# The Labor Market and Macro Volatility: A Nonstationary General-Equilibrium Analysis *

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### Abstract

The evolution of the aggregate labor market is far from smooth. I investigate the success of a macro model in replicating the observed levels of volatility of unemployment and other key variables. I take variations in productivity growth and in exogenous product demand (government purchases plus net exports) as the primary exogenous sources of fluctuations. The macro model embodies new ideas about the labor market, all based on equilibrium—the models I consider do not rest on inefficiency in the use of labor caused by an inappropriate wage. I find that non-standard features of the labor market are essential for understanding the volatility of unemployment. These models include simple equilibrium wage stickiness, where the sticky wage is an equilibrium selection rule. A second model based on modern bargaining theory delivers a different kind of stickiness and has a unique equilibrium. A third model posits fluctuations in matching efficiency that may arise from variations over time in the information about prospective jobs among job-seekers. Reasonable calibrations of each of the three models match the observed volatility of unemployment.

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# 1   Introduction

Observed movements of employment and unemployment are larger than standard general-equilibrium models can comprehend, given the relatively small exogenous shocks that hit the economy. I investigate this topic in a family of models with labor markets that contain amplification mechanisms that may contribute to the understanding of volatility.

The models incorporate some standard features of modern general-equilibrium macroeconomics. Households plan consumption according to a standard Euler equation. Investment faces adjustment cost. With respect to the labor market, I depart from the main tradition of real business cycle and dynamic general equilibrium macro models. That tradition has considered the margin between employment in the market and time spent at home and has neglected time spent looking for work. I follow the other important tradition of modern macroeconomics, the matching model of job search. This line of thought considers the margin between working and looking for work and takes labor-force participation to be inelastic. In place of the Nash-bargain wage determination usually considered in that tradition, I consider two alternatives. First is the equilibrium sticky-wage formulation in Hall (2005c). The wage is less responsive to current conditions, though not so sticky as to create bilaterally inefficient outcomes between employer and worker. Second is the bargaining model of Hall and Milgrom (2005), which replaces non-credible threats to abandon wage bargaining with credible threats to extend bargaining. That model achieves a sticky-wage result by limiting the role of conditions in the labor market in wage determination. I also consider a model where fluctuations in matching efficiency drive unemployment fluctuations. The last model mimics fluctuations arising from changes in the quality of information available to job-seekers that induce alterations in self-selection, following Hall (2005a).

The economy in this paper is explicitly non-stationary. I present essentially exact numerical solutions to a stochastic growth model whose levels of output, consumption, and capital have unit roots, inherited from the unit root in the efficiency of production. Rather than deal with filtered data to remove the unit root, I state the key variables as ratios to the

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capital stock. The model is stationary in these ratios.

I measure volatility in terms of changes in the variables. This approach captures the cyclical component of volatility along with the important movements that are not necessarily associated with any concept of a business cycle. The model and associated measurement approach do not rest on any attempt to separate cyclical movements from other movements.

This paper is a further development of the efforts of Merz (1995), Andolfatto (1996), and Alexopoulos (2004) to incorporate unemployment in general-equilibrium macro modeling. The most important difference from earlier work is in wage determination and the amplification of fluctuations that occurs with equilibrium sticky wages and other non-standard features of the labor market.

I find that departing from the standard view of the labor market is essential for understanding the volatility of unemployment. As Shimer (2005) has demonstrated, the standard Mortensen and Pissarides (1994) setup cannot rationalize movements of unemployment of the observed magnitude as the result of productivity shocks of the observed magnitude. Non-standard labor-market models provide the needed amplification mechanisms. A basic idea common to all of the models—discussed in Hall (2005c)—is that employers determine the level of recruiting effort based on the expected share of the joint value created when they form matches with workers. In times when the share is low, recruiting effort is correspondingly low. Job-finding rates for the unemployed are low and the unemployment rate is high. Matches are formed and retained efficiently, according to the principle of maximizing joint value. Bilateral efficiency implies that the model's view of the labor market is one of economic equilibrium, not the disequilibrium previously associated with amplification mechanisms based on sticky wages.

The model overcomes one of the most persuasive criticisms of real business-cycle models—that the models portray recessions as the result of actual contractions in productivity (see Summers (1986)). Here, in a growing economy, a recession with abnormal unemployment will occur when productivity growth is positive but lower than normal. I also

show that movements in exogenous product demand are important sources of fluctuations. This driving force has received less attention from earlier models.

At reasonable values of the key parameters—intertemporal elasticity of substitution, capital adjustment cost, and wage stickiness—the models account reasonably well for the volatilities of consumption growth, the investment/capital ratio, the capacity/capital ratio, and unemployment.

# 2 Model

## 2.1 Shocks

Serially independent shocks $a_t$ affect productivity $A_t$, which evolves as

$$A_{t+1} = a_t A_t. \tag{1}$$

The shocks are drawn from a discrete set, $\{a_1, ..., a_5\}$ with probabilities $\{\psi_1, ..., \psi_5\}$. Serially correlated shocks $g_t$ affect exogenous product demand. They are stated as ratios to consumption, $c_t$, so the shocks to product demand are $g_t c_t$. They are drawn from a discrete set $\{g_1, g_2, g_3\}$ and obey a first-order Markoff process:

$$\text{Prob}\left[g_{t+1} = g' | g_t = g\right] = \pi_{g,g'} \tag{2}$$

## 2.2 Technology, capital, and consumption

Let $y_t$ be output produced at time $t$, $A_t$ be the efficiency of production, $n_t$ be employment, $k_t$ be capital, $x_t = k_{t+1}/k_t$ be the capital growth ratio (also the investment/capital ratio), $v_t$ be resources expended in recruiting workers, and $c_t$ be consumption. $\tau$ is a parameter controlling adjustment cost. Labor and capital form gross output according to

$$A_t n_t^{\gamma} k_t^{1-\gamma}. \tag{3}$$

Adjustment costs are:

$$\frac{1-\tau}{2\tau} k_t (x_t - 1)^2. \tag{4}$$

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The government purchases part of output and another part leaves the country as net exports. I take the sum of the two to be an exogenous fraction, $g_t$, of consumption. Because the model is nonstationary, the exogenous component of product demand needs to be linked to a variable that grows according to the economy's stochastic trend; consumption is the most suitable choice because it is the most stable.

Capital deteriorates at rate $\delta$, so it follows the law of motion

$$k_{t+1} = (1 - \delta)k_t + A_t n_t^\gamma k_t^{1-\gamma} - \frac{1-\tau}{2\tau} k_t (x_t - 1)^2 - v_t - (1 + g_t)c_t. \tag{5}$$

I let $q_t$ be the market price of installed capital at $t$. Firms solve the atemporal capital-installation problem:

$$\max_{x_t} \ q_t x_t k_t - \frac{1-\tau}{2\tau} k_t (x_t - 1)^2 - x_t k_t. \tag{6}$$

The first-order condition is:

$$x_t = \frac{\tau}{1-\tau}(q_t - 1) + 1, \tag{7}$$

Tobin's investment equation. The Tobin coefficient, $\tau$, controls capital adjustment. If $\tau = 0$, capital does not adjust at all; the economy is the endowment economy of Lucas (1978). If $\tau = 1$, capital adjusts without impediment and $q$ is always one.

Households can buy and sell a claim to a unit of installed capital with price $q_t$. Its return ratio is

$$R_{t+1} = \frac{(1-\delta)q_{t+1} + (1-\gamma)A_{t+1}n_{t+1}^\gamma k_{t+1}^{-\gamma}}{q_t}. \tag{8}$$

Households have an intertemporal elasticity of substitution of $\sigma$. Their intertemporal marginal rate of substitution is

$$m_{t+1} = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\frac{1}{\sigma}}. \tag{9}$$

The marginal rate of substitution prices all economic values in period $t + 1$. I define the present value operator,

$$P_t(X_{t+1}) = E_t\left(m_{t+1}X_{t+1}\right). \tag{10}$$

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Households plan consumption to satisfy the Euler equation,

$$P_t\left(R_{t+1}\right) = 1. \tag{11}$$

This setup makes the implicit assumption that individuals are fully insured for the idiosyncratic risk of unemployment—all individuals consume the same amount, independent of their employment situations. See Merz (1995) for further discussion of this point.

## 2.3 Labor market

The labor market operates according to principles laid out in Hall (2005c), Hall (2005b), and, in a later section, Hall and Milgrom (2005), based on some of the features of Mortensen and Pissarides (1994).

I normalize the labor force at one. The number of workers who lose their jobs each quarter is a constant, $\overline{s}$. Among the job-losers, a fraction $f$ find new jobs immediately. The remainder find jobs after a quarter. Thus the unemployment rate is $u_t = \overline{s}(1 - f_t)$. The separation rate—the fraction of workers who lose their jobs—is

$$s_t = \frac{\overline{s}}{1 - u_t}. \tag{12}$$

This setup mimics actual unemployment reasonably well and has the (substantial) modeling benefit that the unemployment rate is not a state variable. The emphasis on the probability that a job-loser becomes unemployed is realistic—much of the variation in U.S. unemployment arises from fluctuations in the fraction of job-losers who become unemployed—see Hall (2005d).

