Optimal Policy Rules in HANK

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Inequality & stabilization policy

Does \textit{inequality} change optimal \textit{stabilization policy}? If so, how?

- Recently: increased policy interest & fast-growing academic literature.
  E.g.: Bhandari et al. (2021), Acharya et al. (2022), LeGrand et al. (2022), Davila-Schaab (2023), …
  
  a) \textbf{Transmission}: how do instruments affect any given target? (e.g., output & inflation)
  
  b) \textbf{Objectives}: desire to dampen distributional effects of business cycle
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  a) Transmission: how do instruments affect any given target? (e.g., output & inflation)
  b) Objectives: desire to dampen distributional effects of business cycle

- This paper: linear-quadratic approximation to HANK model
  
  o Derive optimal policy rules as forecast target criteria, applicable for all shocks
  o Main benefits of our approach:
    1. Separate role of inequality through transmission vs. objectives
    2. Sufficient statistics for optimal rules
Main results

a) **Dual mandate** \[ \mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \{ \lambda \pi \hat{\pi}^2_t + \lambda y \hat{y}_t^2 \} \]

- Find same rule as in RANK. Optimal \( \{y, \pi\} \) paths are unaffected by inequality.

b) **Ramsey policy** \[ \mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \{ \lambda \pi \hat{\pi}^2_t + \lambda y \hat{y}_t^2 \} \]

- Find: implications of inequality for opt. policy depend on distributional incidence of policy
  - E.g.: MP is progressive in Bhandari et al. (2022) vs. distributionally neutral in Werning (2015).
- Our strategy: infer distributional incidence from rich quantitative model.
  - (i) Monetary policy is close to neutral w.r.t. distribution. \( \Rightarrow \) Optimal policy close to dual-mandate policy.
  - (ii) Stimulus checks have strong distributional effects. \( \Rightarrow \) Complementary to monetary policy.
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b) **Ramsey policy** \[ \mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \{ \lambda_\pi \hat{\pi}_t^2 + \lambda_y \hat{y}_t^2 + \text{inequality term} \} \]

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  \[ \Rightarrow \text{Optimal policy close to dual-mandate policy.} \]

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Model Environment

• We study perfect foresight transitions.

• Optimal stochastic linear-quadratic regulator features certainty equivalence.
Unit continuum of ex-ante identical households $i \in [0, 1]$

- **Consumption-savings problem**
  - Standard preferences:
    \[
    \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [u(c_{it}) - \nu(l_{it})]
    \]
  - Idiosyncratic earnings—allows for unequal exposure to business cycle:
    \[e_{it} = \Phi(\zeta_{it}, m_t, e_t), \quad \int_0^1 e_{it} di = e_t\]
    where $m_t$ is an “inequality shock” (= demand shock) & $e_t$ is aggregate labor income
  - Budget constraint [$a_{it}$ is value of portfolio]:
    \[c_{it} + \text{[cost of asset purchases]} = a_{it} + (1 - \tau_y + \tau_{e,t}) e_{it} + \tau_{x,t}, \quad a_{it} \geq a\]
Households

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- **Labor supply**: intermediated by labor unions
Production & wage-setting

• Supply side structure
  a) Production
    ◦ Intermediate goods are produced using capital and labor: $y_{jt} = Ak_{jt}^\alpha \ell_{jt}^{1-\alpha}$
    ◦ Subject to nominal rigidities. Pay labor & capital, and earn pure profits. A share $1 - \alpha$ of profits goes to labor. Hard-wiring constant labor share.
    ◦ Aggregate capital is fixed at $\bar{k}$
Production & wage-setting

- **Supply side structure**
  
  a) **Production**
  
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  - Aggregate capital is fixed at $\bar{k}$

  b) **Wage-setting**

  - Unions use rep.-agent MRS for wage-setting [= MRS at average, not average MRS]
  
  - Assume uniform labor rationing—everyone works the same amount
  
  - Inequality does not affect labor supply
Production & wage-setting

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  b) **Wage-setting**
  
  - Unions use rep.-agent MRS for wage-setting \([= MRS \text{ at average, not average MRS}]\)
  
  - Assume uniform labor rationing—everyone works the same amount
  
  - Inequality does not affect labor supply
  
- **Standard NKPC**: \( \hat{\pi}_t = \kappa \hat{y}_t + \beta \hat{\pi}_{t+1} + \psi \varepsilon_t \), where \( \varepsilon_t \) is a **cost-push shock.**
Households can trade **three assets:**

1. Capital
2. Short-term nominal bonds
3. Long-term nominal bonds [coupons decline geometrically]
Assets

- Households can trade **three assets**:
  1. Capital
  2. Short-term nominal bonds
  3. Long-term nominal bonds [coupons decline geometrically]

