jan. 27, 2009	104.0	137.8	210.9	251.3	314.9	417.7	363.7
	Jan. 29, 2009	102.4	146.4	227.6	270.5	331.9	423.6	377.9
	Change	**-1.6**	**8.6**	**16.7**	**19.2**	**17.0**	**5.9**	**14.2**
V	Mar. 17, 2009	70.0	107.4	192.1	232.7	305.1	412.9	365.0
	Mar. 18, 2009	67.2	104.1	173.4	211.0	283.5	405.4	359.9
	Change	**-2.8**	**-3.2**	**-18.8**	**-21.7**	**-21.6**	**-7.5**	**-5.1**
VI	Aug. 11, 2009	56.6	130.3	272.0	311.6	367.8	432.5	397.8
	Aug. 13, 2009	54.8	130.2	272.5	311.6	368.4	432.7	397.5
	Change	**-1.7**	**0.0**	**0.5**	**0.1**	**0.6**	**0.2**	**-0.3**
VII	Sep. 22, 2009	55.9	132.0	264.6	302.1	360.9	428.5	391.8
	Sep. 24, 2009	55.4	131.5	262.4	300.4	358.1	424.4	387.8
	Change	**-0.6**	**-0.5**	**-2.2**	**-1.6**	**-2.8**	**-4.1**	**-4.0**
VIII	Nov. 3, 2009	59.3	134.2	269.3	307.5	363.0	426.7	391.7
	Nov. 5, 2009	61.1	137.7	275.8	316.5	372.9	435.7	401.4
	Change	**1.8**	**3.4**	**6.4**	**9.0**	**9.8**	**9.0**	**9.7**
Total net change		**-42.3**	**-31.7**	**-47.8**	**-46.6**	**-46.2**	**-39.2**	**-31.7**

Table 4: **Canadian Nominal Yield Responses on U.S. QE Announcement Dates**
The table reports the two-day response of Canadian nominal yields at seven different maturities around the U.S. QE announcement dates. All numbers are measured in basis points.

nouncement dates in four BEI rates obtained from an estimation of the CLR model using our full sample of daily data through the end of June 2024. On net, medium- and long-term Canadian BEI had a modest negative reaction to those announcements. Thus, Canadian real yields changed almost in lockstep with nominal yields to the U.S. news. This could be an early sign that Canadian long-term inflation expectations remained anchored close to the Bank of Canada's 2 percent inflation target during this period. In contrast, we document softening inflation expectations for the subsequent decade. More importantly, the modest

Event		BEI			
		5-year	10-year	20-year	30-year
I	Nov. 24, 2008	103.7	124.8	135.6	128.6
	Nov. 26, 2008	92.7	113.9	125.0	118.2
	Change	**-10.9**	**-10.8**	**-10.6**	**-10.4**
II	Nov. 28, 2008	86.6	107.8	119.0	112.3
	Dec. 2, 2008	78.1	99.6	111.5	105.1
	Change	**-8.5**	**-8.2**	**-7.5**	**-7.2**
III	Dec. 15, 2008	82.5	105.1	117.1	110.5
	Dec. 17, 2008	83.5	106.9	119.7	113.5
	Change	**1.0**	**1.7**	**2.7**	**3.1**
IV	Jan. 27, 2009	93.1	117.0	129.6	123.3
	Jan. 29, 2009	99.0	122.6	134.5	127.8
	Change	**5.9**	**5.7**	**4.9**	**4.6**
V	Mar. 17, 2009	103.3	129.5	143.5	137.7
	Mar. 18, 2009	108.6	135.2	150.1	144.7
	Change	**5.3**	**5.7**	**6.6**	**7.0**
VI	Aug. 11, 2009	191.9	218.6	227.8	219.4
	Aug. 13, 2009	191.5	218.2	227.5	219.2
	Change	**-0.4**	**-0.4**	**-0.3**	**-0.2**
VII	Sep. 22, 2009	184.8	210.7	219.6	211.1
	Sep. 24, 2009	185.3	211.3	220.2	211.7
	Change	**0.6**	**0.6**	**0.6**	**0.6**
VIII	Nov. 3, 2009	204.2	229.2	236.3	227.0
	Nov. 5, 2009	201.5	226.5	233.4	223.9
	Change	**-2.8**	**-2.7**	**-3.0**	**-3.1**
Total net change		**-9.8**	**-8.6**	**-6.6**	**-5.7**

Table 5: **Canadian BEI Responses on U.S. QE Announcement Dates**
The table reports the two-day response of Canadian BEI at four different maturities around the U.S. QE announcement dates. All numbers are measured in basis points.

negative responses of Canadian BEI suggest that the U.S. QE announcements failed to fully offset disinflationary pressures in North America, as explored in our model-based analysis in the main text.

As with the U.S. case, we begin the analysis of the effects of U.K. QE on Canadian yields with a model-free inspection of the changes in the Canadian yield data around the U.K. event windows. Table 6 shows changes of nominal yields with maturities ranging from one year to thirty years. Except for the first announcement date, Feb. 10, 2009, the entire nominal

Event		Nominal yields						
		1-year	2-year	5-year	7-year	10-year	20-year	30-year
I	Feb. 10, 2009	83.1	122.2	210.7	254.4	325.2	415.5	372.0
	Feb. 11, 2009	83.9	122.7	211.1	252.0	321.0	412.1	368.1
	Change	**0.7**	**0.5**	**0.5**	**-2.4**	**-4.2**	**-3.4**	**-3.9**
II	Mar. 3, 2009	66.9	102.5	189.2	234.7	312.3	416.0	367.9
	Mar. 4, 2009	62.9	98.9	182.0	226.8	302.7	403.5	356.3
	Change	**-4.0**	**-3.6**	**-7.2**	**-7.9**	**-9.6**	**-12.5**	**-11.6**
III	May 6, 2009	45.5	99.7	208.6	253.8	324.6	428.3	393.6
	May 7, 2009	47.2	103.4	216.0	262.1	332.6	434.3	399.4
	Change	**1.7**	**3.6**	**7.4**	**8.3**	**8.0**	**6.0**	**5.8**
IV	Aug. 5, 2009	62.9	141.9	281.8	321.1	375.8	440.1	405.9
	Aug. 6, 2009	61.4	140.5	278.7	317.3	372.2	436.1	402.2
	Change	**-1.5**	**-1.4**	**-3.2**	**-3.9**	**-3.6**	**-4.0**	**-3.8**
V	Nov. 4, 2009	59.6	135.1	271.3	311.2	368.2	431.7	396.8
	Nov. 5, 2009	61.1	137.7	275.8	316.5	372.9	435.7	401.4
	Change	**1.6**	**2.6**	**4.4**	**5.3**	**4.6**	**4.1**	**4.6**
VI	Feb. 3, 2010	59.5	130.2	259.2	305.9	366.8	433.2	404.9
	Feb. 4, 2010	56.8	125.2	252.2	298.9	360.6	428.2	400.3
	Change	**-2.7**	**-5.0**	**-7.0**	**-7.0**	**-6.2**	**-5.0**	**-4.6**
VII	Oct. 5, 2011	83.5	93.0	142.8	180.5	224.8	288.2	279.0
	Oct. 6, 2011	86.1	96.8	149.8	188.7	233.6	294.9	286.0
	Change	**2.7**	**3.7**	**7.0**	**8.2**	**8.8**	**6.7**	**7.1**
Total net change		**-1.5**	**0.4**	**1.9**	**0.0**	**-2.2**	**-8.1**	**-6.4**

Table 6: **Canadian Nominal Yield Responses on U.K. QE Announcement Dates**
The table reports the one-day response of Canadian nominal yields at seven different maturities around the U.K. QE announcement dates. All numbers are measured in basis points.

curve moves in the same direction, with longer horizons generally showing more significant responses. On net, the final row of Table 6 shows that U.K. QE primarily decreased yields on longer maturity bonds while shorter horizon yields were little affected. Additionally, the magnitudes tend to be much smaller than for the U.S. counterpart, indicating that the Bank of England's QE announcements had a more marginal effect on the Canadian yields compared to those generated by the U.S. Federal Reserve's announcements.