Job-seekers receive unemployment benefits at a flow rate $\lambda_t$, financed by a lump-sum tax.

Workers receive compensation with present value $W_t$ from jobs starting in quarter $t$. The Bellman value, $U_t$, associated with being unemployed is the sum of expected unemployment benefits, the wage, $W_{t+1}$, paid by the job to be found next quarter, and the value,

$V_{t+1}$, of being employed at that time, apart from the wages on that job:

$$U_t = \lambda_t + P_t(W_{t+1} + V_{t+1}). \tag{13}$$

The worker may (1) separate next quarter and find another job immediately, (2) separate next quarter and enter unemployment, or (3) remain at work. $V_t$ is the present value, while employed, of these three possible states next quarter:

$$V_t = P_t \left( s_{t+1} f_{t+1}(W_{t+1} + V_{t+1}) + s_{t+1}(1 - f_{t+1})U_{t+1} + (1 - s_{t+1})V_{t+1} \right). \tag{14}$$

I let $Z_t$ be the present value of the marginal product of the worker on a particular job, at the time the job begins. It satisfies the recursion,

$$Z_t = \gamma A_t(1 - u_t)^{\gamma-1} k_t^{1-\gamma} + P_t \left( (1 - s_{t+1})Z_{t+1} \right). \tag{15}$$

Employers devote an amount of output, $v_t$, to recruiting and qualifying workers—$v_t$ includes all expenditures on workers that occur before the terms of a match are set through a wage bargain. Employers' recruiting spending determines the job-finding rate, $f_t$. When employers are looking actively for new workers, jobs are easy to find and $f_t$ is high. The marginal benefit of added recruitment spending in terms of the job-finding rate declines with $v_t$ according to a concave function $\phi(v_t/k_t)$. I scale by capital because recruiting and qualifying workers is more expensive in a more advanced economy.

My treatment of the matching process departs from the standard in the matching literature for reasons of modeling convenience, not substance. Linking the job-finding rate directly to recruiting expenditure can be seen as a reduced form of the usual set-up, where vacancies appear explicitly—see Hall (2005b).

Employers earn a flow of profit equal to the benefit from hiring one new worker (the employer's share of the surplus for a job, $Z_t - W_t$) multiplied by the flow of new hires, $\bar{s}f_t$, less the cost of recruiting the flow of workers, $v_t$. Employers compete for the services of job-seekers up to the point of zero profit. The zero-profit condition is

$$v_t = (Z_t - W_t)\bar{s}f_t. \tag{16}$$

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## 2.4 Stationary model

Before discussing wage determination, I create a stationary version of the model by re-defining the following variables to be their earlier values divided by the capital stock, $k_t$: consumption, $c_t$, recruiting resources, $v_t$, government purchases, $g_t$, and the four labor-market values, $U_t$, $V_t$, $Z_t$, and $W_t$. I also introduce the state variable $z_t = A_t k_t^{-\gamma}$, the ratio of capacity, $A_t k_t^{1-\gamma}$, to capital, $k_t$. In the labor market, I assume that unemployment compensation is a constant fraction of the marginal product of labor: $\lambda_t = \lambda \gamma z_t (1 - u_t)^{\gamma-1} k_t$

The model becomes

$$z_{t+1} = a_{t+1} x_t^{-\gamma} z_t \tag{17}$$

$$x_t = 1 - \delta + z_t (1 - u_t)^\gamma - \frac{1 - \tau}{2\tau}(x_t - 1)^2 - v_t - (1 + g_t)c_t \tag{18}$$

$$x_t = \frac{\tau}{1 - \tau}(q_t - 1) + 1 \tag{19}$$

$$R_{t+1} = \frac{(1 - \delta)q_{t+1} + (1 - \gamma)z_{t+1}(1 - u_{t+1})^\gamma}{q_t} \tag{20}$$

$$m_{t+1} = \beta \left( x_t \frac{c_{t+1}}{c_t} \right)^{-\frac{1}{\sigma}} \tag{21}$$

$$P_t (R_{t+1}) = 1 \tag{22}$$

$$U_t = \lambda \gamma z_t (1 - u_t)^{\gamma-1} + P_t(W_{t+1} + V_{t+1})x_t \tag{23}$$

$$V_t = P_t \left( s_{t+1} f_{t+1}(W_{t+1} + V_{t+1}) + s_{t+1}(1 - f_{t+1})U_{t+1} + (1 - s_{t+1})V_{t+1} \right) x_t \tag{24}$$

$$Z_t = \gamma z_t (1 - u_t)^{\gamma-1} + P_t \left( (1 - s_{t+1})Z_{t+1} \right) x_t \tag{25}$$

$$v_t = (Z_t - W_t)\bar{s} f_t \tag{26}$$

$$f_t = \phi(v_t) \tag{27}$$

$$u_t = \bar{s}(1 - f_t) \tag{28}$$

$$s_t = \frac{\bar{s}}{1 - u_t}. \tag{29}$$

## 2.5 Wages

I will consider the following model of wage determination:

$$W_t = \kappa W^* + (1 - \kappa)\frac{1}{2}(U_t - V_t + Z_t). \tag{30}$$

The quantity $\frac{1}{2}(U_t - V_t + Z_t)$ is the average of the reservation wage of the jobseeker, $U_t - V_t$, and the reservation wage of the employer, $Z_t$. It is the wage set in a symmetric Nash bargain, as in the earlier literature. To the extent that $\kappa > 0$, the wage is sticky compared to the Nash-bargain wage, as in Hall (2005c). It does not respond as much to the immediate ups and downs of productivity. If $\kappa = 0$, the Nash bargain sets the wage. This wage rises when a favorable productivity shock, $a_t$, hits the economy. Output rises more than the capital stock—this is true without capital adjustment costs, but is even more true in their presence. The scarcity raises the marginal product of labor, which is proportional to the state variable $z_t$—see equation (25). The actual wage $W_t$ responds immediately through the Nash bargain. As capital rises to restore the normal output/capital ratio, $z_t$ returns to its normal value and the wage is back to normal in relation to the capital stock.

I set the parameter $W^*$ to the level of the wage/capital ratio in the stationary non-stochastic Nash-bargain model—the normal wage in the flexible-wage model. In the fixed-wage model ($\kappa = 1$), the wage/capital ratio, $W_t$, is constant at the level $W^*$. That is, the wage remains at its normal level in relation to the capital stock rather than rising when productivity growth is strong and falling when it is weak.

Intermediate values of $\kappa$ describe an economy with partial wage stickiness.

Equation (30) may place the wage outside the bargaining set—that is, below the jobseeker's reservation wage $U_t - V_t$ or above the employer's reservation wage, $Z_t$. I modify the equation to keep the wage within the bargaining set. Thus the sticky wage lies within the equilibrium set described in Hall (2005c). Sticky wages never cause inefficient separations—they cause unemployment volatility through their influence on employers' recruiting efforts.

## 2.6 Other issues

I assume that the productivity and exogenous spending shocks each take on one of a finite number of possible values. In this setting, the expectations in the Euler equations are summations rather than integrals.

To solve the model, I represent each endogenous variable as a function of the state variables, $z$ and $g$, with a finite set of parameters (specifically, the coefficients of a Tchebysheff polynomial of order 9). I write the model in terms of the current value of $z$ and its value one year in the future, $z' = ax^{-\gamma}z$. The resulting model is a large system of nonlinear equations in the coefficients of the functions, which can be solved by standard methods. Based on the solution, I represent the entire model as a first-order Markoff process of very high dimension. I calculate the stationary distribution from the Markoff matrix and derive the (essentially) exact standard deviations of the endogenous variables from that distribution. For further details, see Judd (1998) and the appendix to this paper.

Notice that the model embodies a transversality condition that the variables tend toward a stationary point. In other words, a solution corresponds to a terminal condition in the distant future, not a condition that has an appreciable effect on the economy today. The transversality condition takes the form of the requirement that the consumption rule $c_g(z)$ depends only on $z$ and $g$. If the economy faced a terminal condition in finite time, the consumption rule would depend on that condition as well.