- Budget constraint:

  \[ c_{it} + \frac{1}{1 + r_t} a_{it+1} = a_{it} + (1 - \tau_y + \tau_{e,t}) e_{it} + \tau_{x,t}, \quad a_{it} \geq a \]

- Don’t need to model **portfolio choice** (all assets pay same return for \( t = 1, 2, \cdots \))

- Only need **existing date-0 portfolios** when asset prices respond to news
  - We will impute these using data on portfolio composition across net worth levels
    Related to approach in Auclert-Rognlie (2020)
Government & eq’m characterization

- Policymaker sets two **policy instruments**
  1. Short-term nominal rate $i_t$
  2. Uniform lump-sum transfers $\tau_{x,t}$

  **Background:** taxes/transfers $\tau_{e,t}$ adjust to keep long-term budget balance.

- **Perfect-foresight eq’m** [notation: boldface = time paths]

**Equilibrium**

Given paths of shocks $\{m_t, \varepsilon_t\}_{t=0}^\infty$ and government policy instruments $\{i_t, \tau_{x,t}\}_{t=0}^\infty$, paths of aggregate output and inflation $\{y_t, \pi_t\}_{t=0}^\infty$ are part of a linearized equilibrium if and only if

$$
\begin{align*}
\hat{\pi} &= \kappa \hat{y} + \beta \hat{\pi}_{t+1} + \psi \hat{\varepsilon} \quad \text{(NKPC)} \\
\hat{y} &= \tilde{C}_y \hat{y} + \tilde{C}_\pi \hat{\pi} + \tilde{C}_i \hat{i} + \tilde{C}_\tau \hat{\tau}_x + C_m \hat{m} \quad \text{(IS*)}
\end{align*}
$$

McKay and Wolf
Dual Mandate
We first study optimal monetary policy for a dual mandate policymaker.

Why? Relevant in practice & allows us to disentangle role of loss function vs. transmission.
Dual mandate optimal policy problem

- We first study optimal monetary policy for a **dual mandate** policymaker
  Why? Relevant in practice & allows us to disentangle role of loss function vs. transmission.

  - **Loss function** [exogenously assumed]
    \[
    \mathcal{L}^{DM} \equiv \sum_{t=0}^{\infty} \beta^t \left[ \lambda_{\pi} \hat{\pi}_t^2 + \lambda_y \hat{y}_t^2 \right] = \lambda_{\pi} \hat{\pi}' W \hat{\pi} + \lambda_y \hat{y}' W \hat{y} \tag{1}
    \]
    where \( W = \text{diag}(1, \beta, \beta^2, \ldots) \)
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\]  

where \( W = \text{diag}(1, \beta, \beta^2, \ldots) \)

- **Constraint set** [follows from eq’m characterization]

\[
\hat{\pi} = \kappa \hat{y} + \beta \hat{\pi}_{t+1} + \psi \epsilon \quad \text{(NKPC)}
\]

\[
\hat{y} = \bar{C}_y \hat{y} + \bar{C}_\pi \hat{\pi} + \bar{C}_i i + \bar{C}_x \hat{\tau}_x + C_m \hat{m} \quad \text{(IS*)}
\]
Sequence-space representation of dual-mandate policy

- FOC for choice of $i_t$

$$\frac{\partial L^{DM}}{\partial \hat{\pi}} \frac{\partial \hat{\pi}}{\partial i_t} + \frac{\partial L^{DM}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial i_t} = 0$$
Sequence-space representation of dual-mandate policy

- FOC for choice of $i_t$

\[
(\lambda_\pi W\hat{\pi})' \frac{\partial \hat{\pi}}{\partial i_t} + (\lambda_y W\hat{y})' \frac{\partial \hat{y}}{\partial i_t} = 0
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Forecast targeting rule: adjust expected policy path so condition above holds.
Sequence-space representation of dual-mandate policy

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- Same holds at all dates. Transpose and stack equations for all $t$:

\[\Theta'_{\pi,i} \lambda_\pi W\hat{\pi} + \Theta'_{y,i} \lambda_y W\hat{y} = 0\]
Sequence-space representation of dual-mandate policy

• FOC for choice of $i_t$

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• Forecast targeting rule: adjust expected policy path so condition above holds.
Proposition

The optimal monetary policy rule for a dual mandate policymaker can be written as the forecast target criterion

\[ \hat{\pi}_t + \frac{\lambda_y}{\lambda_{\pi} \kappa} (\hat{y}_t - \hat{y}_{t-1}) = 0, \quad \forall t = 0, 1, \ldots \]
Proposition

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(2)

- **Same target criterion as in RANK model**
  
  - Inequality does not affect *optimal* \( \{y, \pi\} \) paths in response to *any* non-policy shock
  