Table 7 shows changes in four BEI rates on the seven U.K. QE announcement days. On net, all BEI rates increased, with the largest increase coming at the thirty-year horizon and the smallest coming at the five-year horizon. Importantly, the bulk of the reaction

Event		BEI			
		5-year	10-year	20-year	30-year
I	Feb. 10, 2009	111.3	136.0	148.5	142.1
	Feb. 11, 2009	112.5	137.2	149.9	143.5
	Change	**1.2**	**1.2**	**1.3**	**1.4**
II	Mar. 3, 2009	91.3	116.6	129.4	123.0
	Mar. 4, 2009	91.1	116.7	129.9	123.7
	Change	**-0.2**	**0.1**	**0.5**	**0.7**
III	May 6, 2009	122.9	151.0	164.8	158.7
	May 7, 2009	126.6	154.9	168.7	162.5
	Change	**3.8**	**3.9**	**3.9**	**3.9**
IV	Aug. 5, 2009	194.5	221.1	230.1	221.6
	Aug. 6, 2009	194.3	220.8	229.8	221.3
	Change	**-0.3**	**-0.3**	**-0.3**	**-0.3**
V	Nov. 4, 2009	202.6	227.7	234.7	225.3
	Nov. 5, 2009	201.5	226.5	233.4	223.9
	Change	**-1.2**	**-1.2**	**-1.3**	**-1.4**
VI	Feb. 3, 2010	216.6	243.4	253.3	245.2
	Feb. 4, 2010	215.5	242.5	252.6	244.6
	Change	**-1.1**	**-0.9**	**-0.7**	**-0.6**
VII	Oct. 5, 2011	181.2	201.8	216.5	211.7
	Oct. 6, 2011	182.5	202.8	217.0	211.9
	Change	**1.2**	**1.0**	**0.5**	**0.2**
Total net change		**3.4**	**3.7**	**3.9**	**3.9**

Table 7: **Canadian BEI Responses on U.K. QE Announcement Dates**
The table reports the one-day response of Canadian BEI at four different maturities around the U.K. QE announcement dates. All numbers are measured in basis points.

is concentrated around the May 2009 announcement that kept the policy rate fixed at 0.5 percent and increased the size of QE by £50 billion to a total of £125 billion. Changes on all other days are close to zero. Overall, the changes are around four basis points, indicating that the U.K. announcements had limited spillover effects on Canadian BEI rates.

E Model Selection in the CLR Model

In this appendix, we go through a careful model selection procedure for the benchmark CLR model similar to the one described in the main text for the augmented CLR-L model.

Alternative Specifications	Goodness of fit statistics			
	$\log L$	k	p-value	BIC
(1) Unrestricted $K^{\mathbb{P}}$	30,674.90	28	n.a.	-61,181.90
(2) $\kappa^{\mathbb{P}}_{31} = 0$	30,674.89	27	0.89	-61,187.88
(3) $\kappa^{\mathbb{P}}_{31} = \kappa^{\mathbb{P}}_{41} = 0$	30,674.80	26	0.67	-61,193.69
(4) $\kappa^{\mathbb{P}}_{31} = \kappa^{\mathbb{P}}_{41} = \kappa^{\mathbb{P}}_{43} = 0$	30,674.68	25	0.62	-61,199.45
(5) $\kappa^{\mathbb{P}}_{31} = \ldots = \kappa^{\mathbb{P}}_{12} = 0$	30,674.29	24	0.38	-61,204.67
(6) $\kappa^{\mathbb{P}}_{31} = \ldots = \kappa^{\mathbb{P}}_{13} = 0$	30,674.08	23	0.52	-61,210.24
(7) $\kappa^{\mathbb{P}}_{31} = \ldots = \kappa^{\mathbb{P}}_{21} = 0$	30,672.35	22	0.06	**-61,212.78**
(8) $\kappa^{\mathbb{P}}_{31} = \ldots = \kappa^{\mathbb{P}}_{14} = 0$	30,665.23	21	< 0.01	-61,204.53
(9) $\kappa^{\mathbb{P}}_{31} = \ldots = \kappa^{\mathbb{P}}_{32} = 0$	30,664.07	20	0.13	-61,208.21
(10) $\kappa^{\mathbb{P}}_{31} = \ldots = \kappa^{\mathbb{P}}_{24} = 0$	30,661.25	19	0.02	-61,208.57
(11) $\kappa^{\mathbb{P}}_{31} = \ldots = \kappa^{\mathbb{P}}_{23} = 0$	30,656.03	18	< 0.01	-61,204.12
(12) $\kappa^{\mathbb{P}}_{31} = \ldots = \kappa^{\mathbb{P}}_{34} = 0$	30,655.57	17	0.34	-61,209.20
(13) $\kappa^{\mathbb{P}}_{43} = \ldots = \kappa^{\mathbb{P}}_{42} = 0$	30,652.57	16	0.01	-61,209.20

Table 8: **Evaluation of Alternative Specifications of the CLR Model**

There are thirteen alternative estimated specifications of the CLR model. Each specification is listed with its maximum log likelihood ($\log L$), number of parameters (k), the p-value from a likelihood ratio test of the hypothesis that it differs from the specification above with one more free parameter, and the Bayesian information criterion (BIC). The period analyzed covers monthly data from January 31, 1991, to June 28, 2024.

For estimates of expected inflation and r^*_t based on our definition, the specification of the mean-reversion matrix $K^{\mathbb{P}}$ is critical. To select the best fitting specification of the CLR model's real-world dynamics, we use a general-to-specific modeling strategy in which the least significant off-diagonal parameter of $K^{\mathbb{P}}$ is restricted to zero and the model is re-estimated. This strategy of eliminating the least significant coefficient is carried out down to the most parsimonious specification, which has a diagonal $K^{\mathbb{P}}$ matrix. As in the main text, the final specification choice is based on the value of the Bayesian information criterion (BIC).

The summary statistics of the model selection process are reported in Table 8. The BIC is minimized by specification (7), which has a $K^{\mathbb{P}}$ matrix given by

$$K^{\mathbb{P}}_{BIC} = \begin{pmatrix} \kappa^{\mathbb{P}}_{11} & 0 & 0 & \kappa^{\mathbb{P}}_{14} \\ 0 & \kappa^{\mathbb{P}}_{22} & \kappa^{\mathbb{P}}_{23} & \kappa^{\mathbb{P}}_{24} \\ 0 & \kappa^{\mathbb{P}}_{32} & \kappa^{\mathbb{P}}_{33} & \kappa^{\mathbb{P}}_{34} \\ 0 & \kappa^{\mathbb{P}}_{42} & 0 & \kappa^{\mathbb{P}}_{44} \end{pmatrix}.$$

The estimated parameters of this preferred specification are reported in Table 9. We note that the nominal and real level factors are very persistent processes with near-unit root dynamics.

