Monetary Policy Expectations at the Zero Lower Bound*

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Abstract

Obtaining monetary policy expectations from the yield curve is difficult near the zero lower bound (ZLB). Standard dynamic term structure models, which ignore the ZLB, can be misleading. Shadow-rate models are better suited for this purpose, because they account for the distributional asymmetry in projected short rates induced by the ZLB. Besides providing better interest rate fit and forecasts, our shadow-rate models deliver estimates of the future monetary policy liftoff from the ZLB that are closer to survey expectations. We also document significant improvements for inference about monetary policy expectations when macroeconomic factors are included in the term structure model.

Keywords: dynamic term structure models, shadow rates, liftoff, macro-finance

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> To support continued progress toward maximum employment and price stability, the Committee expects that a highly accommodative stance of monetary policy will remain appropriate for a considerable time after the asset purchase program ends and the economic recovery strengthens. In particular, the Committee decided to keep the target range for the federal funds rate at 0 to 1/4 percent and currently anticipates that this exceptionally low range for the federal funds rate will be appropriate at least as long as the unemployment rate remains above 6-1/2 percent, inflation between one and two years ahead is projected to be no more than a half percentage point above the Committee's 2 percent longer-run goal, and longer-term inflation expectations continue to be well anchored.
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> — FOMC Statement, December 12, 2012

1 Introduction

Divining future monetary policy actions has been of special interest during the Great Recession and its aftermath. Expectations of future monetary policy actions are commonly obtained from the term structure of interest rates, which captures financial market participants' views regarding the prospective path of the short-term interest rate—the policy instrument of central banks.[1] Gaussian affine dynamic term structure models (DTSMs) are the standard representations in finance that have been used to extract such expectations (e.g., Piazzesi, 2010). However, while these models have provided good empirical representations of yield curves in the past, they appear ill-suited to represent the near-zero interest rates that have prevailed in recent years in several countries. In particular, standard Gaussian DTSMs do not recognize that in the real world, with currency available as an alternative asset, interest rates are bounded below by zero because negative nominal interest rates would lead to riskless arbitrage opportunities.[2] The fact that Gaussian affine DTSMs ignored the zero lower bound (ZLB) previously had little practical effect since interest rates were well above zero. However, as nominal interest rates have fallen to near zero, the lack of an appropriate nonnegativity restriction in Gaussian affine models has become a conspicuous deficiency, and the usefulness of these models appears to have declined substantially.

In light of the theoretical shortcoming of standard Gaussian affine DTSMs near the ZLB, we employ the shadow-rate model proposed by Black (1995) in order to estimate monetary policy expectations.[3] This representation replaces the affine short-rate specification of standard

[1] We focus on expectations of *conventional* monetary policy not central bank asset purchases.

[2] The precise lower bound on interest rates depends on institutional factors and may be slightly negative in the presence of significant costs associated with storing and spending large amounts of currency.

[3] Alternatively, one could consider stochastic-volatility models with square-root processes or Gaussian quadratic models. However, the shadow-rate model has the advantage of matching the canonical Gaussian

DTSMs with an identical affine process for an unobserved shadow short rate. The short rate is determined by the maximum of zero and this shadow short rate. Only a few studies have used shadow-rate DTSMs that respect the ZLB, in large part because the associated nonlinearity makes it difficult to solve for bond prices. In particular, a key advantage of affine DTSMs—namely, analytical, affine bond pricing—is lost. Instead, numerical solution methods are required, so the calculation of model-implied interest rates is computationally very costly. Therefore, only a small set of risk factors are typically used, and full estimation of the model is the exception rather than the rule. However, by taking advantage of advances in computing power and the efficiency of modern Monte Carlo methods, we are able to estimate a variety of shadow-rate DTSMs, obtain accurate approximations of bond prices, and filter latent risk factors with manageable computational cost.

