The Term Structure of Real Rates and Expected Inflation*

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Abstract

Changes in nominal interest rates must be due to either movements in real interest rates or expected inflation, or both. We develop a term structure model with regime switches, time-varying prices of risk and inflation to identify these components of the nominal yield curve. We find that the unconditional real rate curve is fairly flat at 1.44%, but slightly humped. In one regime, the real term structure is steeply downward sloping. Real rates (nominal rates) are pro-cyclical (counter-cyclical) and inflation is negatively correlated with real rates. An inflation risk premium that increases with the horizon fully accounts for the generally upward sloping nominal term structure. We find that expected inflation drives about 80% of the variation of nominal yields at both short and long maturities, but during normal times, all of the variation of nominal term spreads is due to expected inflation and inflation risk.

1 Introduction

The real interest rate and expected inflation are two key economic variables; yet, their dynamic behavior is essentially unobserved. A large empirical literature has yielded surprisingly few generally accepted stylized facts. For example, whereas theoretical research often assumes the real interest rate to be constant, empirical estimates for the real interest rate process vary between constancy (Fama, 1975), mean-reverting behavior (Hamilton, 1985), or a unit root process (Rose, 1988). There seems to be more consensus on the fact that real rate variation, if it exists at all, should only affect the short end of the term structure but that the variation in long-term interest rates is primarily affected by shocks to expected inflation (see, among others, Mishkin, 1990; Fama, 1990). However, Pennacchi (1991) finds the exact opposite result. Another phenomenon that has received wide attention is the Mundell (1963) and Tobin (1965) effect: the correlation between real rates and (expected) inflation appears to be negative (see, for example, Marshall, 1992).

In this article, we establish a comprehensive set of stylized facts regarding real rates, expected inflation and inflation risk premiums, and their role in determining the nominal term structure in the US. To infer the behavior of these variables, we use a model with three distinguishing features. First, we specify a no-arbitrage term structure model with both (nominal) bond yields and inflation data to efficiently identify the real and nominal term structure. Second, our model accommodates regime-switching (RS) behavior but still produces closed-form solutions for the term structure of interest rates. We go beyond the extant RS literature by attempting to identify the real and nominal sources of the regime switches. Third, the model accommodates flexible time-varying risk premiums crucial for matching time-varying bond premia (see, for example, Dai and Singleton, 2002). These features allow our model to produce an excellent fit to the dynamics of inflation and nominal interest rates.

This paper is organized as follows. We briefly describe the related literature in Section 2. Section 3 develops the model, including the derivation of the bond prices implied by the RS term structure model. In Section 4, we briefly describe how to estimate the model with maximum likelihood and detail the specification tests we use to select the best model. In Section 5, we conduct various specification tests and analyze the parameter estimates of the best model. We find that the best-performing model has separate and independent real and inflation regimes. Surprisingly, the inflation process only has a regime-dependent drift; we cannot reject that inflation volatility remains the same across inflation regimes. In contrast, the real factor regimes feature a low mean and low volatility in one regime, and a high mean and high volatility in the other regime.

Section 6 contains the main economic results which can be summarized as follows:

1. Unconditionally, the term structure of real rates assumes a fairly flat shape around 1.44%, with a slight hump, peaking at a 1-year maturity. However, there are some regimes in which the real rate curve is downward sloping.

2. Real rates are quite variable at short maturities but smooth and persistent at long maturities. There is no significant real term spread.

3. The real short rate is negatively correlated with both expected and unexpected inflation, but the statistical evidence for a Mundell-Tobin effect is weak.

4. The model matches an unconditional upward-sloping nominal yield curve by generating an inflation risk premium that is increasing in maturity.

5. Nominal interest rates (spreads) do not behave pro-cyclically (counter-cyclically) across the business cycle but our model-implied real rates do.

6. The decompositions of nominal yields into real yields and expected inflation at various horizons indicate that variation in inflation compensation (expected inflation and inflation risk premiums) explains about 80% of the variation in nominal rates at both short and long maturities.

7. Inflation compensation only explains about 50% of the variation in nominal term spreads at short horizons, but is the main determinant of nominal interest rate spreads at long horizons.

Finally, Section 7 concludes.

2 Related Literature

To better appreciate the relative contribution of our article, we link it to three distinct literatures: (i) the extraction of real rates and expected inflation from nominal yields and realized inflation or inflation forecasts, (ii) the theoretical term structure literature and equilibrium affine models in finance and (iii) the empirical regime-switching literature on interest rates and inflation.

Our approach for identifying real rates and expected inflation differs substantially from the previous literature. First, we develop a no-arbitrage term structure model and use (nominal) term structure data, with inflation data, to identify the real and expected inflation components of nominal interest rates. In contrast, an early literature uses neither term structure data, nor a pricing model to obtain estimates of real rates and expected inflation. Mishkin (1981) and Huizinga and Mishkin (1986) simply project ex-post real rates on instrumental variables. This approach is very sensitive to measurement error and omitted variable bias. The measurement error may spuriously lead to a Mundell-Tobin effect because expected inflation is computed as the difference between nominal and real rates.

Other authors, such as Hamilton (1985), Fama and Gibbons (1982) and Burmeister et al. (1986) use low-order ARIMA models and identify expected inflation and real rates under the assumption of rational expectations using a Kalman filter. Whereas implicit assumptions on the time-series process of the forcing variables in the model and rational expectations are still essential to obtain identification in our model, the use of term structure information with time-varying risk premiums to identify the unobserved components is likely to significantly increase efficiency and mitigate peso problems.

Second, our model accommodates regime switches but still produces closed-form solutions for the term structure of interest rates. The empirical evidence for RS behavior in interest rates is very strong and confirmed in many articles (see, among many others, Hamilton, 1988; Gray, 1996; Sola and Driffill, 1994; Bekaert et al., 2001; Ang and Bekaert, 2002a). However, articles that have used term structure information and a pricing model to obtain estimates of real rates and expected inflation have so far ignored RS behavior. This includes papers by Pennacchi (1991), Sun (1992) and Boudoukh (1993) for US data and Barr and Campbell (1997), Remolona et al. (1998), Evans (1998), Risa (2001), and Buraschi and Jiltsov (2002) for UK data. This is curious, because the early literature implicitly demonstrated the importance of accounting for regime or structural changes. For example, the Huizinga-Mishkin (1986) projections are unstable over the 1979-1982 period, and the slope coefficients of regressions of future inflation onto term spreads in Mishkin (1990) are substantially different pre- and post-1979 (see also Goto and Torous, 2002).

Finally, there are a number of articles that have formulated term structure models accommodating regime switches (see Naik and Lee, 1994; Hamilton, 1988; Boudoukh et al., 1999; Bekaert et al., 2001; Bansal and Zhou, 2002; Bansal et al., 2003; Dai et al., 2003). All of these models are concerned only with nominal interest rate data. Veronesi and Yared (1999) consider real and nominal yields, but their model is a more restrictive formulation than we propose and only allows regime-switching in the long-term mean. The tractability of our proposed model also simplifies estimation in that the likelihood function can be derived and simulation-based estimation methods are unnecessary.

Evans (2003) is most closely related to our article. He formulates a dynamic pricing model with regime switches for UK real and nominal yields and inflation. Evans assumes that real yields are observable. However, in the US, real bonds (Treasury Income Protection Securities or TIPS) have traded only post-1997 and so real rates are unobservable over almost the whole interest rate history of the US.[1] The TIPS market also had considerable liquidity problems during the first few years. Evans (2003) does not separate regimes in real term structure variables and inflation, or accommodate time-varying prices of risk. The most general model we estimate has two separate regime variables each having two possible realizations: the first variable is a factor affecting the real term structure, the second regime variable only affects inflation. Hence, there are four regimes in total. Earlier work has stressed either inflation regimes (Evans and Wachtel, 1993; Evans and Lewis, 1995) or real interest rate regimes (Garcia and Perron, 1996). In this article, we separately identify the contributions of real and nominal factors to regime changes.

3 The Model

We derive a parsimonious model that accommodates regime-switches and is consistent with the dynamics of both term structure and inflation data. From the term structure literature (see, for example, Dai and Singleton, 2000), affine term structure models require three factors to match term structure dynamics. We start with a 3-factor representation of yields, but, while most term structure studies use only unobservable factors, we incorporate an observed factor, inflation, which switches regimes. Ang and Piazzesi (2002) show that incorporating macro factors improves the ability of standard term structure models to fit the dynamics of yields. A second factor represents time-variation in the price of risk. Fisher (1998), Dai and Singleton (2002) and Cochrane and Piazzesi (2002) demonstrate that, in the context of affine models, time-varying prices of risk succesfully capture the dynamics of term premia. Finally, a third factor represents a latent RS term structure factor.

In our first model (Section 3.1 and 3.2), we accommodate two possible regimes, as is customary in the literature on regime switches in nominal short rates (see, for example, Hamilton, 1988; Gray, 1996; Ang and Bekaert, 2002a). However, we also consider models with separate regimes for inflation and a real factor (Section 3.3). Sections 3.4 and 3.5 explore the term structure and inflation risk premium implications of the models.

[1] Even for the UK, observable real interest rates are only available post-1982.

3.1 The Benchmark Regime-Switching Model

Let the state variables $X_t = (q_t\, f_t\, \pi_t)'$, where q_t and f_t are unobserved state variables and π_t is observable inflation, follow the RS process:

$$X_{t+1} = \mu(s_{t+1}) + \Phi X_t + \Sigma(s_{t+1})\varepsilon_{t+1}, \tag{1}$$

where s_t indicates the regime and:

$$\mu(s_t) = \begin{bmatrix} \mu_q \\ \mu_f(s_t) \\ \mu_\pi(s_t) \end{bmatrix}, \quad \Phi = \begin{bmatrix} \Phi_{qq} & 0 & 0 \\ \Phi_{fq} & \Phi_{ff} & 0 \\ \Phi_{\pi q} & \Phi_{\pi f} & \Phi_{\pi\pi} \end{bmatrix}, \quad \Sigma(s_t) = \begin{bmatrix} \sigma_q & 0 & 0 \\ 0 & \sigma_f(s_t) & 0 \\ 0 & 0 & \sigma_\pi(s_t) \end{bmatrix}. \tag{2}$$

While the conditional mean and volatility of f_t and π_t switch regimes, the conditional mean and volatility of q_t does not. The reduced-form process for inflation is quite complex and involves moving average terms. This is important because the autocorrelogram of inflation is empirically well approximated by an ARMA process. Note that the mean-reversion of all variables in Φ is not regime-dependent.

The real short rate is affine in the state variables:

$$r_t = \delta_0 + \delta_1' X_t. \tag{3}$$

The q_t parameter in X_t also determines the price of risk (see below). This means that the time-varying price of risk can directly influence the real rate as it would in any equilibrium model with growth. The model also allows for arbitrary correlation between the real rate and inflation. Note that since f_t and π_t in X_t change across regimes, the real rate process (3) also inherits the RS structure of the state variables.

The regime variable $s_t = 1, 2$ follows a Markov chain with transition probability matrix:

$$\Pi = \begin{bmatrix} p_{11} = Pr(s_t = 1 | s_{t-1} = 1) & p_{12} = 1 - p_{11} \\ p_{21} = 1 - p_{22} & p_{22} = Pr(s_t = 2 | s_{t-1} = 2) \end{bmatrix}. \tag{4}$$

We denote the stable probabilities of the Markov chain implied by Π as $\pi_i = Pr(s_t = i)$. Finally, we specify the real pricing kernel to take the form:

$$\widehat{m}_{t+1} = \log \widehat{M}_{t+1} = -r_t - \frac{1}{2}\lambda_t(s_{t+1})'\lambda_t(s_{t+1}) - \lambda_t(s_{t+1})'\varepsilon_{t+1} \tag{5}$$

where the prices of risk $\lambda_t(s_t)$ are given by:

$$\begin{aligned} \lambda_t(s_t) &= (\gamma_t\ \lambda(s_t)')' \\ \gamma_t &= \gamma_0 + \gamma_1 q_t \\ &= \gamma_0 + \gamma_1 e_1' X_t, \end{aligned} \tag{6}$$

where e_i represents a vector of zero's with a 1 in the ith position and $\lambda(s_t) = (\lambda_f(s_t)\, \lambda_\pi(s_t))'$. In this formulation, the prices of risk of f_t and π_t change across regimes. The variable q_t controls the time-variation of the price of risk in γ_t in (6) and does not switch regimes.[2]

Our model significantly extends several existing specifications. Naik and Lee (1994) and Landén (2000) present models with constant prices of risk and regime switches where bond prices have affine solutions. Veronesi and Yared (1999) use Liptser and Shiryaev (1979) filtering techniques to obtain closed-form solutions for bond prices, but they can only handle switching in the mean $\mu(s_t)$. Bansal and Zhou (2002) and Evans (2003) allow switching in the mean reversion parameters, covariances and means, but both articles use linearizations to obtain approximate analytical bond prices. Bansal and Zhou (2002) also use a discrete-time Cox-Ingersoll-Ross (1985) process with only regime-dependent constant prices of risk, unlike the more flexible, time-varying price of risk q_t our model employs. In Section 3.4, we show that our specification produces closed-form solutions for bond prices, enabling both efficient estimation and the ability to fully characterize real and nominal yields at all maturities without discretization error.

The model has two main caveats. First, Gray (1996), Bekaert et al. (2001) and Ang and Bekaert (2002a) show that mean-reversion of the short rate is significantly different across regimes. Second, Ang and Bekaert (2002b) show that only time-varying transition probabilities (for example, used by Diebold et al., 1994), can reproduce the non-linearities in the short rate drift and volatility functions estimated by Aït-Sahalia (1996), Stanton (1997) and others. If we relax both these constraints, closed-form bond prices are no longer available. While these are important concerns, the numerical difficulties in computing bond prices for these more complex specifications are formidable and the use of term structure information is critical in identifying both the inflation and real rate components in interest rates and the RS parameters. Moreover, our model with a latent term structure factor and a time-varying price of risk, combined with the RS means and variances, is very rich and cannot be identified from inflation and short rate series alone.

An alternative formulation of a RS term structure model by Dai, Singleton and Wang (2003) incorporates regime-dependent mean reversions and state-dependent probabilities under the real measure. However, for bond pricing under the risk-neutral measure, both the mean reversion and the transition probabilities must be constants, exactly as in our formulation. These restrictions must be imposed to obtain closed-form bond prices (see Dai and Singleton, 2003). While it is theoretically possible to allow time-varying transition probabilities under

[2] Allowing γ_t to switch across regimes results in the loss of closed-form solutions for bond prices, which we detail in Section 3.4.

the real measure, our full specification (below) allows for separate regimes in both a real factor and inflation, making state-dependent transition probabilities computationally intractable. (Dai, Singleton and Wang allow for only two regimes.) Furthermore, in their specification, the evolution of the factors in (1) depends on s_t rather than s_{t+1} and they parameterize the factor prices of risk in the pricing kernel in (5) to also depend on s_t, rather than s_{t+1}. In our model, by specifying the dependence on s_{t+1}, the conditional variance of our factors embeds a jump term reflecting the difference in conditional means in the two regimes (see Gray, 1996). In the Dai-Singleton-Wang parameterization, the differences in means across the regimes of the factors in (1) do not enter the conditional variances of the factors. Our results (below) show that the conditional means of inflation significantly differ across regimes, while the conditional variances do not, making the regime-dependent means an important source of inflation heteroskedasticity.

In an extension, detailed in Appendix A, we also consider an alternative RS model with an additional unobserved factor representing expected inflation, which generalizes classic ARMA-models of real rates and expected inflation. To gauge the actual contribution of regime switches, we also estimate single-regime counterparts to the benchmark and stochastic expected inflation models.

3.2 Identification

In a single-regime setting, Dai and Singleton (2000) show that many term structure models with unobserved state variables result in observationally equivalent systems. Hence, restrictions must be imposed on the parameters for identification. In a single-regime Gaussian model, Dai and Singleton show that identification can be accomplished by setting the conditional covariance to be a diagonal matrix and letting the correlations enter through the feedback matrix (Φ), which is parameterized to be lower triangular, which we do here. Note that the process for inflation is influenced by both past inflation, time-varying prices of risk (through q_t) and the term-structure (through f_t).

Since q_t and f_t are latent variables, they can be arbitrarily scaled. Hence, we set $\delta_1 = (\delta_q \, \delta_f \, \delta_\pi)' = (1 \, 1 \, \delta_\pi)'$ in (3). Setting δ_q and δ_f to be constants allows σ_q and σ_f to be estimated. Because q_t is an unobserved variable, estimating μ_q in (2) is equivalent to allowing γ_0 in (6) or δ_0 in (3) to be non-zero. Hence, q_t must have zero mean for identification. Therefore, we set $\mu_q = 0$, since q_t does not switch regimes. Similarly, because we estimate $\lambda_f(s_t)$, we constrain f_t to have zero mean.

Because we only have nominal bonds, it is impossible to identify the instantaneous inflation risk premium in our model. Articles focusing on US data that attempt to identify the instantaneous inflation risk premium, such as Veronesi and Yared (1999) and Buraschi and Jiltsov (2002), obtain identification through the restrictions imposed by economic models. Therefore, we follow the literature and set $\lambda_\pi(s_t) = 0$, so that the instantaneous inflation risk premium is zero. However, because inflation and real rates are correlated, long-term bond yields may still embed inflation risk, as we show below.[3]

The resulting model is theoretically identified from the data but we impose two additional restrictions on the benchmark model. First, we set $\Phi_{12} = 0$ in (2). With this restriction, there are, in addition to inflation factors, two separate and easily identifiable sources of variation in interest rates: a regime-switching factor and a time-varying price of risk. Identifying their relative contribution to interest rate dynamics becomes easy with this restriction and it is not immediately clear how a non-zero Φ_{12} would help enrich the model. Second, we set $\gamma_0 = 0$ in (6) and instead estimate the RS price of risk $\lambda_f(s_t)$. Theoretically, affine models allow the identification of $N-1$ prices of risk with the use of N zero-coupon bonds, but empirically, it is very difficult to accurately pin down more than one constant price of risk (see Dai and Singleton, 2000).

To obtain some intuition on identification, consider a two period real bond. The 2-period term spread in an affine model is given by:

$$\widehat{y}_t^2 - r_t = \frac{1}{2}\left(\mathrm{E}_t(r_{t+1}) - r_t\right) - \frac{1}{4}\mathrm{var}_t\left(r_{t+1}\right) + \frac{1}{2}\mathrm{cov}_t\left(\widehat{m}_{t+1}, r_{t+1}\right). \tag{7}$$

The first term $(\mathrm{E}_t(r_{t+1}) - r_t)$ is an Expectations Hypothesis (EH) term, the second term $\mathrm{var}_t\,(r_{t+1})$ is a Jensen's inequality term and the last term, $\mathrm{cov}_t(\widehat{m}_{t+1}, r_{t+1})$, is the risk premium. In the single-regime affine setting equivalent to our model, this term is given by:

$$\mathrm{cov}_t(-\widehat{m}_{t+1}, r_{t+1}) = \gamma_0\sigma_q + \lambda_f\sigma_f + \gamma_1\sigma_q q_t, \tag{8}$$

which shows that the effects of γ_0 and λ_f are indistinguishable as they both act as constant terms.

The RS model has a considerably more complex expression for the 2-period real term

[3] The literature using UK indexed gilts often attempts to estimate the instantaneous inflation risk premium. While we cannot directly estimate the instantaneous inflation risk premium due to the lack of indexed bonds in the US, we did estimate models with (fixed) slightly negative $\lambda_\pi(s_t)$ parameters. These models simply yield lower real rates and higher inflation premiums than the results we report.

spread:

$$\begin{aligned}\widehat{y}_t^2(i) - r_t =& \frac{1}{2}(\mathrm{E}_t(r_{t+1}|s_t = i) - r_t) - \frac{1}{2}\left(\gamma_0\sigma_q + \gamma_1\sigma_q q_t\right) \\ & - \frac{1}{2}\log\sum_{j=1}^{K} p_{ij}\exp\Big[-\delta_1'\Big(\mu(j) - \mathrm{E}\left[\mu(s_{t+1})|s_t = i\right]\Big) \\ & + \frac{1}{2}\delta_1'\Sigma(j)\Sigma(j)'\delta_1 + \lambda_f(j)\sigma_f(j)\Big],\end{aligned} \tag{9}$$

for K regimes. First, the term spread now switches across regimes, explicitly shown by the dependence of the yield $\widehat{y}_t^2(i)$ on regime $s_t = i$. The EH term $(\mathrm{E}_t(r_{t+1}|s_t = i) - r_t)$ also switches regimes. The time-varying price of risk term, $-\frac{1}{2}(\gamma_0\sigma_q + \gamma_1\sigma_q q_t)$, is the same as in (8) because the process for q_t does not switch regimes. The remaining terms in (9) are non-linear, as they involve the log of the sum of an exponential function of regime-dependent terms, weighted by transition probabilities. Within the non-linear expression, the term $\frac{1}{2}\delta_1'\Sigma(j)\Sigma(j)'\delta_1$ represents a Jensen's inequality term, which is regime-dependent, and $\lambda_f(j)\sigma_f(j)$ represents a RS price of risk term. A new term in (9) that does not have a counterpart in (8) is $-\delta_1'(\mu(j) - \mathrm{E}\left[\mu\left(s_{t+1}\right)|s_t = i\right])$. This is a jump term involving the difference of drifts across regimes. Hence, it is unlikely that adding another constant, γ_0, adds much flexibility to the model.