## 3    Volatility Measures

Table 1 shows data for the years 1949 through 2000 for the volatility measures that I consider. The first is productivity growth, taken as an exogenous driving force of fluctuations. The standard deviation of quarterly growth in total factor productivity is 0.94 percent. The second is the ratio of exogenous spending to consumption, another driving force. Its standard deviation is 4.8 percent.

| Variable | Data source | Units | Value |
|---|---|---|---|
| Productivity growth | Bureau of Labor Statistics, Multi-factor Productivity Index, Private Business | Quarterly standard deviation, percent, inferred as half the annual standard deviation | 0.94 |
| Government purchases plus net exports relative to consumption | Bureau of Economic Analysis, National Income and Product Accounts, Table 1.1.5. Nominal Gross Domestic Product | Standard deviation of quarterly data, percent | 4.83 |
| Consumption growth | Bureau of Economic Analysis, National Income and Product Accounts,Table 1.1.3. Real Gross Domestic Product, Quantity Indexes | Standard deviation of quarterly data, percent | 0.88 |
| Investment/capital ratio | Growth ratio of the real capital stock from Bureau of Economic Analysis, Fixed Asset Tables, Table 1.2. Chain-Type Quantity Indexes for Net Stock of Fixed Assets and Consumer Durable Goods | Standard deviation of annual data, divided by 4, percent | 0.22 |
| Log of capacity/capital ratio | Ratio of productivity index to (capital stock raised to the power 0.7), with constant exponential trend removed | Standard deviation of log of annual data, multiplied by 100 | 4.43 |
| Unemployment | Bureau of Labor Statistics, Unemployment rate, 16 years and older | Standard deviation of quarterly data, percent | 1.57 |

Table 1. Data on macro volatility in the U.S., 1949-2000

The remaining measures are for variables taken to be endogenous in the model. Consumption growth has a quarterly standard deviation of 0.88 percent. The standard deviation of the investment/capital ratio (the growth rate of the capital stock) is 0.22 percent. I take unemployment to be stationary, so I measure its volatility as the standard deviation of its level, 1.57 percent.

# 4  Functional Forms and Parameter Values

In my discussion of the volatility implied by the model, I will start from a base case for the key parameters: I take the wage stickiness coefficient $\kappa$ to be 0.43, implying a wage that responds only partially to the current state of the economy. I take the base value of the intertemporal elasticity of substitution $\sigma$ to be 0.4, somewhat above the value suggested in Hall (1988), in deference to suggestions in the recent literature of downward bias in my estimates. The implied value of the coefficient of relative risk aversion is 2.5, a reasonable value in terms of the discussion in Lucas (1994). Finally, I take the base value of the Tobin coefficient for capital adjustment $\tau$ to be 0.25—see Hall (2004).

I consider the following alternative values of the key parameters: Flexible wages, with $\kappa = 0$, less intertemporal substitution, with $\sigma = 0.2$ (with higher coefficient of relative risk aversion, 5), and faster capital adjustment, with $\tau = 0.5$.

The driving forces for fluctuations in the model are technology growth, $a_t$, and exogenous spending relative to consumption, $g_t$. In both cases, I convert time-series data to discrete values by placing them in bins. I define the bins so that an equal fraction of the observations fall in each bin. In the case of technology growth, I take the deviation from the mean to be half the annual growth rate, to match the standard deviation of the observed annual data. The average values for quarterly productivity growth in the five bins are -0.90, -0.11, 0.26, 0.79, and 1.56 percent.

For exogenous product demand (the ratio of government purchases and net exports to consumption), I work directly with quarterly data. Table 2 describes the distribution. The

| | Transition probabilities from column value to row value | | |
|---|---|---|---|
| Value | 24.5 | 31.3 | 36.7 |
| 24.5 | 93.5 | 6.5 | 0.0 |
| 31.3 | 7.8 | 85.7 | 6.5 |
| 36.7 | 0.0 | 6.5 | 93.5 |

Table 2. Transition frequencies for exogenous spending as a percent of consumption

average value of $g_t$ in each bin are shown in the columns labeled "Value." The table shows the transition frequencies among the three categories.

I take the rate of deterioration of capital and the labor elasticity of the production function to have the standard values $\delta = 0.025$, and $\gamma = 0.7$. I pick the utility discount so that the average return to capital is 5 percent per year: $\beta = 0.9992$.

I take the recruiting technology to have the form

$$f = \phi\left(v/k\right) = \omega\left(v/k\right)^{1/2}. \tag{31}$$

I pick the efficiency parameter of the recruiting technology, $\omega$, by setting up a version of the model with constant, non-stochastic productivity growth at the average rate from the values in Table 2. I prescribe that the unemployment rate be its post-1949 average of 5.5 percent. I take the exogenous separation rate $\bar{s}$ to have the value 10 percent per quarter per month (see Hall (2005d) for a discussion of the measurement of the separation rate). Then I solve the model comprising equations (18) through (30) plus an additional equation that gives the condition for a stationary $z$,

$$\bar{g}x^{-\gamma} = 1 \tag{32}$$

The solution provides the value $\omega = 8.9$.

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# 5 Properties of the Model

The state variables, $z_t$, the capacity/capital ratio, and $g_t$, exogenous spending, capture the current departure of the economy from its steady state. A favorable productivity shock increases capacity immediately and raises $z_t$ correspondingly. Investment responds immediately and capital begins to rise, driving $z_t$ back to its normal level until the next shock occurs. Productivity shocks are permanent in the model. On the other hand, exogenous spending shocks, $g_t$, are persistent but ultimately die out, as ratios to consumption. A strictly permanent spending shock would depress consumption permanently by the same amount and leave all other variables unaffected. In the case where the spending shock dies out over time, investment falls at the time of the shock. The cumulating decline in the capital stock causes $z_t$ to rise gradually in the period after the spending shock. Investment rises later to restore the steady-state value of $z_t$ as spending returns to normal. Thus $z_t$ has an exponential response to productivity impulses and a hump-shaped response to spending impulses.

Table 3 reports the calculated volatilities for a variety of combinations of the key parameters. The base case, in the first column of results, fits the observed volatilities fairly well. The model slightly overstates the volatilities of consumption growth and the investment ratio and matches the volatilities of the capacity/capital ratio and unemployment fairly closely. Of course, the match is the result of adjusting parameter values, so Table 3 is really the result of applying indirect inference informally. As I will show shortly, the exogenous spending shock is an important contributor to the model's ability to explain volatility—in the presence of only the technology shock, the model understates volatility.

The second column of results shows the main point of the paper. Replacing the sticky-wage specification with the standard Nash-bargain model of wage determination, I find vastly too little volatility of unemployment. This finding confirms, in a general-equilibrium setting, Shimer (2005)'s point that the standard matching model cannot generate anything like a realistic account of the movements of unemployment. Notice that the volatilities of

| Parameter | Actual | Base | Flexible wage | Lower inter-temporal substi-tution | Lower adjustment cost |
|---|---|---|---|---|---|
| σ, intertemporal elasticity of substitution | | 0.4 | 0.4 | 0.2 | 0.4 |
| τ, adjustment cost parameter | | 0.25 | 0.25 | 0.25 | 0.5 |
| κ, wage stickiness parameter | | 0.43 | 0 | 0.43 | 0.43 |
| *Standard deviations* | | | | | |
| Consumption growth | 0.88 | 0.94 | 0.98 | 1.03 | 0.90 |
| Investment/capital ratio | 0.22 | 0.22 | 0.23 | 0.28 | 0.23 |
| Log of capacity/capital ratio | 4.43 | 4.59 | 4.79 | 6.10 | 4.53 |
| Unemployment | 1.57 | 1.59 | 0.24 | 1.66 | 1.54 |

Table 3. Actual and calculated volatilities

the other variables are reasonably well matched. One must look in the labor market itself to see that something important is missing from a general-equilibrium model. Standard models can account for the volatilities of consumption growth and the other non-labor variables.

The column of Table 3 headed "Lower intertemporal substitution" considers an economy with an intertemporal elasticity, $\sigma$, of 0.2 in place of the base value of 0.4. This economy has higher consumption volatility because the consumption substitution effect is smaller and offsets the wealth effect less. The volatilities of the investment/capital and capacity/capital ratios are—unrealistically—much higher because consumption absorbs less of each shock.

The column of Table 3 headed "Lower adjustment cost" describes an economy with a higher response of investment to Tobin's $q$. The coefficient, $\tau$, is 0.5 in this economy against 0.25 in the base economy. A lower adjustment cost has remarkably little effect on the volatilities implied by the model.

## 5.1   Endogenous variables as functions of state variables

An instructive way to display the model is in terms of functions such as $u_g(z)$ that show how the key variables depend on the two state variables. These functions, represented as polynomials in $z$, are the results of solving the model, as described in the Appendix.