To uncover monetary policy expectations embedded in the yield curve at the ZLB, we estimate Gaussian affine and shadow-rate DTSMs with a variety of risk factors using U.S. data for the past three decades. Notably, our paper is the first to introduce macroeconomic factors into a shadow-rate model. There is now a sizable literature arguing that a joint macro-finance approach is a very productive research avenue for term structure modeling (e.g., Rudebusch, 2010). However, when the nominal term structure is constrained by the ZLB, the addition of macroeconomic variables to the DTSM information set is likely to be quite beneficial for inference and particularly revealing about the future evolution of the yield curve. Intuitively, the ZLB limits the information content of the yield curve as shorter-term interest rates are pinned at zero. The associated risk factors of a standard yields-only model are also constrained and not as informative. A shadow-rate model partially lifts this veil by allowing shadow interest rates to take on a broader range of movement—even into negative territory. Those negative shadow rates are of course unobservable; however, the correlation of macro variables with observed interest rates away from the ZLB can help identify the shadow yield curve when observed rates are pinned near zero. Thus, at the ZLB, it is likely to be especially useful to take into account macroeconomic data when trying to predict how long the policy rate will remain near zero—as the FOMC clearly stated in the epigraph above. In our analysis, we compare yields-only and macro-finance shadow-rate DTSMs and document the significant benefit of including macro factors.

We begin our analysis with an evaluation of affine and shadow-rate models during the recent period of near-zero interest rates in the United States. During this period, shadow-rate models fit the cross section of yields substantially better than affine models because the latent

DTSM when interest rates are away from the ZLB, so the vast literature and results that are based on the standard model remain intact.

shadow short rate provides an additional degree of model flexibility. In addition, we document that affine models frequently violate the ZLB and produce substantial estimated probabilities of negative future short rates. Shadow-rate models avoid such violations by construction, and they can capture the important distributional asymmetry of expected future short rates at the ZLB. Finally, affine models cannot capture the phenomenon of the short rate remaining near zero for many years at a time, which has been the case in the United States and Japan. Consequently, such models produce inaccurate short-rate forecasts at the ZLB. In contrast, shadow-rate models can accurately represent and forecast near-zero policy rates.

While the use of the estimated current shadow short rate as a measure of the stance of monetary policy has received some attention in academic and policy circles (Bullard, 2012; Krippner, 2013), we find that such estimates are highly model-dependent, in line with the findings of Christensen and Rudebusch (2013).[4] Instead, we find forecasts of future values of the shadow rate to be of much greater interest. Comparing forecasts of the shadow short rate to forecasts of the observable short rate can reveal how tightly the ZLB is binding. The wedge between the two forecast paths reflects the asymmetry induced by the ZLB on the distribution of future short rates. Based on estimates of this wedge, we show that the ZLB was a tighter constraint in 2012 than previously during the ZLB period.

Based on forecasts of future shadow rates we can also address a key policy issue: the anticipated timing for the liftoff of the short rate from the zero bound (as in the epigraph). A common approach to assess the liftoff expectations of financial market participants is to use the horizon at which *forward rates*—the risk-neutral expected future short rates—cross a given threshold, say 25 basis points, as an estimate of the future liftoff date. However, this practice is problematic because it ignores the asymmetry of the distribution of future short rates near the ZLB. One needs to consider the path of *shadow forward rates*, or equivalently the modal path of future short rates, to appropriately estimate the expected duration of the ZLB period. We estimate this duration based on the horizon at which shadow forward rates first exceed 25 basis points. This simple approach leads to a forecast of the policy liftoff which is approximately optimal under absolute error loss, without the need to derive the full distribution of the liftoff horizon. Our estimates closely accord with private-sector and survey forecasts of policy liftoff. The expected duration until liftoff from the ZLB also provides a one-dimensional summary of the stance of monetary policy—though it is a far from a perfect measure, in part because it ignores the anticipated pace of tightening after liftoff.

