# James Hamilton's Comments on "Regime Shifts in a Dynamic Term Structure Model …" By Dai, Singleton, and Wang

$P_t$ = price of security at $t$

$r_t$ = risk-free rate

If investors risk neutral,

$$P_t = E_t(e^{-r_t} P_{t+1})$$

$P_t$ depends on state $\mathbf{x}_t = (\mathbf{y}_t', s_t)'$

"factor" $\mathbf{y}_t \in \Re^N$

$$\mathbf{y}_{t+1} | \mathbf{x}_t, s_{t+1} \sim N(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t \boldsymbol{\Sigma}_t')$$

$\boldsymbol{\mu}_t$ depends on $\mathbf{y}_t$ and $s_t$

$\boldsymbol{\Sigma}_t$ depends only on $s_t$

neither depends on $s_{t+1}$

$P_t$ depends on state $\mathbf{x}_t = (\mathbf{y}_t', s_t)'$

"regime" $s_t \in \{0, 1, 2, \ldots, S\}$

$\pi_t(s_{t+1}) =$ Prob that $t+1$ regime is $s_{t+1}$ given $\mathbf{x}_t$

$P_t$ = price of security at $t$

$r_t$ = risk-free rate

If investors risk neutral,

$$P_t = E_t[e^{-r_t} P_{t+1}(\mathbf{x}_{t+1})]$$

Suppose instead that investors behaved as if risk-neutral with beliefs

$$\mathbf{y}_{t+1} | \mathbf{x}_t, s_{t+1} \sim N(\boldsymbol{\mu}_t^Q, \boldsymbol{\Sigma}_t \boldsymbol{\Sigma}_t')$$

$$\boldsymbol{\mu}_t^Q = \boldsymbol{\mu}_t - \boldsymbol{\Sigma}_t \boldsymbol{\Lambda}_t$$

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e.g., scalar case: $\Lambda_t > 0$

$\uparrow$ risk $\Rightarrow \uparrow \Sigma_t \Rightarrow \mu_t^Q \downarrow$

so investors do not value payoffs correlated with the factor

$\boldsymbol{\Lambda}_t =$ “market price of factor risk”

Likewise, suppose that to evaluate probability of seeing $s_{t+1}$ investors used not the true $\pi_t(s_{t+1})$ but instead

$$\pi_t(s_{t+1}) \exp[-\Gamma_t(s_{t+1})]$$

e.g., if $\Gamma_t(s_{t+1}) > 0,$ investors put less value on payoff when state is $s_{t+1}$

$\Gamma_t =$ "market price of regime risk"

If the market used these distorted probabilities to evaluate

$$P_t = E_t^Q[e^{-r_t} P_{t+1}(\mathbf{x}_{t+1})],$$

the density used to find this expectation would be

$$f_t^Q(\mathbf{x}_{t+1}) = \pi_t^Q(s_{t+1}) f_t^Q(\mathbf{y}_{t+1})$$

$$f_t^Q(\mathbf{y}_{t+1}) = (2\pi)^{-N/2} |\mathbf{\Sigma}_t|^{-1} \times$$

$$\exp\left[ -\frac{(\mathbf{y}_{t+1} - \boldsymbol{\mu}_t^Q)'(\mathbf{\Sigma}_t \mathbf{\Sigma}_t')^{-1}(\mathbf{y}_{t+1} - \boldsymbol{\mu}_t^Q)}{2} \right]$$

$$
\begin{aligned}
&(\mathbf{y}_{t+1}-\boldsymbol{\mu}_t^Q)'(\boldsymbol{\Sigma}_t\boldsymbol{\Sigma}_t')^{-1}(\mathbf{y}_{t+1}-\boldsymbol{\mu}_t^Q) \\
&\quad = (\mathbf{y}_{t+1}-\boldsymbol{\mu}_t+\boldsymbol{\Sigma}_t\boldsymbol{\Lambda}_t)'(\boldsymbol{\Sigma}_t\boldsymbol{\Sigma}_t')^{-1}\times \\
&\qquad (\mathbf{y}_{t+1}-\boldsymbol{\mu}_t+\boldsymbol{\Sigma}_t\boldsymbol{\Lambda}_t) \\
&\quad = (\mathbf{y}_{t+1}-\boldsymbol{\mu}_t)'(\boldsymbol{\Sigma}_t\boldsymbol{\Sigma}_t')^{-1}(\mathbf{y}_{t+1}-\boldsymbol{\mu}_t) \\
&\qquad +\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t+2\boldsymbol{\Lambda}_t'\boldsymbol{\Sigma}_t^{-1}(\mathbf{y}_{t+1}-\boldsymbol{\mu}_t)
\end{aligned}
$$