$K^{\mathbb{P}}$	$K^{\mathbb{P}}_{\cdot,1}$	$K^{\mathbb{P}}_{\cdot,2}$	$K^{\mathbb{P}}_{\cdot,3}$	$K^{\mathbb{P}}_{\cdot,4}$	$\theta^{\mathbb{P}}$		Σ
$K^{\mathbb{P}}_{1,\cdot}$	0.0719	0	0	0.0992	0.0017	$\Sigma_{1,1}$	0.0058
	(0.0687)			(0.0899)	(0.0799)		(0.0001)
$K^{\mathbb{P}}_{2,\cdot}$	0	0.1680	-0.3207	0.2526	-0.0216	$\Sigma_{2,2}$	0.0114
		(0.1570)	(0.1152)	(0.1570)	(0.0105)		(0.0003)
$K^{\mathbb{P}}_{3,\cdot}$	0	0.4468	1.2864	-0.9426	-0.0029	$\Sigma_{3,3}$	0.0226
		(0.2433)	(0.2745)	(0.3562)	(0.0058)		(0.0009)
$K^{\mathbb{P}}_{4,\cdot}$	0	-0.1291	0	-0.0018	0.0501	$\Sigma_{4,4}$	0.0041
		(0.0492)		(0.0528)	(0.0100)		(0.0001)

Table 9: **Estimated Parameters in the Preferred CLR Model**
The estimated parameters for the mean-reversion matrix $K^{\mathbb{P}}$, the mean vector $\theta^{\mathbb{P}}$, and the volatility matrix Σ in the CLR model preferred according to the BIC. The Q-related parameter is estimated at $\lambda = 0.3113$ (0.0035) and $\alpha^R = 0.6955$ (0.0079). The maximum log likelihood value is 30672.35. The numbers in parentheses are the estimated standard deviations.

On the other hand, the common curvature factor is a volatile process that reverts to mean fairly quickly, while the common slope factor has dynamic properties in between the two extremes. Finally, the yield decompositions from the preferred CLR model are examined in online Appendix G.1.2.

F Model Selection in the Shadow-Rate B-CLR-L Model

In this appendix, we go through a careful model selection procedure for the shadow-rate B-CLR model similar to the one described in the main text for the CLR-L model.

To select the best fitting specification of the B-CLR-L model's real-world dynamics, we use a general-to-specific modeling strategy in which the least significant off-diagonal parameter of $K^{\mathbb{P}}$ is restricted to zero and the model is re-estimated. This strategy of eliminating the least significant coefficient is carried out down to the most parsimonious specification, which has a diagonal $K^{\mathbb{P}}$ matrix. The final specification choice is based on the value of the Bayesian information criterion (BIC) as in Christensen et al. (2014).[9]

The summary statistics of the model selection process are reported in Table 10. The BIC

[9]The Bayesian information criterion is defined as $\text{BIC} = -2\log L + k \log T$, where k is the number of model parameters and T is the number of data observations. We have 402 nominal yield and 343 real yield monthly observations. We follow CLR and interpret T as referring to the longest data series and fix it at 402.

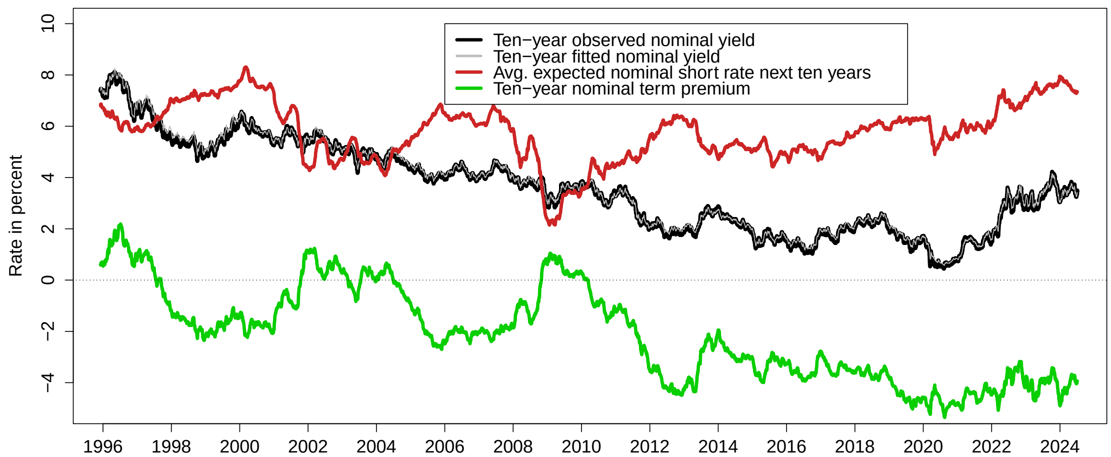
Alternative specifications	Goodness of fit statistics			
	$\log L$	k	p-value	BIC
(1) Unrestricted $K^{\mathbb{P}}$	32,083.72	58	n.a.	-63,819.65
(2) $\kappa^{\mathbb{P}}_{12} = 0$	32,083.66	57	0.73	-63,825.52
(3) $\kappa^{\mathbb{P}}_{12} = \kappa^{\mathbb{P}}_{34} = 0$	32,083.67	56	1.00	-63,831.54
(4) $\kappa^{\mathbb{P}}_{12} = \kappa^{\mathbb{P}}_{34} = \kappa^{\mathbb{P}}_{15} = 0$	32,083.51	55	0.57	-63,837.22
(5) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{13} = 0$	32,083.14	54	0.39	-63,842.47
(6) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{21} = 0$	32,083.14	53	1.00	-63,848.47
(7) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{43} = 0$	32,082.56	52	0.28	-63,853.30
(8) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{45} = 0$	32,081.52	51	0.15	-63,857.22
(9) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{35} = 0$	32,078.84	50	0.02	-63,857.86
(10) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{23} = 0$	32,077.79	49	0.15	-63,861.75
(11) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{31} = 0$	32,077.69	48	0.65	-63,867.55
(12) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{32} = 0$	32,077.19	47	0.32	-63,872.55
(13) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{53} = 0$	32,077.05	46	0.60	-63,878.26
(14) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{52} = 0$	32,076.10	45	0.17	-63,882.36
(15) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{51} = 0$	32,076.05	44	0.75	-63,888.26
(16) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{54} = 0$	32,076.02	43	0.81	**-63,894.19**
(17) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{24} = 0$	32,072.71	42	0.01	-63,893.57
(18) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{25} = 0$	32,063.79	41	< 0.01	-63,881.73
(19) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{42} = 0$	32,056.80	40	< 0.01	-63,873.74
(20) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{41} = 0$	32,056.19	39	0.27	-63,878.52
(21) $\kappa^{\mathbb{P}}_{12} = \ldots = \kappa^{\mathbb{P}}_{14} = 0$	32,049.10	38	< 0.01	-63,870.33

Table 10: **Evaluation of Alternative Specifications of the B-CLR-L Model**
There are twenty-one alternative estimated specifications of the B-CLR-L model. Each specification is listed with its maximum log likelihood ($\log L$), number of parameters (k), the p-value from a likelihood ratio test of the hypothesis that it differs from the specification above with one more free parameter, and the Bayesian information criterion (BIC). The period analyzed covers monthly data from January 31, 1991, to June 28, 2024.

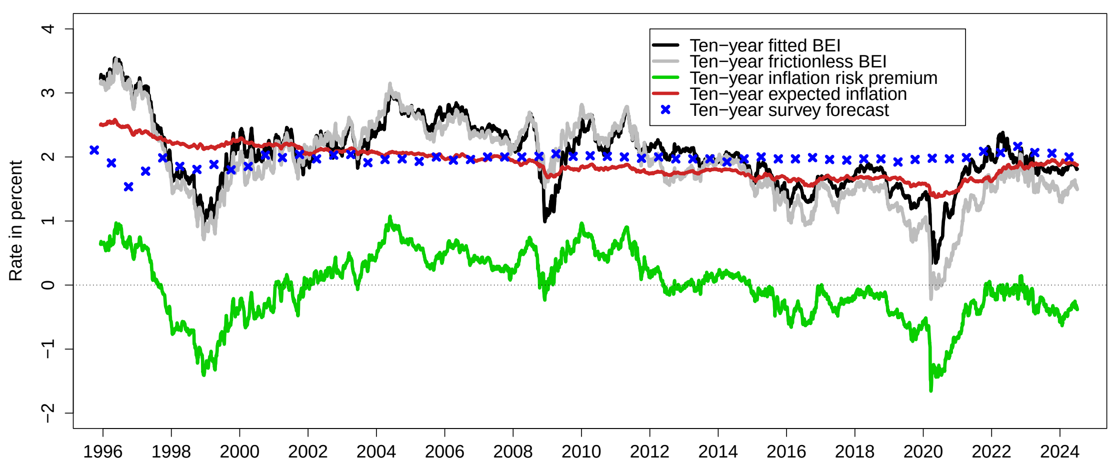
is minimized by specification (16), which has a $K^{\mathbb{P}}$ matrix given by