As noted above, some existing studies have used shadow-rate term structure models to analyze yield curves in the proximity or presence of the ZLB. Bomfim (2003) employs a two-

[4]In essence, such estimates heavily depend on the associated DTSM's fit at the short end of the yield curve.

factor model to estimate the probability of the future policy rate hitting the ZLB during the 2002–2003 period. Ueno et al. (2006) analyze Japanese interest rates over the period 2001–2006 using a one-factor model, for which Gorovoi and Linetsky (2004) have derived an analytical solution, and Ichiue and Ueno (2007) apply a two-factor model to the same data. Kim and Singleton (2012), henceforth KS, estimate two-factor models using Japanese yield data and demonstrate the good performance of shadow-rate models in comparison to alternative DTSM specifications that incorporate the ZLB. Christensen and Rudebusch (2013) estimate one-, two- and three-factor models on the Japanese data and document the sensitivity of shadow-rate estimates to the model specification. Two studies have considered the recent U.S. experience: Krippner (2012) calibrates a restricted two-factor model to obtain estimates of the shadow short rate, and Ichiue and Ueno (2013) compare term premium estimates obtained from approximate shadow-rate and affine Gaussian models. Our study goes beyond these papers in several ways, most notably by incorporating macroeconomic information, assessing the expected liftoff from the ZLB, and providing evidence on how tight the ZLB is binding during particular periods.

The paper is structured as follows. Section 2 lays out our econometric framework. In Section 3 we evaluate affine and shadow-rate models in terms of cross-sectional fit, violations of the ZLB, and short-rate forecasts. Section 4 discusses shadow short rates and the asymmetry in the distribution of future short rates due to the ZLB constraint. Section 5 discusses estimation of the future liftoff date and compares model-implied and alternative estimates of the duration of the ZLB period. Section 6 concludes.

2 Dynamic term structure models

In this section, we describe our Gaussian affine and shadow-rate DTSM specifications and the constraint played by the ZLB on interest rates. Both the affine and shadow-rate DTSMs we use impose an absence of arbitrage—in the sense that the cross section of bond yields is consistent with their time series behavior, after allowing for a risk adjustment. This assumption implies the existence of a risk-neutral probability measure ($\mathbb{Q}$) in addition to the real-world probability measure ($\mathbb{P}$). Asset prices are discounted expected future payoffs under $\mathbb{Q}$. As usual, our models assume a low-dimensional vector of state variables, X_t, contains all of the information at time t that is relevant for investors. We also outline our empirical implementation, which uses monthly U.S. data.

2.1 Gaussian affine models

The canonical affine Gaussian DTSM (in which the risk factors are homoskedastic) is based on three assumptions:

1. **Short rate:** The short-term interest rate—the one-month rate in our context—is affine in the N risk factors X_t, i.e.,
$$r_t = \delta_0 + \delta_1' X_t, \tag{1}$$
for scalar δ_0 and N-vector δ_1.

2. **Risk-neutral distribution:** Under $\mathbb{Q}$, the risk factors are assumed to follow a Gaussian vector autoregression (VAR),
$$X_t = \mu^{\mathbb{Q}} + \phi^{\mathbb{Q}} X_{t-1} + \Sigma \varepsilon_t^{\mathbb{Q}}, \tag{2}$$
where Σ is lower triangular and $\varepsilon_t^{\mathbb{Q}}$ is an $i.i.d.$ standard normal random vector under $\mathbb{Q}$.

3. **Physical or real-world distribution:** Under $\mathbb{P}$, X_t also follows a Gaussian VAR,
$$X_t = \mu + \phi X_{t-1} + \Sigma \varepsilon_t, \tag{3}$$
where ε_t is an $i.i.d.$ standard normal random vector under $\mathbb{P}$.[5]

Note that these assumptions imply the existence of an essentially-affine stochastic discount factor as in Duffee (2002). The price of a bond with a maturity of m periods is determined by

$$P_t^m = E_t^{\mathbb{Q}}\left[\exp\left(-\sum_{i=0}^{m-1} r_{t+i}\right)\right]. \tag{4}$$

In an affine model, this expectation can be found analytically, and it is exponentially affine in the risk factors. Model-implied yields therefore are affine functions of the factors, and the same holds for forward rates, expected short rates, and risk-neutral yields. Term premia are defined as the difference between model-implied yields and risk-neutral yields. The details are well-known, but for completeness, we summarize them in Appendix A.

