COMMENT ON HALL AND REIS

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1. Introduction

I thought it might be useful to test Hall and Reis' idea for using indexed payments on reserves in the context of a cash-in-advance model so that we can be explicit about the differences between forces for arbitrage in the asset markets and forces for arbitrage in the goods market as referenced on page 7 in their paper in discussing the determination of the price level under a gold standard or similarly, under a fixed exchange rate.

I focus on their third implementation of their proposal — the nominal payment on reserves process discussed in section 1.4.

To keep the model simple, I will stick with a constant velocity cash in advance model. I assume that the central bank pegs the nominal interest rate. I allow for the possibility of price level indeterminacy by including a sunspot variable in the initial period. After that, the economy is deterministic.

I prove that under the policy proposed by Hall and Reis for paying interest on reserves, the price level, the nominal interest rate, and the inflation rate are indeterminate. The logic of this result is as follows. Different expectations of the initial price level correspond to different nominal payoffs on reserves. Since the Hall-Reis policy is to peg the price of reserves, different expectations of the initial price level correspond to different nominal interest rates. These different expectations of the initial price level and nominal interest rates are validated in equilibrium by different expectations of the inflation rate.

I start this note with a description of the model. I next review the standard result on indeterminacy of the initial price level under a fixed nominal interest rate. I then present the result on the Hall-Reis proposal.

2. The Model

Consider a standard cash in advance model in which the endowment of goods is constant. Time is discrete with $t = 0, 1, 2, \ldots$.

The only uncertainty in the model is over an initial sunspot variable that we use to index alternative equilibria. Specifically, in time 0, let z be the realization of some public sunspot. Let this sunspot be a discrete random variable. Let $f(z_0)$ denote the probability that $z = z_0$ is realized at $t = 0$.

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Trade takes place in two locations: the asset market and the goods market. In the asset market, each agent manages her portfolio of financial assets, receives the proceeds from the sale of her endowment in the goods market in the form of currency being returned to the asset market, pays any lump sum taxes and or transfers using reserves held in the asset market, and equips herself with a new inventory of currency to take into the goods market to purchase consumption. In the goods market, the agent sells her endowment for currency and uses currency to purchase consumption from other agents.

Allocations: We assume that each agent's endowment is constant at Y. Hence, in equilibrium, total consumption is always equal to Y. An allocation of consumption is a sequence for each possible realization of z_0, $\{c_t(z_0)\}_{t=0}^{\infty}$. Agents have preferences over allocations given by

$$\sum_{z_0} \sum_{t=0}^{\infty} \beta^t \log(c_t(z_0)) f(z_0) \tag{1}$$

Feasibility requires

$$c_t(z_0) = Y \tag{2}$$

for all t and all z_0.

Cash in Advance: The agent uses currency to purchase consumption in the goods market. Let $m_t(z_0)$ be the amount of currency obtained by the agent in the asset market in period t. In the goods market, the agent faces cash flow constraints

$$m_t(z_0) \geq P_t(z_0) c_t(z_0) \tag{3}$$

for all t, z_0.

Standard Interest Rate Peg We first assume that the central bank pursues a policy of pegging the short term nominal interest rate paid on reserves to some constant $\bar{i} > 0$. It is also convenient to write the price of a security at t that pays off one dollar in currency in the asset market at $t+1$ as $\bar{q} = 1/(1+\bar{i}) < 1$.

Trade in the Asset Market: The agent starts with initial nominal wealth in the asset market W_0. We assume that this is equal to the outstanding amount of interest bearing reserves plus currency — the only financial assets in positive net supply. The government finances interest on reserves by imposing lump sum taxes and allows for growth in the stock of reserves through lump sum transfers at dates $t \geq 1$. The net tax is paid in reserves and is denoted T_t for $t \geq 1$. Agents trade in the asset market as follows. At $t = 0$ agents sell a portion of their nominal assets W_0 for currency, $m_0(z_0)$ and retain the remainder $A_0(z_0) = W_0 - m_0(z_0)$ invested in interest bearing reserves.