Finally, we impose one restriction not necessary for identification, but for efficiency gains. The mean level of the real short rate in (3) is determined by the mean level of inflation multiplied by δ_π and the constant term δ_0. We set δ_0 to match the mean of the nominal short rate in the data, improving the fit of the model.

3.3 Incorporating Different Real and Inflation Regimes

The two regime specification in (4) restricts the real rate and inflation to share the same regimes. To incorporate the possibility of different real and inflation regimes, we introduce two different regime variables $s_t^r \in \{1,2\}$ for the f_t process and $s_t^\pi \in \{1,2\}$ for the inflation process. Because f_t is a real factor, we refer to s_t^r as the real regime variable. Nevertheless, the reduced form model for the real rate incorporates both the inflation and real regimes, since inflation is one of the factors entering the real short rate (3). The δ_π parameter controls how a switch in the inflation regime impacts the real rate. For example, a monetary authority actively following a Taylor rule implies that high inflation brings about high real rates. Moreover, because f_t enters the conditional mean of inflation, the inflation process is affected by the real regime as well. For example, monetary-policy induced increases in real rates could ward off higher inflation next period through a negative $\Phi_{\pi f}$ coefficient.

To incorporate the effects of s_t^π and s_t^r, we define an aggregate regime variable $s_t \in \{1, 2, 3, 4\}$ to account for all possible combinations of $\{s_t^r, s_t^\pi\} = \{(1,1), (1,2), (2,1), (2,2)\}$, following Hamilton (1994). The full transition probability matrix is of dimension 4×4 and has 10 additional parameters relative to the benchmark model. To reduce the number of parameters, we investigate three restricted cases: independent regimes and two cases of correlated regimes.

We briefly outline the correlated cases, but provide full details in Appendix B. In Case A, we specify the current real regime to depend on the contemporaneous realization of the inflation regime and on the past real regime. This is consistent with monetary policy changing real rates in response to inflation shocks, as is the case in a Taylor rule. One short-coming of the Case A specification is that it cannot capture periods when monetary policy has successfully used real rate increases to stave off a regime of high inflation. In constrast, in Case B, future inflation regimes depend on the stance of the monetary authority's real regime as well as the current inflation environment.

3.4 Bond Prices

Real Bond Prices

In our model, the real zero coupon bond price of maturity n conditional on regime $s_t = i$, $\widehat{P}_t^n(s_t = i)$, is given by:

$$\widehat{P}_t^n(i) = \exp(\widehat{A}_n(i) + \widehat{B}_n X_t), \tag{10}$$

where $\widehat{A}_n(i)$ is dependent on regime $s_t = i$, $\widehat{B}_n$ is a $1 \times N$ vector and N is the total number of factors in the model, including inflation. The expressions for $\widehat{A}_n(i)$ and $\widehat{B}_n$ are given in Appendix C. Since the real bond prices are given by (10), it follows that the real yields $\widehat{y}_t^n(i)$ are affine functions of X_t, conditional on regime i:

$$\hat{y}_t^n(i) = -\frac{\log(\widehat{P}_t^n)}{n} = -\frac{1}{n}(\widehat{A}_n(i) + \widehat{B}_n X_t). \tag{11}$$

The technical innovation in deriving (10) is to recognize that the $\widehat{B}_n$ parameter does not switch for two reasons. First, Φ remains constant across regimes. Second, the time-varying price of risk parameter γ_1 also does not switch across regimes. If these parameters become regime-dependent, closed-form bond solutions are no longer possible.

The pricing implications of (11), together with the assumed dynamics of X_t in (1) imply that the autoregressive dynamics of both inflation and bond yields are assumed constant over time but the drifts vary through time and the shocks are heteroskedastic. This puts our model in the middle of a recent debate between Cogley and Sargent (2001 and 2002) and Sims (1999 and

2001) regarding the dynamics of inflation, short rates and other macro-variables. Cogley and Sargent argue that the conduct of monetary policy has changed through time, whereas Sims finds little evidence of movements in the parameter values and instead stresses the heteroskedasticity of the shocks. The economic content of our pricing equations (11) is that although we do not allow for systematic changes in monetary policy following Bernanke and Mihov (1998) and Sims (1999), we do allow for a changing drift in the policy equation, and heteroskedastic shocks.

While the expressions for $\widehat{A}_n(i)$ and $\widehat{B}_n$ are complex, some intuition can be gained on how the prices of risk affect each term. The constant price of risk γ_0 enters only the constant term in the yields $\widehat{A}_n(s_t)$, but affects the term in all regimes. A more negative γ_0 causes long maturity yields to be, on average, higher than short maturity yields, as is true in an affine model. The regime-dependent prices of risk $\lambda(s_t)$ also only affect the $\widehat{A}_n(s_t)$ terms. Unlike the γ_0 term, the $\lambda(s_t)$ term enters non-linearly, still affecting the unconditional average shape of the real yield curve. In addition, since the $\lambda(s_t)$ terms differ across regimes, $\lambda(s_t)$ also controls the regime-dependent level of the yield curve away from the unconditional shape of the yield curve. Thus, the model can accommodate an upward sloping yield curve in one regime but a downward sloping yield curve in another regime, thereby capturing the switching signs of term premiums documented by Boudoukh et al. (1999). The prices of risk affect the time-variation in the yields through the parameter γ_1. This term only enters the $\widehat{B}_n(s_t)$ terms. A more negative γ_1 means that long-term yields are higher, and respond more to shocks in the price of risk factor q_t.

Nominal Bond Prices

To compute nominal bond prices and yields, we formulate the nominal pricing kernel in the standard way as $M_{t+1} = \widehat{M}_{t+1} P_{t+1}/P_t$:

$$m_{t+1} = \log M_{t+1} = -r_t - \frac{1}{2}\lambda_t(s_{t+1})'\lambda_t(s_{t+1}) - \lambda_t(s_{t+1})'\varepsilon_{t+1} - e_N' X_{t+1}, \tag{12}$$

where e_N is a vector of zeros with a 1 in the last position, pulling π_t from the $N \times 1$ vector X_t. This allows nominal bond prices to be written as:

$$P_t^n(i) = \exp(A_n(i) + B_n X_t), \tag{13}$$

where the scalar $A_n(i)$ is dependent on regime $s_t = i$ and B_n is an $1 \times N$ vector. Since nominal bond prices are given by (13), it follows that the nominal yields $y_t^n(i)$ are affine functions of X_t, conditional on regime i:

$$y_t^n(i) = -\frac{\log(P_t^n)}{n} = -\frac{1}{n}(A_n(i) + B_n X_t). \tag{14}$$

Appendix D shows that the only difference between the $\widehat{A}_n(i)$ and $\widehat{B}_n$ terms for real bond prices and the $A_n(i)$ and B_n terms for nominal bond prices are due to terms that select inflation from X_t. Positive inflation shocks decrease nominal bond prices.

3.5 Expected Inflation and Inflation Risk

The difference in the pricing of nominal and real bonds is at the core of this article and we now discuss the relationship between the two at length. We call the difference between the nominal and real yield the "inflation compensation." The inflation compensation differs from actual expected inflation through three channels: (i) a Jensen's inequality term, (ii) a convexity bias and (iii) an inflation risk premium. The first two effects are present in nominal bonds of all maturities but are of second order importance. We discuss them first in the context of a one period nominal bond. The third effect is not present in a one period bond because we set $\lambda_\pi(s_t) = 0$. However, the model allows for arbitrary correlation between real and inflation factors and this leads to inflation risk premiums in long-term bonds. As we show below, the same parameters that are critically important in driving a potential Mundell-Tobin effect are also important in determining the magnitude and sign of the inflation risk premium.

The Components of the Short Rate

We break the nominal yield into the real rate plus expected inflation implied from bond yields by taking the difference between (14) and (11):

$$y_t^1 = r_t + \pi_{t,1}^e,$$

where expected inflation implied from the nominal bond price, $\pi_{t,1}^e$, over the next period is given by:[4]

$$\pi_{t,1}^e(i) = -\log\left[\sum_{j=1}^{K} p_{ij} \exp\left(-\mu_\pi(j) + \frac{1}{2}\sigma_\pi^2(j) + \sigma_\pi(j)\lambda_\pi(j)\right)\right] + e_N' \Phi X_t. \qquad (15)$$

The last term in the exponential represents the instantaneous inflation risk premium, which is zero by assumption in our model. The $\frac{1}{2}\sigma_\pi^2(j)$ term is the standard Jensen's inequality term, while $-\mu_\pi(s_t)$ represents the non-linear, regime-dependent part of expected inflation. The last term $e_N'\Phi X_t$ represents the time-varying part of expected inflation, which does not switch across regimes.

[4] For a single-regime affine model, $\pi_{t,1}^e$ is given by $\left(\mu_\pi - \frac{1}{2}\sigma_\pi^2 - \sigma_\pi\lambda_\pi\right) + e_N'\Phi X_t$.

We can compare $\pi^e_{t,1}$ in (15) with the corresponding expression for $\mathrm{E}_t(\pi_{t+1})$, the actual expected inflation implied from the assumed factor dynamics (1):

$$\begin{aligned} \mathrm{E}_t(\pi_{t+1}|s_t = i) &= e_N' \mathrm{E}[\mu(s_{t+1})|s_t = i] + e_N' \Phi X_t \\ &= \left(\sum_{j=1}^{K} p_{ij} \mu_\pi(j) \right) + e_N' \Phi X_t. \end{aligned} \tag{16}$$

The constant terms in (15) for $\pi^e_{t,1}$ and (16) for $\mathrm{E}_t(\pi_{t,1})$ are different. The bond-price implied inflation term ($\pi^e_{t,1}$) reflects both a Jensen's inequality term $\frac{1}{2}\sigma^2_\pi(s_t)$ and a non-linear term, driven by taking the log of a sum, weighted by transition probabilities. Because $\exp(.)$ is a convex function, Veronesi and Yared (1999) call this non-linearity effect through the regime-dependent means of inflation $\mu_\pi(s_t)$ a "convexity bias." Like the Jensen's term, this also makes $\pi^e_{t,1} < \mathrm{E}_t(\pi_{t+1})$. Note that $\pi^e_{t,1}$ in (15) and $\mathrm{E}_t(\pi_{t+1})$ in (16) have the same time-varying inflation forecast component, $e_N' \Phi X_t$. Hence, to investigate how expected inflation covaries with different variables, for example, real rates, we can look at either $\pi^e_{t,1}$ or $\mathrm{E}_t(\pi_{t+1})$ for a 1-quarter horizon.[5]

The Correlation between Real Rates and Inflation

The model allows for arbitrary correlation between real rates and unexpected and expected inflation. To gain some intuition, let us focus on conditional covariances and derive them for an affine model. First, δ_π primarily drives the correlation between real rates and unexpected inflation. That is, $\mathrm{cov}_t(r_{t+1}, \pi_{t+1}) = \delta_\pi \sigma^2_\pi$. The Mundell-Tobin effect concerns the (negative) correlation between real rates and expected inflation, which is given by:

$$\mathrm{cov}_t(r_{t+1}, \mathrm{E}_{t+1}(\pi_{t+2})) = \Phi_{\pi q} \sigma^2_q + \Phi_{\pi f} \sigma^2_f + \delta_\pi \Phi_{\pi\pi} \sigma^2_\pi.$$

Note that $\delta_\pi < 0$ is not a sufficient condition to obtain a Mundell-Tobin effect. In our RS model, $\mathrm{cov}_t(r_{t+1}, \mathrm{E}_{t+1}(\pi_{t+2}))$ switches across regimes because the variances are regime-dependent. In addition, the correlation between real rates and expected inflation also depends on the regime-dependent means, similar to Gray (1996).

In recent macro-economic studies, the presence or absence of a Mundell-Tobin effect is linked to the central bank's reaction function (see, for example, Clarida, Gali and Gertler, 2000). An aggressive or activist central bank should induce a positive correlation between real rates and expected inflation, and vice versa. From this perspective, the (controversial) evidence in favor of time-varying monetary policy behavior (see Cogley and Sargent, 2002; Goto and Torous,

[5] For longer horizons, $n > 1$, the time-varying components for $\pi^e_{t,n}$ and $\mathrm{E}_t(\pi_{t+n,n})$ differ.

2002; Cho and Moreno, 2002) can only be accommodated by the RS model, but not by an affine model.

It is tempting to conclude that if a Mundell-Tobin effect exists, nominal bonds should be less risky than real bonds. After all, it implies that, when a bad shock is experienced (an increase in real rates), the holders of nominal bonds experience a countervailing effect, namely a decrease in expected inflation increasing bond prices. This intuition is not correct as we now discuss.

The Inflation Risk Premium

The dynamics of the factors in (1) allow us to compute the actual expected inflation $\mathrm{E}_t(\pi_{t+n,n})$ over n periods, where:

$$\mathrm{E}_t(\pi_{t+n,n}) = \frac{1}{n}\mathrm{E}_t(\pi_{t+1} + \cdots + \pi_{t+n}),$$

and $\pi_{t+n,n} \equiv (\pi_{t+1} + \cdots + \pi_{t+n})/n$ and $\mathrm{E}_t(\pi_{t+1,1}) \equiv \mathrm{E}_t(\pi_{t+1})$. Long-horizon forecasts of inflation from (1) are given by implied long-horizon forecasts from a RS-VAR. Hence, we refer to $\mathrm{E}_t(\pi_{t+n,n})$ as the RS-VAR implied inflation forecast.

The RS-VAR expected inflation $\mathrm{E}_t(\pi_{t+n,n})$ is different from the inflation compensation, π^e_{t+n}. We can now decompose nominal yields as real yields plus the inflation compensation, and decompose π^e_{t+n} into actual expected inflation and a remainder term:

$$\begin{aligned} y^n_t &= \hat{y}^n_t + \pi^e_{t,n} \\ &= \hat{y}^n_t + \mathrm{E}_t(\pi_{t+n,n}) + \varphi_{t,n} \end{aligned} \tag{17}$$

The $\varphi_{t,n}$ term reflects an inflation risk premium and the two non-linear Jensen's and convexity bias terms.

To obtain intuition on the determinants of the inflation risk premium in our model, we investigate the inflation risk premium embedded in a two period bond in a simple affine model. The real bond yield embeds an EH term and a risk premium due to the conditional covariance between the real kernel and the real rate. For a nominal bond, it is the covariance between the nominal kernel and inflation that matters: if inflation is high (purchasing power is low) when the pricing kernel realization (marginal utility in an equilibrium model) is high, nominal bonds are risky. The two period pricing kernel depends on real rates both through its conditional mean and through the innovations. Interestingly, the effects are likely opposite. High real rates are likely to decrease the conditional mean of the pricing kernel but, if the price of risk is negative, positive shocks to the real rate should increase marginal utility. By splitting inflation into unexpected and expected inflation, we can decompose $\varphi_{t,2}$ into four components (ignoring the Jensen's inequality term):

$$\varphi_{t,2} = \frac{1}{2}\left[-\text{cov}_t(r_{t+1}, \mathrm{E}_{t+1}(\pi_{t+2})) - \text{cov}_t(r_{t+1}, \pi_{t+1})\right.$$
$$\left.+\text{cov}_t(\widehat{m}_{t+1}, \mathrm{E}_{t+1}(\pi_{t+2})) + \text{cov}_t(\widehat{m}_{t+1}, \pi_{t+1})\right] \quad (18)$$

Our previous intuition was wrong in that the first two terms reveal that a negative correlation between real rates and both expected and unexpected inflation contributes to a positive risk premium. Nevertheless, a Mundell-Tobin effect does not necessarily imply a positive inflation risk premium because of the two last terms working through the innovations of the pricing kernel. In our model, the last term is zero by assumption, but the third term is not and may swamp the others. In particular, for the affine specification:

$$\varphi_{t,2} = -\frac{1}{2}[\delta_\pi \sigma_\pi^2(1 + \Phi_{\pi\pi}) + \Phi_{\pi q}(\sigma_q^2 + \gamma_1 \sigma_q q_t) + \Phi_{\pi f}(\sigma_f^2 + \lambda_f \sigma_f)]. \quad (19)$$

Hence the time-variation in the inflation risk premium depends on q_t, and the mean premium depends on the same parameters that drive the correlation between real rates and inflation. In particular, if the correlation between real rates and inflation is zero (requiring $\delta_\pi = \Phi_{\pi,q} = \Phi_{\pi,f} = 0$), the inflation risk premium is also zero. Nevertheless, the price of risk λ_f plays a role in determining the inflation risk premium whereas it does not play a role in determining the correlation between real rates and expected inflation. Naturally, the RS model has a richer expression for the inflation risk premium as the variances and $\lambda_f(s_t)$ switch across regimes. The RS inflation premium also depends on regime-dependent means.

Variance Decompositions

To assess the relative importance of real rates and the inflation compensation , we compute the population variances for real rates and priced expected inflation and look at the relative contribution of each component to the variance of the nominal yield (see also Risa, 2001):

$$\begin{aligned}
\tau_{\hat{y}_t^n} &= \frac{\text{cov}\left(y_t^n, \hat{y}_t^n\right)}{\text{var}\left(y_t^n\right)} = \frac{\text{var}\left(\hat{y}_t^n\right) + \text{cov}\left(\hat{y}_t^n, \pi_{t,n}^e\right)}{\text{var}\left(y_t^n\right)} \\
\tau_{\pi_{t,n}^e} &= \frac{\text{cov}\left(y_t^n, \pi_{t,n}^e\right)}{\text{var}\left(y_t^n\right)} = \frac{\text{var}\left(\pi_{t,n}^e\right) + \text{cov}\left(\hat{y}_t^n, \pi_{t,n}^e\right)}{\text{var}\left(y_t^n\right)}.
\end{aligned} \quad (20)$$

We provide an analogous decomposition to (20) for nominal term spreads.

4 Econometrics

To estimate the RS term structure model, we follow standard practice and specify a set of yields that are measured without error to extract the unobserved factors (see Chen and Scott, 1993). The other yields are specified to be measured with error and provide over-identifying restrictions for the term structure model. We use 4-, 12- and 20-quarter maturity zero-coupon yield data from CRSP and the 1-quarter rate from the CRSP Fama risk-free rate file as our yields. Our inflation data is constructed using the Consumer Price Index – All Urban Consumers (CPI-U, seasonally adjusted, 1982-84=100), from the Bureau of Labor Statistics. All data are sampled at the quarterly frequency over the period 1952:Q2 to 2000:Q4. The disadvantages in using monthly inflation data motivates the use of quarterly data. Monthly CPI figures are very seasonal and create a timing problem because prices are collected over the course of the month. The use of a quarterly frequency mitigates both problems. For the benchmark model, we specify the 1-quarter and 20-quarter yields to be measured without error. For the RS model where inflation has a stochastic mean, we additionally specify the 4-quarter yield to be measured without error.

We detail how to compute the likelihood function in Appendix E. The likelihood is not simply the likelihood of the yields measured without error multiplied by the likelihood of the measurement errors, which would be the case in a standard affine model estimation. Since we have regime variables, these must be integrated out of the likelihood function. Our model implies a RS-VAR for inflation and yields with complex cross-equation restrictions resulting from the no-arbitrage assumptions.

Because RS models may yield several local optima, an extensive parameter search was conducted, including using randomized starting values around the local maxima. The global optimum for the benchmark model is very stable to such starting value perturbations.

We report two specification tests of the models, an in-sample serial correlation test for first and second moments in scaled residuals, and an unconditional moment test. The latter is particularly important because we want to decompose the variation of nominal yields of various horizons into real and expected inflation components. A well-specified model should imply unconditional means, variances and autocorrelograms of term spreads, yields and inflation close to the sample moments. These tests are outlined in Appendix F.

5 Estimation Results

5.1 Specification Tests

Panel A of Table 1 reports the specification tests for the residuals for six models. Model I is the single regime counterpart of the benchmark RS model, described in Section 3.1, while Model II is the single regime counterpart to the RS model with stochastic inflation. Model III represents the benchmark RS model with two regimes, whereas Model IV is the model with two independent real and inflation regime variables. We do not report the tests for the models with correlated regime variables, as the test statistics look almost identical to those for Model IV. Model V (VI) is the stochastic inflation RS model with one regime variable (two independent regime variables). We use this nomenclature throughout the remainder of the paper.