$$K^{\mathbb{P}}_{BIC} = \begin{pmatrix} \kappa^{\mathbb{P}}_{11} & 0 & 0 & \kappa^{\mathbb{P}}_{14} & 0 \\ 0 & \kappa^{\mathbb{P}}_{22} & 0 & \kappa^{\mathbb{P}}_{24} & \kappa^{\mathbb{P}}_{25} \\ 0 & 0 & \kappa^{\mathbb{P}}_{33} & 0 & 0 \\ \kappa^{\mathbb{P}}_{41} & \kappa^{\mathbb{P}}_{42} & 0 & \kappa^{\mathbb{P}}_{44} & 0 \\ 0 & 0 & 0 & 0 & \kappa^{\mathbb{P}}_{55} \end{pmatrix}.$$

Thus, compared with the preferred specification of the CLR-L model reported in the main text, there is a single difference, namely $\kappa^{\mathbb{P}}_{24}$ eliminated in step 17. Notably, this is the first parameter to be eliminated in the CLR-L model selection, which raises concerns about the relevance of keeping this particular parameter. Comfortingly, the first 15 parameters elimi-



(a) Ten-year nominal yield decomposition



(b) Ten-year BEI decomposition

Figure 7: **Ten-Year Nominal Yield and BEI Decompositions**

nated are collectively statistically insignificant, and even individually so with the exception of the elimination of $\kappa_{35}^{\mathbb{P}}$ in step 9, which is significant at the 5 percent level. Thus, the insignificance of these parameters seem to be real and apply to both the CLR-L model and the shadow-rate B-CLR-L model. Finally, the last four parameters to be eliminated are identical across the two model selection procedures, and even the sequencing of their elimination is the same. Thus, the significance of these four parameters is confirmed in both models.

Now, to determine whether we should keep $\kappa_{24}^{\mathbb{P}}$ as part our preferred specification for

$K^{\mathbb{P}}$	$K^{\mathbb{P}}_{\cdot,1}$	$K^{\mathbb{P}}_{\cdot,2}$	$K^{\mathbb{P}}_{\cdot,3}$	$K^{\mathbb{P}}_{\cdot,4}$	$K^{\mathbb{P}}_{\cdot,5}$	$\theta^{\mathbb{P}}$		Σ
$K^{\mathbb{P}}_{1,\cdot}$	0.5848	0	0	-0.7129	0	0.0857	σ_{11}	0.0059
	(0.0452)			(0.0545)		(0.0037)		(0.0001)
$K^{\mathbb{P}}_{2,\cdot}$	0	0.4904	0	0	0.0392	-0.0138	σ_{22}	0.0137
		(0.0606)			(0.0136)	(0.0047)		(0.0003)
$K^{\mathbb{P}}_{3,\cdot}$	0	0	0.6213	0	0	-0.0219	σ_{33}	0.0215
			(0.0593)			(0.0054)		(0.0007)
$K^{\mathbb{P}}_{4,\cdot}$	-0.5279	-0.1852	0	0.6609	0	0.0588	σ_{44}	0.0041
	(0.0464)	(0.0309)		(0.0503)		(0.0036)		(0.0001)
$K^{\mathbb{P}}_{5,\cdot}$	0	0	0	0	0.4463	-0.0106	σ_{55}	0.2867
					(0.0580)	(0.0553)		(0.0172)

Table 11: **Estimated Dynamic Parameters of the Preferred B-CLR-L Model**
The table shows the estimated parameters of the $K^{\mathbb{P}}$ matrix, $\theta^{\mathbb{P}}$ vector, and diagonal Σ matrix for the preferred B-CLR-L model. The estimated value of λ is 0.4426 (0.0048), while $\alpha^R = 0.7952$ (0.0182), $\kappa^{\mathbb{Q}}_{liq} = 15.49$ (0.0617), and $\theta^{\mathbb{Q}}_{liq}$ = -0.0014 (0.0001). The maximum log likelihood value is 32,072.71. The numbers in parentheses are the estimated parameter standard deviations.

the B-CLR-L model, we examine its ten-year nominal yield and BEI decompositions. The breakdown of the ten-year nominal yield for the full sample is shown in Figure 7(a). We note the absurdly high level of the average expected nominal short rates over the next ten years. They are practically uniformly above the observed ten-year nominal yields. We add that the 5yr5yr real yield decomposition is equally nonsensical and not shown. In contrast, its decomposition of the ten-year BEI shown in Figure 7(b) is reasonable and qualitatively similar to the decomposition implied by our preferred B-CLR-L model and shown in Section 6 in the main text. Combined we take these results as evidence that the specification above is misspecified. This divergence in results also underscores the importance of eliminating irrelevant superfluous parameters. These observations make us prefer the specification identified in the CLR-L model selection also for the shadow-rate B-CLR-L model.

The estimated parameters from our preferred B-CLR-L model are reported in Table 11. Overall, the estimated parameters are qualitatively very similar to those reported in Section 5.2 of the main text for the preferred CLR-L model. Given that the ZLB never appears to be severely constraining the behavior of Canadian nominal bond yields during our sample period, this consistency in the estimated parameters across the preferred CLR-L and B-CLR-L models seems reasonable.

G Robustness Checks

In this appendix, we provide details on some of the robustness checks we have performed. In the first set of exercises, we examine the effects on our results of altering various model assumptions. We then examine the realism of the short-term real rates generated by our preferred CLR-L model. We end the section with a study of the real-time performance of our preferred CLR-L model, including a real-time out-of-sample inflation forecast exercise structured to match the monthly Consensus Forecasts surveys.

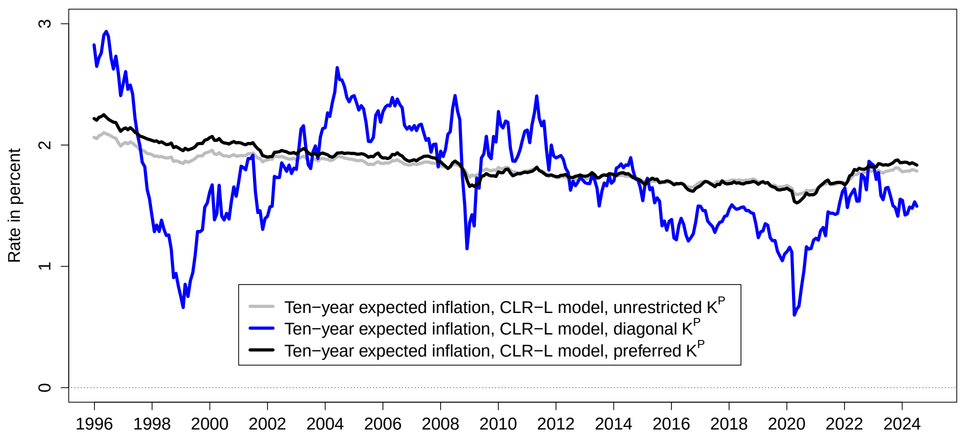
G.1 Model Assumptions and Implementation