At the start of period $t = 1$, the agent has initial nominal wealth in the asset market equal to $W_1(z_0)$ which is equal to principal plus interest on reserves $(1+\bar{i})A_0(z_0)$ less taxes $T_1(z_0)$ plus currency obtained from the sale of the agent's endowment in the goods market in the previous period $P_0(z_0)Y$ plus any unspent currency brought home from the goods market in the previous period $m_0(z_0) - P_0(z_0)c_0(z_0)$.

This sequence of events is then repeated each period. This gives the following constraints on the evolution of the agents' portfolio of currency and reserves.

$$A_t(z_0) = W_t(z_0) - m_t(z_0) \tag{4}$$

for all t and z_0 with initial condition W_0 fixed.

$$W_{t+1}(z_0) = A_t(z_0)(1+\bar{i}) - T_{t+1}(z_0) + P_t(z_0)(Y - c_t(z_0)) + m_t(z_0) \tag{5}$$

Agents cannot sell reserves short, so we require

$$A_t(z_0) \geq 0 \tag{6}$$

for all t and z_0.

We impose a date $t = 0$ budget constraint on the agent in the asset market. Specifically, for each realization of z_0, we write the date $t = 0$ budget constraint of the agent restricting his purchase of nominal claims in the asset market as

$$W_0 - m_0(z_0) = \sum_{t=1}^{\infty} \bar{q}^t \left[m_t(z_0) + T_t(z_0) - P_{t-1}(z_0)(Y - c_{t-1}(z_0)) - m_{t-1}(z_0)\right] \tag{7}$$

This constraint is obtained in the standard way by repeated substitution out for $A_t(z_0)$ and $W_{t+1}(z_0)$ using 4 and 5 and requiring that

$$\lim_{t\to\infty} \bar{q}^t W_{t+1}(z_0) = 0$$

for each z_0.

Fiscal Policy: Lump sum taxes are set to finance the interest on reserves and net growth in the stock of reserves plus currency at a fixed rate of $\beta(1+\bar{i}) - 1$. At $t \geq 1$ this is

$$T_t(z_0) = \bar{i}A_{t-1}(z_0) + (\beta(1+\bar{i}) - 1)\, W_{t-1}(z_0) \tag{8}$$

Equilibrium Supply of Currency and Reserves Note that when goods markets clear, we can substitute the government budget constraints 8 into the equations describing the evolution of reserves and currency 4 and 5 to get that reserves plus currency evolve according to

$$W_{t+1}(z_0) = \beta(1+\bar{i})W_t(z_0) \tag{9}$$

This calculation establishes that in all equilibria (regardless of the specification of asset markets) the sum of currency and reserves at the end of asset market trading grows at a constant rate under these monetary and fiscal policies.

Definition of Equilibrium: An equilibrium is an allocation $\{c_t(z_0)\}_{t=0}^{\infty}$ that is feasible as in equation 2 for all t and z_0, a collection of sequences of price levels $\{P_t(z_0)\}_{t=0}^{\infty}$, a collection of sequences of currency and reserve holdings $\{m_t(z_0), A_t(z_0), W_{t+1}(z_0)\}_{t=0}^{\infty}$, and taxes $\{T_t(z_0)\}_{t=0}^{\infty}$, such that for all z_0:

(1) The cash flow constraints 3 are satisfied,

(2) The stocks of currency and reserves satisfy constraints 4, 5, and 6
(3) The agent's budget constraint in the asset market 7 is satisfied,
(4) The government budget constraints 8 are satisfied,
(5) The allocation and the stocks of currency and reserves maximize the agent's utility 1 subject to the cash flow constraints 3 and the budget constraints in the asset market 6 and 7.