The residuals of the yields and spreads are well behaved for all models. In stark contrast, only two of the six models pass the specification tests for the inflation residuals: Models IV and VI. Both these models have two separate regime variables for a real factor and inflation. Panel B reports goodness-of-fit tests of four sets of moments: the mean and variance of the spread and the long rate (recall that all models fit the mean of the short rate by construction in the estimation procedure), the mean and variance of inflation, and the autocorrelogram of the spread and the autocorrelogram of inflation. Model IV is the only model that passes all of the specification tests. This is important, because we need to accurately match the sample moments of yields and inflation in the data in order to make reliable inferences about unobserved real rates and expected inflation.

It is informative to consider how good the fit is by examining the reported moments in Panel C. Model IV has a phenomenal fit: of the 13 moments reported, Model IV comes within 1 standard error of 9 data moments, and is comfortably within two standard errors of the remaining 4. All the other models produce some moments that lie outside 2 standard error bounds of the data estimate. In particular, the inflation autocorrelogram is perhaps the most difficult to match. Inflation features a relatively low first order autocorrelation coefficient with very slowly decaying higher-order autocorrelations. All our models feature complex reduced-form inflation models with MA components that can theoretically match this pattern. However, only Model IV produces a good fit.

5.2 Parameter Estimates

To conserve space, we report the parameter estimates for all the models in the Appendix. Here we focus on Model IV, the benchmark RS inflation model with independent real and inflation regimes, which is the best performing model.

Table 2 reports the relative contributions of the different factors driving the short rate, long yield, and term spread and inflation dynamics in the model. The price of risk factor q_t is relatively highly correlated with both inflation and the nominal short rate, but shows little correlation with the nominal spread. In other words, q_t is a level factor. The RS term structure factor f_t is most highly correlated with the nominal spread, in absolute value, so f_t is a slope factor. The factor f_t is also less variable and less persistent than q_t. Hence, f_t does not play a large role in the dynamics of the real rate, only accounting for 7% of its variation. The most variable factor is inflation and it accounts for 47% of the variation of the real rate. Inflation is negatively correlated with the real short rate, at -39% and q_t is positively correlated with the real short rate (42%). The model produces a 68% (-41%) correlation between inflation and the nominal short rate (nominal 5-year spread), which matches the data correlation of 70% (-37%) almost perfectly.

The Φ parameters link the term structure factors directly to the conditional mean of inflation and hence to expected inflation. In our model, the equation governing π_t is:

$$\pi_{t+1} = \mu_\pi(s_t^\pi) + 0.50q_t + 0.85f_t + 0.59\pi_t + \sigma_\pi(s_t^\pi)\varepsilon_{t+1}^\pi. \tag{21}$$

Hence, expected inflation moves positively with q_t, and so moves positively with the real short rate, but expected inflation moves negatively with the term spread since f_t enters positively. The effect of past inflation is positive. These numbers cannot be compared with the data because the q_t and f_t factors are unobserved. However, we can project inflation onto the short rate, the spread and past inflation both in the data and in the model, and we report the results on the last two lines of Table 2. When the short rate increases by 1%, the model signals an increase in expected inflation of 34 basis points. A 1% increase in the spread predicts a 13 basis point decrease in expected inflation. These patterns are consistent with what is observed in the data, but the response to an increase in the spread is slightly stronger in the data. Past inflation has a coefficient of 0.57 and the data coefficient is 0.56, and is hence almost exactly matched by the model.

Inflation enters the real short rate equation (3) with a negative coefficient, $\delta_\pi = -0.54$, with a standard error of 0.05, inducing a negative covariance between inflation and real rates. However, the large and positive feedback parameters of Φ in the inflation equation (21) may offset the negative δ_π coefficient in driving a potential Mundell-Tobin effect. We discuss this further in Section 6.1.

Table 3 reports some parameter estimates for Models III and IV that illustrate the importance of regimes and at the same time characterize what differentiates one regime from another. Panel A reports Model III, where real rates and inflation have the same two regimes. The first real rate regime is characterized by a low f_t mean and low standard deviation. However, only the standard deviations are significantly different across regimes. For the inflation process, both the mean and standard deviations differ across the two regimes at the 5% level. The first regime has a higher drift of inflation but lower volatility of innovations in inflation. The prices of risk for the f_t factor are border-line significantly different across the two regimes.

In Panel B of Table 3, we estimate separate independent real and inflation regimes. For the f_t factor, the inference is the same as in Panel A. However, for the inflation process, there is no significant difference across regimes in the innovation variances. This does not mean that inflation is homoskedastic in this model. The significantly different regime-dependent means of f_t affect the conditional variance, so that there is heteroskedastic inflation across real regimes. The first inflation regime is a high inflation regime while the second inflation regime is a low inflation regime, and the difference in drift is significant at the 1% level. Finally, the prices of risk for the f_t factor do not differ significantly across the two regimes, but both are significantly different from zero. Hence, the addition of the separate regimes for real rates and inflation has important consequences for the behavior of real rates and expected inflation that are not picked up by restricting the estimation to only one regime variable.

We next examine the smoothed regime probabilities over the sample period. We start with the benchmark one regime variable-two regimes model (Model III) in Figure 1, because this model has been the main focus of the RS literature, applied to nominal interest rates. The second regime is a regime of a high mean-high variability f_t factor, but low mean inflation factor. The high variability regime occurs from 1973-1975, in 1979-1982 and briefly after the oil shock in the recession of 1975. This is consistent with other studies. For example, Garcia and Perron (1996) find that a high inflation regime prevails during 1973-1982 and that the real rate switches to a more volatile regime after 1973 to the end of their sample in 1986. The second regime also aligns exactly with the monetary targeting period from 1979-1982 found by RS studies on the nominal rate (see, for example, Gray, 1996; Ang and Bekaert, 2002a).

In Figure 2, we plot the smoothed regime probabilities for Model IV, for the two separate and independent regime variables. The top panel graphs the probability of the real regime variable being in the high variability-high mean regime. The 1979-1982 period is still associated with this regime, but it starts a bit later and there is a relapse into this regime around 1985. The period just prior to the 1960 recession, the 4 years before the 1970 recession, and the two years before the 1975 recession and the 1975 recession itself are also classified as being in the high real rate regime. This regime briefly reappears in 1995. Overall the real regime variable spends around 21% of the sample period in the high mean-high variability regime.

The inflation regimes look very different, indicating the potential importance of separating the real and inflation regime variables. Over the sample, we spend most of the time in inflation regime 1, the high inflation regime. Some exceptions include the period after the 1955 and 1970 recessions, a brief period between the 1980 and 1982 recessions and a more extensive period in the mid-1980's and finally the early 1990's. Some previous attempts at identifying inflation regimes include Evans and Wachtel (1993) and Evans and Lewis (1995). However, these models are not directly comparable as they feature a random walk component in one regime (with no drift) and an AR(1) model in the other. The random walk regime has the most variable innovations and dominates the 1968-1983 period. During this time, their regime variable makes a few dips that coincide rather well with the dips in our regime probability of being in the high inflation regime. However, we classify 1955-1968 as a high inflation regime and Evans and Lewis classify this period as a low inflation variability regime. It is perhaps better to characterize the high inflation regime as a "normal regime" and the low inflation regime as a "deflation regime". The model accommodates rapid decreases in inflation by a transition to the second regime.

In Model IV, the two regime variables are independent. The models with correlated real and inflation regimes (Cases A and B) have similar qualitative properties of the transition probability matrices. Given the similarities between Model IV and Cases A and B, it is no surprise that a likelihood ratio test fails to reject independence of the two regime variables from both Cases A and B at the 10% level. Consequently, we focus our attention on Model IV.

6 The Term Structure of Real Rates and Expected Inflation

We summarize the behavior of real short rates, 1-quarter ahead expected inflation and nominal short rates in Table 4 before examining the term structure of each component in detail. The first regime is a low real rate-high inflation regime, where we spend 77% of the time in population. In this regime, both real rates and inflation are not very volatile. The second regime has a high mean real short rate (2.20%), combined with low expected inflation (both $\pi^e_{t,1}$ and $\mathrm{E}_t(\pi_{t+1})$ are 2.48%). This is similar to regime 4, except both real rates and expected inflation are much more volatile in regime 4. Regime 3 has slightly higher real rates than regime 1, but is still a low real rate regime. However, both real rate and inflation volatility are high. Regime 3 also has the highest expected inflation mean and volatility. Note that the moments of $\pi^e_{t,1}$ are near-identical to $\mathrm{E}_t(\pi_{t+1})$ (and so are not reported), indicating that the Jensen's inequality and convexity bias discussed in Section 3.4 are economically not important.

Hence, we can summarize our regime characteristics as:

		Real Rates	Inflation	% Time
$s_t = 1$	$s^r_t = 1, s^\pi_t = 1$	Low and Stable	High and Stable	77%
$s_t = 2$	$s^r_t = 1, s^\pi_t = 2$	High and Stable	Low and Stable	10%
$s_t = 3$	$s^r_t = 2, s^\pi_t = 1$	Low and Volatile	High and Volatile	11%
$s_t = 4$	$s^r_t = 2, s^\pi_t = 2$	High and Volatile	Low and Volatile	2%

All the levels (low or high) and variability (stable or volatile) are relative statements, so caution must be taken in the interpretation. Because of the dependence of the real rate on inflation, the terminology “high and low real rate regime” based on the regimes in f_t is misleading. The means of real rates are driven mostly by the inflation regime, while the volatility of both real rates and inflation is driven by the real regime. The negative covariance between inflation and real rates (for example, in regimes 2 and 4, inflation is high but real rates are low) is due to the negative δ_π coefficient in the real short rate equation (3).

6.1 The Behavior of Real Yields

The Real Rate Term Structure

We examine the real term structure in Figure 3. To facilitate comparison with the previous literature, the top panel of Figure 3 graphs the regime-dependent real term structure for the two-regime benchmark model (Model III). Every point on the curve for regime i represents the expected real zero coupon bond yield conditional on regime i, ($\mathrm{E}[\hat{y}_t^n(i)|s_t = i]$).[6] The normal regime has a fairly flat, but slightly humped real term structure with a peak at a maturity around 6 quarters. The second regime has a strongly downward-sloping real yield curve, with a real short rate of 2.7%. Hence, the behavior uncovered by previous RS models seems to be primarily driven by real rate variation. Unconditionally, the term structure is also fairly flat, but is slightly hump-shaped, starting at a rate of about 1.7%, increasing to just over 1.8% at 4-quarters and then declining to around 1.7% at 20-quarters. Our RS term structure model is very flexible and can easily produce a variety of shapes of the real yield curve, including flat, inverse-humped, upward-sloping or downward-sloping yield curves. Hence, the estimated flat, but slightly humped, shape of the real term structure is not due to any restrictions imposed by the model.

In the bottom panel of Figure 3, we show the real term structure of Model IV. In this model, regime 1 corresponds to the normal low real rate-high inflation regime. Like Model III in the top panel, the real term structure is also fairly flat with a slight hump-shape peaking at 4-quarters. Both regimes 2 and 4, which are the low inflation regimes, have downward sloping real yield curves but they are not as steep as regime 1 for Model III. These regimes may reflect periods where an activist monetary policy achieved disinflation through high real rates. Finally, regime 3, a low real rate-high inflation and volatile regime, has a very humped, non-linear, real term structure. Unconditionally, the real rate curve does not look very different from that obtained

[6] It is also possible to compute the more extreme case $\mathrm{E}[\hat{y}_t^n(i)|s_t = i, \forall t]$, that is, assuming that the process never leaves regime i. These curves have similar shapes to the ones shown in the figures but lie at different levels. Appendix G details the computation of these conditional moments.

in the two-regime benchmark model but real rates are, on average, 20 to 30 basis points lower. Figure 3 uncovers our first stylized fact:

Stylized Fact 1 *Unconditionally, the term structure of real rates assumes a fairly flat shape around 1.44%, with a slight hump, peaking at a 1-year maturity. However, there are some regimes in which the real rate curve is downward sloping.*

Characteristics of Real Rates

From the top panel of Figure 4, we see that the real short rate exhibits considerable short-term variation, sometimes decreasing and increasing sharply. Some of this variation may have genuine economic causes, for example the action of monetary policy. Note, for example, the sharp decreases of real rates in the 1958 and 1975 recessions and the sharp increases after the two oil shocks. In fact, standard error bands arising from parameter uncertainty (not shown) are very tight. It is also striking that, consistent with the older literature, real rates are generally low from the 1950's until 1980. The sharp increase in the early 1980's up to almost 8% was temporary, but it took until the 1990's before real rates reached the low levels of a little below 2%. This is still slightly higher than the level of the real rate before 1980. In the bottom panel of Figure 4, we graph the 5-year real yield. Not surprisingly, the 5-year real rate shows much less time-variation, but the same secular effects that drive the variation of the short real rate are visible. Given these patterns, it is not surprising that the Garcia-Perron (1996) model, which allowed for a finite number of possible ex-ante real rates, provided a reasonable fit to the data (up until the end of their sample in 1986), although it is clear that it misses some important variation and would have a hard time generating the gradual decrease of real rates since the 1980's.

Table 5 reports a number of unconditional characteristics of real yields. The unconditional standard deviation of the real short rate (20-quarter real yield) is 1.60% (0.61%). These moments solidly reject the hypothesis that the real short rate is constant, but at long horizons real yields are much more stable and persistent. This is clearly shown by the autocorrelations of the real short rate and 20-quarter real rate, which are 61% and 94%, respectively. Hence:

Stylized Fact 2 *Real rates are quite variable at short maturities but smooth and persistent at long maturities. There is no significant real term spread.*

The mean of the 20-quarter real term spread is only 2 basis points. The standard error is only 27 basis points, so that the real term structure cannot account for the 93 basis points nominal term spread in the data.

The Correlation of Real Rates and Inflation

There has been much interest in the correlation between real rates and expected inflation (the Mundell-Tobin effect). Older empirical studies such as Huizinga and Mishkin (1986) and Fama and Gibbons (1982) report negative correlations, but their analysis suffers from potential measurement error bias and implicitly assumes a zero inflation risk premium. Hence, their instrumented real rates may partially embed inflation risk premiums. Pennachi (1991), using a two factor affine model of real rates and expected inflation, finds that the conditional correlation between the two is negative. However, the evidence is less uniform than this discussion suggests. Barr and Campbell (1997) use UK interest rates and find that the unconditional correlation between real rates and inflation is small but positive, whereas the correlation between the change in the real rate and changes in expected inflation is strongly negative. Using an empirical RS model, Goto and Torous (2002) claim that the negative relation between expected inflation and real rates in the US has switched signs since 1981.

Table 5, Panel B reports conditional and unconditional correlations of real rates and unexpected and expected inflation. At the one quarter horizon, the conditional correlation of real rates with both inflation and expected inflation is negative in all regimes and hence also unconditionally. As expected, the effect is more negative for actual inflation than for expected inflation. The differences across regimes are not large in economic terms and the correlations are overall not significantly different from zero. Consequently, we do not find strong statistical evidence for a Mundell-Tobin effect:

Stylized Fact 3 *The real short rate is negatively correlated with both expected and unexpected inflation, but the statistical evidence for a Mundell-Tobin effect is weak.*

These results may have the following monetary policy interpretation. On average, positive inflation shocks lead to lower real rates, which smooths the behavior of nominal interest rates. However, the weak relation with expected inflation suggests that, in some instances, monetary policy may have been activist, raising real rates in response to expected inflation shocks. The fact that the correlation between expected inflation and real rates is least negative in the volatile real rate regimes 3 and 4 is consistent with this interpretation. The correlations of longer horizon real rates robustly turn positive, although they are again not very precisely estimated. The positive signs at long horizons result from the positive effect of the Φ coefficients dominating the negative effect of the δ_π coefficient.

6.2 The Behavior of Expected Inflation

The Term Structure of Expected Inflation

Table 6 reports some characteristics of expected inflation ($\pi^e_{t,n}$ and $\mathrm{E}_t(\pi_{t+n,n})$), differentiating the moments across regimes.[7] We focus first on the inflation compensation estimates $\pi^e_{t,n}$. The first inflation regime, $s^\pi_t = 1$ is a high inflation regime with a mean of 4.22% as opposed to 2.54% in the second inflation regime. If we look at expected inflation across the real rate regimes, the first real rate regime $s^r_t = 1$ (where the real rate is stable), is associated with lower expected inflation, 3.96%, than the second real rate regime (where the real rate is volatile) at 4.42%.

Perhaps the most striking feature in Table 6 is the upward sloping term structure of $\pi^e_{t,n}$ in all regimes. In particular, the inflation compensation is 91 basis points in the first inflation regime and 146 basis points in the second inflation regime. Of course, if $\pi^e_{t,n}$ truly reflects expected inflation, this could not occur. Hence, we uncover the next important stylized fact:

Stylized Fact 4 *The model matches an unconditional upward-sloping nominal yield curve by generating an inflation risk premium that is increasing in maturity.*

The bottom half of Table 6 reports expected inflation from the RS-VAR, $\mathrm{E}_t(\pi_{t+n,n})$. For this measure of expected inflation, we always approach the unconditional mean of inflation as n increases, in all regimes. Recall that there are three sources of differences between $\pi^e_{t,n}$ and $\mathrm{E}_t(\pi_{t+n,n})$ in our model. Our estimates reveal that the first two sources, the usual Jensen's term and a convexity bias, are trivially small in magnitude (at most a few basis points) and so cannot account for the large spread in $\pi^e_{t,n}$. Instead the upward slope in the term structure of inflation compensation is due to an inflation risk premium.

The Inflation Risk Premium

The third panel of Table 6 reports statistics on the inflation risk premium $\varphi_{t,n}$. The one-quarter inflation premiums are slightly negative because of the Jensen's term and the convexity bias.[8] The inflation risk premium is higher in the variable real rate regime 2, reaching 1.14% at the 20-quarter horizon, but the differences between the two real regimes are small. There is

[7] The 1-quarter ahead forecasts of expected inflation produced by the RS-VAR, $\mathrm{E}_t(\pi_{t+1})$ rarely exceed 2% in absolute value and are on average -2 basis points, so the model produces unbiased forecast errors. We defer a more detailed analysis of the inflation forecasting performance of our model to future work.

[8] If $\lambda_\pi(s_t)$ were negative, a positive one-quarter ahead expected risk premium would result, but with only nominal yields, this parameter cannot be identified. We feel that it is unlikely that there is a large inflation risk premium at the one quarter horizon, because it would imply that real rates are even lower than our current estimates.

stronger regime-dependence when we investigate the inflation regimes. In the high-expected inflation regime 1, the inflation risk premium is also high (1.04% at the 20-quarter horizon), whereas the deflation regime has a lower inflation risk premium (0.47% at the 20-quarter horizon). Unconditionally, the inflation risk premium is 97 basis points at the 20-quarter horizon, which is statistically significantly different from zero. The variability of the risk premium (not reported) also increases with horizon, reaching 34 basis points at the 20-quarter horizon.

Figure 5 provides some intuition on which parameters have the largest effect on the unconditional 20-quarter inflation risk premium. The risk premium is not very sensitive to δ_π or $\Phi_{\pi q}$. However, increasing the persistence of the inflation process either through $\Phi_{\pi\pi}$ or $\Phi_{\pi f}$ considerably increases $\varphi_{t,n}$. Increasing these parameters would also turn the slightly negative correlation between expected inflation and real rates into a positive correlation. The effect of persistence is also stronger than the effect of the price of risk $\lambda_f(s_t^r)$. Making the price of risk more negative naturally increases the inflation risk premium, but this would cause the model to grossly over-estimate the nominal term spread.

Figure 6 graphs the 20-quarter inflation risk premium over time. The inflation risk premium has decreased in every recession, except for the 1981-83 recession, coinciding with monetary targeting. After the 1953-54 recession, the inflation risk premium was almost zero. The general trend is that the premium steadily rose from the 1950's throughout the 1960's and 1970's before entering a very volatile period during the monetary targeting period from 1979 to the early 1980's. It is then that the premium reached a peak of 2.1%. Whereas the trend since then has been downward, there have been large swings in the premium. From a temporary low of 60 basis points in the mid-eighties it shot up to 1.3%, coinciding with the halting of the large dollar appreciation of the early 1980's, and then dropped to around 40 basis points in 1993. In 1995 the premium shot up to 1.3% at the same time the Fed started to raise interest rates. During the late 1990's bull market inflation risk premiums were fairly stable and averaged around 80 basis points.

6.3 Nominal Term Structure

In their two regime model, Bansal and Zhou (2002) show that one regime displays the normal situation of an upward sloping nominal yield curve and the other regime displays a fairly flat nominal yield curve. The top panel of Figure 7 confirms this pattern for Model III, which features only two regimes. However, the much better performing Model IV, shown in the bottom panel of Figure 7, with four regimes, does not produce a downward-sloping or flat nominal yield curve in any regime. Recall that in some regimes the real rate curve is downward sloping, but the upward sloping term structure of priced expected inflation completely counteracts this effect.