G.1.1 Sensitivity to $\mathbb{P}$-Dynamics

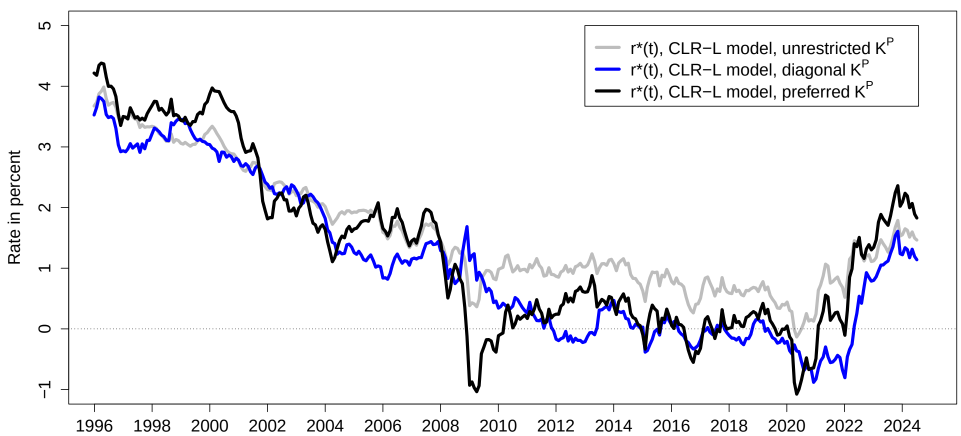
To assess the sensitivity of our ten-year expected inflation and r_t^* estimates to the specification of the mean-reversion matrix $K^{\mathbb{P}}$, we compare them in Figure 8 to the corresponding estimates from the CLR-L models with unrestricted and diagonal $K^{\mathbb{P}}$ matrix, respectively. In panel (a) of the figure, we note the clear sensitivity of the ten-year expected inflation to the choice of $K^{\mathbb{P}}$ specification. Importantly, both of the alternative estimates suggest that long-term expected inflation is even lower at the end of our sample than estimated by the preferred CLR-L model. Thus, we view our results regarding the decline in long-term inflation expectations to be conservative based on this evidence. In contrast, as can be seen from Figure 8(b), our r_t^* estimate is not overly sensitive to this model choice given that all three series are close to each other throughout the sample.

G.1.2 Effect of Liquidity Adjustment

The effect on our estimates from accounting for liquidity premiums in the RRB prices is the subject of Figure 9. In panel (a), the black line is the estimate of the ten-year expected inflation from the preferred CLR-L model, and the grey line is the estimate from the preferred CLR model described in Appendix E, which does not account for time-varying liquidity effects in RRB prices. While the estimate from the CLR-L model indicates a persistent gradual decline in investors' long-term inflation expectations from an original level slightly above two percent in the mid-1990s, the estimate from the CLR model is very erratic starting from a level well above two percent in the mid-1990s before dropping to near zero in the late 1990s. It then reverses that decline in the early 2000s and stays elevated until 2011 when it starts



(a) Ten-year expected inflation

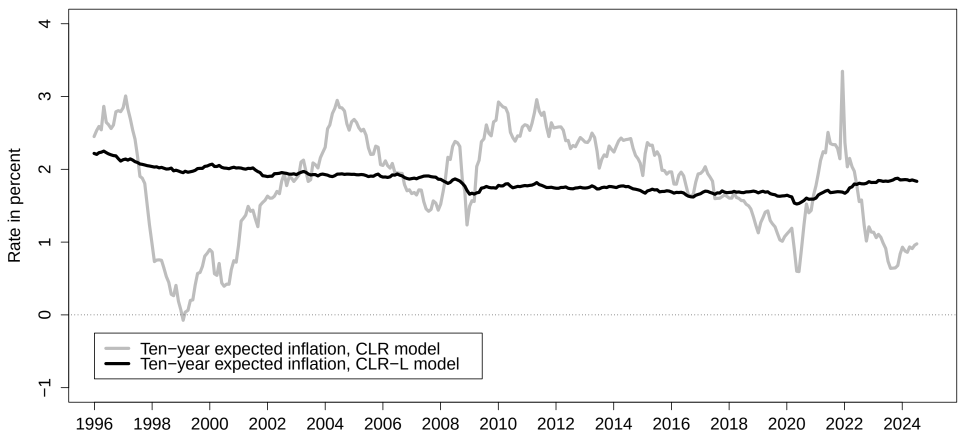


(b) r_t^*

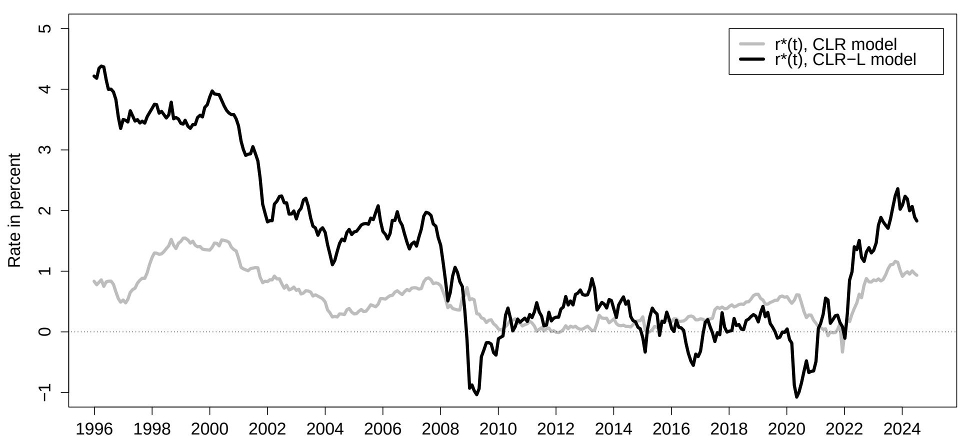
Figure 8: **Sensitivity of Ten-Year Expected Inflation and r* Estimates to $K^{\mathbb{P}}$ Specification**

a persistent downward trend that leaves it close to one percent by the end of our sample. In panel (b), the black line is the estimate of r_t^* from the preferred CLR-L model, and the grey line is the r_t^* estimate from the preferred CLR model. Here, it is also the case that accounting for the liquidity premiums in RRB prices leads to notable differences in the long-term profile of the r_t^* estimates.

Overall, we find the differences in the estimates from the two models to be sizable. More importantly, we consider the estimates from the CLR model to be unconvincing thanks to their high volatility and unusual time-series patterns. This underscores the importance of the



(a) Ten-year expected inflation



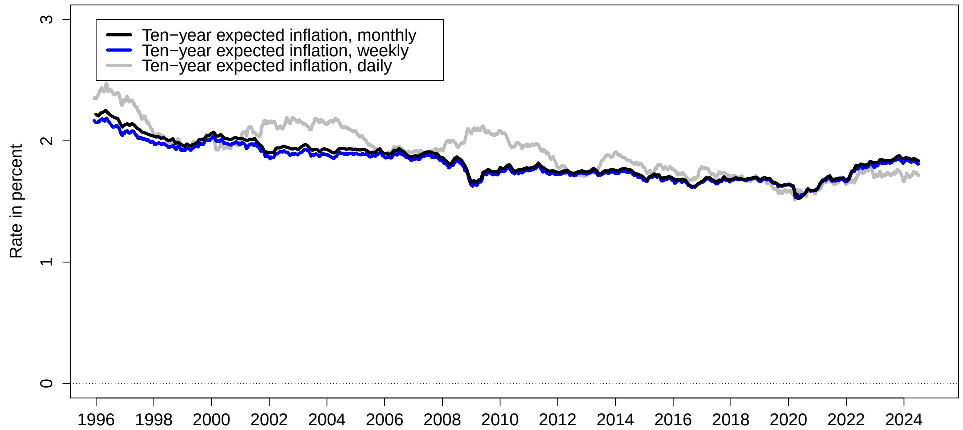
(b) r_t^*

Figure 9: **Effect of Liquidity Adjustment on Estimates of Ten-Year Expected Inflation and r***