Conclusion:

$$f_t^Q(\mathbf{y}_{t+1}) = f_t(\mathbf{y}_{t+1}) \times$$

$$\exp\left[-(1/2)\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t - \boldsymbol{\Lambda}_t'\boldsymbol{\Sigma}_t^{-1}(\mathbf{y}_{t+1} - \boldsymbol{\mu}_t)\right]$$

$$\pi_t^Q(s_{t+1}) = \pi_t(s_{t+1}) \exp[-\Gamma_t(s_{t+1})]$$

$$
\begin{aligned}
f_t^Q(\mathbf{x}_{t+1}) &= f_t^Q(\mathbf{y}_{t+1})\pi_t^Q(s_{t+1}) \\
&= f_t(\mathbf{x}_{t+1}) \exp(z_{t+1})
\end{aligned}
$$

$$
\begin{aligned}
z_{t+1} &= -(1/2)\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t - \boldsymbol{\Lambda}_t'\boldsymbol{\Sigma}_t^{-1}(\mathbf{y}_{t+1} - \boldsymbol{\mu}_t) \\
&\quad - \Gamma_t(s_{t+1})
\end{aligned}
$$

Can we justify the pricing rule?

$$P_t = E_t^Q(e^{-r_t} P_{t+1})$$

$$= E_t(e^{-r_t} e^{z_{t+1}} P_{t+1})$$

$$= E_t(M_{t+1} P_{t+1})$$

for $M_{t+1} = e^{-r_t} e^{z_{t+1}}$

e.g., $M_{t+1} = \beta U'(c_{t+1})/U'(c_t)$

$$\boldsymbol{\mu}_t^Q = \boldsymbol{\mu}_t - \boldsymbol{\Sigma}_t \boldsymbol{\Lambda}_t$$

parameterization:

$$\boldsymbol{\Lambda}_t = \boldsymbol{\Sigma}_t^{-1}[\boldsymbol{\lambda}_0(s_t) + \boldsymbol{\lambda}_Y(s_t)\mathbf{y}_t]$$

implies

$$\boldsymbol{\mu}_t^Q = \boldsymbol{\mu}_t - \boldsymbol{\lambda}_0(s_t) - \boldsymbol{\lambda}_Y(s_t)\mathbf{y}_t$$

assume:

$$\boldsymbol{\mu}_t = \boldsymbol{\lambda}_Y(s_t)\mathbf{y}_t + \mathbf{d}^*(s_t) + \mathbf{D}\mathbf{y}_t$$

$$E_t^Q(\mathbf{y}_{t+1}) = \mathbf{d}(s_t) + \mathbf{D}\mathbf{y}_t$$

market acts as if $\mathbf{y}_t \sim VAR(1)$

with regime-shift intercept

and variance

$$r_{t,n} = A_n(s_t) + \mathbf{B}_n^{'}\mathbf{y}_t$$

where $A_n(j), \mathbf{B}_n$ are known functions of other params

If there are $N = 3$ factors, then $N = 3$ interest rates,

$$\hat{\mathbf{R}}_t = (r_{t,6}, r_{t,24}, r_{t,120})'$$

could be used to calculate factors $\mathbf{y}_t$ from $\hat{\mathbf{R}}_t$ and regimes:

$$\mathbf{y}_t = \mathbf{B}^{-1}[\hat{\mathbf{R}}_t - \mathbf{A}(s_t)]$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_6' \\ \mathbf{B}_{24}' \\ \mathbf{B}_{120}' \end{bmatrix}$$

$$\mathbf{A}(s_t) = \begin{bmatrix} A_6(s_t) \\ A_{24}(s_t) \\ A_{120}(s_t) \end{bmatrix}$$

$$\hat{\mathbf{R}}_{t+1} = \mathbf{c}(s_t) + \mathbf{C}\hat{\mathbf{R}}_t + \mathbf{v}_{t+1}$$

$$\mathbf{v}_{t+1} \sim N(\mathbf{0}, \mathbf{V}(s_t))$$

VAR(1) with regime-switch intercept and variance assume $m$ other interest rates

$\tilde{\mathbf{R}}_t$ priced with error:

$$m = 1$$

$$\tilde{\mathbf{R}}_t = r_{t,60}$$

$$\tilde{R}_{t+1} = A_{60}(s_{t+1}) + \mathbf{B}_{60}^{'}\mathbf{y}_{t+1} + u_{t+1}$$

$$u_{t+1} \sim N(0, \Omega(s_{t+1}))$$

$$\tilde{R}_{t+1} = A_{60}(s_{t+1})$$

$$+ \mathbf{B}_{60}^{'}\mathbf{B}^{-1}[\hat{\mathbf{R}}_{t+1} - \mathbf{A}(s_{t+1})] + u_{t+1}$$

$$\begin{bmatrix} \hat{\mathbf{R}}_{t+1} \\ \tilde{R}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{c}(s_t) \\ c(s_{t+1}) \end{bmatrix} + \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{d}' & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{R}}_t \\ \tilde{R}_t \end{bmatrix} + \tilde{\mathbf{v}}_{t+1}$$

$$\tilde{\mathbf{v}}_{t+1} \sim N(\mathbf{0}, \tilde{\mathbf{V}}(s_t, s_{t+1}))$$

Testable implications:

(1) last column of VAR coeffs $=$ 0

(guide for choosing $\hat{\mathbf{R}}_t$ vs. $\tilde{\mathbf{R}}_t$)

(2) VAR(1) vs. VAR(2)

(3) forecasting (particularly large $n$)

(4) fit for other $r_{t,n}$

What are factors?

two slopes and a butterfly

What are regimes?

recessions

Why not add industrial production growth to observation vector? (driven by both factors and regimes)