Figure 1 shows $u_g(z)$ for the intermediate value of exogenous spending, $g$. The difference is dramatic between the response of unemployment in the sticky- and flexible-wage models. The curve at the bottom of this and later figures shows the marginal probability distribution of the capacity/capital ratio, $z$. In the sticky-wage model, the wage remains close to its normal level in relation to the capital stock even when a disappointment in productivity growth causes the capacity/capital ratio to drop below normal. In this case, the wage is higher than the flexible wage and employers have correspondingly less incentive to deploy resources in attracting new workers. The job-finding rate is lower than normal
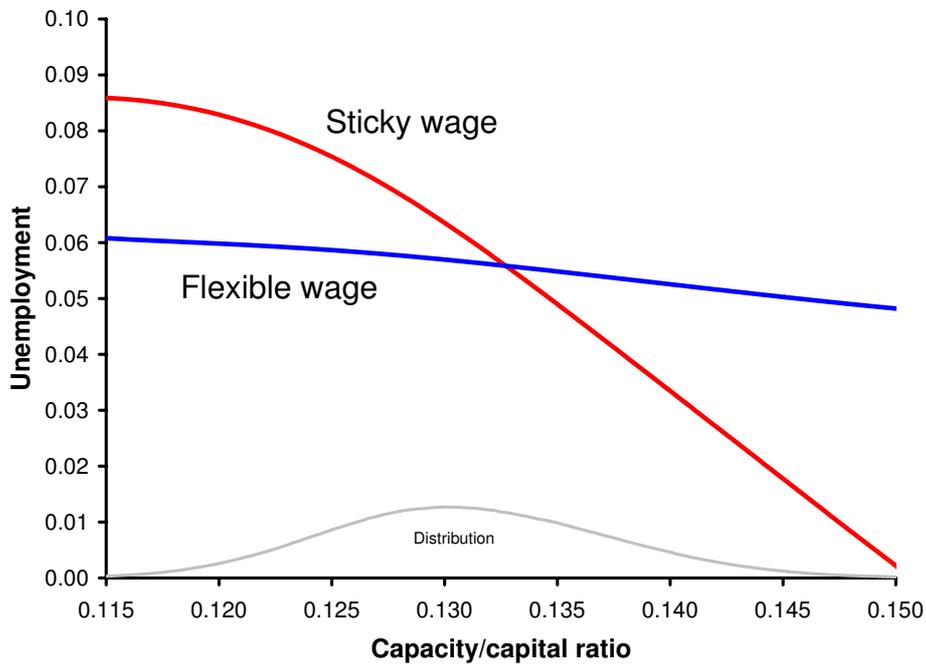
Figure 1. Unemployment responses, base case and flexible wage

and unemployment is high. The reverse holds for good news in productivity, which brings forth high recruiting effort and low unemployment.

By contrast, Figure 1 shows that unemployment is barely responsive to shifts in the capacity/capital ratio when wages are flexible. Adjustments of the wage result in only small changes in the incentive to hire and corresponding unemployment rate. The figure confirms, in a full dynamic general-equilibrium setting, the point of Shimer (2005) that the Mortensen-Pissarides model with Nash-bargain wage setting cannot explain the volatility of unemployment given the known amplitude of the driving force.

Figure 1 makes it clear that the flexible-wage model cannot come close to matching the observed volatility of unemployment. From Table 1, unemployment has a standard deviation of 1.57 percentage points. The model—with base-case parameters except for full wage flexibility ($\kappa = 0$)—implies a standard deviation of unemployment of only 0.24 percentage points. Accordingly, I devote most of the attention in the discussion to the
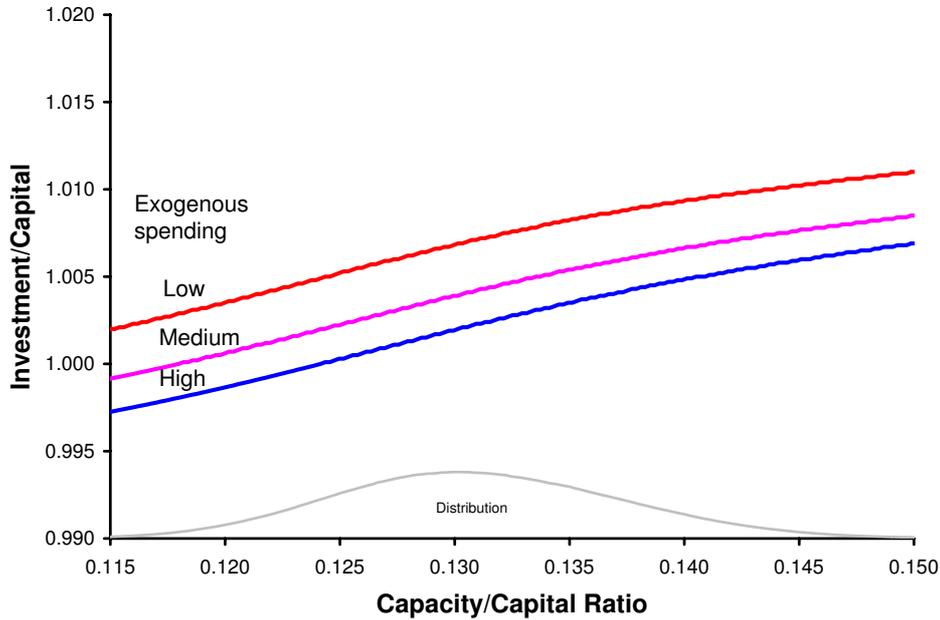
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Figure 2. Investment as a function of the state variables

sticky-wage case ($\kappa = 0.43$).

Figure 2 shows one of the key relationships in the model, the value of investment (measured by the growth factor for capital, $x$) as a function of the capacity/capital ratio, $z$, and the level of exogenous spending, $g$. Investment responds to spending because spending shocks, though highly persistent, are not permanent. The economy responds to spending shocks in part by deferring investment until the time when the shock begins to subside.

Figure 3 shows consumption as a function of the capacity/capital ratio $z$ and the discrete state of exogenous spending. Because the spending shocks are quite persistent, consumption responds significantly to the shocks.

The solution of the model includes a function $W_g(z)$ that describes the wage as a function of the capacity/capital ratio, $z$, and the discrete state of exogenous spending, $g$. The model requires that the wage lie within the bargaining set $[U - V, Z]$. The model solution also supplies the functions $U_g(z)$, $V_g(z)$, and $Z_g(z)$. Figure 4 shows the relation between
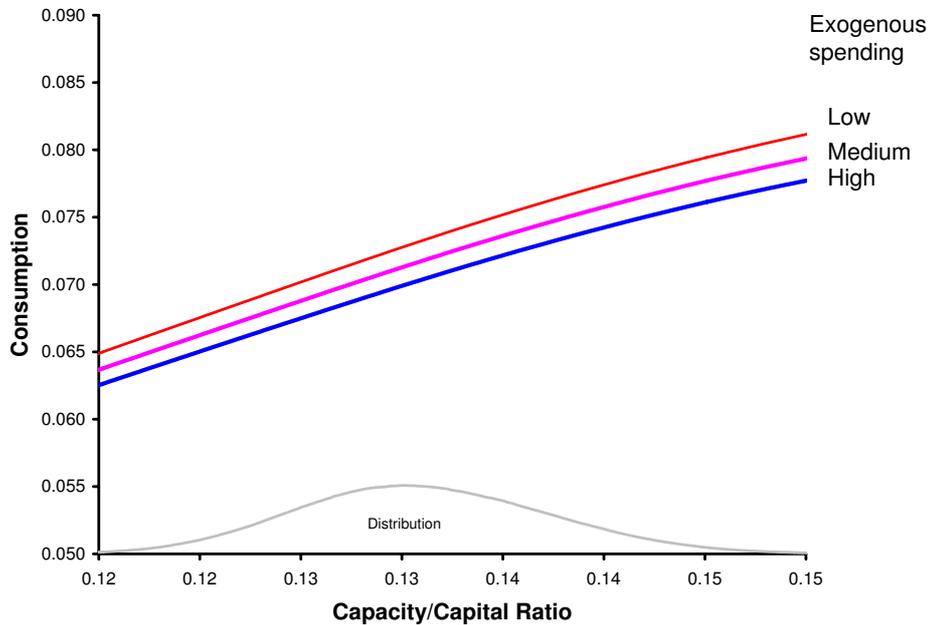
Figure 3. Consumption as a function of the state variables

the actual wage $W(z)$ and the boundaries of the bargaining set, for the intermediate value of exogenous spending, $g = g_2$. The actual wage reaches the upper boundary at the lowest values of $z$, but, as the distribution at the bottom of the figure shows, this outcome is extremely rare. In bad times, employers gain hardly any of the surplus from a job match and consequently put almost no resources into recruitment. The unemployment rate rises to serious recession levels.