  - Demand block only matters *residually* for sequence of interest rates needed to achieve the optimal \( \{y, \pi\} \) paths?
Ramsey Problem
We consider a **social welfare function** with Pareto weights

\[
\mathcal{V}^{HA} = \sum_{t=0}^{\infty} \beta^t \int \varphi(\zeta) \left[ u(\omega_t(\zeta) c_t) - \nu(l_t) \right] d\Gamma(\zeta)
\]

(3)

- \(\zeta\) is the idiosyncratic history of a household, \(\varphi(\zeta)\) is a Pareto weight on the utility of households with history \(\zeta\), and \(\omega_t(\zeta)\) is the time-\(t\) consumption share of such households.
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**Objective:** evaluate (3) to second-order using first-order approximation of eq’m
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**Objective**: evaluate (3) to second-order using first-order approximation of eq’m

**Our approach**: ensure efficient steady state [as in Woodford (2003)]

- Assumptions: production subsidy + back out weights \(\varphi(\zeta)\)
- Our SWF will capture cyclical insurance motive, not long-run redistribution
Full Ramsey problem

The problem then fits into **linear-quadratic form**:

- **Loss function**: to second order, social welfare function $\mathcal{V}^{HA}$ is proportional to $-\mathcal{L}^{HA}$

  \[
  \mathcal{L}^{HA} = \sum_{t=0}^{\infty} \beta^t \left[ \lambda_{\pi} \hat{\pi}_t^2 + \lambda_y \hat{y}_t^2 + \int \lambda_{\omega(\zeta)} \hat{\omega}_t(\zeta)^2 d\Gamma(\zeta) \right]
  \]

  inequality term

  \[
  = \lambda_{\pi} \hat{\pi}'W\hat{\pi} + \lambda_y \hat{y}'W\hat{y} + \int \lambda_{\omega(\zeta)} \hat{\omega}(\zeta)'W\hat{\omega}(\zeta) d\Gamma(\zeta)
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\]

- **Computation**: stabilizing consumption distribution = stabilizing prices

McKay and Wolf
Ramsey Problem

Optimal Monetary Policy
Proposition

The optimal monetary policy rule for a Ramsey planner with loss $L^{HA}$ can be written as the forecast target criterion

$$
\Theta'_{\pi,i} \lambda_{\pi} W \hat{\pi} + \Theta'_{y,i} \lambda_{y} W \hat{y} \quad + \quad \int \Theta'_{\omega(\zeta),i} \lambda_{\omega(\zeta)} W \hat{\omega}(\zeta) d\Gamma(\zeta) = 0
$$

- **Dual-mandate criterion**
- **Effects of instrument on consumption shares**

So does inequality affect the optimal policy rule?
Optimal Ramsey monetary policy rule

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- dual-mandate criterion
- effects of instrument on consumption shares

- So does **inequality** affect the optimal policy rule?
  - No iff policy does not affect consumption shares ($\Theta_{\omega(\zeta),i} = 0$) [e.g. as in Werning (2015)]
  - Yes in prior work: large distributional effects that can offset effects of business-cycle shocks Bhandari et al. (2021): rate cut offsets distributional effects of cost-push shock.
Optimal Ramsey monetary policy rule

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- **dual-mandate criterion**
- **effects of instrument on consumption shares**

So does inequality affect the optimal policy rule?

- What do we know about $\Theta_{\omega(\zeta),i}$?

⇒ **Our strategy**: use data on household balance sheets to discipline distributional effects.
Key calibration points

1. Income cyclicality: labor income more cyclical for low-income workers

\[ e_{it} = \Phi(\zeta_{it}, m_t, e_t) \] calibrated as in Guvenen et al. (2022)
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   - e.g. typical middle-class household is long capital (housing) and short bonds (mortgage)
Consumption effects of monetary policy

- **Model**
  - Simulate expansionary monetary shock,
  - plot initial cons. change by wealth
  - find rather small distr. effects
Consumption effects of monetary policy

**Model**
- Simulate expansionary monetary shock,
- plot initial cons. change by wealth
  ⇒ find rather small distr. effects

**Empirical evidence**
- Holm et al. ('21): U-shaped effect
- Coibion et al. ('17): progressive effect
- Chang & Schorfheide ('22): regressive effect
• **Dual mandate**: cut rates to perfectly stabilize aggregate demand and so \( \{y, \pi\} \)


• **Ramsey policy**: similar, since monetary policy is ill-suited to offset the distr. incidence

⇒ stabilizing consumption at the bottom would imply large overshooting of $y$ and $\pi$
Application: distributional shock