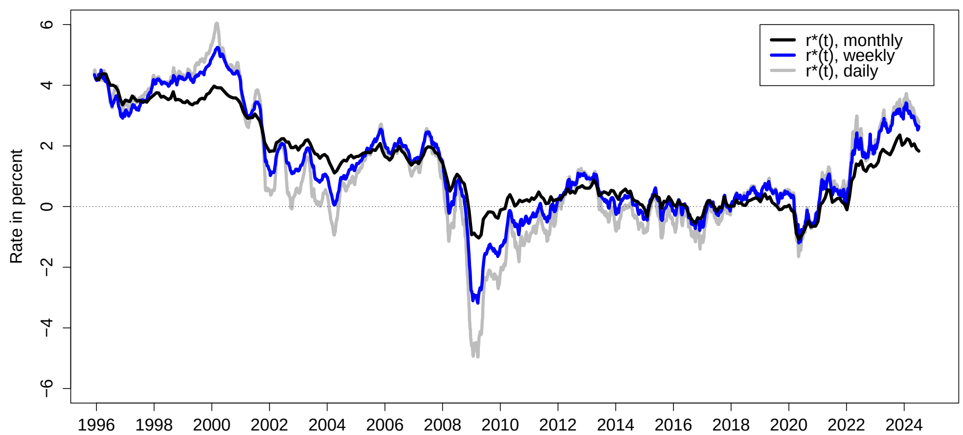
RRB liquidity premium adjustment provided within the CLR-L model.

G.1.3 Sensitivity to Data Frequency

In this section, we re-estimate the preferred CLR-L model using daily and weekly data instead of the monthly frequency considered throughout most of the paper. Figure 10 compares the ten-year expected inflation and r_t^* from the three estimations. In terms of the estimated paths for the ten-year expected inflation, they are very similar and insensitive to the choice of data frequency, while there is somewhat more dispersion in the estimated paths for r_t^*. However,



(a) Ten-year expected inflation



(b) r_t^*

Figure 10: **Sensitivity of Ten-Year Expected Inflation and r* Estimates to Data Frequency**

they are relatively close outside of two brief periods in 1998-1999 and 2009-2010, and they all point to a persistent trend lower from 1995 to 2020, which is somewhat more pronounced with daily and weekly data than at monthly frequency. And this also applies to the sharp reversal during the last two years of the sample. Importantly, this demonstrates that the CLR-L model can be robustly implemented at the data frequencies normally considered in financial studies, including our own analysis of unconventional monetary policy spillovers in Section 6 in the main text using daily data. Furthermore, the results from the monthly data considered throughout the paper are somewhat more stable and hence conservative.

G.1.4 Alternative Model Dynamics

As an additional robustness check, we analyze the sensitivity of our r_t^* estimate to the shadow-rate model alternative. Given that a notable fraction of our sample is near the ZLB, we briefly explore whether our r_t^* estimate is robust to accounting for the ZLB through the shadow-rate extension of the CLR-L model. As already described, we adopt an approach inspired by Black (1995) and replace r_t^N in equation (4) by $r_t^N = \max\left(L_t^N + S_t, 0\right)$ and solve for nominal yields using the approximation in Christensen and Rudebusch (2015), but this B-CLR-L model is otherwise identical to the CLR-L model described in the main text.

As shown in Figure 11, the two r_t^* estimates from the preferred CLR-L and B-CLR-L models are relatively close to each other and highly positively correlated. In contrast, the estimated ten-year inflation from the preferred shadow-rate B-CLR-L model is characterised by a more material and persistent decline than observed within our preferred CLR-L model.

To better understand the differences between the two models' decompositions, we note that the modified functional forms fitted to the data implies very similar estimated state variables as shown in Figure 7 in the main text. Instead, the differences can be traced back to differences in the estimated dynamic parameters that entail that long-term inflation expectations are somewhat more volatile in the B-CLR-L model, while the natural real rate estimates are barely distinguishable. To establish this, we use the B-CLR-L model and combine it with the estimated parameters from the CLR-L model. As shown in Figure 11, this produces results barely distinguishable from those delivered by the CLR-L model.

As a corollary, we also note that a shadow-rate specification matters little for measuring the stance of monetary policy as detailed in Section 5.4.2 in the main text given the small sensitivity of our r_t^* estimate to that alternative.

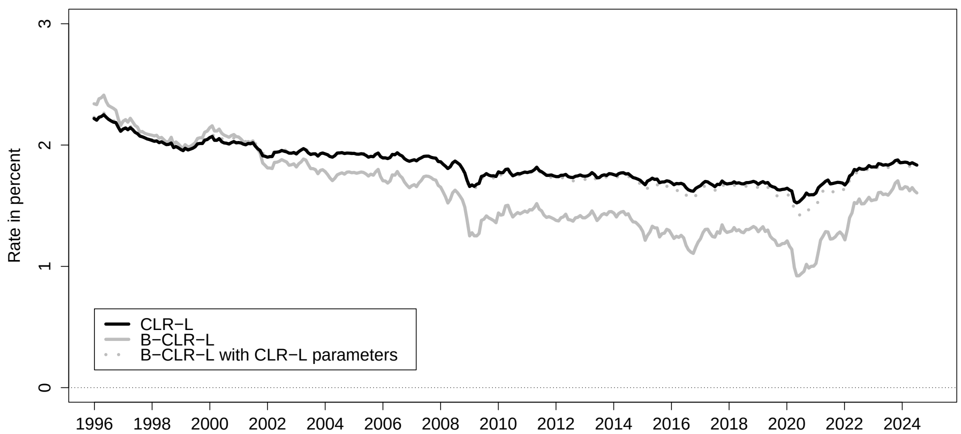
Overall, we take these results to support our choice to focus on the workhorse Gaussian model class for our general assessment in the main text.

G.2 Realism of the Model-Implied Real Rates

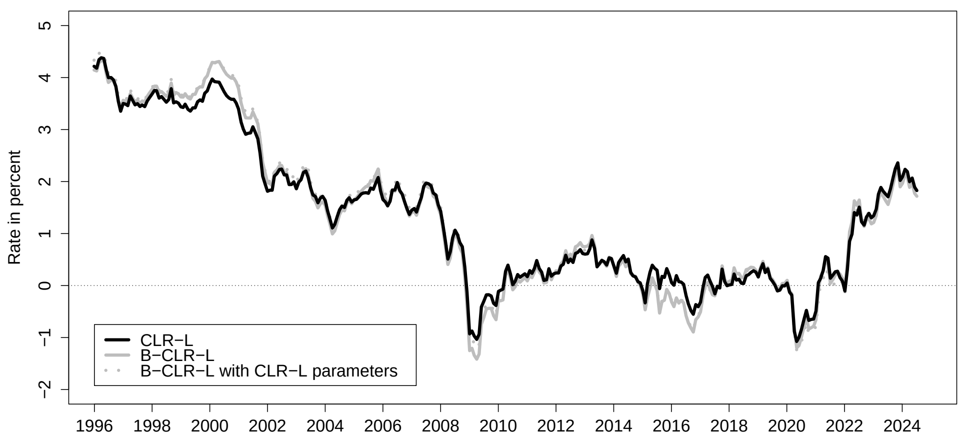
In this section, to build further confidence around our r_t^* estimate, we aim to assess the realism of the model-implied short-term real rates, which is the key building block in our finance-based definition of the natural rate.[10]

To begin, for each month from January 2000 to June 2024, we combine the twelve-month

[10] We thank Min Wei for suggesting this exercise.



(a) Ten-year expected inflation



(b) r_t^*

Figure 11: **Sensitivity of Estimates of Ten-Year Expected Inflation and r* to Alternative Models**

ahead Consensus Forecasts of the three-month Canadian government bond yield with their forecasts of CPI inflation for the coming calendar year to produce a survey-based measure of the three-month real yield expected to prevail 12 months ahead. Although there is a mild mismatch between the two survey forecast horizons, any potential issues arising from that discrepancy are significantly tempered by the general stability of the survey inflation forecasts.[11]

To produce the matching model output, we calculate the model-implied three-month real

[11]This stability is visible in Figure 14.