3. Multiplicity of Equilibrium

We now establish that there are multiple equilibria under this interest rate rule. This is the standard one-dimensional indeterminacy of the initial price level under an nominal interest rate rule. Each equilibrium has the same nominal interest rate and inflation rate. These equilibria correspond to different conversions of reserves to currency. Because fiscal policy rebates any inflation tax revenue lump sum, the fiscal theory of the price level does not apply.

Proposition 1: Let $P_t(z_0)$ be a collection of price levels in the interval $[0, W_0/Y]$, one distinct number for each possible realization of z_0. Define subsequent price levels

$$P_{t+1}(z_0) = \left(\beta(1+\bar{i})\right)^t P_0(z_0)$$

Then the feasible allocation $c_t(z_0) = Y$ for all z_0 and $t \geq 0$, together with price levels as defined above, money holdings $m_t(z_0) = P_t(z_0)Y$, reserves $A_t(z_0), W_{t+1}(z_0)$ computed from 4 and 5, and taxes computed from 8 comprise an equilibrium.

Proof: Consider each item in the definition of equilibrium.

(1) The cash flow constraints 3 are satisfied by construction.
(2) The stocks of currency and reserves satisfy constraints 4 and 5 by construction.
(3) The agent's budget constraint in the asset market 7 is satisfied given that equations 4 and 5 are satisfied and $\beta < 1$,
(4) The government budget constraints 8 are satisfied, and finally
(5) It is straightforward to show that the allocation and the stocks of currency and reserves maximize the agent's utility 1 subject to the cash flow constraints 3 and the budget constraints in the asset market 7.

4. The Hall-Reis Proposal (version 3)

Consider now a different specification of monetary policy than the fixed interest rate rule, one that follows the Hall-Reis proposal. Under our old policy, reserves sold in the asset market at time t paid a fixed nominal rate of return $1/\bar{q} = (1+\bar{i})$ in the asset market at $t+1$ independent of z_0.

Indexing of Reserves: Now let the payoff to reserves at $t+1$ be indexed to the price level. Given the timing of this cash-in-advance model, the most recent price level available to be used for indexing is the price level at $t-1$ (since the price level at t has not been realized yet).

Let the payoff on reserves sold at $t \geq 0$ in the asset market at $t+1$ be given by

$$(1 + i_t(z_0))\frac{P_t(z_0)}{P_t^*}$$

where, for $t \geq 1$

$$P_t^* = \left(\frac{\beta}{\bar{q}}\right)^t P_0^*$$

and where $1 + i_t(z_0) = 1/q_t(z_0)$ is the nominal interest rate in the asset market in period t on discount bounds issued by the Treasury. This corresponds to equation (9) in the Hall and Reis paper.

A Central Bank: Now that we are going to have both Treasury Bonds and Interest Bearing Reserves, we need to be a bit more specific about the operations of the central bank as distinct from those of the Treasury. We assume that the Central Bank has no direct power to tax. It can only trade Treasury bonds for reserves. We assume that the central bank faces a portfolio constraint — it cannot issue Treasury bonds.

The Balance Sheets of Households and the Central Bank: We assume that the consolidated debt of the Treasury and the Central Bank at the start of period $t = 0$ is $\bar{B}_0$. This debt corresponds to the initial nominal assets of the households W_0 in equation 4. We will specify fiscal policy below such that this consolidated debt of the Treasury and Central Bank has a constant real value and hence grows in line with the nominal economy.