- **Ramsey policy**: similar, since monetary policy is ill-suited to offset the distr. incidence
  \[ \Rightarrow \text{stabilizing consumption at the bottom would imply large overshothing of } y \text{ and } \pi \]
**Joint fiscal-monetary**: stimulus checks provide agg. & cross-sectional stabilization

⇒ monetary policy at the Ramsey optimum barely responds
Stimulus check incidence

Consumption change on impact

% of steady state income vs. Wealth percentile
Conclusions
Conclusions

• How does inequality affect optimal stabilization policy?
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a) Dual mandate

  ◦ Same $y$ & $\pi$ outcomes
  ◦ Optimal rate paths unlikely to change much
Conclusions

• How does inequality affect optimal stabilization policy?

a) Dual mandate
   ○ Same $y$ & $\pi$ outcomes
   ○ Optimal rate paths unlikely to change much

b) Ramsey policy
   ○ Deviate from dual mandate policy iff monetary policy has substantial distributional effects
   ○ Our reading of evidence & model: exposure to MP is fairly uniform
   ○ Fiscal policy is much better-suited for cyclical insurance
Appendix
• Unit continuum of unions $k$, demand $\ell_{ikt}$ units from household $i$. Total union labor supply is $\ell_{kt} \equiv \int_0^1 e_{it} \ell_{ikt} \, dt$.

• Total output is

$$y_t = \left( \int_k \ell_{kt}^\frac{\varepsilon_t}{\varepsilon_t-1} \, dk \right)^{\frac{\varepsilon_t}{\varepsilon_t-1}}$$

• The price index of the labor aggregate is

$$w_t = \left( \int_k w_{kt}^{1-\varepsilon_t} \, dk \right)^{1/(1-\varepsilon_t)}$$

and demand for labor from union $k$ is

$$\ell_{kt} = \left( \frac{W_{kt}}{w_t} \right)^{-\varepsilon_t} y_t.$$
• Union problem: choose the reset wage $w^*$ and $l_{kt}$ to maximize

$$\sum_{s \geq 0} \beta^s \theta^s \left[ u_c(c_{t+s})(1-\tau_y) \frac{\bar{\epsilon} \Xi}{(\bar{\epsilon} - 1)(1-\tau_y)} \frac{w^* l_{kt} - \nu \ell (l_{t+s}) l_{kt}}{\rho_{t+s}} \right]$$

subject to labor demand constraint

$\Xi$ is subsidy-related steady-state wedge, see loss function proof.

• This gives

$$\hat{\pi}_t = \kappa \hat{y}_t + \beta \hat{\pi}_{t+1} + \psi \hat{\epsilon}_t$$

where $\kappa \equiv (\phi + \gamma) \frac{(1-\theta)(1-\beta \theta)}{\theta}$, $\phi \equiv \frac{\nu \ell (l)}{\nu_t (l)}$ and $\psi \equiv -\frac{\kappa}{(\phi + \gamma)(\epsilon - 1)}$

• Aggregating production gives $y_t = \frac{\ell_t}{d_t}$ where $l_t \equiv \int_0^1 \int_0^1 e_{it} l_{ikt} d i d k$ and $d_t$ captures efficiency losses
Equilibrium characterization

- NKPC is as in original optimality condition. Proof combines all other optimality and market-clearing conditions to get (IS*)

- Consumption-savings problem gives aggregate consumption function. Using output market-clearing, $e_{it} w_t l_{it} = e_{it} y_t$, we get

$$\hat{y} = C_y \hat{y} + C_r \hat{r} + C_x \hat{x} + C_e \hat{e} + C_m \hat{m}$$

- Write relationships between asset prices and rates of return as

$$\hat{r}_0 = r_0(\hat{\pi}_0, \hat{y}_0, \hat{q}_0), \quad \hat{r}_{+1} = r_{+1}(i, \hat{\pi}), \quad \hat{q} = q(\hat{\pi}_{+1}, \hat{y}_{+1}, \hat{r}_{+1})$$

- From the government budget constraint we get

$$\hat{\tau}_e = \tau_e(\hat{y}, \hat{\tau}_x, \hat{\pi}, \hat{q})$$
Equilibrium characterization

- Plugging the asset pricing and gov’t budget relations into the consumption function:

\[ \hat{y} = C_y\hat{y} + C_r\hat{r}(\hat{y}, \hat{\pi}, \hat{i}) + C_x\hat{\tau}_x + C_e\hat{\tau}_e(\hat{y}, \hat{\pi}, \hat{i}, \hat{\tau}_x) + C_m m \]

and so

\[ \hat{y} = \left[ C_y + C_r R_y + C_e T_y \right] \hat{y} + \left[ C_r R_\pi + C_e T_\pi \right] \hat{\pi} + \left[ C_r R_i + C_e T_i \right] \hat{i} + \left[ C_x + C_e T_x \right] \hat{\tau}_x + C_m m \]