For Model IV, the first regime (low real rate-high inflation regime) displays a nominal yield curve that is nicely upward sloping, with the slope flattening out for longer maturities, matching the unconditional term structure and the data almost exactly. In the second regime, where the inflation drift is lower, the yield curve is steeply upward sloping but rates are lower than in the first regime because of lower expected inflation. In the third regime, the term structure is steeply upward sloping at the short end but then becomes flat and slightly downward sloping for maturities extending beyond 6 quarters. Nominal interest rates are the highest in this regime because expected inflation is high in this regime, as the level of real rates is about the same in this regime as in regime 1. In the 4th regime, the term structure is J-shaped with rates below the unconditional curve. This is a regime where the real interest rate curve is downward sloping, but at a high level. Inflation compensation, however, is low in this regime (making nominal yields low), and is upward sloping, which starts to counteract the downward real slope at maturities longer than 1 year, causing the slight dip between 1 and 4-quarters.

Interest rates are often associated with the business cycle. According to the conventional wisdom, interest rates are pro-cyclical and spreads counter-cyclical (see, for example, Fama, 1990). Table 7 shows that this is incorrect. In fact, interest rates are overall larger during recessions. However, when we focus on real rates, the conventional story is right:

Stylized Fact 5 *Nominal interest rates (spreads) do not behave pro-cyclically (counter-cyclically) across the business cycle but our model-implied real rates do.*

This can only be the case if expected inflation is counter-cyclical. The table shows that this is indeed the case, with expected inflation being strongly counter-cyclically, reaching 5.05% in recessions but only 3.66% in expansions. Veronesi and Yared (1999) also find that real rates are pro-cyclical.

One interesting fact that Table 7 illustrates is that recessions are characterized by more uncertainty, in the sense that all interest rates, spreads and inflation are more volatile in recessions than they are in expansions. Whereas these are simply empirical facts, our model shows that this is also the case for real rates, real rate spreads and expected inflation spreads. The bottom part of the table lists the proportions of each regime realized through the whole sample, compared with the proportions of each regime realized in expansions and recessions. The normal regime 1 occurs much more during expansions. In comparison, the volatile real rate and inflation regimes 3 and 4 occur much more often during recessions.

6.4 Variance Decompositions

Table 8 reports the population variance decomposition (20) of the nominal yield into real and priced expected inflation variation produced by Model IV. We also report the variance decompositions conditional on each regime.[9] The results are striking:

Stylized Fact 6 *The decompositions of nominal yields into real yields and expected inflation at various horizons indicate that variation in inflation compensation (expected inflation and inflation risk premiums) explains about 80% of the variation in nominal rates at both short and long maturities.*

This is at odds with the folklore wisdom that expected inflation primarily affects long-term bonds (see, among others, Fama, 1975; Mishkin, 1981). However, this result is consistent with Pennacchi (1991), who identifies expected inflation from survey data. The decomposition shows little variation across regimes. Expected inflation is slightly less important at short horizons, but the long-horizon decomposition is essentially unchanged.

Looking at the variance decomposition of nominal term spreads, Table 8 shows that, unconditionally, inflation accounts for 52% of the 4-quarter term spread and 83% of the 20-quarter term spread. For term spread changes, inflation shocks dominate at the long-end of the yield curve. In the normal regime 1, inflation shocks account for almost all (99%) of the movements of the long term spread. In regimes 3 and 4, inflation accounts for relatively little of movements in term spreads. In these regimes, real rates are very volatile, and expected inflation accounts for only 11% of the variation in the 4-quarter term spread, increasing to 60% for the 20-quarter term spread. Hence, the conventional wisdom that inflation is more important for the long end of the yield curve holds, not for the level of yields, but for term spreads. The later work of Mishkin (1990 and 1992) finds evidence consistent with our findings as his regressions use inflation changes and term spreads, rather than yield levels. Hence, we have our last stylized fact:

Stylized Fact 7 *Inflation compensation only explains about 50% of the variation in nominal term spreads at short horizons, but is the main determinant of nominal interest rate spreads at long horizons.*

Finally, we have also split up the contribution of inflation compensation into its expected inflation and inflation risk premium components, but do not report the results for brevity.

[9] We do not split up priced expected inflation further into actual expected inflation and the inflation risk premium because for all of our decompositions, the bulk of the variation in the inflation compensation comes from actual expected inflation rather than risk premiums.

Expected inflation is the dominant component, suggesting that shocks to expected inflation die out very slowly. This result is consistent with the findings of Gürkaynak, Sack and Swanson (2003), who find strong sensitivity of long-term US forward rates to macroeconomic and monetary policy news releases. They can replicate such behavior in a model where private agents adjust their expectations of long-run inflation rate to macro and policy surprises, and inflation follows a reduced-form ARMA process, as in our model.

7 Conclusion

In this article, we develop a term structure model that embeds regime switches in both real and nominal factors, and which incorporates time-varying price of risks. The model that provides the best fit with data has independent real and inflation regimes. This four-regime model is substantially different in its implications for the term structure than the standard two-regime model. While the regimes for a real factor and inflation are independent, real rates and both expected and unexpected inflation are negatively correlated.

We find that the real rate curve is fairly flat but slightly humped, with an average real rate of 1.44% and a 20-quarter spread of not even 2 basis points. The real short rate has a variability of 1.60% and has an autocorrelation of 61%. In some regimes, the real rate curve is steeply downward sloping. The yield curve of expected inflation implied by bond yields is steeply upward sloping. This is due to an upward sloping inflation risk premium, which is unconditionally 97 basis points on average.

The standard view that interest rates are pro-cyclical and spreads counter-cyclical is typically based on real economic effects. However, in the data, nominal interest rates are counter-cyclical. Our model generates real rates that are entirely consistent with the standard view. Although lower, real rates are substantially more variable in recessions. We find that expected inflation accounts for 80% of the variation in nominal yields at both short and long maturities. However, nominal term spreads are primarily driven by changes in expected inflation, particularly during normal times.

We extract unobservable real rates from nominal yields and inflation. Observable real rates would increase our confidence about our results. Unfortunately, the US Treasury only started issuing Treasury Inflation Protection Securities (TIPS) from January 1997 onwards, and the market was very illiquid during its first few years. Roll (2003) provides a detailed analysis of the TIPS, stressing some institutional features that might make TIPS imperfect indicators of true real rates. Nevertheless, it is interesting to note that our results are qualitatively consistent with Roll's findings, over the very short sample period since TIPS began trading. Roll finds that the nominal yield curve is more steeply sloped than the real curve, which is also mostly fairly flat over our over-lapping sample periods. Roll also shows direct evidence of an inflation premium that increases with maturity.

Our work here is only the beginning of a research agenda. First, our model uses term structure information in an efficient way to generate expected inflation. Hence, it is likely that we have constructed an attractive inflation-forecasting model. Simple approaches that use term structure information without no-arbitrage restrictions to forecast inflation have not proved successful (see Stock and Watson, 1999). Second, our model would allow us to link the often discussed deviations from the Expectations Hypothesis (Campbell and Shiller, 1991, for example) to deviations from the Fisher hypothesis (Mishkin, 1992). Finally, although we have made one step in the direction of identifying the economic sources of regime switches in interest rates, more could be done. In particular, a more explicit examination of the role of business cycle variation and changes in monetary policy as sources of the regime switches is an interesting topic for further research.

Appendix

A A Regime-Switching Model with Stochastic Expected Inflation

In a final extension, motivated by the ARMA-model literature (see Fama and Gibbons, 1982; Hamilton, 1985), we allow inflation to be composed of a stochastic expected inflation term plus a random shock:

$$\pi_{t+1} = w_t + \sigma_\pi \varepsilon^\pi_{t+1},$$

where $w_t = \mathrm{E}_t[\pi_{t+1}]$ is the one-period-ahead expectation of future inflation. This can be accomplished in our framework by expanding the state variables to $X_t = (q_t\, f_t\, w_t\, \pi_t)'$ which follow the dynamics of equation (1), except now:

$$\mu(s_t) = \begin{bmatrix} \mu_q \\ \mu_f(s_t) \\ \mu_w(s_t) \\ 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} \Phi_{qq} & 0 & 0 & 0 \\ \Phi_{fq} & \Phi_{ff} & 0 & 0 \\ \Phi_{wq} & \Phi_{wf} & \Phi_{ww} & \Phi_{w\pi} \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{A-1}$$

and $\Sigma(s_t)$ is a diagonal matrix with $(\sigma_q\, \sigma_f(s_t)\, \sigma_w(s_t)\, \sigma_\pi(s_t))'$ on the diagonal. Note that both the variance of inflation and the process of expected inflation are regime-dependent. Moreover, past inflation affects current expected inflation through $\Phi_{w\pi}$.

The real short rate and the regime transition probabilities are the same as in the benchmark model (4). The real pricing kernel also takes the same form as (5) with one difference. The regime-dependent part of the prices of risk in equation (6) is now given by:

$$\lambda(i) = (\lambda_f(i)\ \lambda_w(i)\ \lambda_\pi(i))',$$

but we set $\lambda_w(i) = 0$ so that there is no correlation between the real pricing kernel and any inflation shocks.

B Modelling Separate Real and Inflation Regimes

We detail the independent and correlated real rate and inflation regime specifications (Cases A and B) of Section 3.3. Table A-4 reproduces the 4×4 transition probability matrices implied by the independent model and Cases A and B.

Independent Regimes

In the first case, we impose the restriction that the inflation and the real regimes evolve independently. In other words:

$$Pr\left[s^r_{t+1} = j, s^\pi_{t+1} = k | s^r_t = m, s^\pi_t = n\right] = Pr\left[s^r_{t+1} = j | s^r_t = m\right] \times Pr\left[s^\pi_{t+1} = k | s^\pi_t = n\right]. \tag{B-1}$$

Equation (B-1) gives rise to a restricted transition probability matrix Π_0:

	$[s_{t+1} = 1]$	$[s_{t+1} = 2]$	$[s_{t+1} = 3]$	$[s_{t+1} = 4]$
$[s_t = 1]$	$p^r p^\pi$	$p^r(1-p^\pi)$	$(1-p^r)\, p^\pi$	$(1-p^r)(1-p^\pi)$
$[s_t = 2]$	$p^r(1-q^\pi)$	$p^r q^\pi$	$(1-p^r)(1-q^\pi)$	$(1-p^r)\, q^\pi$
$[s_t = 3]$	$(1-q^r)\, p^\pi$	$(1-q^r)(1-p^\pi)$	$q^r p^\pi$	$q^r(1-p^\pi)$
$[s_t = 4]$	$(1-q^r)(1-q^\pi)$	$(1-q^r)\, q^\pi$	$q^r(1-q^\pi)$	$q^r q^\pi$

(B-2)

The independent case is very parsimonious, adding only two parameters to the benchmark model.

Correlated Regimes Case A

In this case, we decompose the joint transition probability of real rates and inflation regimes as:

$$\begin{aligned} &Pr\left[s^r_{t+1} = j, s^\pi_{t+1} = k | s^r_t = m, s^\pi_t = n\right] \\ &\quad = Pr\left[s^r_{t+1} = j | s^\pi_{t+1} = k, s^r_t = m, s^\pi_t = n\right] \times Pr\left[s^\pi_{t+1} = k | s^r_t = m, s^\pi_t = n\right] \\ &\quad = Pr\left[s^r_{t+1} = j | s^\pi_{t+1} = k, s^r_t = m\right] \times Pr\left[s^\pi_{t+1} = k | s^\pi_t = n\right] \end{aligned} \tag{B-3}$$

In the last line, we assume that the past inflation regime does not determine the contemporaneous correlation of the real rate and inflation regime. Mathematically, we assume that $Pr\left[s^r_{t+1}=j|s^\pi_{t+1}=k,s^r_t=m,s^\pi_t=n\right]=Pr\left[s^r_{t+1}=j|s^\pi_{t+1}=k,s^r_t=m\right]$. We also assume that $Pr\left[s^\pi_{t+1}=k|s^\pi_t=n,s^r_t=m\right]=Pr\left[s^\pi_{t+1}=k|s^\pi_t=n\right]$, or that past real rates do not influence future inflation regime realizations.

In (B-3), we parameterize $Pr\left[s^r_{t+1}=j|s^\pi_{t+1}=k,s^r_t=m\right]$ as $p^{``j",``m"}$, where:

$$j=\begin{cases} A & \text{if } s^r_{t+1}=s^\pi_{t+1}=1 \\ B & \text{if } s^r_{t+1}=s^\pi_{t+1}=2 \end{cases}$$

$$m=\begin{cases} A & \text{if } s^r_t=1 \\ B & \text{if } s^r_t=2. \end{cases}$$

The "j"-component captures (potentially positive) correlation between inflation and real rate regimes. The "m"-component captures persistence in real rate regimes.

With this notation, the transition probability matrix Π_1 assumes the form:

	$[s_{t+1}=1]$	$[s_{t+1}=2]$	$[s_{t+1}=3]$	$[s_{t+1}=4]$
$[s_t=1]$	$p^{AA}p^\pi$	$\left(1-p^{BA}\right)(1-p^\pi)$	$\left(1-p^{AA}\right)p^\pi$	$p^{BA}(1-p^\pi)$
$[s_t=2]$	$p^{AA}(1-q^\pi)$	$\left(1-p^{BA}\right)q^\pi$	$\left(1-p^{AA}\right)(1-q^\pi)$	$p^{BA}q^\pi$
$[s_t=3]$	$p^{AB}p^\pi$	$\left(1-p^{BB}\right)(1-p^\pi)$	$\left(1-p^{AB}\right)p^\pi$	$p^{BB}(1-p^\pi)$
$[s_t=4]$	$p^{AB}(1-q^\pi)$	$\left(1-p^{BB}\right)q^\pi$	$\left(1-p^{AB}\right)(1-q^\pi)$	$p^{BB}q^\pi$

(B-4)

This model has four additional parameters relative to the benchmark model. We can test the null of independent real rate and inflation regimes versus correlated regimes by:

$$H_0: p^{BA}=1-p^{AA} \text{ and } p^{BB}=1-p^{AB}.$$

Correlated Regimes Case B

In Case B, we condition $Pr\left[s^r_{t+1}=j,s^\pi_{t+1}=k|s^r_t=m,s^\pi_t=n\right]$ as:

$$\begin{aligned} &Pr\left[s^r_{t+1}=j,s^\pi_{t+1}=k|s^r_t=m,s^\pi_t=n\right] \\ &\quad=Pr\left[s^r_{t+1}=j|s^\pi_{t+1}=k,s^r_t=m,s^\pi_t=n\right]\times Pr\left[s^\pi_{t+1}=k|s^r_t=m,s^\pi_t=n\right] \\ &\quad=Pr\left[s^r_{t+1}=j|s^r_t=m\right]\times Pr\left[s^\pi_{t+1}=k|s^r_t=m,s^\pi_t=n\right]. \end{aligned} \tag{B-5}$$

Here, we assume that $Pr\left[s^r_{t+1}=j|s^\pi_{t+1}=k,s^r_t=m,s^\pi_t=n\right]=Pr\left[s^r_{t+1}=j|s^r_t=m\right]$. Economically, future real regimes depend only on current real regimes, while future inflation regimes depend on both the stance of the monetary authority's real regime as well as the current inflation environment.

In (B-5), we parameterize $Pr\left[s^\pi_{t+1}=k|s^r_t=m,s^\pi_t=n\right]$ as $p^{``j",``m"}$, where:

$$j=\begin{cases} A & \text{if } s^\pi_{t+1}=s^r_t=1 \\ B & \text{if } s^\pi_{t+1}=s^r_t=2 \end{cases}$$

$$m=\begin{cases} A & \text{if } s^\pi_t=1 \\ B & \text{if } s^\pi_t=2. \end{cases}$$

Then the transition probability matrix Π_2 assumes the form:

	$[s_{t+1}=1]$	$[s_{t+1}=2]$	$[s_{t+1}=3]$	$[s_{t+1}=4]$
$[s_t=1]$	p^rp^{AA}	$p^r\left(1-p^{AA}\right)$	$(1-p^r)\left(1-p^{BA}\right)$	$(1-p^r)p^{BA}$
$[s_t=2]$	p^rp^{AB}	$p^r\left(1-p^{AB}\right)$	$(1-p^r)\left(1-p^{BB}\right)$	$(1-p^r)p^{BB}$
$[s_t=3]$	$(1-q^r)p^{AA}$	$(1-q^r)\left(1-p^{AA}\right)$	$q^r\left(1-p^{BA}\right)$	q^rp^{BA}
$[s_t=4]$	$(1-q^r)p^{AB}$	$(1-q^r)\left(1-p^{AB}\right)$	$q^r\left(1-p^{BB}\right)$	q^rp^{BB}

(B-6)

We refer to this parameterization as the partially-correlated regimes Case B. We can test Case B against the null of the independent model by:

$$H_0: p^{BA}=1-p^{AA} \text{ and } p^{BB}=1-p^{AB}.$$

C Real Bond Prices

Let N_1 be the number of unobserved state variables in the model ($N_1 = 3$ for the stochastic inflation model, $N_1 = 2$ otherwise) and $N = N_1 + 1$ be the total number of factors including inflation. The following proposition describes how our model implies closed-form real bond prices.

Proposition C.1 *Let $X_t = (q_t\, f_t\, \pi_t)'$ or $X_t = (q_t\, f_t\, w_t\, \pi_t)'$ follow (1), with the real short rate (3) and real pricing kernel (5) with prices of risk (6). The regimes s_t follow a Markov chain with transition probability matrix $\Pi = \{p_{ij}\}$. Then the real zero coupon bond price for period n conditional on regime i, $\widehat{P}_t^n(s_t = i)$, is given by:*

$$\widehat{P}_t^n(i) = \exp(\widehat{A}_n(i) + \widehat{B}_n X_t). \tag{C-1}$$

The scalar $\widehat{A}_n(i)$ is dependent on regime $s_t = i$ and $\widehat{B}_n$ is a $1 \times N$ vector that is partitioned as $\widehat{B}_n = [\widehat{B}_{nq}\, \widehat{B}_{nx}]$, where $\widehat{B}_{nq}$ corresponds to the q variable and $\widehat{B}_{nx}$ corresponds to the other variables in X_t. The coefficients $\widehat{A}_n(i)$ and $\widehat{B}_n$ are given by:

$$\begin{aligned}
\widehat{A}_{n+1}(i) = &- \left(\delta_0 + \widehat{B}_{nq}'\sigma_q\gamma_0\right) + \log\sum_j \pi_{ij}\exp\Big(\widehat{A}_n(j) + \widehat{B}_n\mu(j) \\
&- \widehat{B}_{nx}\Sigma_x(j)\lambda(j) + \frac{1}{2}\widehat{B}_n\Sigma(j)\Sigma(j)'\widehat{B}_n'\Big) \\
\widehat{B}_{n+1} = &- \delta_1' + \widehat{B}_n\Phi - \widehat{B}_{nq}\sigma_q\gamma_1 e_1',
\end{aligned} \tag{C-2}$$

where e_i denotes a vector of zero's with a 1 in the ith place and $\Sigma_x(i)$ refers to the lower $N_1 \times N_1$ matrix of of $\Sigma(i)$ corresponding to the non-q_t variables in X_t. The starting values for $\widehat{A}_n(i)$ and $\widehat{B}_n$ are:

$$\begin{aligned}
\widehat{A}_1(i) &= -\delta_0 \\
\widehat{B}_1 &= -\delta_1'.
\end{aligned} \tag{C-3}$$

Proof:

We first derive the initial values in (C-3):

$$\begin{aligned}
P_t^1(i) &= \sum_j p_{ij}\mathrm{E}_t\left[\widehat{M}_{t+1}|S_{t+1} = j\right] \\
&= \sum_j p_{ij}\exp\left(-r_t - \frac{1}{2}\lambda_t(j)'\lambda_t(j) - \lambda_t(j)'\varepsilon_{t+1}\right) \\
&= \exp(-\delta_0 - \delta_1' X_t)
\end{aligned} \tag{C-4}$$

Hence:

$$\widehat{P}_t^1(i) = \exp(\widehat{A}_1(i) + \widehat{B}_1 X_t),$$

where $A_1(i)$ and B_1 take the form in (C-3).