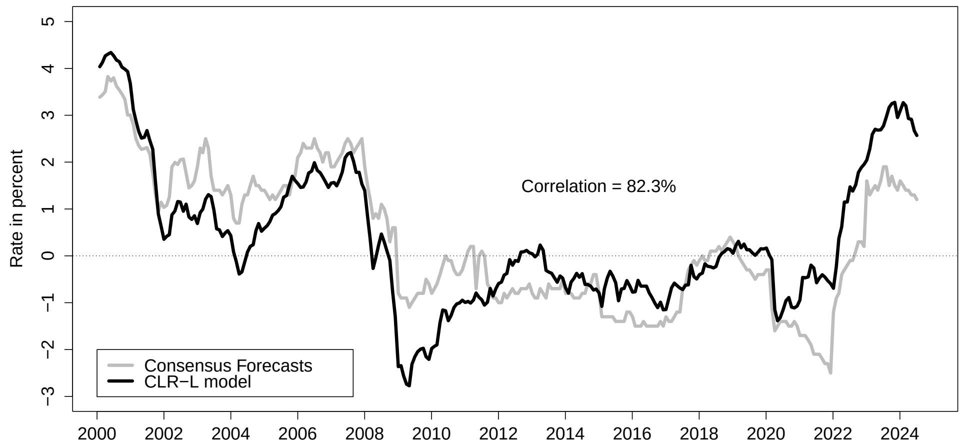


Figure 12: **Model-Implied and Survey-Based Expected Three-month Real Yields**

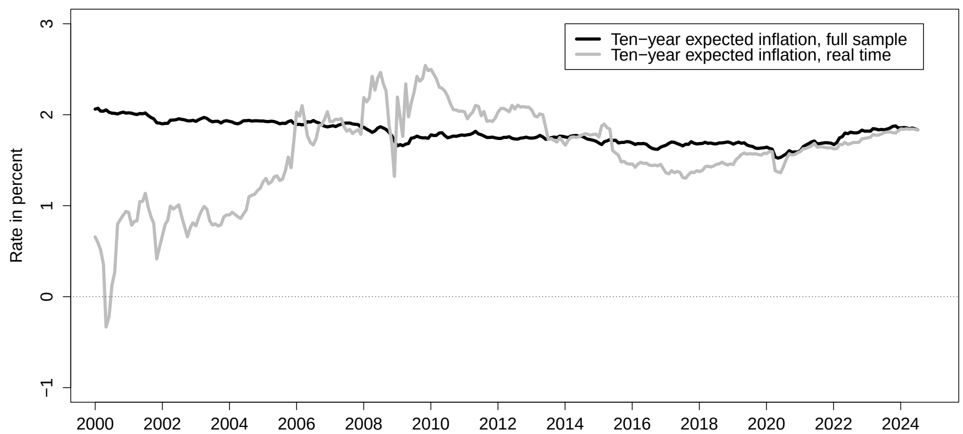
yield expected to prevail 12 months later. As a consequence, the model-implied real yield expectations are for a horizon exactly identical to that of the survey-based real yield forecasts.

Figure 12 shows these survey-based and model-implied expected three-month real yield series. We note both their general closeness and very high positive correlation (82%). Based on this evidence, we conclude that our model is able to generate realistic levels of shorter-term real yields that closely track the information in the Consensus Forecasts surveys even though none of that information is used in the model estimation.

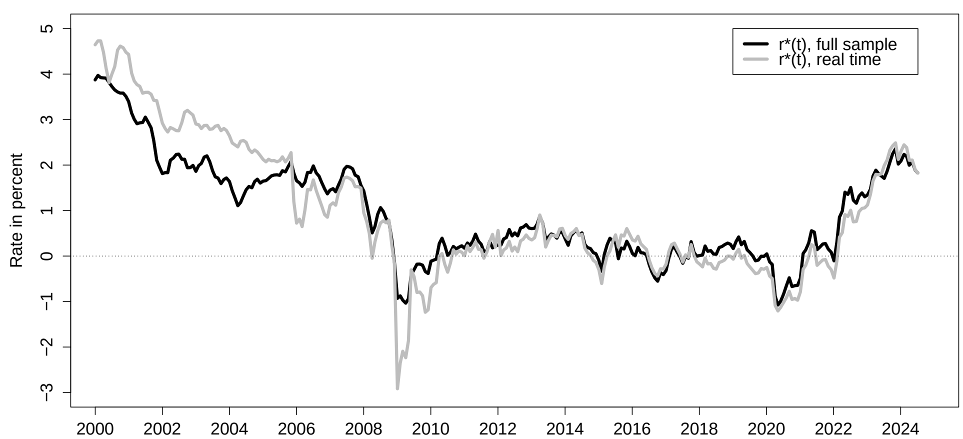
G.3 Real-Time Analysis

A well-known criticism of macro-based estimates of both expected inflation and the natural real rate is that they can exhibit significant variation as additional and revised data become available.

All else equal, finance-based estimates should be less subject to this line of criticism as the key model input, namely the observed bond prices, are available in real time and not subject to any revisions. However, finance-based estimates could still vary as the sample length increases, for example the estimated persistence of the state variables may change, and this could be particularly relevant for our sample period during which the general level of interest rates declined for multiple decades. To dispel such concerns, we estimate the preferred CLR-L model in real time starting in December 1999 through June 2024. This



(a) Ten-year expected inflation



(b) r_t^*

Figure 13: **Accuracy of Real-Time Estimates of Ten-Year Expected Inflation and $\mathbf{r}^*$**

allows us to generate real-time estimates of the ten-year expected inflation and r_t^* and compare them to the corresponding full sample "look back" estimates, which is done in Figure 13. Although we do see some discrepancies between the estimates as we go back through time, these results show that both the ten-year expected inflation and the r_t^* estimates from our preferred CLR-L model are reliable in real time and can be used for policy analysis, which is indeed very encouraging. Importantly, this exercise also provides support for our choice to rely on full-sample estimates in our analysis of the international spillover effects of unconventional monetary policy in Section 6 of the main text.

Full sample: December 31, 1999-November 30, 2022

Model	Mean	RMSE	MAE
Consensus Forecasts	12.89	127.48	84.37
Fitted BEI	42.29	153.85	102.34
Random walk	3.50	170.39	121.73
CLR-L model	72.34	160.82	114.07

Normal period: January 31, 2005-December 31, 2019

Model	Mean	RMSE	MAE
Consensus Forecasts	-31.25	68.82	55.21
Fitted BEI	-8.08	75.77	62.42
Random walk	-10.20	108.78	82.48
CLR-L model	-2.28	73.68	61.20

Table 12: **Summary Statistics of CPI Inflation Forecast Errors**
This table reports the mean forecasting errors (Mean), the root mean squared forecasting errors (RMSE), and the mean absolute forecasting errors (MAE). The preferred CLR-L model forecasts are computed from monthly recursive estimations. The forecast errors are reported as the true value minus the model-implied prediction, and all numbers are reported in annual basis points.

In addition to a tight fit to the data, these results show that real-time output from the CLR-L model is relatively stable. To explore whether these desirable properties allow the model to forecast out of sample, we examine its ability to project future inflation. We structure the forecast exercise to match the Consensus Forecasts survey of professional forecasters at monthly frequency. At the start of each month, the professional forecasters are asked about their expectations for the change in the CPI for the coming calendar year in addition to their expectations about the change for the current calendar year. To have a series of pure forecasts not distorted by incoming realizations, we focus on the monthly survey forecasts of CPI inflation over the coming calendar year. We then use the real-time model estimation results from the end of December 1999 to the end of November 2022 to generate the matching model-implied CPI inflation forecast. This has the advantage that the model-implied forecasts reflect information available at the end of each month and therefore lag the official survey dates by between one and two weeks. Thus, this exercise is by design conservative and slightly handicaps the model.