- This has verified all eq’m relations, giving sufficiency of (NKPC) and (IS*)
Optimal dual mandate rule: proof

- FOCs of optimal policy problem are

\[ \lambda_\pi W\hat{\pi} + \Pi_\pi W\phi_\pi - \tilde{\phi}'_\pi W\phi_y = 0 \]

\[ \lambda_y W\hat{y} - \Pi_y W\phi_\pi + (I - \tilde{\phi}'_y)W\phi_y = 0 \]

\[ -\tilde{\phi}'_i W\phi_y = 0. \]

- Guess that \( \phi_y = 0 \). Then we get

\[ \lambda_\pi \hat{\pi} + \lambda_y W^{-1}\Pi'_\pi (\Pi'_y)^{-1}W\hat{y} = 0 \]

which can re-written to give the stated relation

- Remains to verify the guess that \( \phi_y = 0 \)
Optimal dual mandate rule: proof

- Consider some arbitrary \((m, \varepsilon)\), and let \((\hat{y}^*, \hat{\pi}^*)\) denote the solution of the system (NKPC) + dual mandate rule given \((m, \varepsilon)\).

- Plugging into the consumption function:

\[
\hat{y}^* - \tilde{C}_y \hat{y}^* - \tilde{C}_\pi \hat{\pi}^* - C_m m = \tilde{C}_i \hat{i}
\]

\[\text{demand target}\]

- Remains to show that we can find \(\hat{i}^*\) such that this relation holds.
Optimal dual mandate rule: proof

- Supply term has NPV
  \[ \sum_{t=0}^{\infty} \left( \frac{1}{1 + \bar{r}} \right)^t \bar{y} \hat{y}_t \]

- Aggregating household budget constraints we get that
  \[ \sum_{t=0}^{\infty} \left( \frac{1}{1 + \bar{r}} \right)^t \bar{c} \hat{c}_t = \sum_{t=0}^{\infty} \left( \frac{1}{1 + \bar{r}} \right)^t \left\{ (1 + \bar{r}) \bar{a}_t \bar{r} + (1 - \tau_y) \bar{y} \hat{y}_t + \bar{x} \hat{x}_t + \bar{e} \hat{e}_t \right\} \]
  
  Doing the same for the gov’t budget constraint:
  \[ \sum_{t=0}^{\infty} \left( \frac{1}{1 + \bar{r}} \right)^t \left\{ (1 + \bar{r}) \bar{a}_t + \bar{x} \hat{x}_t + \bar{e} \hat{e}_t \right\} = \sum_{t=0}^{\infty} \tau_y \bar{y} \hat{y}_t \]

- Thus the two have the same NPV. Then the stated condition is sufficient to ensure implementability.
Ramsey loss function

**Proposition**

To second order, the social welfare function (3) is proportional to $-\mathcal{L}^{HA}$, given as

$$
\mathcal{L}^{HA} \equiv \sum_{t=0}^{\infty} \beta^t \left[ \hat{\pi}_t^2 + \frac{\kappa}{\bar{\varepsilon}} \hat{y}_t^2 + \frac{\kappa \gamma}{(\gamma + \phi)\bar{\varepsilon}} \int \frac{\hat{\omega}_t(\zeta)^2}{\bar{\omega}(\zeta)} d\Gamma(\zeta) \right]
$$

(4)

where $\hat{\omega}_t(\zeta) = \omega_t(\zeta) - \bar{\omega}(\zeta)$ and $\bar{\omega}(\zeta)$ is the steady-state consumption share of an individual with history $\zeta$. 
Ramsey planner loss function: proof

- Write planner per-period utility flow as

\[
U_t = \int \varphi(\zeta) \left( \bar{c}^e_{\hat{c}t} \omega_t(\zeta) \right)^{1-\gamma} \frac{1}{1 - \gamma} d\Gamma(\zeta) - \nu \left( \bar{t} \bar{c}^e_{\hat{c}t} \right)
\]  \hspace{1cm} (5)

- Objective: find 2nd-order approximation to \( U_t \) that depends only on 2nd-order terms

- Preliminary definitions
  - Steady state needs to equalize marginal utility of consumption across histories:
    \[
    \varphi(\zeta) \bar{c}^{1-\gamma} \bar{\omega}(\zeta)^{-\gamma} = \bar{u}_c \bar{c} \quad \forall \zeta
    \]
  - Imposing that consumption shares integrate to 1 yields
    \[
    \int \varphi(\zeta)^{1/\gamma} d\Gamma(\zeta) = \bar{c} \bar{u}_c^{1/\gamma}
    \]
Ramsey planner loss function: proof