## 5.2  Impulse response functions

The impulse response function is a standard tool for understanding the properties of dynamic models. I define the impulse response function as the difference between the mean of a variable following an impulse and the stationary mean of the variable. I characterize the impulse as a shift in the distribution relative to the stationary distribution. For the exogenous spending impulse, the alternative distribution is the one obtained by applying
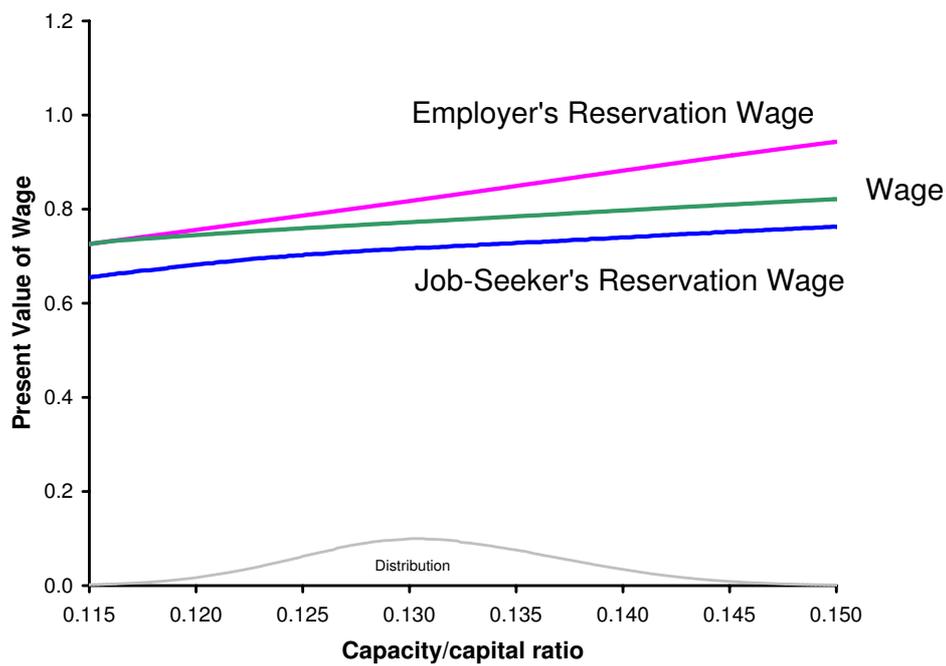
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Figure 4. Reservation and actual wages as functions of $z$, for middle level of spending

a special transition matrix to the stationary distribution. The special matrix alters the exogenous transition matrix $\pi_{i,i'}$ by subtracting a small number from a diagonal element and adding the same number to the element to the right. The effect is an upward shift in the discrete state $g$. For the technology shock, the alternative distribution is the density evaluated at $z - .01$. The effect is to shift the capacity/capital ratio upward by 0.01.

I multiply the discretized stationary joint distribution of $z$ and $g$ by the special transition matrix to find the distribution of the two variables immediately after the impulse. I then multiply that distribution by the standard transition matrix to find the distribution in the period following the impulse. I continue the process for 20 quarters. From the distribution of $z$ and $g$ in each period, I calculate the distributions of the other endogenous variables and calculate their means.

Figure 5 shows the responses of unemployment over time to a positive productivity impulse, for the flexible-wage and sticky-wage models. The scale is omitted from the vertical axis, as the size of the impulse is arbitrary and the purpose of this and other similar figures is to compare responses of different models to the same impulse. In both cases, unemployment falls immediately and gradually returns to normal (shown as a fine horizontal line). The magnitude of the effect is far greater for the sticky-wage case. With flexible wages, the incentive for job-creation, $Z - W$, hardly improves when $Z$ rises, because $W$ rises almost as much. With sticky wages, $W$ rises as well, but not as much, so the incentive increases, spending on recruiting rises, and the labor market tightens. In both cases, the dynamics of $z$ control the return to normal. The immediate effect of the technology impulse is a jump in $z$—the impulse raises the numerator of $z$ while leaving its denominator, the capital stock, unchanged. As capital accumulates following the shock, $z$ returns to normal and all variables, including the unemployment rate, also return to normal.

Figure 6 shows the responses of unemployment to an increase in exogenous spending. Again, the response is much more dramatic for the sticky-wage model. In that model, the initial effect of the impulse is to increase unemployment, because the increase in spending
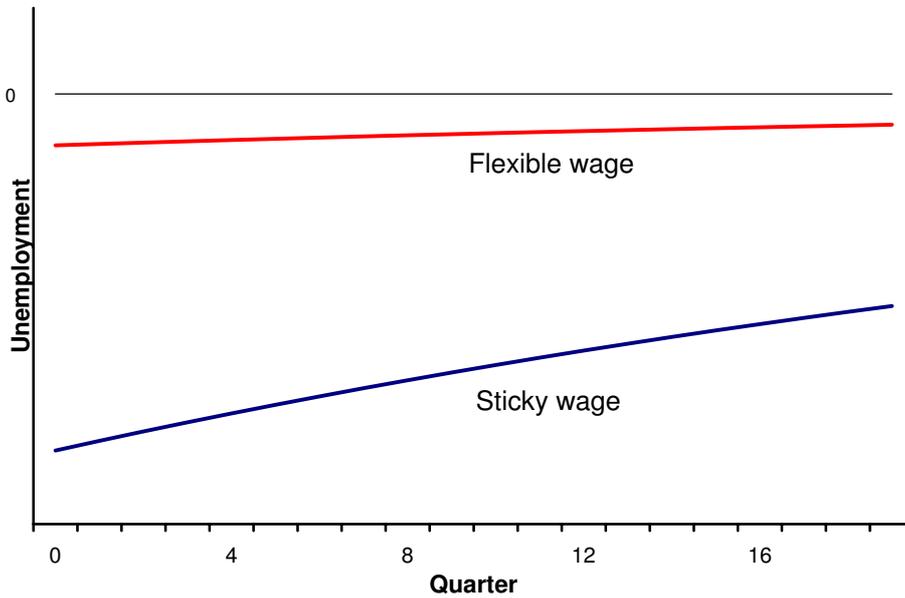
Figure 5. Response of unemployment to a productivity shock

lowers the present value of future productivity, $Z$, immediately because it raises the expected future discount rate. This effect would be even stronger if labor supply were somewhat elastic. I explore these discount-rate effects below. After three quarters, a stronger effect from rising $z$ overcomes the initial effect and unemployment begins to fall below its normal level. Eventually, unemployment turns around and returns to its normal level. The process is prolonged because the spending shock is highly persistent.

## 5.3 The role of variations in the discount rate

One of the important contributions of general-equilibrium analysis of matching models of the labor market is to assess the role of variations in the discount rate. Recall that the present value of a worker's productivity over the duration of a job, measured as of the beginning of the job, is

$$Z_t = \gamma z_t (1 - u_t)^{\gamma-1} + P_t \left( (1 - s_{t+1}) Z_{t+1} \right) x_t. \tag{33}$$
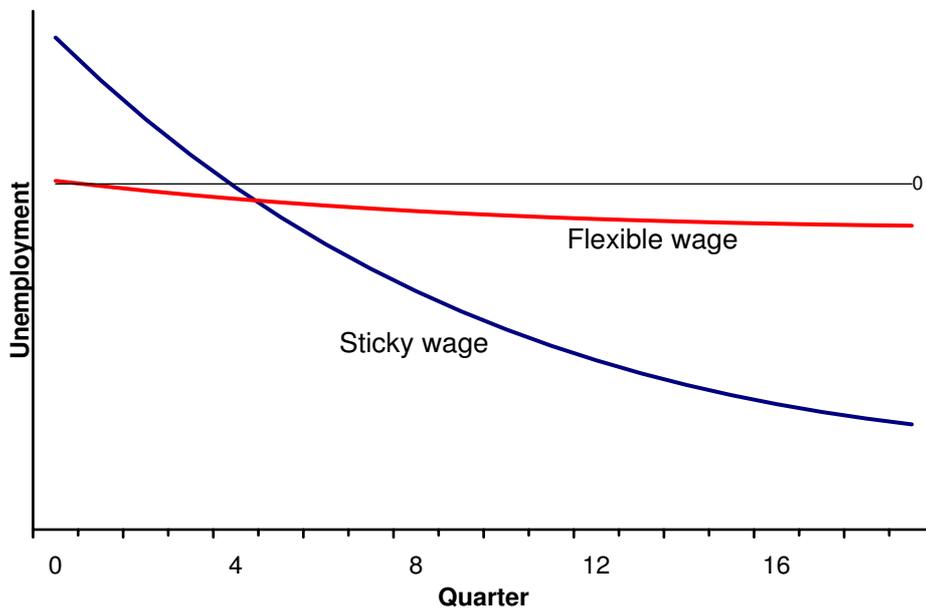
Figure 6. Response of unemployment to a spending shock

A higher discount rate lowers the second term on the right-hand side. The equilibrium value of $Z$ is sensitive to the future discount rate over the roughly three-year period that the job will last.

To measure the role of variations in the discount rate, I create a version of the model in which the discount applied to the three labor-market variables, $Z$, $U$, and $V$, is constant rather than endogenous. I fix the marginal rate of substitution $m$ at its stationary value from the calibration. In this model, unemployment is substantially more volatile—the standard deviation of unemployment is 3.9 percent, compared to 1.6 percent in actuality and in the base case.