Importantly, a Gaussian model implies that interest rates can turn negative with non-zero probability, and during times of near-zero interest rates, violations of the ZLB can be quite

[5] The assumption of stationary VAR dynamics over the entire sample might appear problematic, since the onset of the ZLB period could alter the dynamics of the risk factors. A more flexible specification involving structural break or regime-shifting dynamics is beyond the scope of this paper.

prevalent. First, both the model-implied risk-neutral short-rate expectations, i.e., forward rates, as well as real-world short-rate expectations can turn negative.[6] Second, the probabilities of negative future short rates can become substantial. In Section 3.2, we document recent violations of the ZLB by various estimated affine models.

2.2 Shadow-rate models

Following Black (1995), our shadow-rate DTSMs are specified exactly like standard affine models except the affine short-rate equation (1) is replaced by a shadow-rate specification:

$$r_t = \max(s_t, 0), \quad s_t = \delta_0 + \delta_1' X_t. \tag{5}$$

The shadow short rate, s_t, is modeled as affine Gaussian, and the short rate is equal to s_t or zero, whichever is larger, ensuring that the short rate cannot turn negative. The ZLB on nominal interest rates is typically motivated by the presence of physical currency.[7] Since the storage and use of large amounts of physical currency can incur significant transaction costs, the ZLB has been violated at times in the past when interest rates have dipped into negative territory but remained close to zero. We could account for this fact by specifying a slightly negative lower bound in equation (5). On the other hand, short-term interest rates in practice typically remain slightly above zero, which would be an argument in favor of a slightly positive *effective* lower bound. Neither of these alternative specifications are likely to materially affect our results, and we specify the lower bound to be exactly zero.

One possible interpretation of the shadow short rate is that this would be the short-term interest rate if the ZLB was absent. Since this counterfactual may be hard to imagine, we tend to favor a different interpretation: The shadow rate is a simple statistical device that enables us to incorporate the ZLB in a straightforward fashion into a DTSM, because it can reasonably be modeled as Gaussian. One of the key advantages of shadow-rate models is that, except for the alternative short-rate equation, they are the same as affine Gaussian models, a mainstay of term structure analysis. Therefore, a shadow-rate model retains many of the features and advantages of an affine Gaussian model, and away from the ZLB, it essentially matches the standard Gaussian DTSM exactly. Hence, the vast literature of past results with the canonical model are still applicable. Another major advantage of shadow-rate models is

[6] Throughout this paper, we refer to $\mathbb{Q}$-measure expectations of future short rates as forward rates. These differ from the actual forward rates, which can be contracted by simultaneously buying and selling bonds of different maturities, by a convexity term.

[7] Fisher (1896) pointed out that investors would rather store their wealth in currency than to lend it at a loss.

that in contrast to other tractable non-Gaussian models that respect the ZLB constraint, such as square-root diffusion (Cox-Ingersoll-Ross) models and quadratic models, the probability of a zero future short rate is non-zero. This becomes crucial when addressing the issue of the duration of near-zero policy rates and the time of future liftoff, as we do in this paper.