At each date t, the household chooses holdings of reserves, currency, and Treasury Debt according to

$$W_t(z_0) = m_t(z_0) + A_t(z_0) + q_t(z_0)B_{t+1}(z_0) \tag{10}$$

with the initial condition that $W_0(z_0) = W_0$. The nominal holdings of each agent then evolve according to

$$W_{t+1}(z_0) = \frac{1}{q_t(z_0)}\frac{P_t(z_0)}{P_t^*}A_t(z_0) + B_{t+1}(z_0) - T_{t+1}(z_0) + \tag{11}$$

$$P_t(z_0)(Y - c_t(z_0)) + m_t(z_0)$$

We impose that the household cannot short any government liability so we have bounds 6 and

$$B_{t+1}(z_0) \geq 0 \tag{12}$$

We assume that the Central Bank creates currency and reserves by purchasing Treasury Bonds. This implies that the outstanding debt of the Treasury at the end of asset trade in t is

$$q_t(z_0)B_{t+1}(z_0) = \bar{B}_t(z_0) - m_t(z_0) - A_t(z_0) \tag{13}$$

where $\bar{B}_t(z_0)$ is the consolidated debt of the Treasury and Central Bank at the start of period t and is equal to $\bar{B}_0$ at $t = 0$.

Fiscal Policy

The consolidated debt of the Treasury and the Central Bank at the start of period $t+1$ is

$$\bar{B}_{t+1}(z_0) = m_t(z_0) + \frac{1}{q_t(z_0)} \frac{P_t(z_0)}{P_t^*} A_t(z_0) + B_{t+1}(z_0) - T_{t+1}(z_0)$$

Assume that taxes are set so that this debt grows with the nominal economy at the price path intended by the central bank. Specifically, assume that

$$\bar{B}_t = \left(\frac{\beta}{\bar{q}}\right)^t \bar{B}_0 \tag{14}$$

independent of z_0.

Note that in any equilibrium, we then have

$$W_{t+1}(z_0) = \bar{B}_{t+1} \tag{15}$$

Equilibrium

Now consider the specification of equilibrium. The feasibility condition 2 and the cash flow constraints 3 are unchanged. The constraints 4 and 5 are replaced by 10 and 11. The bounds 6 are augmented with 12. We will not use the date 0 budget constraint 7 directly. Instead we will use the sequence constraints and bounds above. With the specification of fiscal policy 14 and the bond market clearing condition 15, we will have that a date zero budget constraint is well defined.

Definition of Equilibrium: An equilibrium is an allocation $\{c_t(z_0)\}_{t=0}^{\infty}$ that is feasible as in equation 2 for all t and z_0, a collection of sequences of price levels and discount prices for Tbills $\{P_t(z_0), q_t(z_0)\}_{t=0}^{\infty}$, a collection of sequences of currency, reserve holdings, Tbill holdings, and total nominal wealth $\{m_t(z_0), A_t(z_0), B_{t+1}(z_0), W_{t+1}(z_0)\}_{t=0}^{\infty}$, and taxes $\{T_t(z_0)\}_{t=0}^{\infty}$, such that for all z_0:

(1) The cash flow constraints 3 are satisfied,
(2) The stocks of currency and reserves satisfy constraints 10 and 11
(3) Bounds 6 and 12 are satisfied
(4) The government budget constraints 14 are satisfied, so that 15 are also satisfied
(5) The allocation and the stocks of currency, reserves, Tbills, and total financial wealth maximize the agent's utility 1 subject to the cash flow constraints 3 and the budget constraints in the asset market 10 and 11 and bounds 6 and 12.

The question we seek to answer is whether it is possible to have an equilibrium under this set of monetary and fiscal policies such that $P_0(z_0)$ differs across z_0.

Necessary Conditions: Consider the first order conditions for optimality for the agent.

Let $\delta_t(z_0)$ be the Lagrange multipliers on the cash flow constraints 3. The first order conditions for optimality of $c_t(z_0)$ are

$$\delta_t(z_0) = \beta^t \frac{f(z_0)}{P_t(z_0)c_t(z_0)} \tag{16}$$

for all $t \geq 0$ and z_0.