Endogenous discounting reduces the volatility of unemployment. When a positive productivity shock hits the economy, the return to capital rises immediately. Consumption growth rises to satisfy the consumption Euler equation. Both immediate and future discount rates are higher. The value of $Z$ is correspondingly lower than it would be with a

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constant discount rate. The reward to recruiting, $Z - W$, is also lower than it would be with a constant rate. This factor offsets the direct stimulus from higher productivity in raising $Z$, which therefore rises less than it would with a constant discount rate.

In the case of a positive shock to exogenous spending, the reason for attenuation of the effect on unemployment through the endogenous discount rate is similar, but not quite the same. Because the spending shock has no effect on productivity and no initial effect on the capital stock, the effects of the discount rate arise entirely through expectations of a higher future discount rate. The spending shock results in a decline in investment, which drives up the return as the shortfall in capital cumulates. The effect that is immediate in the case of a productivity shock occurs over time for a spending shock.

## 5.4 Contributions of the two driving forces

Table 4 compares the standard deviations of the key variables in the full model (corresponding to the first column of results in Table 3) and in structurally identical models that have only the technology shock and only the exogenous spending shock. I find the exact solutions to the modified models, so that these results apply to economies where participants know about the nature of the shocks. The left column of results repeats the standard deviations from Table 3 for the full model. The middle column shows the effects of the technology shock alone by setting the exogenous spending shock to zero. The right-hand column shows the effect of the spending shock alone.

The spending shock is a little more important than the productivity shock for the investment/capital ratio, while the technology shock is a little more important for consumption growth and the capacity/capital ratio and rather more important for unemployment.

# 6 Credible Threats in Wage Bargaining

Hall and Milgrom (2005) point out that the standard model in the Mortensen-Pissarides tradition imputes threat points in the wage bargaining process that are not credible. A

| Standard deviations | Actual | Base | Productivity shock alone | Exogenous spending shock alone |
|---|---|---|---|---|
| Consumption growth | 0.88 | 0.94 | 0.73 | 0.60 |
| Investment/capital ratio | 0.22 | 0.22 | 0.14 | 0.17 |
| Log of capacity/capital ratio | 4.43 | 4.59 | 3.66 | 2.37 |
| Unemployment | 1.57 | 1.59 | 1.34 | 0.73 |

Table 4. Contributions of technology and spending shocks

job-seeker and an employer, having found each other, enjoy a valuable joint surplus that would be dissipated if one of the parties walked away from bargaining. Rather than walk away, a party can take a step to create a successful match. The sequential-offer bargaining model of Binmore, Rubinstein and Wolinsky (1986) provides a coherent framework for studying credible threats. At each step in bargaining, one party may accept an offer or make a counter-offer. The choice involves balancing the possibility of improving the terms of the bargain by making a counter-offer and the cost of the delay resulting from making the counter-offer rather than accepting the pending offer.

In the simplest model of the wage bargain based on alternating offers and credible threats, the wage is

$$W_t = \frac{1}{2}\left(\frac{b_t}{r_t} - V_t + Z_t\right) \tag{34}$$

Here $b$ is the net joint bias in the job-seeker's favor resulting from delay—the sum of the job-seeker's benefit from spending time bargaining rather than working and the employer's cost of extending the bargaining process. During bargaining delay, the job-seeker avoids burdensome work and continues to draw unemployment benefits. The employer incurs costs of time devoted to the wage negotiations. The variable $r$ denotes the short-term

discount rate applicable to the bias. Notice that the capitalized value $b_t/r_t$ replaces $U_t$ in the earlier flexible-wage specification based on the Nash bargain with non-credible threat points. Although the wage is disconnected from current conditions in the labor market as measured by $U_t$, it remains sensitive to the value of the rest of the job-seeker's career, $V_t$. This value influences the wage bargain because one of the consequences of delay in making the bargain is to delay the receipt of $V_t$, which occurs at the moment the job-seeker makes the bargain and begins work.

The model with completely isolated bargaining has unrealistic implications for volatility. Hall and Milgrom (2005) present a version of the model in which bargaining is less isolated from conditions in the market because there is a hazard, $\mu$, that the productive opportunity will disappear during the bargaining process. The unique equilibrium in the bargaining game becomes:

$$W_t = \frac{1}{2}\left(\frac{b_t}{r_t + \mu} + Z_t + \frac{\mu}{r_t + \mu}U_t - V_t\right) \tag{35}$$

In the GE model incorporating credible threats, I pick the parameter $\mu$ to match the observed volatility of unemployment approximately. The value is 1.3, corresponding to a likelihood of 0.5 that the bargaining process would end after 2.3 weeks, should the job-seeker and employer fail to reach a bargain. In the equilibrium of the bargaining game, the parties make a bargain instantly, so the possibility of bargaining for a period followed by departure of the job-seeker to another employer never occurs.

In the stochastic growth model of this paper, the bargaining bias $b_t$ might reasonably be constant in relation to the capital stock or constant in relation to the marginal product of labor. I split the difference by making it a Cobb-Douglas combination of the two, with equal elasticities of 0.5. I pick the value of $b$ so that, in the non-stochastic steady state, the wage is the same in this model as in the base case. I also need to extend the model by adding the short-term interest rate $r_t$ as another endogenous variable. See Hall and Milgrom (2005) for more details on these points.

| Standard deviations | Actual | Base--sticky-wage model | Credible threats in wage bargaining |
|---|---|---|---|
| Consumption growth | 0.88 | 0.94 | 1.00 |
| Investment/capital ratio | 0.22 | 0.22 | 0.22 |
| Log of capacity/capital ratio | 4.43 | 4.59 | 4.83 |
| Unemployment | 1.57 | 1.59 | 1.54 |

Table 5. Volatilities, base model and credible-bargaining model

Table 5 shows the volatilities implied by the model with credible threats, with $\mu = 1.3$. The model matches the observed values about as closely as does the base model with sticky wages. Figures 7 and 8 show the responses of unemployment to productivity and spending impulses when equation (35) governs wages, with $\mu = 1.3$, in comparison to the response for the earlier sticky-wage model. The credible-bargaining model responds much less to productivity and much more to spending. The reason for the smaller response to productivity is easy to explain—the value of $\mu$ implies a tighter connection between labor-market conditions and the wage. In equation (35), the coefficient on $U$ is $\frac{1.3}{0.025+1.3} = 0.990$. With respect to fluctuations in productivity, the credible-bargaining model almost has the same amount of wage flexibility as the Nash-bargain model.

To explain the effect of higher exogenous spending, I examine the share of the surplus accruing to employers:

$$Z_t - W_t = \frac{1}{2}\left(Z_t - \frac{b_t}{r_t + \mu} - \frac{\mu}{r_t + \mu}U_t + V_t\right). \tag{36}$$

This quantity determines employer recruiting effort. Higher interest rates reduce $Z_t + V_t$ and thus reduce $Z_t - W_t$, reduce recruiting effort, and raise unemployment. But this effect is offset by similar movements in $U_t$ in the opposite direction and the coefficient on
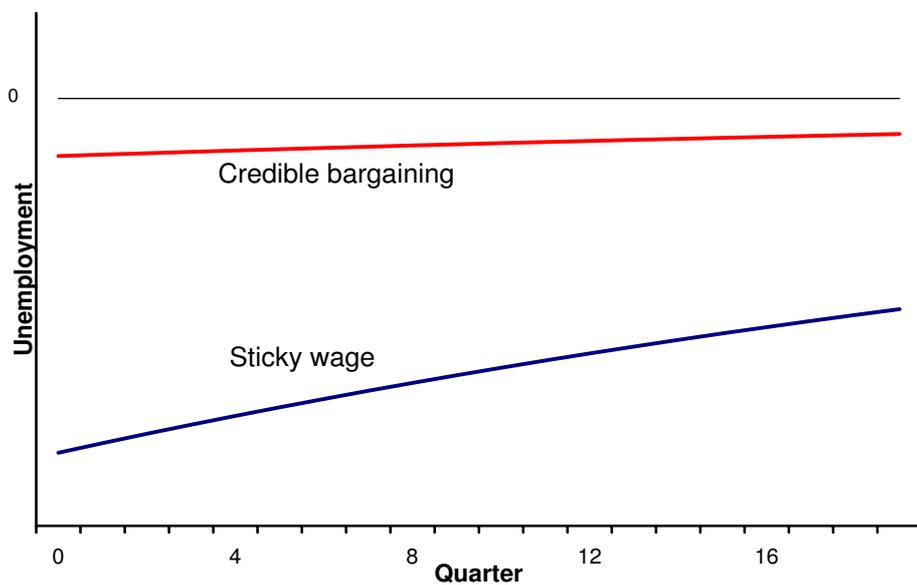
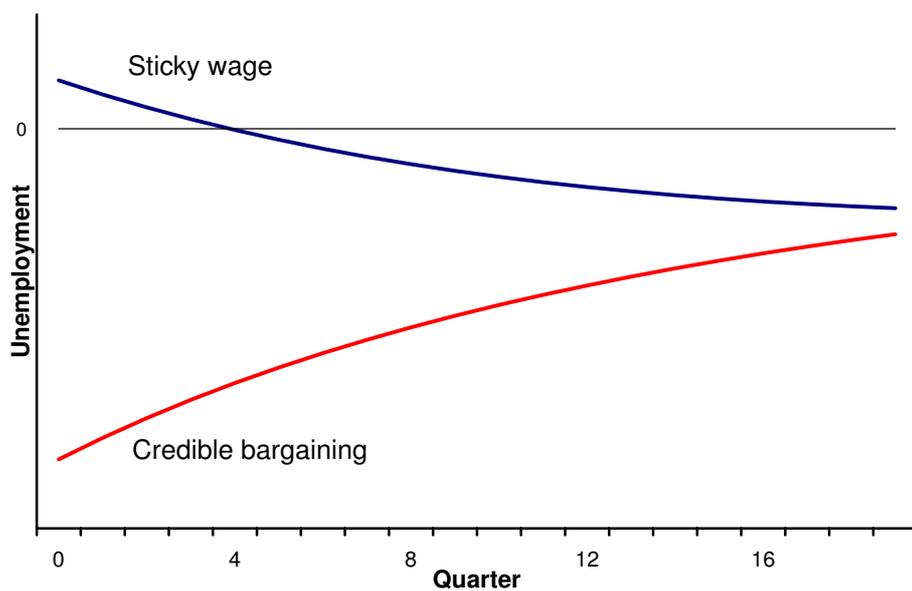Figure 7. Response of unemployment to productivity impulse, sticky-wage and credible bargaining models