We prove the recursion (C-2) by induction. We assume that (C-1) holds for maturity n and examine $\widehat{P}_t^{n+1}(i)$:

$$\begin{aligned}
\widehat{P}_t^{n+1}(i) &= \sum_j p_{ij}\mathrm{E}_t\exp\left[-r_t - \frac{1}{2}\lambda_t(j)'\lambda_t(j) - \lambda_t(j)'\varepsilon_{t+1} + \widehat{A}_n(j) + \widehat{B}_n X_{t+1}\right], \\
&= \sum_j p_{ij}\mathrm{E}_t\exp\Big[-\delta_0 - \delta_1' X_t - \frac{1}{2}\lambda_t(j)'\lambda_t(j) - \lambda_t(j)'\varepsilon_{t+1} + \widehat{A}_n(j) \\
&\quad + \widehat{B}_n(\mu(j) + \Phi X_t + \Sigma(j)\varepsilon_{t+1})\Big]
\end{aligned} \tag{C-5}$$

Evaluating the expectation, we have:

$$
\begin{aligned}
\widehat{P}_t^{n+1}(i) =& \sum_j p_{ij} \exp\left[-\delta_0 - \delta_1' X_t - \frac{1}{2}\lambda_t(j)' \lambda_t(j) + \widehat{A}_n(j) + \widehat{B}_n \mu(j)\right. \\
& \left. + \widehat{B}_n \Phi X_t + \frac{1}{2}\left(\widehat{B}_n \Sigma(j) - \lambda_t(j)'\right)\left(\widehat{B}_n \Sigma(j) - \lambda_t(j)'\right)'\right] \\
=& \exp\left[-\delta_0 + \left(\widehat{B}_n \Phi - \delta_1'\right) X_t\right] \\
& \times \sum_j p_{ij} \exp\left[\widehat{A}_n(j) + \widehat{B}_n \mu(j) - \widehat{B}_n \Sigma(j) \lambda_t(j) + \frac{1}{2}\widehat{B}_n \Sigma(j) \Sigma(j) \widehat{B}_n'\right]
\end{aligned} \tag{C-6}
$$

But we can write:

$$
\begin{aligned}
\widehat{B}_n \Sigma(j) \lambda_t(j) &= [\widehat{B}_{nq} \; \widehat{B}_{nx}] \begin{bmatrix} \sigma_q(\gamma_0 + \gamma_1 e_1' X_t) \\ \Sigma_x(j) \lambda(j) \end{bmatrix} \\
&= \widehat{B}_{nq} \sigma_q (\gamma_0 + \gamma_1 e_1' X_t) + \widehat{B}_{nx} \Sigma_x(j) \lambda(j).
\end{aligned} \tag{C-7}
$$

Expanding and collecting terms, we can write:

$$
\widehat{P}_t^n(i) = \exp(\widehat{A}_n(i) + \widehat{B}_n X_t),
$$

where $\widehat{A}_n(i)$ and $\widehat{B}_n$ take the form of (C-2). ■

D Nominal Bond Prices

Following the notation of Appendix C, let N_1 be the number of unobserved state variables in the model ($N_1 = 3$ for the stochastic inflation model, $N_1 = 2$ otherwise) and $N = N_1 + 1$ be the total number of factors including inflation. The following proposition describes how our model implies closed-form nominal bond prices.

Proposition D.1 *Let $X_t = (q_t \, f_t \, \pi_t)'$ or $X_t = (q_t \, f_t \, w_t \, \pi_t)'$ follow (1), with the real short rate (3) and real pricing kernel (5) with prices of risk (6). The regimes s_t follow a Markov chain with transition probability matrix $\Pi = \{p_{ij}\}$. Then the nominal zero coupon bond price for period n conditional on regime i, $P_t^n(s_t = i)$, is given by:*

$$
P_t^n(i) = \exp(A_n(i) + B_n X_t), \tag{D-1}
$$

where the scalar $A_n(i)$ is dependent on regime $s_t = i$ and B_n is an $N \times 1$ vector:

$$
\begin{aligned}
A_{n+1}(i) =& -\left(\delta_0 + B_{nq}' \sigma_q \gamma_0\right) + \log \sum_j \pi_{ij} \exp\Bigg(A_n(j) + (B_n - e_N')\mu(j) \\
& - \left(B_{nx} - e_{N_1}'\right)\Sigma_x(j)\lambda(j) + \frac{1}{2}(B_n - e_N')\Sigma(j)\Sigma(j)(B_n - e_N')'\Bigg) \\
B_{n+1} =& -\delta_1' + (B_n - e_N')\Phi - B_{nq}\sigma_q\gamma_1 e_1',
\end{aligned} \tag{D-2}
$$

where e_i denotes a vector of zero's with a 1 in the ith place, $A(i)$ is a scalar dependent on regime $s_t = i$, B_n is a row vector, which is partitioned as $B_n = [B_{nq} \, B_{nx}]$, where B_{nq} corresponds to the q variable and $\Sigma_x(i)$ refers to the lower $N_1 \times N_1$ matrix of of $\Sigma(i)$ corresponding to the non-q_t variables in X_t. The starting values for $A_n(i)$ and B_n are:

$$
\begin{aligned}
A_1(i) &= -\delta_0 + \log \sum_j \pi_{ij} \exp\left(-e_N' \mu(j) + \frac{1}{2} e_N' \Sigma(j) \Sigma(j)' e_N + e_{N_1}' \Sigma_x(j) \lambda(j)\right) \\
B_1 &= -(\delta_1' + e_N' \Phi).
\end{aligned} \tag{D-3}
$$

Proof:

We first derive the initial values (D-3) by directly evaluating:

$$
\begin{aligned}
P_t^1(i) &= \sum_j p_{ij} E_t\left[\widehat{M}_{t+1} | S_{t+1} = j\right] \\
&= \sum_j p_{ij} \exp\left(-r_t - \frac{1}{2}\lambda_t(j)'\lambda_t(j) - \lambda_t(j)'\varepsilon_{t+1} - e_N'(\mu(j) + \Phi X_t + \Sigma(j)\varepsilon_{t+1})\right) \\
&= \exp(-\delta_0 - \delta_1' X_t - e_N'\Phi X_t) \\
&\quad \times \sum_j p_{ij} \exp\left(-e_N'\mu(j) - e_N'\Sigma(j)\varepsilon_{t+1} - \frac{1}{2}\lambda_t(j)'\lambda_t(j) - \lambda_t(j)'\varepsilon_{t+1}\right) \\
&= \exp(-\delta_0 - \delta_1' X_t - e_N'\Phi X_t) \\
&\quad \times \sum_j p_{ij} \exp\left(-e_N'\mu(j) + \frac{1}{2}e_N'\Sigma(j)\Sigma(j)'e_N + e_N'\Sigma(j)\lambda_t(j)\right).
\end{aligned} \tag{D-4}
$$

Note that $e_N'\Sigma(j)\lambda_t(j) = e_{N_1}'\Sigma_x(j)\lambda(j)$. Hence:

$$
P_t^1(i) = \exp(A_1(i) + B_1 X_t)
$$

where $A_1(i)$ and B_1 are given by (D-3).

To prove the general recursion we use proof by induction:

$$
\begin{aligned}
P_t^{n+1}(i) &= \sum_j p_{ij} E_t\Bigg[\exp\left(-r_t - \frac{1}{2}\lambda_t(j)'\lambda_t(j) - \lambda_t(j)'\varepsilon_{t+1} - e_N' X_{t+1}\right) \\
&\qquad \exp(A_n(j) + B_n X_{t+1})\Bigg] \\
&= \sum_j p_{ij} E_t\Bigg[\exp\Bigg(-\delta_0 - \delta_1' X_t - \frac{1}{2}\lambda_t(j)'\lambda_t(j) - \lambda_t(j)'\varepsilon_{t+1} + A_n(j) \\
&\qquad + (B_n - e_N')(\mu(j) + \Phi X_t + \Sigma(j)\varepsilon_{t+1})\Bigg)\Bigg] \\
&= \sum_j p_{ij} \exp\Bigg(-\delta_0 - \delta_1' X_t - \frac{1}{2}\lambda_t(j)'\lambda_t(j) + A_n(j) + (B_n - e_N')\mu(j) \\
&\qquad + (B_n - e_N')\Phi X_t + \frac{1}{2}\left((B_n - e_N')\Sigma(j) - \lambda_t(j)'\right)\left((B_n - e_N')\Sigma(j) - \lambda_t(j)'\right)'\Bigg) \\
&= \exp(-\delta_0 + ((B_n - e_N')\Phi - \delta_1')X_t) \sum_j p_{ij} \exp\Bigg(A_n(j) + (B_n - e_N')\mu(j) \\
&\qquad - (B_n - e_N')\Sigma(j)\lambda_t(j) + \frac{1}{2}(B_n - e_N')\Sigma(j)\Sigma(j)(B_n - e_N')'\Bigg)
\end{aligned} \tag{D-5}
$$

Now note that:

$$
\begin{aligned}
(B_n - e_N')\Sigma(j)\lambda_t(j) &= (B_n - e_N')\begin{bmatrix} \sigma_q(\gamma_0 + \gamma_1 e_1' X_t) \\ \Sigma_x(j)\lambda(j) \end{bmatrix} \\
&= \begin{bmatrix} B_{nq} \\ B_{nx} - e_{N_1}' \end{bmatrix} \begin{bmatrix} \sigma_q(\gamma_0 + \gamma_1 e_1' X_t) \\ \Sigma_x(j)\lambda(j) \end{bmatrix} \\
&= B_{nq}\sigma_q(\gamma_0 + \gamma_1 e_1' X_t) + \left(B_{nx} - e_{N_1}'\right)\Sigma_x(j)\lambda(j)
\end{aligned} \tag{D-6}
$$

where $B_n = [B_{nq}\ B_{nx}]$.

Hence, collecting terms and substituting (D-6) into (D-5), we have:

$$
P_t^{n+1}(i) = \exp[A_{n+1}(i) + B_{n+1} X_t],
$$

where: $A_n(i)$ and B_n are given by (D-2).

Note that the $A_n(i)$ term allows for an inflation premium, captured through $\Sigma_x(s_t)$ and $\lambda(s_t)$, but this is zero under our formulation. ■

E Likelihood Function

We specify the set of nominal yields without measurement error as Y_{1t} ($N_1 \times 1$) and the remaining yields as Y_{2t} ($N_2 \times 1$). There are as many yields measured without error as there are latent factors in X_t. The complete set of yields are denoted as $Y_t = (Y_{1t}'\, Y_{2t}')'$ with dimension $M \times 1$, where $M = N_1 + N_2$. Note that the total number of factors in X_t is $N = N_1 + 1$, since the last factor, inflation, is observable.

Given the expression for nominal yields in (13), the yields observed without error and inflation, $Z_t = (Y_{1t}'\, \pi_t)'$, take the form:

$$Z_t = \mathcal{A}_1(s_t) + \mathcal{B}_1 X_t, \tag{E-1}$$

where:

$$\mathcal{A}_1(s_t) = \begin{bmatrix} \mathcal{A}_n(s_t) \\ 0 \end{bmatrix} \qquad \mathcal{B}_1 = \begin{bmatrix} \mathcal{B}_n \\ e_N' \end{bmatrix}, \tag{E-2}$$

where $\mathcal{A}_n(s_t)$ is the $N_1 \times 1$ vector stacking the $-A_n(s_t)/n$ terms for the N_1 yields observed without error, and $\mathcal{B}_n$ is a $N_1 \times N$ matrix which stacks the $-B_n/n$ vectors for the two yields observed without error. Then we can invert for the unobservable factors:

$$X_t = \mathcal{B}^{-1}(Z_t - \mathcal{A}_1(s_t)) \tag{E-3}$$

Substituting this into (E-1) and using the dynamics of X_t in (1), we can write:

$$Z_t = c(s_t, s_{t-1}) + \Psi Z_{t-1} + \Omega(s_t)\epsilon_t, \tag{E-4}$$

where:

$$\begin{aligned} c(s_t, s_{t-1}) &= \mathcal{A}_1(s_t) + \mathcal{B}_1\mu(s_t) - \mathcal{B}_1\Phi\mathcal{B}_1^{-1}\mathcal{A}_1(s_{t-1}) \\ \Psi &= \mathcal{B}_1\Phi\mathcal{B}_1^{-1} \\ \Omega(s_t) &= \mathcal{B}_1\Sigma(s_t) \end{aligned}$$

Note that our model implies a RS-VAR for the observable variables with complex cross-equation restrictions.

The yields Y_{2t} observed with error have the form:

$$Y_{2t} = \mathcal{A}_2(s_t) + \mathcal{B}_2 X_t + u_t, \tag{E-5}$$

where $\mathcal{A}_2$ and $\mathcal{B}_2(s_t)$ follow from Proposition D.1 and u is the measurement error, $u_t \sim N(0, V)$, where V is a diagonal matrix. We can solve for u_t in equation (E-5) using the inverted factor process (E-3). We assume that u_t is uncorrelated with the errors ε_t in (1).

Following Hamilton (1994), we redefine the states s_t^* to count all combinations of s_t and s_{t-1}, with the corresponding re-defined transition probabilities $p_{ij}^* = p(s_{t+1}^* = i | s_t^* = j)$. We re-write (E-4) and (E-5) as:

$$\begin{aligned} Z_t &= c(s_t^*) + \Psi Z_{t-1} + \Omega(s_t^*)\epsilon_t, \\ Y_{2t} &= \mathcal{A}_2(s_t^*) + \mathcal{B}_2 X_t + u_t. \end{aligned} \tag{E-6}$$

Now the standard Hamilton (1989 and 1994) and Gray (1996) algorithms can be used to estimate the likelihood function. Since (E-6) gives us the conditional distribution $f(\pi_t, Y_t^1 | s_t^* = i, I_{t-1})$, we can write the likelihood as:

$$\begin{aligned} \mathcal{L} &= \prod_t \sum_{s_t^*} f(\pi_t, Y_{1t}, Y_{2t} | s_t^*, I_{t-1}) Pr(s_t^* | I_{t-1}) \\ &= \prod_t \sum_{s_t^*} f(Z_t | s_t^*, I_{t-1}) f(Y_{2t} | \pi_t, Y_{1t}, s_t^*, I_{t-1}) Pr(s_t^* | I_{t-1}) \end{aligned} \tag{E-7}$$

where:

$$f(Z_t | s_t^*, I_{t-1}) = (2\pi)^{-(N_1+1)/2} |\Omega(s_t^*)\Omega(s_t^*)'|^{-1/2} \\ \exp\left(-\frac{1}{2}(Z_t - c(s_t^*) - \Psi Z_{t-1})'[\Omega(s_t^*)\Omega(s_t^*)']^{-1}(Y_{2t} - c(s_t^*) - \Psi Z_{t-1})\right)$$

is the probability density function of Z_t conditional on s_t^* and

$$f(Y_{2t}|\pi_t, Y_{1t}, s_t^*, I_{t-1}) =$$
$$(2\pi)^{-N_2/2}|V|^{-1/2} \exp\left(-\frac{1}{2}(Y_{2t} - \mathcal{A}_2(s_t^*) - \mathcal{B}_2 X_t)' V^{-1}(Y_{2t} - \mathcal{A}_2(s_t^*) - \mathcal{B}_2 X_t\right)$$

is the probability density function of the measurement errors conditional on s_t^*.

The ex-ante probability $Pr(s_t^* = i|I_{t-1})$ is given by:

$$Pr(s_t^* = i|I_{t-1}) = \sum_j p_{ji}^* Pr(s_{t-1}^* = j|I_{t-1}), \tag{E-8}$$

which is updated using:

$$\begin{aligned} Pr(s_t^* = j|I_t) &= \frac{f(Z_t, s_t^* = j|I_{t-1})}{f(Z_t|I_{t-1})} \\ &= \frac{f(Z_t|s_t^* = j, I_{t-1})Pr(s_t^* = j|I_{t-1})}{\sum_k f(Z_t|s_t^* = k, I_{t-1})Pr(s_t^* = k|I_{t-1})} \end{aligned}$$

An alternative way to derive the likelihood function is to substitute (E-3) into (E-5). We then obtain a RS-VAR with complex cross-equation restrictions for all variables in the system $(Z_t' \, Y_{2t}')'$.

F Specification Tests

Residual Tests

We report two tests on in-sample scaled residuals ϵ_t of yields and inflation. The scaled residuals ϵ_t are not the same as the shocks ε_t in (1). For a variable x_t, the scaled residual is given by $\epsilon_t = (x_t - \mathrm{E}_{t-1}(x_t))/\sqrt{\mathrm{var}_{t-1}(x_t)}$, where x_t are yields or inflation. The conditional moments are computed using our RS model and involve ex-ante probabilities $p(s_t = i|I_{t-1})$. Following Bekaert and Harvey (1997), we use a GMM test for serial correlation in scaled residuals ϵ_t:

$$E[\epsilon_t \, \epsilon_{t-1}] = 0. \tag{F-1}$$

We also test for serial correlation in the second moments of the scaled residuals:

$$E[((\epsilon_t)^2 - 1)\,((\epsilon_{t-1})^2 - 1)] = 0. \tag{F-2}$$

Moment Tests

To enable comparison across several non-nested models of how the moments implied from various models compare to the data, we introduce the point statistic:

$$H = (h - \bar{h})' \Sigma_h^{-1} (h - \bar{h}), \tag{F-3}$$

where $\bar{h}$ are sample estimates of unconditional moments, h are the unconditional moments from the estimated model, and Σ_h is the covariance matrix of the sample estimates of the unconditional moments, estimated by GMM (Newey-West, 1987). In this comparison, the moments implied by various models are compared to the data, with the data sampling error Σ_h held constant across the models. The moments we consider are the first and second moments of term spreads and long yields; the first and second moments of inflation; the autocorrelogram of term spreads; and the autocorrelogram of inflation.

Equation (F-3) ignores the sampling error of the moments of the model, implied by the uncertainty in the parameter estimates, making our moment test informal. However, this allows the same weighting matrix, computed from the data, to be used across different models, similar to Hansen and Jagannathan (1997). If parameter uncertainty is also taken into account, we might fail to reject, not because the model accurately pins down the moments, but because of the large uncertainty in estimating the model parameters.

G Computing Moments of the Regime-Switching Model

The formulae given here assume that there are K regimes $s_t = 1, \dots K$. Timmermann (2000) provides explicit formulae for a similar formulation of (1), except that the conditional mean of X_{t+1} depends on $\mu(s_{t+1}) + \Phi(X_t - \mu(s_t))$ rather than on $\mu(s_{t+1}) + \Phi X_t$. In Timmermann's set-up, $\mathrm{E}(X_t|s_t)$ is trivially $\mu(s_t)$, whereas in our model the computation is more complex.