Let $\eta_t(z_0)$ be the Lagrange multipliers on the constraints 10 and $\lambda_{t+1}(z_0)$ be the Lagrange multipliers on the constraints 11. The first order conditions for optimality of $m_t(z_0)$ are

$$\eta_t(z_0) = \delta_t(z_0) + \lambda_{t+1}(z_0) \tag{17}$$

Let $\mu_t(z_0)$ be the Lagrange multipliers on the constraints 6. The first order conditions for optimality of $A_t(z_0)$ are

$$\eta_t(z_0) = \mu_t(z_0) + \lambda_{t+1}(z_0)\frac{1}{q_t(z_0)}\frac{P_t(z_0)}{P_t^*} \tag{18}$$

Let $\nu_t(z_0)$ be the Lagrange multipliers on the constraints 12. The first order conditions for optimality of $B_{t+1}(z_0)$ are

$$\eta_t(z_0) = \nu_t(z_0) + \lambda_{t+1}(z_0)\frac{1}{q_t(z_0)} \tag{19}$$

The first order conditions with respect to $W_{t+1}(z_0)$ give

$$\eta_{t+1}(z_0) = \lambda_{t+1}(z_0) \tag{20}$$

The conditions 17 and 20 imply that

$$\eta_t(z_0) \geq \eta_{t+1}(z_0)$$

and, hence, from 16 and imposing 2, we have a lower bound in inflation corresponding to the zero lower bound on nominal interest rates

$$\frac{P_{t+1}(z_0)}{P_t(z_0)} \geq \beta \tag{21}$$

The first order conditions 18, 19, and 20 imply that

$$\eta_t(z_0) \geq \frac{1}{q_t(z_0)} \max\{\frac{P_t(z_0)}{P_t^*}, 1\}\eta_{t+1}(z_0)$$

with $A_t(z_0) = 0$ if

$$\frac{P_t(z_0)}{P_t^*} < 1$$

and $B_{t+1}(z_0) = 0$ if

$$\frac{P_t(z_0)}{P_t^*} > 1.$$

If this condition is a strict inequality, we have that both $A_t(z_0)$ and $B_{t+1}(z_0)$ are equal to zero. In this case, we have that $m_t(z_0) = \bar{B}_t$. That is, the consolidated liabilities of the Treasury and the Central Bank consist entirely of currency.

Using 16 and 20 together with the feasibility condition 2 we have the following difference equation that equilibrium prices must solve

$$\frac{P_{t+1}(z_0)}{P_t(z_0)} \geq \frac{\beta}{q_t(z_0)} \max\{\frac{P_t(z_0)}{P_t^*}, 1\} \tag{22}$$

This expression must hold as an equality if $P_{t+1}(z_0) < \bar{B}_{t+1}/Y$. In the event that this expression holds as a strict inequality, we must have $P_{t+1}(z_0) = \bar{B}_{t+1}/Y$ corresponding to the price level being determined by the equilibrium supply of currency $m_{t+1}(z_0) = \bar{B}_{t+1}$.

Note from expression 22 that expression 21 must be true whenever

$$\frac{1}{q_t(z_0)} \frac{P_t(z_0)}{P_t^*} > 1$$

This expression is the nominal interest rate on reserves and hence is a floor on the equilibrium interest rate.

Multiplicity of Equilibria We have multiple equilibria under this Hall-Reis Rule. We show this result by construction. We focus on constructing equilibria in which the equilibrium inflation rate is constant over time and across realizations of z_0. We construct those equilibria in the next proposition. It appears that there exist other equilibria in which the inflation rate evolves over time as well. We construct those equilibria following the next proposition.

Proposition: There are multiple equilibria with constant interest and inflation rates corresponding to any positive choice of P_0^*. These equilibria are indexed by the initial value of the price level $P_0(z_0) > 0$.

Proof: Set P_0^* and choose $P_0(z_0) \in (0, \bar{B}/Y]$. Set the rest of prices according to

$$P_{t+1}(z_0) = \left(\frac{\beta}{\bar{q}}\right)^t P_0(z_0)$$

so that $P_t(z_0)/P_t^*$ is constant. Set $c_t(z_0) = Y$ to satisfy 2. Set $W_{t+1}(z_0) = \bar{B}_{t+1}$ to satisfy 15.