Figure 8. Response of unemployment to spending impulse, sticky-wage and credible bargaining models

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$U_t$, as noted earlier, is close to one. The positive effect of the higher interest rate in the denominator of the coefficients multiplying $b_t$ and $U_t$ dominates. This effect is surprisingly powerful.

In the credible-bargaining framework, fluctuations in unemployment arise primarily from changes in exogenous spending and only secondarily from fluctuations in productivity. Traditional Keynesian macroeconomics had the same perspective. Although the credible-bargaining model has the traditional property, the mechanism is rather different. When high exogenous product demand drives up short-term rates temporarily, job-seekers are at a disadvantage compared to normal times because extending bargaining is more costly. The wage bargain shifts in favor of employers, who respond by recruiting more aggressively and lowering the unemployment rate.

# 7 Fluctuations in Matching Efficiency

In the standard model, an exogenous decline in matching efficiency, modeled as a lower value of the parameter $\omega$ in equation (31), triggers a response in the labor market that resembles a recession. Absent an explanation for such a decline in matching efficiency, however, this account of a recession has little interest. Hall (2005a) describes a model that generates fluctuations of this kind, based on changes in the extent of self-selection among job applicants. When self-selection is high, matching efficiency is also high because employers enjoy a high yield from evaluating applicants.

In the model, a job-seeker has private information about her probability of qualifying for a particular job opening. A job-seeker sets a cutoff probability $p^*$ and incurs an application cost $k_W$ to apply for a job where the perceived probability of acceptance meets the cutoff. The employer incurs an evaluation cost $k_E$ and evaluates all applicants as long as the expected payoff from a hire covers that cost. The parties split the surplus equally—the model embodies the standard Nash bargain based on non-credible threats to disclaim the potential relationship.

The expected probability that an applicant is qualified—given that applicants use their private information and do not apply unless they know the probability to be at least $p^*$—is a function $\chi(p^*)$. Job-seekers set $p^*$ according to

$$k_W = p^* \frac{1}{2} S, \tag{37}$$

where $S = Z - U + V$ is the surplus. Employers adjust their job postings to satisfy the zero-profit condition,

$$k_E = \chi(p^*) \frac{1}{2} S. \tag{38}$$

The ratio of the two defines a simple condition determining the cutoff:

$$\frac{\chi(p^*)}{p^*} = \frac{k_E}{k_W}. \tag{39}$$

Changes in the distribution of the information available to job-seekers and changes in the costs $k_W$ and $k_E$ cause shifts in the cutoff probability $p^*$. The ratio of the costs is unlikely to be an important driving force. But movements in $p^*$ resulting from changes in information may be a potent driving force. When job-seekers become better informed, $p^*$ rises dramatically.

Fluctuations arising from self-selection operate as follows: An event occurs, such as a reallocation of labor from a shrinking sector to growing sectors. In consequence, the typical job-seeker is less well informed about about the likelihood of qualifying for a given job opening. Job-seekers lower their cutoff probabilities. Employers perceive a lowering in the fraction of applicants who are qualified and reduce their recruiting efforts correspondingly. Job-seekers respond by further decreasing their application cutoffs. The positive feedback is less than complete and the labor market reaches a new equilibrium because the surplus rises as the market slackens—job-seekers' opportunity costs $U - V$ are lower with higher unemployment and the surplus is thus higher.

The stochastic GE framework of this paper cannot embed the full self-selection model. Instead, I treat changes in the cutoff probability as generating changes in the matching-efficiency parameter $\omega$. I double the number of discrete states indexed by $g$ to include two

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| Standard deviations | Actual | Base | Flexible wages and variations in matching efficiency |
|---|---|---|---|
| Consumption growth | 0.88 | 0.94 | 0.98 |
| Investment/capital ratio | 0.22 | 0.22 | 0.23 |
| Log of capacity/capital ratio | 4.43 | 4.59 | 4.87 |
| Unemployment | 1.57 | 1.59 | 1.63 |

Table 6. Volatilities with variation in matching efficiency

independent levels of $\omega$. In the low-information state, I subtract a constant from $\omega$ and in the high-information state, I add the same constant. In the results reported below, the constant is 1.7. Thus $\omega$ moves back and forth from 7.2 to 10.6 instead of holding at its constant value of 8.9, as in the other models in this paper. I choose the switching probability to match the observed persistence of unemployment.

Table 6 compares the volatility of an economy with variations in matching efficiency to the volatilities of the actual U.S. economy and the base version of the model from earlier in the paper. The model with flexible wages and variations in matching efficiency is about as successful as the base model in matching the volatilities of the key macro variables.

# 8    Concluding Remarks

The equilibrium sticky-wage model, when incorporated in a complete dynamic stochastic macro model, gives a reasonable account of the volatility of key macro variables, including particularly the unemployment rate. A similar model with flexible wages set by a Nash bargain falls far short of matching the actual volatility of unemployment.

Not only does the model with equilibrium wage stickiness match the standard deviation

of unemployment, but it is capable of explaining rare episodes when the unemployment rate exceeds 8 percent. As Figure 1 shows, such episodes occur about once every 40 years. They are the result of an unlucky succession of productivity disappointments that result in a capacity/capital ratio that is well below its normal level. Wages fail to adjust fully, and the economy finds itself far to the left in Figure 4, with little of the anticipated surplus from employment accruing to employers. They put correspondingly little effort into finding new workers, the job-finding rate is low, and unemployment is high. The situation gradually corrects itself, as the capacity/capital ratio moves back to normal.

An alternative model of wage determination based on credible threats and sequential offers yields wage stickiness and unemployment volatility similar to that found with the equilibrium sticky-wage specification. The alternative model supplies an answer to the primary objection to the equilibrium sticky-wage model, that it describes only an indeterminate equilibrium. The outcome of sequential bargaining is unique.

A third model capable of explaining observed unemployment volatility adopts the standard flexible-wage specification, but invokes shifts in matching efficiency. These shifts may be the result of changes in the amount of information job-seekers have about their likelihood of qualifying for jobs. Well-informed job-seekers self-select, thereby raising the yield from employers' recruiting efforts and raising matching efficiency.

The results in this paper suggest that modern ideas about friction in the labor market can explain the volatility of employment and unemployment. Extensions of this investigation would look at properties of the variables beyond their standard deviations—covariances across variables and over time. Further, a full view of labor-market volatility probably involves a mixture of the mechanisms explored here and others as well—see Hall (2005d) for a review of other research with this goal. Mechanisms involving nominal frictions may also prove important in a full explanation of the movements of unemployment. Further, I have considered only a narrow range of driving forces. Monetary and other shocks belong in a fuller model.

# References

Alexopoulos, Michelle, "Unemployment and the Business Cycle," *Journal of Monetary Economics*, 2004, *51* (2), 277–298.

Andolfatto, David, "Business Cycles and Labor-Market Search," *American Economic Review*, 1996, *86* (1), 112–132.

Binmore, Ken, Ariel Rubinstein, and Asher Wolinsky, "The Nash Bargaining Solution in Economic Modeling," *RAND Journal of Economics*, Summer 1986, *17* (2), pp. 176–188.

Hall, Robert E., "Intertemporal Substitution in Consumption," *Journal of Political Economy*, April 1988, *96* (2), 339–357.