Conditional First Moments $\mathrm{E}(X_t|s_t)$

Starting from (1), and taking expectations conditional on s_{t+1}, we have:

$$\mathrm{E}(X_{t+1}|s_{t+1}) = \mathrm{E}(\mu(s_{t+1})|s_{t+1}) + \Phi\mathrm{E}(X_t|s_{t+1}) \tag{G-1}$$

To evaluate $\mathrm{E}(X_t|s_{t+1})$ we use Bayes Rule:

$$\mathrm{E}(X_t|s_{t+1} = i) = \sum_{j=1}^{K} \mathrm{E}(X_t|s_t = j) Pr(s_t = j|s_{t+1} = i). \tag{G-2}$$

The probability $Pr(s_t = j|s_{t+1} = i)$ is the transition probability of the 'time-reversed' Markov chain that moves backward in time. These backward transition probabilities are given by:

$$Pr(s_t = j|s_{t+1} = i) \triangleq b_{ij} = p_{ji}\left(\frac{\pi_j}{\pi_i}\right),$$

where $p_{ji} = Pr(s_{t+1} = i|s_t = j)$ are the forward transition probabilities in (4) and $\pi_i = Pr(s_t = i)$ is the stable probability of regime i. Denote the backward transition probability matrix as $B = \{b_{ij}\}$

Using the backward transition probabilities, (G-1) can be rewritten:

$$\mathrm{E}(X_{t+1}|s_{t+1} = i) = \mu(i) + \Phi\sum_{j=1}^{K} \mathrm{E}(X_t|s_t = j) b_{ji}. \tag{G-3}$$

Assuming stationarity, that is $\mathrm{E}(X_{t+1}|s_{t+1} = i) = \mathrm{E}(X_t|s_t = i)$, and defining the $K \times 1$ vectors:

$$\vec{\mathrm{E}}(X_t|s_t) = \begin{bmatrix} \mathrm{E}(X_t|s_t = 1) \\ \vdots \\ \mathrm{E}(X_t|s_t = K) \end{bmatrix} \quad \text{and} \quad \vec{\mu}(s_t) = \begin{bmatrix} \mu(1) \\ \vdots \\ \mu(K) \end{bmatrix},$$

we can write:

$$\vec{\mathrm{E}}(X_t|s_t) = \vec{\mu}(s_t) + \Phi\vec{\mathrm{E}}(X_t|s_t)B'.$$

Hence, we can solve for $\vec{\mathrm{E}}(X_t|s_t)$ as:

$$\vec{\mathrm{E}}(X_t|s_t) = (I - B \otimes \Phi)^{-1}\vec{\mu}(s_t) \tag{G-4}$$

Conditional Second Moments $\mathrm{E}(X_t X_t'|s_t)$

Starting from (1), we can write:

$$X_{t+1}X_{t+1}' = (\mu(s_{t+1}) + \Phi X_t + \Sigma(s_{t+1})\varepsilon_{t+1})(\mu(s_{t+1}) + \Phi X_t + \Sigma(s_{t+1})\varepsilon_{t+1})', \tag{G-5}$$

and taking expectations conditional on s_{t+1}, we have:

$$\begin{aligned} \mathrm{E}(X_{t+1}X_{t+1}'|s_{t+1}) = {} & \mu(s_{t+1})\mu(s_{t+1})' + \Sigma(s_{t+1})\Sigma(s_{t+1})' \\ & + \mu(s_{t+1})(\Phi\mathrm{E}(X_t|s_{t+1})' + \Phi\mathrm{E}(X_t|s_{t+1})\mu(s_{t+1})' + \Phi\mathrm{E}(X_t X_t|s_{t+1})\Phi'. \end{aligned} \tag{G-6}$$

We can evaluate the term $\mathrm{E}(X_t|s_{t+1})$ from (G-2). Hence, we can define an $N \times N$ matrix $G(i)$:

$$G(i) = \mu(i)\mu(i)' + \Sigma(i)\Sigma(i)' + \mu(i)(\Phi\mathrm{E}(X_t|s_{t+1} = i)' + \Phi\mathrm{E}(X_t|s_{t+1} = i)\mu(i)'. \tag{G-7}$$

Substituting (G-7) into (G-6) and using Bayes' Rule, we have:

$$\begin{aligned}
\mathrm{E}(X_{t+1}X_{t+1}'|s_{t+1} = i) &= G(i) + \sum_{j=1}^{K} \Phi[\mathrm{E}(X_tX_t'|s_t = j)Pr(s_t = j|s_{t+1} = i)]\Phi' \\
&= G(i) + \sum_{j=1}^{K} b_{ij}\Phi\mathrm{E}(X_tX_t'|s_t = j)\Phi'
\end{aligned}$$

Taking vec's of both sides, we obtain:

$$\text{vec}(\mathrm{E}(X_{t+1}X_{t+1}'|s_{t+1} = i)) = G(i) + (\Phi \otimes \Phi)\sum_{j=1}^{K} \text{vec}(\mathrm{E}(X_{t+1}X_{t+1}'|s_{t+1} = i))b_{ij} \tag{G-8}$$

If we define the $KN^2 \times 1$ vectors:

$$\vec{\mathrm{E}}(X_tX_t'|s_t) = \begin{bmatrix} \text{vec}(\mathrm{E}(X_{t+1}X_{t+1}'|s_{t+1} = 1)) \\ \vdots \\ \text{vec}(\mathrm{E}(X_{t+1}X_{t+1}'|s_{t+1} = K)) \end{bmatrix} \quad \text{and} \quad \vec{G} = \begin{bmatrix} \text{vec}(G(1)) \\ \vdots \\ \text{vec}(G(K)) \end{bmatrix}$$

we can write (G-8) as:

$$\vec{\mathrm{E}}(X_tX_t'|s_t) = \vec{G} + (\Phi \otimes \Phi)\vec{\mathrm{E}}(X_tX_t'|s_t)B'.$$

Hence, we can solve for $\vec{\mathrm{E}}(X_tX_t'|s_t)$ as:

$$\vec{\mathrm{E}}(X_tX_t'|s_t) = (I_{KN^2} - B \otimes (\Phi \otimes \Phi))^{-1}\vec{G}. \tag{G-9}$$

Unconditional Moments

The first unconditional moment $\mathrm{E}(X_t)$ is solved simply by taking unconditional expectations of (1), giving

$$\mathrm{E}(X_t) = (I - \Phi)^{-1}\sum_{i=1}^{K} \pi_i\mu(i). \tag{G-10}$$

To solve the second unconditional moment var(X_t), we use:

$$\begin{aligned}
\text{var}(X_t) &= \mathrm{E}(X_tX_t') - \mathrm{E}(X_t)\mathrm{E}(X_t)' \\
&= \mathrm{E}(\mathrm{E}(X_tX_t'|s_t)) - \mathrm{E}(X_t)\mathrm{E}(X_t)' \\
&= \sum_{i=1}^{K} \{\text{var}(X_t|s_t = i) + \mathrm{E}(X_t|s_t = i)\mathrm{E}(X_t|s_t = i)'\}\,\pi_i - \mathrm{E}(X_t)E(X_t)'
\end{aligned} \tag{G-11}$$

Moments of Yields

Bond yields are affine functions of X_t, from Propositions C.1 and D.1. Hence, they can be written as $Y_t = A+BX_t$ for some choice of A and B. Then, moments of Y_t are given by:

$$\begin{aligned}
\mathrm{E}(Y_t|s_t) &= A + B\mathrm{E}(X_t|s_t) \\
\text{var}(Y_t|s_t) &= B\text{var}(X_t|s_t)B'
\end{aligned} \tag{G-12}$$

$$\begin{aligned}
\mathrm{E}(Y_t) &= A + B\mathrm{E}(X_t) \\
\text{var}(Y_t) &= B\text{var}(X_t)B'
\end{aligned} \tag{G-13}$$

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Table 1: Residual and Moment Tests

Panel A: Residual Tests

		Models					
		I	II	III	IV	V	VI
1-qtr yield	Serial Correlation	0.30	0.70	0.35	0.05	0.60	0.24
	Heteroskedasticity	0.26	0.15	0.07	0.08	0.07	0.09
5-year spread	Serial Correlation	0.20	0.30	0.05	0.62	0.83	0.14
	Heteroskedasticity	0.21	0.28	0.14	0.09	0.18	0.06
Inflation	Serial Correlation	0.00**	0.03*	0.02*	0.28	0.00**	0.41
	Heteroskedasticity	0.14	0.11	0.55	0.28	0.59	0.26

Panel B: χ^2 Tests on Moments (p-values)

	Models					
	I	II	III	IV	V	VI
Mean/var spread and long rate	0.43	0.00**	0.03*	0.86	0.00**	0.00**
Mean/var inflation	0.00**	0.19	0.54	0.71	0.52	0.00**
1,5,10 autocorrelations spread	0.25	0.33	0.03*	0.37	0.34	0.18
1,5,10 autocorrelations inflation	0.04*	0.00**	0.04*	0.16	0.00**	0.00**

Panel C: Model-Implied and Sample Moments

		Models						Data	
		I	II	III	IV	V	VI	Moment	SE
stdev y_t^1		3.15	3.30†	2.93	3.10	3.32	5.07†	2.86	0.15
mean y_t^{20}		6.46	6.59	6.25	6.46	5.80†	4.71†	6.39	0.20
stdev y_t^{20}		2.60	2.54	2.28†	2.66	2.68	3.62†	2.74	0.14
mean $y_t^{20} - y_t^1$		1.00	1.13†	0.78†	1.00	0.34	-0.75†	0.93	0.07
stdev $y_t^{20} - y_t^1$		1.27†	1.48†	1.52†	1.09	1.55†	2.83†	1.03	0.05
mean π_t		3.68	4.02	3.74	4.03	4.21	5.30	3.88	0.23
stdev π_t		4.87†	3.72†	3.38	3.47	3.37	5.22†	3.19	0.16
Spread Autocorrelations	$\rho(1)$	0.81	0.76	0.87†	0.79	0.77	0.79	0.73	0.05
	$\rho(5)$	0.36	0.36	0.47	0.29	0.29	0.43	0.32	0.12
	$\rho(10)$	0.14	0.16	0.22	0.09	0.10	0.18	-0.04	0.12
Inflation Autocorrelations	$\rho(1)$	0.90†	0.49†	0.73	0.78	0.37†	0.61†	0.79	0.06
	$\rho(5)$	0.57	0.27†	0.31†	0.46	0.17†	0.32†	0.59	0.10
	$\rho(10)$	0.34	0.18	0.18	0.33	0.10†	0.20	0.36	0.09

Panel A reports p-values of scaled residual tests in (F-1) and (F-2). The first entry reports the p-value of a GMM-based test of $\mathrm{E}(\epsilon_t \epsilon_{t-1}) = 0$ and the second row reports the p-value of a GMM-based test of $\mathrm{E}[(\epsilon_t^2 - 1)(\epsilon_{t-1}^2 - 1)] = 0$. Panel B reports p-values of goodness-of-fit χ^2 tests (equation (F-3)) for various moments implied by different models. The long rate refers to the 20-quarter nominal rate y_t^{20} and the spread refers to $y_t^{20} - y_t^5$. For each line, the moments specified in the first column are used in the H point-statistic. Panel C reports moments of 5-year spreads and inflation π_t implied by various models, compared with the sample estimates in data and standard errors in the last two columns, computed using GMM with 4 Newey-West (1987) lags. We denote the ith correlation as $\rho(i)$ at a quarterly frequency. Means and standard deviations are in percent. In Panels A and B, p-values less than 0.05 (0.01) are denoted by * (**). In Panel C, moments outside plus or minus two standard errors of the data moment are denoted by †. Models I and II are the single-regime equivalents of the inflation and expected inflation models in Sections 3.1 and A, respectively. Model III denotes the benchmark RS model with two regimes. Model IV denotes the benchmark RS model with separate and independent real and inflation regimes. The results of the inflation model allowing for correlated real and inflation regimes (Case I and II) are almost identical to Model IV and so are omitted. Model V denotes the RS expected inflation model with two regimes. Model VI denotes the RS expected inflation model with separate and independent real and inflation regimes.

Table 2: Factor Behavior

	Stdev	Auto	Contribution to Real Rate Variance	Correlation with Various Variables: Inflation	Nominal Short Rate	Nominal Spread	Real Short Rate	Real Spread
q	1.78	0.98	0.47	0.59	0.89	-0.12	0.42	-0.08
	(0.62)	(0.01)	(0.35)	(0.12)	(0.06)	(0.05)	(0.22)	(0.01)
f	0.72	0.93	0.07	0.25	0.44	-0.98	0.16	-0.26
	(0.21)	(0.06)	(0.09)	(0.08)	(0.12)	(0.01)	(0.18)	(0.15)
π	3.47	0.80	0.46	1.00	0.68	-0.41	-0.39	0.64
	(0.43)	(0.06)	(0.36)	–	(0.09)	(0.06)	(0.29)	(0.11)
Data π	3.19	0.77			0.70	-0.37		

Projection of Inflation on Lagged Instruments

	Inflation	Nominal Short Rate	Nominal Spread
Model	0.57	0.34	-0.13
	(0.06)	(0.07)	(0.17)
Data	0.56	0.26	-0.40
	(0.06)	(0.07)	(0.15)

The table reports various unconditional moments of the three factors: the time-varying price of risk factor q_t, the regime-switching factor f_t and inflation π_t, from the benchmark model with independent real and inflation regimes (Model IV). The short rate refers to the 1-quarter nominal yield and the spread refers to the 20-quarter nominal term spread. The row labelled 'Data π' refers to actual inflation data. The numbers between parentheses are standard errors reflecting parameter uncertainty from the estimation, computed using the delta-method. The variance decomposition of the real rate is computed as cov$(r_t, z_t)/$var(r_t), with z_t respectively q_t, f_t and $\delta_\pi \pi_t$. The last row reports multivariate projection coefficients of inflation on the lagged short rate, spread and inflation implied by the model.

Table 3: Selected Regime-Switching Parameters

	Regime 1	Regime 2	P-value Test of Equality
Panel A: Model III			
$\mu_f(s_t) \times 100$	-0.016	0.096	0.09
	(0.015)	(0.088)	
$\sigma_f(s_t) \times 100$	0.077	0.258	0.00**
	(0.020)	(0.099)	
$\mu_\pi(s_t) \times 100$	0.453	0.199	0.00**
	(0.078)	(0.120)	
$\sigma_\pi(s_t) \times 100$	0.400	0.973	0.03*
	(0.018)	(0.381)	
λ_f	-0.482	-0.005	0.05*
	(0.104)	(0.353)	
Panel B: Model IV			
$\mu_f(s_t^r) \times 100$	-0.006	0.039	0.10
	(0.004)	(0.023)	
$\sigma_f(s_t^r) \times 100$	0.078	0.246	0.00**
	(0.020)	(0.020)	
$\mu_\pi(s_t^\pi) \times 100$	0.435	0.219	0.00**
	(0.079)	(0.095)	
$\sigma_\pi(s_t^\pi) \times 100$	0.479	0.471	0.88
	(0.027)	(0.052)	
λ_f	-0.523	0.335	0.23
	(0.100)	(0.157)	

We report selected parameters from the benchmark model with two regimes (Panel A, Model III) and independent real and inflation regimes (Panel B, Model IV). All parameters are unscaled and not annualized. The p-values relate to Wald χ^2 tests of parameter equality across regimes. P-values less than 0.05 (0.01) are denoted by * (**).

Table 4: Real Rates, Expected Inflation, Nominal Rates Across Regimes

		Regime			
		$s_t = 1$	$s_t = 2$	$s_t = 3$	$s_t = 4$
Real Short Rates	Mean	1.31	2.20	1.50	2.39
		(0.43)	(0.53)	(0.40)	(0.50)
	Std Dev	1.54	1.53	1.82	1.81
		(0.23)	(0.27)	(0.28)	(1.03)
Inflation Compensation $\pi^e_{t,1}$	Mean	4.16	2.48	4.62	2.94
		(0.42)	(0.65)	(0.48)	(0.71)
	Std Dev	2.78	2.78	3.17	3.17
		(0.52)	(0.52)	(0.54)	(0.54)
Nominal Short rate	Mean	5.47	4.68	6.12	5.32
		(0.06)	(0.24)	(0.26)	(0.35)
	Std Dev	2.99	2.99	3.72	3.72
		(0.80)	(0.80)	(0.69)	(0.69)

We report means and standard deviations for real short rates, 1-quarter expected inflation and nominal short rates implied by the benchmark model with independent real and inflation regimes (Model IV), across each of the four regimes. The regime $s_t = 1$ corresponds to $(s^r_t = 1, s^\pi_t = 1)$, $s_t = 2$ to $(s^r_t = 1, s^\pi_t = 2)$, $s_t = 3$ to $(s^r_t = 2, s^\pi_t = 1)$ and $s_t = 4$ to $(s^r_t = 2, s^\pi_t = 2)$. Standard errors reported in parentheses are computed using the delta-method.

Table 5: Characteristics of Real Rates

Panel A: Unconditional Moments

Maturity Qtrs	Mean	Stdev	Auto
1	1.44	1.60	0.61
	(0.42)	(0.23)	(0.08)
4	1.59	1.00	0.73
	(0.43)	(0.26)	(0.13)
12	1.52	0.67	0.89
	(0.42)	(0.34)	(0.10)
20	1.46	0.61	0.94
	(0.43)	(0.36)	(0.06)
Spread 20-1	0.02	1.29	0.57
	(0.27)	(0.18)	(0.06)

Panel B: Conditional Correlations with Actual and Expected Inflation

Maturity	Regime $s_t = 1$		Regime $s_t = 2$		Regime $s_t = 3$		Regime $s_t = 4$		Unconditional	
Qtrs	Actual	Expected	Actual	Expected	Actual	Expected	Actual	Expected	Actual	Expected
1	-0.40	-0.11	-0.41	-0.13	-0.41	-0.01	-0.39	-0.04	-0.40	-0.13
	(0.31)	(0.27)	(0.29)	(0.26)	(0.35)	(0.28)	(0.30)	(0.26)	(0.29)	(0.31)
4	-0.21	0.03	-0.24	-0.01	-0.29	0.03	-0.29	-0.01	-0.23	0.00
	(0.44)	(0.37)	(0.42)	(0.37)	(0.56)	(0.50)	(0.49)	(0.45)	(0.43)	(0.45)
12	0.17	0.30	0.11	0.24	0.14	0.29	0.08	0.21	0.12	0.30
	(0.42)	(0.35)	(0.42)	(0.36)	(0.56)	(0.53)	(0.53)	(0.51)	(0.43)	(0.45)
20	0.33	0.42	0.27	0.36	0.35	0.45	0.27	0.36	0.28	0.45
	(0.32)	(0.27)	(0.33)	(0.29)	(0.44)	(0.41)	(0.43)	(0.41)	(0.35)	(0.36)

The table reports various moments of the real rate, implied from the benchmark model IV with independent real rate and inflation regimes. Panel A reports the unconditional mean, standard deviation and autocorrelation of real yields of various maturity in quarters. Panel B reports the correlation of real yields with actual and unexpected inflation implied from the model. We report the conditional correlation of real yields with actual inflation corr$(\hat{y}^n_{t+1}, \pi_{t+1,n}|s_t)$, and the conditional correlation of real yields with expected inflation corr$(\hat{y}^n_{t+1}, \mathrm{E}_{t+1}(\pi_{t+1+n,n})|s_t)$. The regime $s_t = 1$ corresponds to $(s^r_t = 1, s^\pi_t = 1)$, $s_t = 2$ to $(s^r_t = 1, s^\pi_t = 2)$, $s_t = 3$ to $(s^r_t = 2, s^\pi_t = 1)$ and $s_t = 4$ to $(s^r_t = 2, s^\pi_t = 2)$. Standard errors reported in parentheses are computed using the delta-method.

Table 6: Regime-Dependent Means of Inflation Compensation

Qtrs	Real Rate Regimes $s_t^r = 1$	Real Rate Regimes $s_t^r = 2$	Inflation Regimes $s_t^\pi = 1$	Inflation Regimes $s_t^\pi = 2$	Unconditional
Inflation Compensation $\pi^e_{t,n}$					
1	3.96	4.42	4.22	2.54	4.02
	(0.42)	(0.49)	(0.42)	(0.42)	(0.74)
4	4.17	4.96	4.49	2.67	4.27
	(0.39)	(0.50)	(0.39)	(0.61)	(0.40)
12	4.71	5.43	4.99	3.42	4.80
	(0.39)	(0.47)	(0.39)	(0.49)	(0.39)
20	4.94	5.42	5.13	4.00	5.00
	(0.41)	(0.47)	(0.42)	(0.44)	(0.41)
RS-VAR Expected Inflation $\mathrm{E}_t(\pi_{t+n,n})$					
1	3.97	4.42	4.22	2.54	4.03
	(0.42)	(0.49)	(0.42)	(0.66)	(0.42)
4	3.95	4.53	4.20	2.75	4.03
	(0.42)	(0.52)	(0.42)	(0.64)	(0.42)
12	3.97	4.41	4.13	3.27	4.03
	(0.42)	(0.48)	(0.42)	(0.54)	(0.42)
20	3.99	4.28	4.09	3.54	4.03
	(0.42)	(0.45)	(0.42)	(0.48)	(0.42)
Expected Inflation Risk Premium $\varphi_{t,n}$					
1	-0.00	-0.00	-0.00	-0.00	-0.00
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
4	0.22	0.42	0.29	-0.08	0.24
	(0.08)	(0.16)	(0.10)	(0.06)	(0.09)
12	0.74	1.01	0.86	0.15	0.77
	(0.26)	(0.37)	(0.29)	(0.17)	(0.27)
20	0.95	1.14	1.04	0.47	0.97
	(0.35)	(0.42)	(0.37)	(0.26)	(0.35)

The table reports means and standard deviations of expected inflation and inflation risk premiums (inflation compensation), conditional on the real regime (s_t^r) or inflation regime (s_t^π), implied from the benchmark model IV with independent real and inflation regimes. Moments computed conditional on the real regime variable s_t^r (inflation regime s_t^π) integrate out the effect of s_t^π (s_t^r). Standard errors reported in parentheses are computed using the delta-method.