The summary statistics of the 276 monthly forecast errors from this exercise are reported in the top panel of Table 12. Note that the CLR-L model lags behind fitted BEI and the survey

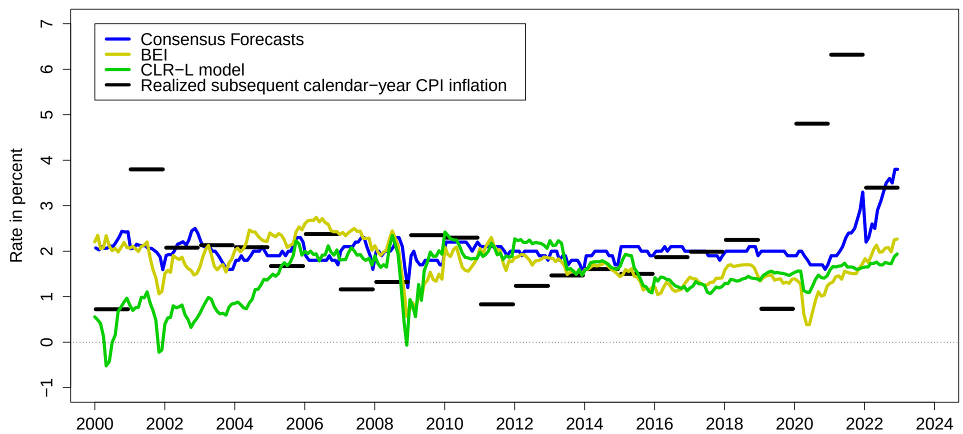


Figure 14: **CPI Inflation Forecasts and Realizations**

forecasts, although it is competitive relative to the random walk assumption as measured by RMSE and MAE.

In comparing the forecast series, Figure 14 shows that the survey forecasts are very stable, even at the short calendar-year-ahead horizon examined here, except for the last two years of the forecast exercise. In contrast, the CLR-L model-implied forecasts exhibit somewhat more variation and mostly consistent with the changes in the corresponding fitted BEI rates. However, we note the CLR-L model's very low inflation forecasts in the 2000-2004 period, which really stands out in the early years, and the significant uptick in the survey forecasts in the 2021-2022 period.

Hence, the CLR-L model is off relative to the survey forecasts early on in the sample when there is only a limited number of real return bonds available, and again during the spell of high inflation in the 2022-2023 period. The latter period causes the model problems due to the fact that BEI rates in Canada and elsewhere around the world barely registered the spell of high inflation. This is also evident in Figure 12 in the main text.

Therefore, to examine the model performance in more normal times, we also report the summary statistics for the in-between fifteen-year period from January 2005 to December 2019. These results are shown in the bottom panel of Table 12. During this period the preferred CLR-L model performs better than both the random walk and fitted BEI, and its performance is only slightly inferior to that of the survey forecasts. Although the preferred

CLR-L model produces a lower mean forecast error than the survey forecast, its RMSE and MAE remain a notch higher.

Overall, we take these observations to imply that the preferred CLR-L model is able to generate what appears to be reasonably realistic short-term inflation dynamics even though our sample is dominated by long-term real return bonds and the model does not get to observe short-term real yields with less than two years to maturity until after December 2019.

G.4 Summary

To summarize, based on the outcomes from the presented set of robustness checks, we consider the output from our preferred CLR-L model to be robust and representative. As a consequence, we feel comfortable relying on the output from this model in our main analysis.

References

Andreasen, Martin M., Jens H. E. Christensen, and Simon Riddell, 2021, "The TIPS Liquidity Premium," *Review of Finance*, Vol. 25, No. 6, 1639-1675.

Andreasen, Martin M., Jens H. E. Christensen, and Glenn D. Rudebusch, 2019, "Term Structure Analysis with Big Data: One-Step Estimation Using Bond Prices," *Journal of Econometrics*, Vol. 212, 26-46.

Black, Fisher, 1995, "Interest Rates as Options," *Journal of Finance*, Vol. 50, No. 7, 1371-1376.

Cheridito, Patrick, Damir Filipović, and Robert L. Kimmel, 2007, "Market Price of Risk Specifications for Affine Models: Theory and Evidence," *Journal of Financial Economics*, Vol. 83, No. 1, 123-170.

Chernov, Mikhail and Philippe Mueller, 2012, "The Term Structure of Inflation Expectations," *Journal of Financial Economics*, Vol. 106, No. 2, 367-394.

Christensen, Jens H. E., Francis X. Diebold, and Glenn D. Rudebusch, 2011, "The Affine Arbitrage-Free Class of Nelson-Siegel Term Structure Models," *Journal of Econometrics*, Vol. 164, No. 1, 4-20.

Christensen, Jens H. E., Jose A. Lopez, and Glenn D. Rudebusch, 2010, "Inflation Expectations and Risk Premiums in an Arbitrage-Free Model of Nominal and Real Bond Yields," *Journal of Money, Credit and Banking*, Supplement to Vol. 42, No. 6, 143-178.

Christensen, Jens H. E., Jose A. Lopez, and Glenn D. Rudebusch, 2014, "Do Central Bank Liquidity Facilities Affect Interbank Lending Rates?," *Journal of Business and Economic Statistics*, Vol. 32, No. 1, 136-151.

Christensen, Jens H. E., Jose A. Lopez, and Glenn D. Rudebusch, 2016, "Pricing Deflation Risk with U.S. Treasury Yields," *Review of Finance*, Vol. 20, No. 3, 1107-1152.

Christensen, Jens H. E. and Glenn D. Rudebusch, 2012, "The Response of Interest Rates to U.S. and U.K. Quantitative Easing," *Economic Journal*, Vol. 122, F385-F414.

Christensen, Jens H. E. and Glenn D. Rudebusch, 2015, "Estimating Shadow-Rate Term Structure Models with Near-Zero Yields," *Journal of Financial Econometrics*, Vol. 13, No. 2, 226-259.

Christensen, Jens H. E. and Glenn D. Rudebusch, 2019, "A New Normal for Interest Rates? Evidence from Inflation-Indexed Debt," *Review of Economics and Statistics*, Vol. 101, No. 5, 933-949.

Dai, Qiang and Kenneth J. Singleton, 2000, "Specification Analysis of Affine Term Structure Models," *Journal of Finance*, Vol. 55, No. 5, 1943-1978.

Duffee, Gregory R., 2002, "Term Premia and Interest Rate Forecasts in Affine Models," *Journal of Finance*, Vol. 57, No. 1, 405-443.

Duffie, Darrell and Rui Kan, 1996, "A Yield-Factor Model of Interest Rates," *Mathematical Finance*, Vol. 6, No. 4, 379-406.

Joslin, Scott, Kenneth Singleton, and Haoxiang Zhu, 2011, "A New Perspective on Gaussian Dynamic Term Structure Models," *Review of Financial Studies*, Vol. 24, No. 3, 926-970.

Litterman, R. and J. A. Scheinkman, 1991, "Common Factors Affecting Bond Returns," *Journal of Fixed Income*, Vol. 1, 62-74.

Merton, Robert C., 1974, "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance*, Vol. 29, No. 2, 449-470.

Nelson, Charles R. and Andrew F. Siegel, 1987, "Parsimonious Modeling of Yield Curves," *Journal of Business*, Vol. 60, No. 4, 473-489.

Rudebusch, Glenn D. and Eric Swanson, 2011, "The Bond Premium in a DSGE Model with Long-Run Real and Nominal Risks," *American Economic Journals: Macroeconomics*, Vol. 4, No. 1, 105-43.

Singleton, Kenneth J., 2006, *Empirical Dynamic Asset Pricing.* Princeton University Press.

Svensson, Lars E. O., 1995, "Estimating Forward Interest Rates with the Extended Nelson-Siegel Method," *Quarterly Review*, No. 3, Sveriges Riksbank, 13-26.