- Preliminary definitions
  - Can recover consumption shares as a function of planner weights:
    \[ \bar{\omega}(\zeta) = \frac{\varphi(\zeta)^{1/\gamma}}{\int \varphi(\zeta)^{1/\gamma} d\Gamma(\zeta)} \quad \forall \zeta \]
  - For future reference define
    \[ \Xi \equiv \left( \int \varphi(\zeta)^{1/\gamma} d\Gamma(\zeta) \right)^\gamma = \varphi(\zeta)\bar{\omega}(\zeta)^{-\gamma} \quad \forall \zeta \]
- Now can begin with first-order terms:
  - For \( c_t \) we get
    \[ \frac{\partial U}{\partial c_t} = \int \varphi(\zeta)(\bar{c}\bar{\omega}(\zeta))^{1-\gamma} d\Gamma(\zeta) \]
    \[ = \bar{c}^{1-\gamma}\Xi \]
Ramsey planner loss function: proof

- Now can begin with first-order terms:
  - For $\ell_t$ we have
    \[
    \frac{\partial U}{\partial \ell_t} = -\nu_\ell(\ell)\ell.
    \]
    Set union subsidy so that $\Xi \bar{c}^{-\gamma} = \nu_\ell$
  - For consumption shares $\omega_t(\zeta)$ we have
    \[
    \frac{\partial U}{\partial \omega_t(\zeta)} = \phi(\zeta)\bar{c}^{1-\gamma}\bar{\omega}(\zeta)^{-\gamma}d\Gamma(\zeta)
    \]
    \[
    = \bar{c}^{1-\gamma}\Xi d\Gamma(\zeta)
    \]
Ramsey planner loss function: proof

- Next consider second-order terms:
  - For level & split of consumption we have
    \[
    \frac{\partial^2 U_t}{\partial \tilde{c}_t^2} = (1 - \gamma)\bar{c}^{1-\gamma}
    \]
    \[
    \frac{\partial U_t}{\partial \omega_t(\zeta)^2} = -\gamma\bar{c}^{1-\gamma} \frac{\bar{\Xi}}{\bar{\omega}(\zeta)} d\Gamma(\zeta)
    \]
    \[
    \frac{\partial^2 U_t}{\partial \tilde{c}_t \partial \omega_t(\zeta)} = (1 - \gamma)\bar{c}^{1-\gamma} d\Gamma(\zeta)
    \]
  - For hours worked we have
    \[
    \frac{\partial^2 U}{\partial \tilde{\ell}_t^2} = -\nu_{\ell\ell}(\tilde{\ell})\tilde{\ell}^2 - \nu_{\ell}(\tilde{\ell})\tilde{\ell}
    \]
Ramsey planner loss function: proof

- We can now put everything together:

\[
U_t \approx \bar{U} + \bar{c}^{1-\gamma} \Xi \hat{c}_t - \nu_{l}(\ell) \bar{\ell} \ell_t \\
+ \frac{1}{2} (1 - \gamma) \Xi \bar{c}^{1-\gamma} \hat{c}_t^2 - \frac{1}{2} \left[ \nu_{\ell l}(\ell) \bar{\ell}^2 + \nu_{\ell}(\ell) \bar{\ell} \right] \ell_t^2 - \frac{1}{2} \gamma \bar{c}^{1-\gamma} \Xi \int \frac{\hat{\omega}(\zeta)^2}{\hat{\omega}(\zeta)} d\Gamma(\zeta) \\
+ \bar{c}^{1-\gamma} \Xi \int \hat{\omega}_t(\zeta) d\Gamma(\zeta) + (1 - \gamma) \bar{c}^{1-\gamma} \Xi \hat{c}_t \int \hat{\omega}_t(\zeta) d\Gamma(\zeta)
\]

Terms in last row are zero.

- Can now write this as

\[
U_t \approx \bar{U} + \bar{c}^{1-\gamma} \Xi \hat{c}_t - \nu_{l}(\ell) \bar{\ell} \left( \hat{c}_t + \hat{d}_t \right) \\
+ \frac{1}{2} (1 - \gamma) \Xi \bar{c}^{1-\gamma} \hat{c}_t^2 - \frac{1}{2} (\phi + 1) \nu_{\ell}(\ell) \bar{\ell} (\hat{c}_t + \hat{d}_t)^2 - \frac{1}{2} \gamma \bar{c}^{1-\gamma} \Xi \int \frac{\hat{\omega}(\zeta)^2}{\hat{\omega}(\zeta)} d\Gamma(\zeta)
\]