Table 7: Conditional Moments Across Business Cycles

	Maturity (Qtrs)	Mean Expansion	Mean Recession	Std Dev Expansion	Std Dev Recession
Real Rates $\hat{y}_t^n$	1	1.64	1.25	1.32	2.33
		(0.20)	(0.20)	(0.05)	(0.08)
	20	1.47	1.59	0.71	0.95
		(0.41)	(0.39)	(0.21)	(0.28)
Real Spread $\hat{y}_t^{20} - \hat{y}_t^1$		-0.17	0.34	0.95	1.65
		(0.28)	(0.29)	(0.07)	(0.15)
Nominal Rates y_t^n	1	5.30	6.29	2.47	4.15
		(0.05)	(0.11)	(0.22)	(0.37)
	20	6.26	7.12	2.42	3.83
		(0.20)	(0.21)	(0.23)	(0.35)
Nominal Spread $y_t^{20} - y_t^1$		0.96	0.83	1.00	1.14
		(0.19)	(0.19)	(0.26)	(0.26)
Inflation Compensation $\pi_{t,n}^e$	1	3.66	5.05	2.26	3.70
		(0.18)	(0.16)	(0.15)	(0.25)
	20	4.79	5.53	1.81	3.02
		(0.39)	(0.39)	(0.38)	(0.58)
Inflation Compensation Spread $\pi_{t,20}^e - \pi_{t,1}^e$		1.13	0.48	1.36	2.19
		(0.31)	(0.31)	(0.10)	(0.14)
Actual Inflation		3.54	5.43	2.76	4.40
Ex-post Real Rate		1.74	0.86	1.99	3.61
Ex-post Real Spread		2.70	1.69	2.32	4.01

Regime Realizations Across Business Cycles

	$s_t = 1$	$s_t = 2$	$s_t = 3$	$s_t = 4$
Whole Sample	0.66	0.16	0.12	0.06
Expansions	0.70	0.15	0.10	0.05
Recessions	0.51	0.17	0.20	0.11

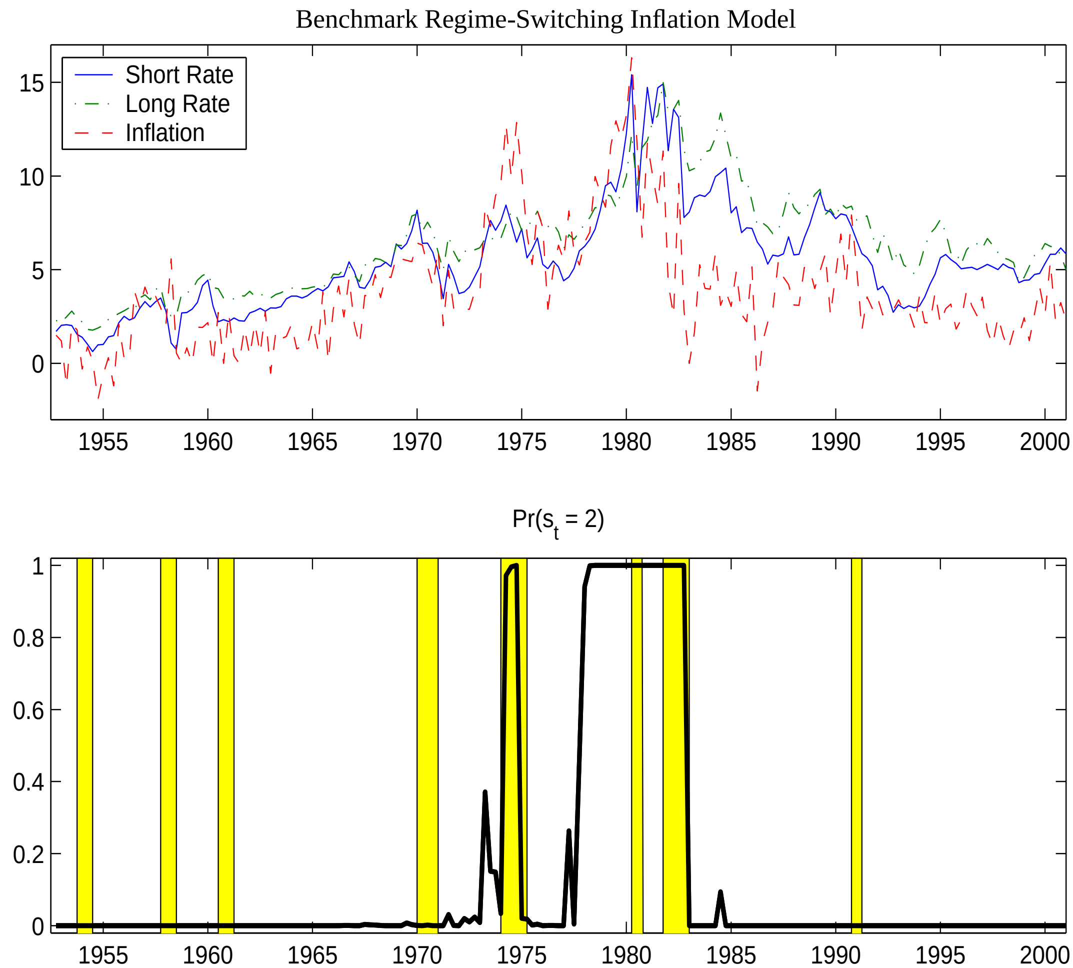
The table reports various sample moments of real rates, nominal rates and expected inflation implied from yield-curve forecasts ($\pi_{t,n}^e$) from Model IV, conditional on expansions and recessions, as defined by the NBER. Standard errors reported in parentheses are computed using the delta-method on sample moments. The ex-post real rate (spread) is the nominal rate (spread) minus actual inflation over the sample. The second part of the table reports the proportions of each regime (the number of periods assigned to be in regime $s_t = i$ divided by the total number of observations) across the whole sample, and conditional on NBER expansions and recessions. The regime classification uses smoothed $Pr(s_t = i|I_T)$ probabilities.

Table 8: Variance Decomposition of Nominal Yields and Spreads

	Regime 1		Regime 2		Regime 3		Regime 4		Unconditional	
n	Real	Infl	Real	Infl	Real	Infl	Real	Infl	Real	Infl
Variance Decomposition of Nominal Yields										
1	0.20	0.80	0.20	0.80	0.26	0.74	0.26	0.74	0.21	0.79
	(0.11)	(0.11)	(0.11)	(0.11)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)
20	0.21	0.79	0.21	0.79	0.20	0.80	0.20	0.80	0.21	0.79
	(0.11)	(0.11)	(0.11)	(0.11)	(0.11)	(0.11)	(0.11)	(0.11)	(0.11)	(0.11)
Variance Decomposition of Nominal Yield Spreads										
4	0.21	0.79	0.21	0.79	0.89	0.11	0.89	0.11	0.48	0.52
	(0.19)	(0.19)	(0.19)	(0.20)	(0.12)	(0.11)	(0.12)	(0.11)	(0.16)	(0.16)
20	0.01	0.99	0.01	0.99	0.40	0.60	0.40	0.60	0.17	0.83
	(0.22)	(0.21)	(0.22)	(0.22)	(0.16)	(0.16)	(0.16)	(0.16)	(0.19)	(0.19)

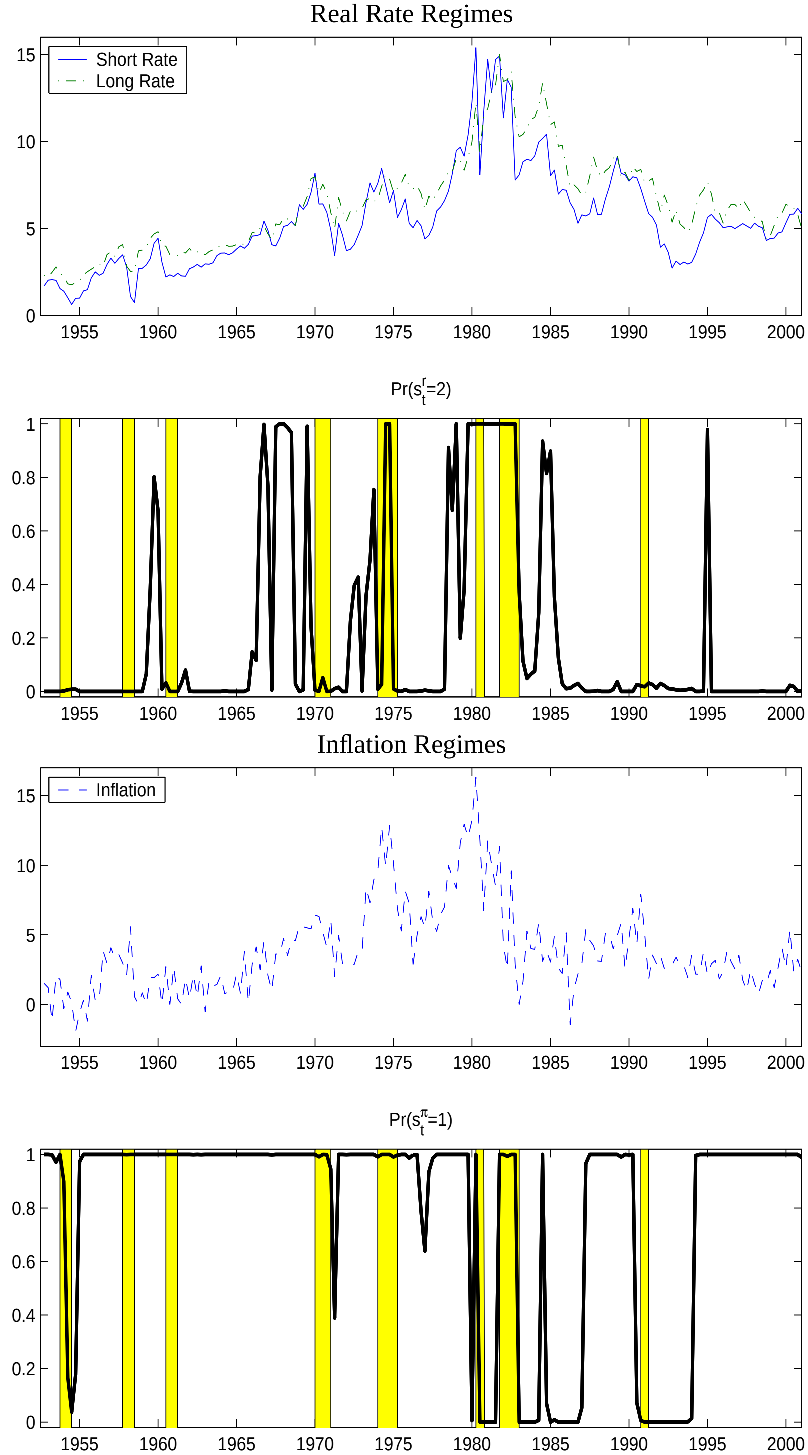
The table reports variance decompositions of nominal yields and nominal yield spreads into real rate ($\tau_{\hat{y}_t^n}$), denoted by 'Real,' and inflation compensation ($\tau_{\pi^e_{t,n}}$), denoted by 'Infl,' components, defined in equation (20) using the inflation compensation $\pi^e_{t,n}$. The regime $s_t = 1$ corresponds to $(s^r_t = 1, s^\pi_t = 1)$, $s_t = 2$ to $(s^r_t = 1, s^\pi_t = 2)$, $s_t = 3$ to $(s^r_t = 2, s^\pi_t = 1)$ and $s_t = 4$ to $(s^r_t = 2, s^\pi_t = 2)$. Standard errors reported in parentheses are computed using the delta-method on population moments.

Figure 1: Smoothed Regime Probabilities: Benchmark Model III



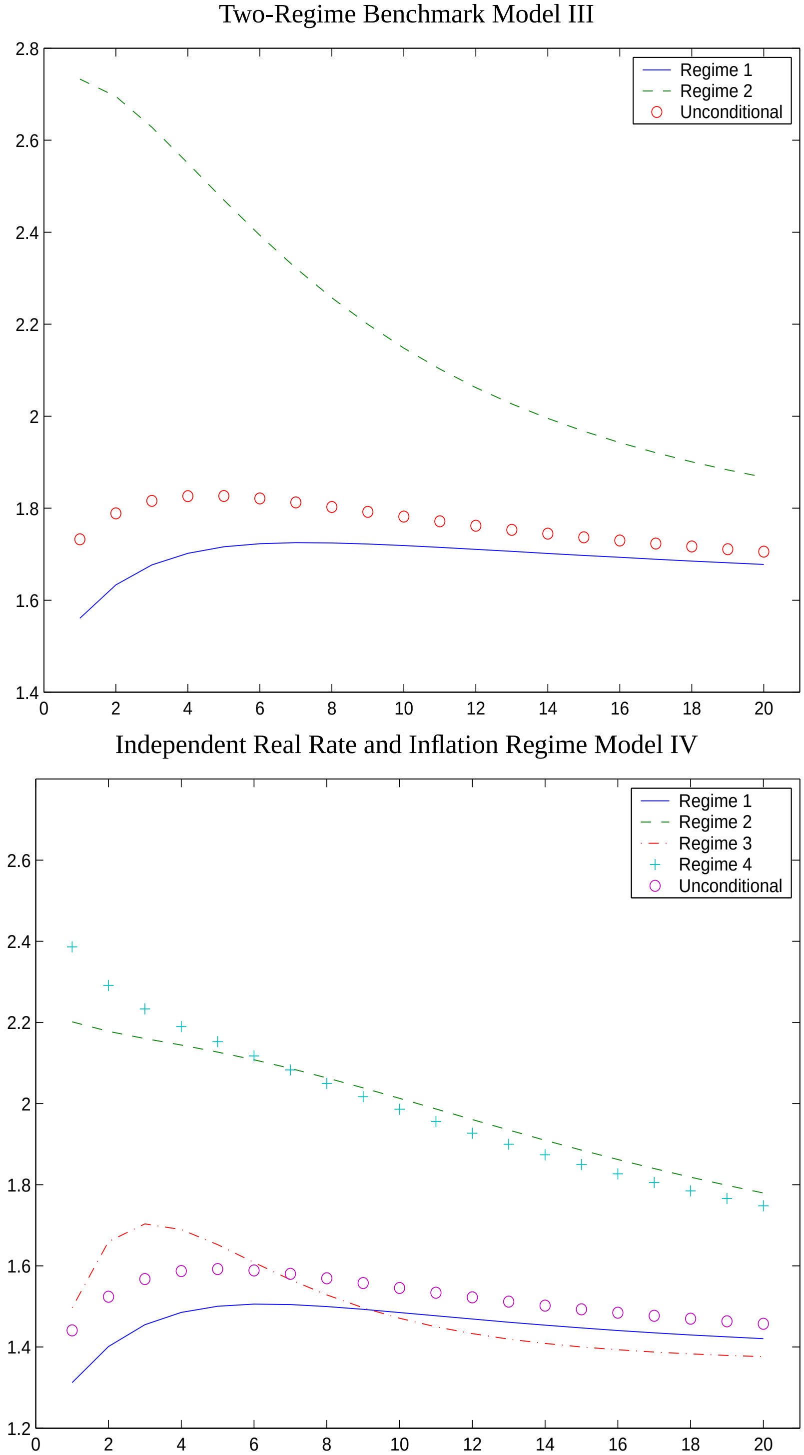
The bottom graph shows the smoothed probabilities of the second regime, $Pr(s_t = 2|I_T)$, of the benchmark model using information over the whole sample, along with short (1-quarter), long (20-quarter) yields and inflation shown in the top panel. NBER recessions are indicated by shaded bars.

Figure 2: Smoothed Regime Probabilities: Independent Real Rate and Inflation Regimes



The figure displays smoothed probabilities, using information over the whole sample, from the benchmark model with separate and independent real rate and inflation regimes (Model IV). The top panel shows the smoothed probabilities of the second real rate regime, $Pr(s^r_t = 2|I_T)$, along with short (1-quarter) and long (20-quarter) yields. In the bottom panel, the smoothed probabilities of the first inflation regime $Pr(s^\pi_t = 1|I_T)$ are shown, together with realized quarterly inflation. NBER recessions are indicated by shaded bars.

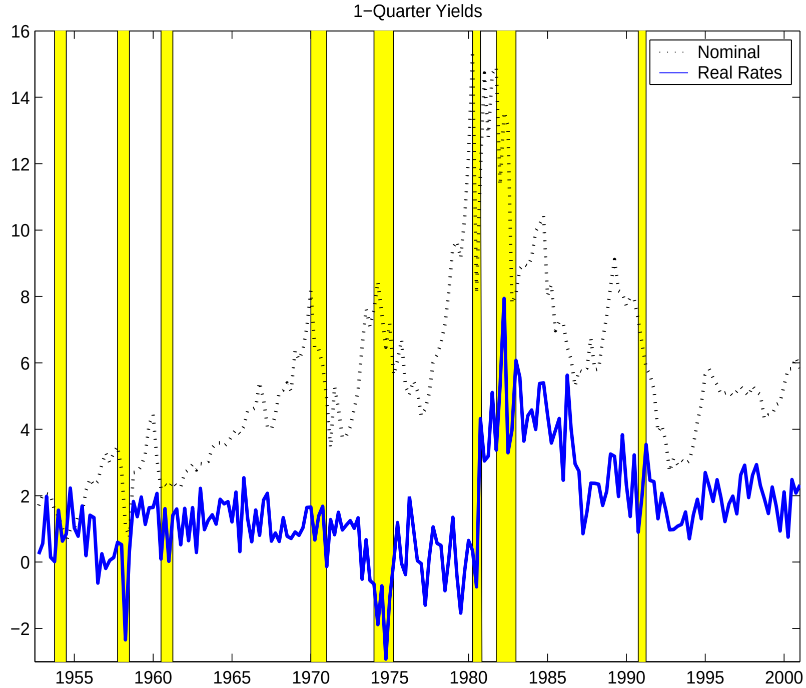
Figure 3: Real Term Structure



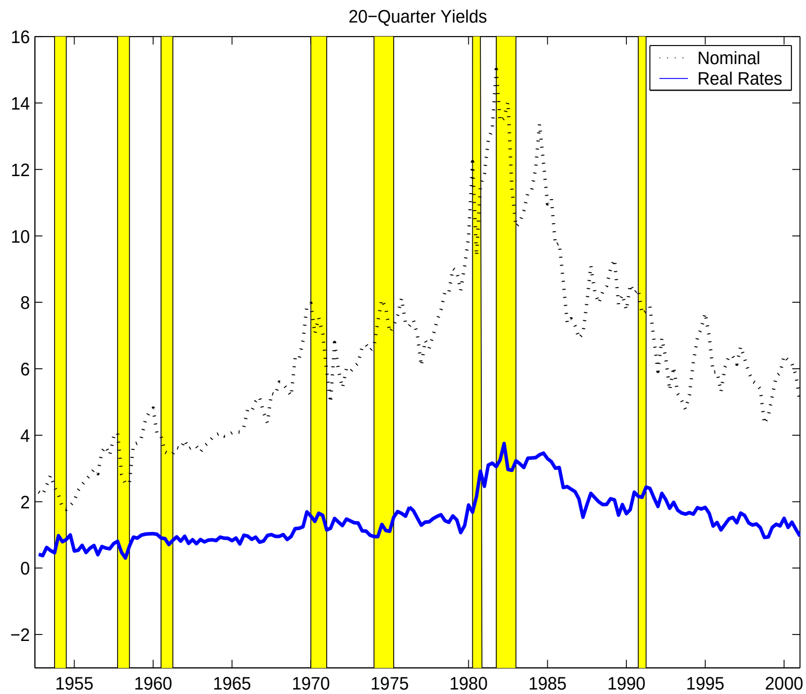
The figure graphs the real yield curve, conditional on each regime and the unconditional real yield curve. The top panel displays the benchmark model with two regimes (Model III) and the bottom panel displays the benchmark model with separate and independent real rate and inflation regimes (Model IV). For Model IV, the regime $s_t = 1$ corresponds to $(s_t^r = 1, s_t^\pi = 1)$, $s_t = 2$ to $(s_t^r = 1, s_t^\pi = 2)$, $s_t = 3$ to $(s_t^r = 2, s_t^\pi = 1)$ and $s_t = 4$ to $(s_t^r = 2, s_t^\pi = 2)$. The x-axis displays maturities in quarters of a year.