Construct the rest of the equilibrium as follows.

If $P_0(z_0) < P_0^*$, set $q_t(z_0) = \bar{q}$ for all t. Expression 22 is satisfied as an equality by construction. This equilibrium will have no excess reserves held in equilibrium (set $A_t(z_0) = 0$). All nominal wealth will be held as money and Tbills. The money holdings are $m_t(z_0) = P_t(z_0)Y$. The Tbill holdings are the residual of the outstanding joint liabilities of the Treasury and the central bank

$$B_{t+1}(z_0) = \bar{B}_{t+1} - m_t(z_0).$$

Note that in this equilibrium, all nominal quantities grow at the same constant rate $\beta/\bar{q}$. The nominal interest rate is given by $(1 + i_t(z_0)) = 1/\bar{q}$.

If $P_0(z_0) > P_0^*$ again set

$$q_t(z_0) = \bar{q}\frac{P_t(z_0)}{P_0^*}$$

for all t. Equation 22 is satisfied by construction. This equilibrium will have no Tbills held in equilibrium ($B_{t+1}(z_0) = 0$). All nominal wealth will be held as money and reserves. The money holdings are $m_t(z_0) = P_t(z_0)Y$. The reserve holdings are the residual of the outstanding joint liabilities of the Treasury and the central bank. The nominal interest rate in equilibrium is the nominal interest rate paid on reserves $(1 + i_t) = 1/\bar{q}$. The price $q_t(z_0)$ is a shadow price for TBills at which the demand for such bills is zero. That is, primary auctions of Tbills fail at this price in equilibrium.

This completes the proof of the proposition.

In all of the equilibria that we constructed in this proposition, the inflation rate and the effective nominal interest rate (the maximum of the nominal rates of return on reserves and TBills) are the same. Note that we can also construct additional equilibria with different inflation rates and (effective) nominal interest rates.

For example, construct an equilibrium by choosing $P_0(z_0) < P_0^*$ and set $q_t(z_0) = q^* \in (\bar{q}, 1)$. Construct the remaining price levels using

$$P_t(z_0) = \left(\frac{\beta}{q^*}\right)^t P_0(z_0)$$

In these equilibria, $P_t(z_0)/P_t^* < 1$ for all t. Hence, expression 22 is satisfied by construction and $A_t(z_0) = 0$ for all t. Households hold their nominal wealth entirely in the forms of currency and TBills. Since

$$\frac{P_{t+1}(z_0)}{P_t(z_0)} < \frac{\beta}{\bar{q}}$$

we have that these portfolios are feasible and that equilibrium inflation and nominal interest rates are lower than was targeted by the monetary and fiscal policies.

Alternatively, construct an equilibrium by choosing $P_t(z_0) > P_0^*$ and set $q_t(z_0) = q^* < \bar{q}\frac{P_0^*}{P_t(z_0)}$. Construct the remaining price levels using the difference equation

$$P_t(z_0) = \frac{\beta}{q^*}\frac{P_t(z_0)}{P_t^*}P_t(z_0)$$

until the period T such that the value of $P_T(z_0)$ implied by this formula first exceeds $\bar{B}_T/Y$. Then set $P_{T+k}(z_0) = \bar{B}_{T+k}/Y$ for $k \geq 0$. In this equilibria, TBill holdings $B_{t+1}(z_0) = 0$. Money holdings are given by $m_t(z_0) = P_t(z_0)Y$. Note that $A_{T+k}(z_0) = 0$ for $k \geq 0$. This equilibrium has inflation and nominal interest rates that exceed the targets set by the Central Bank until such time as all of the liabilities of the Treasury and Central Bank are converted into currency. At that point, the price level evolves according to the standard quantity theory with constant velocity.