_____ , "Measuring Factor Adjustment Costs," *Quarterly Journal of Economics*, August 2004, *119* (3), 899–927.

_____ , "The Amplification of Unemployment Fluctuations through Self-Selection," March 2005. NBER Working Paper 11186.

_____ , "Employment Efficiency and Sticky Wages: Evidence from Flows in the Labor Market," *Review of Economics and Statistics*, August 2005, *87* (3), 397–407.

_____ , "Employment Fluctuations with Equilibrium Wage Stickiness," *American Economic Review*, March 2005, *95* (1), 50–65.

_____ , "Job Loss, Job Finding, and Unemployment in the U.S. Economy over the Past Fifty Years," *NBER Macroeconomics Annual*, 2005. Forthcoming.

_____ and Paul R. Milgrom, "The Limited Influence of Unemployment on the Wage Bargain," April 2005. NBER Working Paper 11245.

Judd, Kenneth, *Numerical Methods in Economics*, Cambridge: MIT Press, 1998.

Lucas, Deborah, "Asset Pricing with Undiversifiable Risk and Short Sales Constraints: Deepening the Equity Premium Puzzle," *Journal of Monetary Economics*, 1994, *34* (3), 325–342.

Lucas, Robert E., "Asset Prices in an Exchange Economy," *Econometrica*, November 1978, *46* (6), 1429–1445.

Merz, Monika, "Search in the Labor Market and the Real Business Cycle," *Journal of Monetary Economics*, 1995, *36* (2), 269–300.

Mortensen, Dale T. and Christopher Pissarides, "Job Creation and Job Destruction in the Theory of Unemployment," *Review of Economic Studies*, July 1994, *61* (3), pp. 397–415.

Shimer, Robert, "The Cyclical Behavior of Equilibrium Unemployment and Vacancies," *American Economic Review*, 2005, *95* (1), 24–49.

Summers, Lawerence H., "Some Skeptical Observations on Real Business Cycle Theory," *Quaterly Review*, Fall 1986, *10*, 21–27. Federal Reserve Bank of Minneapolis.

# Appendix: Computing the Solution

I express the model as functional equations in the state variables, $z$ and $g$, and their future values, $z'$ and $g'$. Thus $c_g(z)$ is consumption when the current draw of the spending shock is $g$ and the output/capacity ratio is $z$. The spending shock takes on $M$ possible values. The future value of $z$ depends on the next draw of the productivity shock, $a'$:

$$z'_{g,a'} = a' x_g(z)^{-\gamma} z. \tag{40}$$

The future value of a variable such as consumption is $c_{g'}(z'_{g,a'})$. To avoid cluttered notation, I will write $z'$ for $z'_{g,a'}$ when the meaning is clear. The present value operator is

$$P_g(X) = \sum_{a'} \sum_{g'} \psi_{a'} \pi_{g,g'} m_{g,g'}(z, z') X_{g,g'}(z, z'). \tag{41}$$

The marginal rate of substitution is

$$m_{g,g'}(z, z') = \beta \left( x_g(z) \frac{c_{g'}(z')}{c_g(z)} \right)^{-\frac{1}{\sigma}}. \tag{42}$$

The return is

$$R_{g,g'}(z, z') = \frac{(1 - \delta) q_{g'}(z') + (1 - \gamma) z' (1 - u_{g'}(z'))^{\gamma}}{q_g(z)}. \tag{43}$$

The core functions in the model are $x_g(z)$, $v_g(z)$, $U_g(z)$, $V_g(z)$, and $Z_g(z)$. The conditions defining these functions are:

$$P_g \left( R_{g,g'}(z, z') \right) = 1 \tag{44}$$

$$U_g(z) = \lambda \gamma z (1 - u_g(z))^{\gamma - 1} + P_g(W_g(z') + V_{g'}(z')) x_g(z) \tag{45}$$

$$V_g(z) = P_g \left( s_{g'}(z') f_{g'}(z') (W_{g'}(z') + V_{g'}(z')) + s_{g'}(z')(1 - f_{g'}(z')) U_{g'}(z') + (1 - s_{g'}(z')) V_{g'}(z') \right) x_g(z) \tag{46}$$

$$Z_g(z) = \gamma z (1 - u_g(z))^{\gamma - 1} + P_g \left( (1 - s_{g'}(z')) Z_{g'}(z') \right) x_g(z) \tag{47}$$

$$v_g(z) = (Z_g(z) - W_g(z))\overline{s}f_g(z). \tag{48}$$

The other equations of the model are:

$$W_g(z) = \kappa W^* + (1 - \kappa)\frac{1}{2}(U_g(z) - V_g(z) + Z_g(z)) \tag{49}$$

$$c_g(z) = \frac{1 - \delta + z(1 - u_g(z))^\gamma - \frac{1-\tau}{2\tau}(x_g(z) - 1)^2 - v_g(z) - x_g(z)}{1 + g} \tag{50}$$

$$q_g(z) = 1 + \frac{1 - \tau}{\tau}(x_g(z) - 1) \tag{51}$$

$$f_g(z) = \phi(v_g(z)) \tag{52}$$

$$s_g(z) = \frac{\overline{s}}{1 - u_g(z)} \tag{53}$$

$$u_g(z) = \overline{s}(1 - f_g(z)). \tag{54}$$

The model has 11 endogenous variables, $x$, $U$, $V$, $Z$, $v$, $W$, $c$, $q$, $f$, $s$, and $u$. I associate each with an equation, starting with $x$ and equation (44) and ending with $u$ and equation (54).

I represent the core functions as Tchebysheff polynomials, as Judd recommends. Let $[a, b]$ include the support of $z$ and let $h(z) = (2z - a - b)/(b - a)$, a function that maps $[a, b]$ into $[-1, 1]$. The polynomials are

$$T_0(h(z)) = 1 \tag{55}$$

$$T_1(h(z)) = h(z) \tag{56}$$

$$T_{i+1}(h(z)) = 2h(z)T_i(h(z)) - T_{i-1}(h(z)) \tag{57}$$

I represent the functions $x_g(z)$, $v_g(z)$, $U_g(z)$, $V_g(z)$, and $Z_g(z)$ as polynomials of order $N$, such as

$$x_g(z) = \sum_{j=0}^{N} \theta_{x,g,j} T_j(h(z)). \tag{58}$$

For $L$ equally spaced values of $z$ in the interval $[a, b]$, with $L$ substantially larger than $N$, the sum of squares of the values of the vector of conditions, equations (44) through (48),
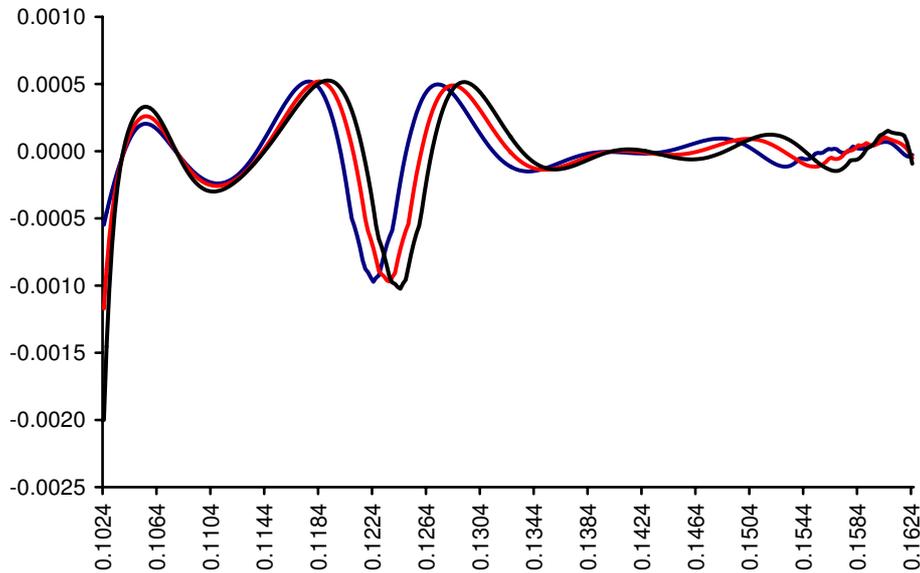
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Figure A. Solution errors

over the 5 equations, $L$ evaluation points, and $M$ discrete states, defines a norm of the departures of the current values of the $\theta$ coefficients from representing a solution to the model. I find the values of the $5MN$ $\theta$ coefficients that minimize the norm. I substitute out other functions such as $u_g(z)$, using the other equations of the model.

To verify the accuracy of the solution and the polynomial approximations, I calculate the values of the 5M-vector at 301 equally spaced points in the support of $z$. Figure A shows the values for the third equation, the one governing $U$.

To find the distribution of $z$, I use 301 equally spaced bins spanning $[a, b]$. This creates a discrete state variable with $301M$ values. I compute the complete transition matrix implied by the model from each of the values to each of the other values. I then solve for the stationary probabilities by matrix inversion. I compute the impulse response functions from the same transition matrix.