McKay and Wolf
Ramsey planner loss function: proof

- Set union subsidy so that the $\widehat{c}_t$ terms cancel. We thus have

$$U_t \approx \bar{U} - \nu(\bar{e})\bar{e}\delta_t - \frac{1}{2} \nu(\bar{e})\bar{e} (\gamma + \phi) \hat{y}_t^2 - \frac{1}{2} \gamma \nu(\bar{e})\bar{e} \int \frac{\hat{w}(\zeta)^2}{\bar{w}(\zeta)} d\Gamma(\zeta)$$

- Finally follow standard steps to express $d_t$ in terms of the history of inflation. After standard steps we get

$$\sum_{t=0}^{\infty} \beta^t U_t \approx -\nu(\bar{e})\bar{e} \sum_{t=0}^{\infty} \beta^t \left[ \frac{\theta \bar{e}}{2(1-\theta)(1-\beta\theta)} \hat{\pi}_t^2 + \frac{1}{2} (\gamma + \phi) \hat{y}_t^2 + \frac{\gamma}{2} \int \frac{\hat{w}(\zeta)^2}{\bar{w}(\zeta)} d\Gamma(\zeta) \right]$$

$$= -\frac{\nu(\bar{e})\bar{e} \theta \bar{e}}{2(1-\theta)(1-\beta\theta)} \sum_{t=0}^{\infty} \beta^t \left[ \hat{\pi}_t^2 + \frac{\kappa}{\bar{e}} \hat{y}_t^2 + \frac{\kappa \gamma}{(\gamma + \phi) \bar{e}} \int \frac{\hat{w}(\zeta_t)^2}{\bar{w}(\zeta_t)} d\Gamma(\zeta_t) \right] ,$$
Getting the $\Omega$’s: computational details

- **Idea:** can obtain fluctuations in consumption shares as a function of fluctuations in a small number of inputs to the consumption-savings problem.

- **Formally,** let $\mathbf{x} \equiv (\mathbf{r}', \mathbf{y}', \mathbf{\tau}'_x, \mathbf{\tau}'_e, \mathbf{m}')'$ be the stacked sequences of inputs to the household problem. Then can show that there is symmetric matrix $Q$ such that

$$
\sum_{t=0}^{\infty} \beta^t \int \frac{\hat{\omega}_t(\zeta, \mathbf{x})^2}{\bar{\omega}(\zeta)} d\Gamma(\zeta) = \mathbf{x}'Q\mathbf{x} + \mathcal{O}(||\mathbf{x}||^3)
$$

- **Key step** is to show that $\hat{\omega}_t(\zeta, \mathbf{x}) \approx \Omega_t(\zeta)\hat{\mathbf{x}}$ which yields

$$
\frac{\hat{\omega}_t(\zeta^t, \mathbf{x})^2}{\bar{\omega}(\zeta^t)} = \mathbf{x}' \frac{\Omega_t(\zeta^t)\Omega_t(\zeta^t)}{\bar{\omega}(\zeta^t)} \hat{\mathbf{x}} + \mathcal{O}(||\hat{\mathbf{x}}||^3)
$$

and so

$$
\sum_{t=0}^{\infty} \beta^t \int \frac{\hat{\omega}_t(\zeta^t, \mathbf{x})^2}{\bar{\omega}(\zeta^t)} d\Gamma(\zeta^t) = \mathbf{x}' \left( \sum_{t=0}^{\infty} \beta^t \int Q_t(\zeta^t) d\Gamma(\zeta^t) \right) \hat{\mathbf{x}} + \mathcal{O}(||\hat{\mathbf{x}}||^3)
$$

\[\equiv Q\]
Getting the $\Omega$’s: computational details

- We obtain $\Omega_t(\zeta)$ using sequence-space methods + simulation [see paper for details]

- Given $Q$, we have a finite-dimensional but non-diagonal LQ problem
  - The objective function can be written as
    \[ L = \frac{1}{2} x' P x, \]
  - We then get the FOC
    \[ \Theta'_{xz} P x = 0 \]
  - and the corresponding optimal instrument path
    \[ z^* = -(\Theta'_{x,z} P \Theta_{x,z})^{-1} \times (\Theta'_{x,z} P \Theta_{x,e} \cdot \varepsilon) \]
More on model calibration

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>CRRA</td>
<td>1.2</td>
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<tr>
<td>$\phi$</td>
<td>Frisch elasticity</td>
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<td>$\beta$</td>
<td>Discount factor</td>
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<tr>
<td>$\kappa$</td>
<td>Phillips curve slope</td>
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<tr>
<td>$\alpha$</td>
<td>Capital share</td>
<td>36%</td>
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<tr>
<td>$\delta$</td>
<td>Depreciation rate</td>
<td>1%</td>
</tr>
<tr>
<td>$a/\bar{y}$</td>
<td>Borrowing limit</td>
<td>-0.27</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Bond duration</td>
<td>0.025</td>
</tr>
<tr>
<td>$\bar{\tau}_x$</td>
<td>Steady state transfer</td>
<td>$0.17 \times GDP$</td>
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### Income and wealth distribution