Figure 4: Time-Series of Real versus Nominal Yields

Real versus Nominal 1-Quarter Yields

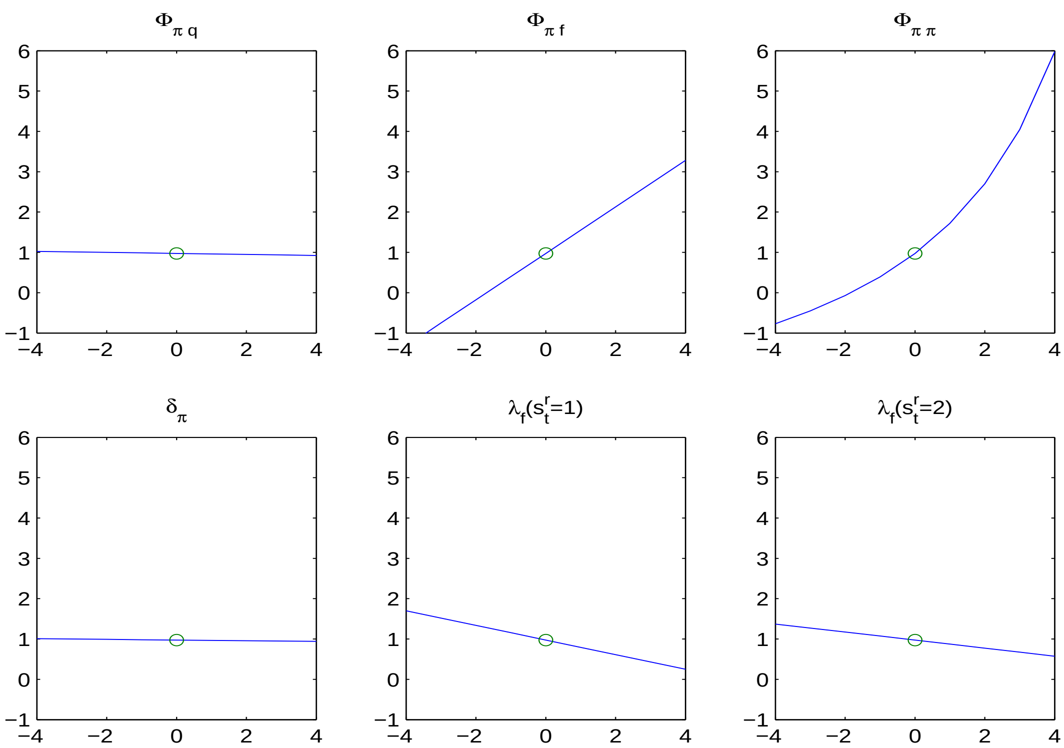


Real versus Nominal 20-Quarter Yields



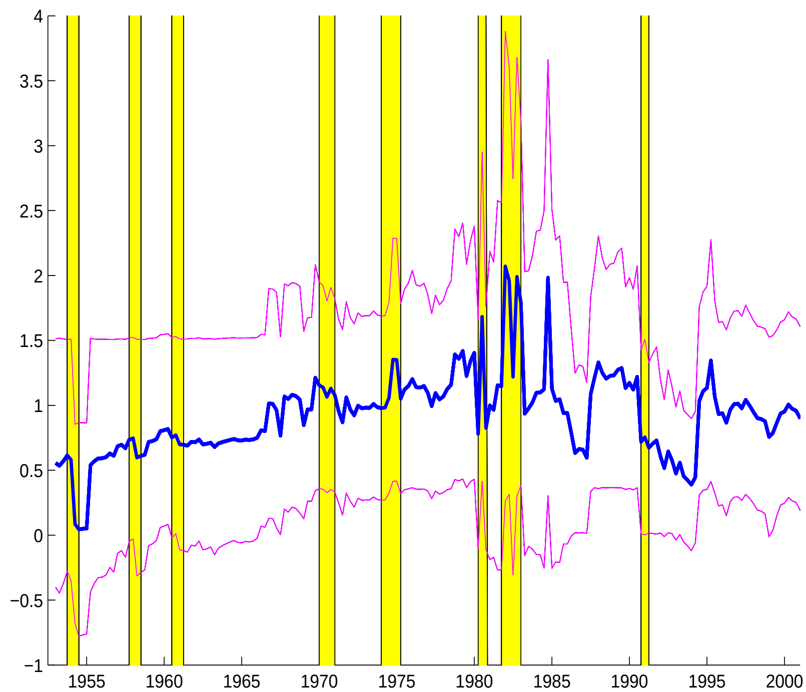
The figure graphs the time-series of real and nominal 1-quarter yields (top panel) and real and nominal 20-quarter yields (bottom panel) from the benchmark model with independent real rate and inflation regimes (Model IV). NBER recessions are indicated by shaded bars.

Figure 5: Comparative Statics of 20-Quarter Inflation Risk Premiums



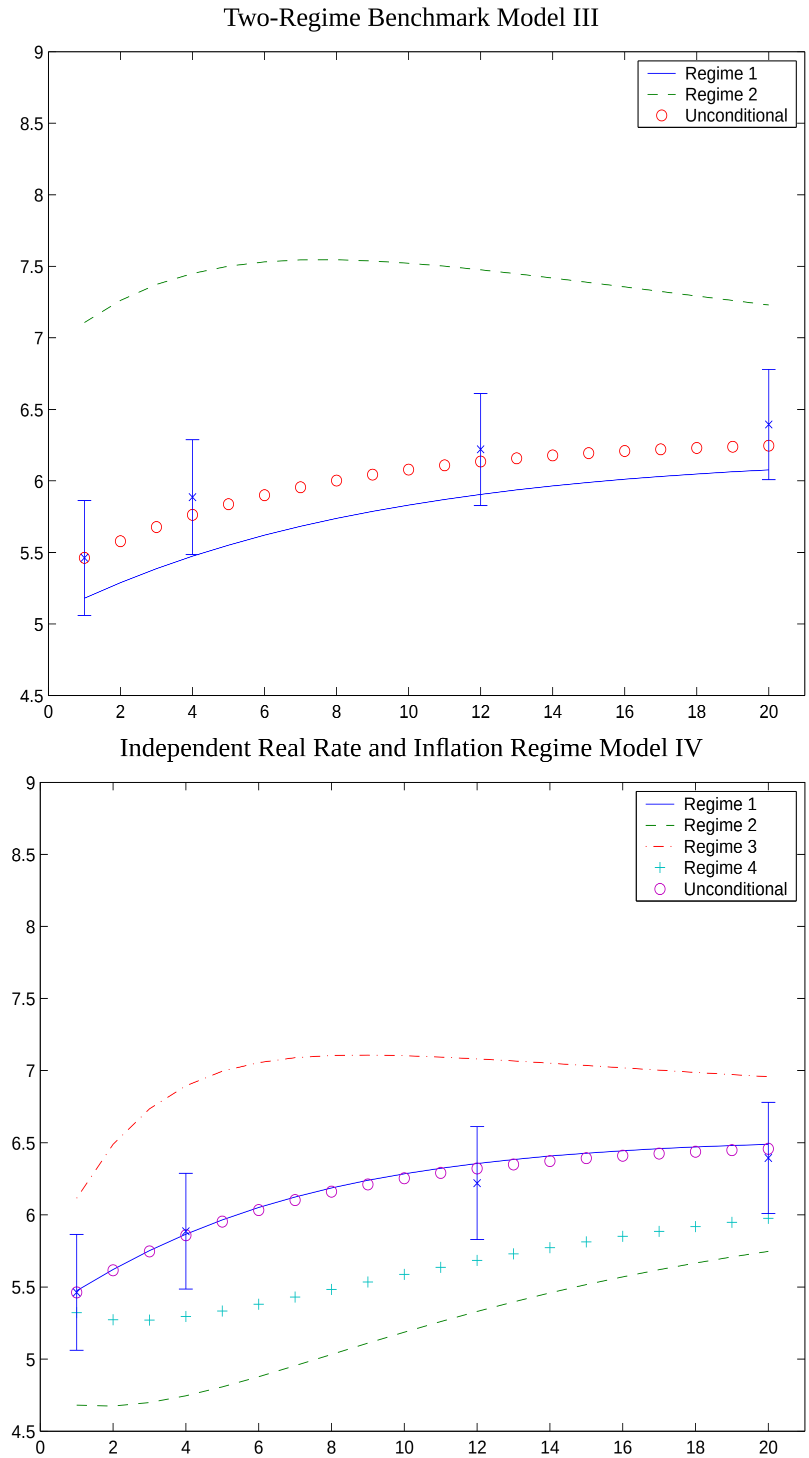
In each plot, we show the unconditional 20-quarter inflation risk premium as a function of various parameters of Model IV. The units on the y-axis are in percentage terms, and we alter the value of each parameter on the x-axis by up to ± 4 standard errors of the estimates of each parameter. The circle represents the baseline case at the estimated parameter values, of 97 basis points.

Figure 6: Time-Series of 20-Quarter Inflation Risk Premiums



The figure graphs the time-series of the 20-quarter inflation risk premium, with 2 SE bounds. NBER recessions are indicated by shaded bars.

Figure 7: Nominal Term Structure



The figure graphs the nominal yield curve, conditional on each regime and the unconditional nominal yield curve. The top panel displays the benchmark model with two regimes (Model III) and the bottom panel displays the benchmark model with separate and independent real rate and inflation regimes (Model IV). For Model IV, the regime $s_t = 1$ corresponds to $(s^r_t = 1, s^\pi_t = 1)$, $s_t = 2$ to $(s^r_t = 1, s^\pi_t = 2)$, $s_t = 3$ to $(s^r_t = 2, s^\pi_t = 1)$ and $s_t = 4$ to $(s^r_t = 2, s^\pi_t = 2)$. The x-axis displays maturities in quarters of a year. Average yields from data are represented by 'x', with 95% confidence intervals represented by vertical bars.

Appendix Tables

Table A-1: Single Regime Models

		Inflation Model I				**Expected Inflation Model II**			
		q	f	π		q	f	w	π
$\mu' \times 100$		0.000	0.000	0.158		0.000	0.000	0.579	0.000
				(0.045)				(0.089)	
		q	f	π		q	f	w	π
Φ	q	0.962	0.000	0.000	q	0.971	0.000	0.000	0.000
		(0.017)				(0.015)			
	f	0.000	0.669	0.000	f	0.000	0.841	0.000	0.000
			(0.032)				(0.012)		
	π	0.262	1.381	0.828	w	0.452	0.920	0.490	-0.065
		(0.066)	(0.655)	(0.019)		(0.081)	(0.116)	(0.083)	(0.049)
					π	0.000	0.000	1.000	0.000
		q	f	π		q	f	w	π
$\sigma \times 100$		0.116	0.095	0.493		0.119	0.165	0.371	0.577
		(0.011)	(0.027)	(0.026)		(0.008)	(0.011)	(0.027)	(0.033)
δ_0		0.010				0.010			
		(0.001)				(0.006)			
		q	f	π		q	f	w	π
δ_1'		1.000	1.000	-0.607		1.000	1.000	-0.496	-0.101
				(0.028)					(0.023)
		q	f	π		q	f	w	π
λ'			-0.314	0.000			-0.265	0.000	0.000
			(0.000)				(0.074)		
γ_1		-26.8				-19.6			
		(15.1)				(13.15)			
Std Dev $\times 100$ of Measurement Errors									
y_t^4		0.079				-			
		(0.005)							
y_t^{12}		0.039				0.028			
		(0.002)				(0.001)			

The left (right)-hand columns report parameter estimates for the single-regime equivalents of the Inflation and Expected Inflation Models, outlined in Sections 3.1 and A, respectively.

Table A-2: Benchmark Model III

		Regime 1			Regime 2		
		q	f	π	q	f	π
$\mu(s_t)' \times 100$		0.000	-0.016	0.453	0.000	0.096	0.199
			(0.015)	(0.078)		(0.088)	(0.120)
		q	f	π			
Φ	q	0.962	0.000	0.000			
		(0.020)					
	f	0.000	0.784	0.000			
			(0.010)				
	π	0.459	1.034	0.555			
		(0.143)	(0.522)	(0.061)			
		q	f	π	q	f	π
$\sigma(s_t) \times 100$		0.100	0.077	0.400	0.100	0.258	0.973
		(0.011)	(0.020)	(0.018)	(0.011)	(0.099)	(0.381)
δ_0		0.009					
		(0.001)					
		q	f	π			
δ_1'		1.000	1.000	-0.516			
				(0.059)			
		q	f	π	q	f	π
$\lambda(s_t)'$			-0.482	0.000		-0.005	0.000
			(0.104)			(0.353)	
γ_1		-42.2					
		(21.2)					
		$s_{t+1} = 1$	$s_{t+1} = 2$				
Π	$s_t = 1$	0.979	0.021				
		(0.006)	(0.006)				
	$s_t = 2$	0.121	0.879				
		(0.032)	(0.032)				
Std Dev $\times 100$ of Measurement Errors							
y_t^4		0.076					
		(0.005)					
y_t^{12}		0.039					
		(0.003)					

The table reports estimates of the Benchmark Regime-Switching Inflation Model, where real rates and inflations have the same regimes. The stable probabilities of regime 1 and 2 are 0.853 and 0.146, respectively.

Table A-3: Benchmark Model with Independent Real Rate and Inflation Regimes IV

		Regime 1			Regime 2
$\mu_f(s_t^r) \times 100$		-0.006			0.039
		(0.004)			(0.023)
$\mu_\pi(s_t^\pi) \times 100$		0.435			0.219
		(0.079)			(0.095)
		q	f	π	
Φ	q	0.976	0.000	0.000	
		(0.015)			
	f	0.000	0.759	0.000	
			(0.012)		
	π	0.499	0.851	0.593	
		(0.135)	(0.479)	(0.058)	
$\sigma_q \times 100$		0.096			
		(0.010)			
$\sigma_f(s_t^r) \times 100$		0.078			0.246
		(0.020)			(0.020)
$\sigma_\pi(s_t^\pi) \times 100$		0.479			0.471
		(0.027)			(0.052)
δ_0		0.009			
		(0.001)			
		q	f	π	
δ_1'		1.000	1.000	-0.536	
				(0.052)	
$\lambda_f(s_t)$		-0.523			0.335
		(0.100)			(0.157)
γ_1		-19.0			
		(16.2)			
		$s_{t+1}=1$	$s_{t+1}=2$	$s_{t+1}=3$	$s_{t+1}=4$
Π	$s_t=1$	0.930	0.022	0.047	0.001
		(0.019)	(0.008)	(0.015)	(0.001)
	$s_t=2$	0.162	0.790	0.008	0.040
		(0.038)	(0.037)	(0.003)	(0.013)
	$s_t=3$	0.319	0.007	0.659	0.015
		(0.072)	(0.003)	(0.070)	(0.006)
	$s_t=4$	0.056	0.271	0.115	0.559
		(0.019)	(0.061)	(0.028)	(0.068)
Std Dev $\times 100$ of Measurement Errors					
y_t^4		0.053			
		(0.003)			
y_t^{12}		0.026			
		(0.001)			

The table reports estimates of the Regime-Switching Inflation Model with independent regimes in real rates and inflation. The regime $s_t = 1$ corresponds to $(s_t^r = 1, s_t^\pi = 1)$, $s_t = 2$ to $(s_t^r = 1, s_t^\pi = 2)$, $s_t = 3$ to $(s_t^r = 2, s_t^\pi = 1)$ and $s_t = 4$ to $(s_t^r = 2, s_t^\pi = 2)$. The stable probabilities of regime 1 to 4 are 0.769, 0.103, 0.113 and 0.015.

Table A-4: Testing Independent versus Correlated Real Rate and Inflation Regimes

Panel A: Transition Probabilities from Independent Real Rate and Inflation Regimes (Model IV)

	$s_{t+1}=1$	$s_{t+1}=2$	$s_{t+1}=3$	$s_{t+1}=4$
$s_t=1$	0.930	0.022	0.047	0.001
	(0.019)	(0.008)	(0.015)	(0.001)
$s_t=2$	0.162	0.790	0.008	0.040
	(0.038)	(0.037)	(0.003)	(0.013)
$s_t=3$	0.319	0.007	0.659	0.015
	(0.072)	(0.003)	(0.070)	(0.006)
$s_t=4$	0.056	0.271	0.115	0.559
	(0.019)	(0.061)	(0.028)	(0.068)

Panel B: Transition Probabilities from Correlated Real Rate and Inflation Regimes Case A

	$s_{t+1}=1$	$s_{t+1}=2$	$s_{t+1}=3$	$s_{t+1}=4$
$s_t=1$	0.927	0.023	0.050	0.000
	(0.021)	(0.008)	(0.017)	(0.003)
$s_t=2$	0.166	0.825	0.009	0.000
	(0.039)	(0.107)	(0.004)	(0.118)
$s_t=3$	0.342	0.004	0.635	0.019
	(0.077)	(0.005)	(0.075)	(0.007)
$s_t=4$	0.061	0.145	0.114	0.679
	(0.021)	(0.152)	(0.028)	(0.158)

Panel C: Transition Probabilities from Correlated Real Rate and Inflation Regimes Case B

	$s_{t+1}=1$	$s_{t+1}=2$	$s_{t+1}=3$	$s_{t+1}=4$
$s_t=1$	0.946	0.017	0.032	0.005
	(0.018)	(0.009)	(0.013)	(0.003)
$s_t=2$	0.180	0.782	0.008	0.028
	(0.043)	(0.041)	(0.005)	(0.011)
$s_t=3$	0.252	0.005	0.640	0.103
	(0.081)	(0.003)	(0.069)	(0.043)
$s_t=4$	0.048	0.208	0.167	0.577
	(0.020)	(0.067)	(0.067)	(0.101)

Likelihood ratio test for independent versus Case A p-value = 0.251
Likelihood ratio test for independent versus Case B p-value = 0.125

The table reports parameter estimates of the transition probability matrix of the Benchmark Model with independent real rate and inflation regimes (Model IV) in Panel A (equation (B-2)) and correlated real rate and inflation regimes Case A (Panel B) using the formulation (B-4). In Panel C, we report the correlated real rate and inflation regimes Case B using the formulation (B-6). The regime $s_t = 1$ corresponds to $(s_t^r = 1, s_t^\pi = 1)$, $s_t = 2$ to $(s_t^r = 1, s_t^\pi = 2)$, $s_t = 3$ to $(s_t^r = 2, s_t^\pi = 1)$ and $s_t = 4$ to $(s_t^r = 2, s_t^\pi = 2)$. Standard errors reported in parenthesis are computed using the delta-method.

Table A-5: Regime-Switching Expected Inflation Model V

		Regime 1				Regime 2			
		q	f	w	π	q	f	w	π
$\mu(s_t)' \times 100$		0.000	0.009	0.477	0.000	0.000	-0.092	1.396	0.000
			(0.009)	(0.056)			(0.008)	(0.178)	
		q	f	w	π				
Φ	q	0.968	0.000	0.000	0.000				
		(0.014)							
	f	0.000	0.850	0.000	0.000				
			(0.012)						
	w	0.390	0.756	0.518	-0.134				
		(0.064)	(0.133)	(0.075)	(0.032)				
	π	0.000	0.000	1.000	0.000				
		q	f	w	π	q	f	w	π
$\sigma(s_t) \times 100$		0.116	0.110	0.176	0.551	0.116	0.302	0.650	0.551
		(0.009)	(0.010)	(0.022)	(0.034)	(0.009)	(0.034)	(0.121)	(0.034)
δ_0		0.009							
		(0.001)							
		q	f	w	π				
δ_1'		1.000	1.000	-0.377	-0.113				
				(0.068)	(0.021)				
		q	f	w	π	q	f	w	π
$\lambda(s_t)'$			-0.195	0.000	0.000		-0.414	0.000	0.000
			(0.099)				(0.323)		
γ_1		-24.1							
		(12.7)							
		$s_{t+1} = 1$	$s_{t+1} = 2$						
Π	$s_t = 1$	0.984	0.016						
		(0.014)	(0.014)						
	$s_t = 2$	0.171	0.829						
		(0.104)	(0.104)						
Std Dev $\times 100$ of Measurement Errors									
y_t^4		0.028							
		(0.001)							

The table reports estimates of the Regime-Switching Expected Inflation Model, where real rates and inflations have the same regimes. The stable probabilities of regime 1 and 2 are 0.913 and 0.087, respectively.

Table A-6: Expected Inflation Model with Independent Real Rate and Inflation Regimes VI

		Regime 1				Regime 2
$\mu_f(s_t^r) \times 100$		-0.029				0.059
		(0.010)				(0.020)
$\mu_\pi(s_t^\pi) \times 100$		0.518				1.557
		(0.063)				(0.211)
		q	f	w	π	
Φ	q	0.987	0.000	0.000	0.000	
		(0.012)				
	f	0.000	0.834	0.000	0.000	
			(0.016)			
	w	0.429	1.141	0.499	-0.015	
		(0.094)	(0.161)	(0.047)	(0.021)	
	π	0.000	0.000	1.000	0.000	
$\sigma_q \times 100$		0.086				
		(0.008)				
$\sigma_\pi \times 100$		0.576				
		(0.031)				
$\sigma_f(s_t^r) \times 100$		0.077				0.288
		(0.009)				(0.057)
$\sigma_w(s_t^\pi) \times 100$		0.147				0.576
		(0.015)				(0.031)
δ_0		0.005				
		(0.001)				
		q	f	w	π	
δ_1'		1.000	1.000	-0.326	-0.039	
				(0.063)	(0.015)	
$\lambda_f(s_t)$		0.081				0.283
		(0.116)				(0.393)
γ_1		-19.1				
		(14.4)				
		$s_{t+1} = 1$	$s_{t+1} = 2$	$s_{t+1} = 3$	$s_{t+1} = 4$	
Π	$s_t = 1$	0.863	0.098	0.035	0.004	
		(0.023)	(0.017)	(0.010)	(0.002)	
	$s_t = 2$	0.517	0.444	0.021	0.018	
		(0.070)	(0.067)	(0.006)	(0.007)	
	$s_t = 3$	0.072	0.008	0.826	0.093	
		(0.020)	(0.003)	(0.026)	(0.017)	
	$s_t = 4$	0.043	0.037	0.495	0.425	
		(0.013)	(0.012)	(0.067)	(0.066)	
Std Dev $\times 100$ of Measurement Errors						
y_t^{12}		0.025				
		(0.001)				

The table reports estimates of the Regime-Switching Expected Inflation Model with independent regimes in real rates and inflation. The regime $s_t = 1$ corresponds to $(s_t^r = 1, s_t^\pi = 1)$, $s_t = 2$ to $(s_t^r = 1, s_t^\pi = 2)$, $s_t = 3$ to $(s_t^r = 2, s_t^\pi = 1)$ and $s_t = 4$ to $(s_t^r = 2, s_t^\pi = 2)$. The stable probabilities of regime 1 to 4 are 0.567, 0.107, 0.274 and 0.052.