<table>
<thead>
<tr>
<th></th>
<th>Wealth</th>
<th></th>
<th>Income</th>
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<tbody>
<tr>
<td></td>
<td>Data</td>
<td>Model</td>
<td>Data</td>
<td>Model</td>
</tr>
<tr>
<td>Top 1%</td>
<td>37</td>
<td>27</td>
<td>17</td>
<td>20</td>
</tr>
<tr>
<td>Top 5%</td>
<td>65</td>
<td>66</td>
<td>32</td>
<td>32</td>
</tr>
<tr>
<td>Top 10%</td>
<td>76</td>
<td>82</td>
<td>43</td>
<td>44</td>
</tr>
<tr>
<td>Top 25%</td>
<td>91</td>
<td>96</td>
<td>64</td>
<td>60</td>
</tr>
<tr>
<td>Top 50%</td>
<td>99</td>
<td>101</td>
<td>84</td>
<td>77</td>
</tr>
</tbody>
</table>

**Table:** Shares (%) of wealth and income concentrated in the top x% of the distribution. Data are from the 2019 Survey of Consumer Finance.
Factor structure of Volcker recession

McKay and Wolf
Factor structure of Volcker recession
## Household portfolios

<table>
<thead>
<tr>
<th>Category</th>
<th>Total</th>
<th>Top 1%</th>
<th>Next 9%</th>
<th>Next 40%</th>
<th>Bottom 50%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real estate and durables</td>
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<td>24</td>
<td>48</td>
<td>72</td>
<td>23</td>
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<tr>
<td>Equity and mutual funds</td>
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<td>101</td>
<td>66</td>
<td>23</td>
<td>2</td>
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<tr>
<td>Currency, deposits, and similar</td>
<td>60</td>
<td>16</td>
<td>23</td>
<td>19</td>
<td>2</td>
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<tr>
<td>Govt. and corp. bonds and similar</td>
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<td>10</td>
<td>11</td>
<td>7</td>
<td>1</td>
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<td>Pension assets</td>
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<td>6</td>
<td>63</td>
<td>58</td>
<td>4</td>
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<tr>
<td>Mortgage liabilities</td>
<td>49</td>
<td>2</td>
<td>12</td>
<td>24</td>
<td>11</td>
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<tr>
<td>Consumer credit and loans</td>
<td>24</td>
<td>1</td>
<td>2</td>
<td>8</td>
<td>12</td>
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<tr>
<td>Net worth excluding pension assets</td>
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<td>147</td>
<td>135</td>
<td>89</td>
<td>4</td>
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<tr>
<td>Capital</td>
<td>419</td>
<td>157</td>
<td>135</td>
<td>101</td>
<td>25</td>
</tr>
<tr>
<td>Short-term bonds</td>
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<td>1</td>
<td>7</td>
<td>-3</td>
<td>-16</td>
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<tr>
<td>Long-term bonds</td>
<td>-33</td>
<td>-11</td>
<td>-8</td>
<td>-9</td>
<td>-5</td>
</tr>
<tr>
<td>Total</td>
<td>374</td>
<td>147</td>
<td>135</td>
<td>89</td>
<td>4</td>
</tr>
</tbody>
</table>
Calibration of household portfolios

- **Household portfolios**
  - We classify SCF assets and liabilities into bundles of capital, short-term bonds, and long-term bonds
    - $1 equity = $1.32 capital - $0.20 long-term bonds - $0.12 short-term bonds
    - $1 mortgage balance = -$0.50 long-term bonds - $0.50 short-term bonds
    - $1 consumer credit = -$1 short-term bonds
    - $1 currency or deposits = $1 short-term bonds
  - We then impute portfolio for households in our model as a function of their net worth
  - These portfolio positions will matter at date 0, through revaluation effects

- **Pension assets**
  - We treat pensions as part of the government
  - Returns earned on these assets are then paid out slowly through taxes
Application: distributional shock

Consumption change on impact

Percent change relative to steady state

Quantile of wealth distribution

p20  p40  p60  p80
Quantitative illustration: supply shock

• \( \{y, \pi\} \) paths agree exactly. What about interest rates?
Quantitative illustration: supply shock

- \( \{y, \pi\} \) paths agree exactly. What about interest rates?
  - Could in principle disagree substantially. But we have emp. evidence on \( i \rightarrow \{y, \pi\} \)
  - Limiting th’m [McKay-Wolf]: optimal \( i \) path can in principle be fully characterized using empirical evidence on the propagation of monetary policy shocks