Of particular interest then are the conditional moments of the future shadow rate at horizon h:

$$\begin{aligned}
\tilde{\mu}_t^h &= E(s_{t+h}|X_t) = \delta_0 + \delta_1' E(X_{t+h}|X_t) = \delta_0 + \delta_1' \left[(1-\phi^h)E(X_t) + \phi^h X_t\right] \\
(\tilde{\sigma}^h)^2 &= Var(s_{t+h}|X_t) = \delta_1' Var(X_{t+h}|X_t)\delta_1 = \delta_1' \left[\sum_{i=0}^{h-1} \Psi_i \Sigma \Sigma' \Psi_i'\right] \delta_1,
\end{aligned}$$

where Ψ_i is the i-th coefficient matrix in the Wold representation for X_t (for a VAR(1), $\Psi_i = \phi^i$), and the conditional variance $(\tilde{\sigma}^h)^2$ is time-invariant because X_t is homoskedastic. Importantly, note that these moments are are identical to the conditional moments of the future short rate in the affine model, but in the shadow-rate model, the conditional mean of the future short rate is:

$$\begin{aligned}
E(r_{t+h}|X_t) &= P(s_{t+h} > 0)E(s_{t+h}|X_t, s_{t+h} > 0) \\
&= N(\tilde{\mu}_t^h/\tilde{\sigma}^h)\left[\tilde{\mu}_t^h + \tilde{\sigma}^h \frac{n(-\tilde{\mu}_t^h/\tilde{\sigma}^h)}{1 - N(-\tilde{\mu}_t^h/\tilde{\sigma}^h)}\right] \\
&= N(\tilde{\mu}_t^h/\tilde{\sigma}^h)\tilde{\mu}_t^h + \tilde{\sigma}^h n(-\tilde{\mu}_t^h/\tilde{\sigma}^h),
\end{aligned}$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution, and $n(\cdot)$ is its density function.[8] The second line follows from well-known results about truncated normal distributions. Using the formulas given above, one can easily calculate the path of expected future short rates for any given value of X_t. By using the $\mathbb{Q}$-measure parameters instead of μ and ϕ to calculate the conditional moments, the forward curve can also be obtained analytically—an attractive feature of shadow-rate models that we use below.

Since yields contain not only average $\mathbb{Q}$-measure expectations of future short rates but also convexity, there is no simple closed-form solution for yields and bond prices in a shadow-rate model. Hence the need arises for numerical solution methods.[9] We use Monte Carlo simulations to evaluate the expectation in equation (4), which is a flexible and reliable method.

[8]This formula is also given in the appendix of KS.

[9]Krippner (2012) has proposed expressing forward rates in a shadow-rate context as the sum of shadow forward rates and an option effect. This can lead to quasi-analytical solutions if the value of the option effect can be derived using an option pricing framework. Christensen and Rudebusch (2013) perform the necessary derivations for their affine Nelson-Siegel model and are able to apply this approach empirically.

Importantly, the computational cost of Monte Carlo simulation does not substantially increase with a higher number of risk factors. Alternative numerical methods, such as approximating the solution to the fundamental partial differential equation (as in KS), become prohibitively costly for more than two risk factors. Details about the implementation and evidence of the accuracy of our method are in Appendix B.

2.3 Risk factors

A key modeling choice is which risk factors to include in the DTSM. We estimate both "yields-only" models, where the risk factors in X_t reflect only information in the yield curve, and "macro-finance" models, where X_t also includes macroeconomic variables. To ensure identification, we use the canonical form of Joslin et al. (2011), henceforth JLS.

We estimate yields-only affine and shadow-rate models using both two-factor and three-factor specifications. We denote the affine models by YA(2) and YA(3) and the shadow-rate models by YZ(2) and YZ(3). The yield factors are taken as the first N principal components of the observed yields. They are assumed to be measured with error, so that the true yield factors are latent and need to be filtered.[10] Our affine yields-only models correspond to the RKF model specification in JLS.

Because the focus of this paper is on current and expected future monetary policy, macroeconomic variables are likely to be very informative to include in the modeling framework—especially during the ZLB period when shorter-term interest rates are constrained near zero. Accordingly, we estimate macro-finance DTSMs that include measures of inflation and economic activity in addition to the yield factors. We estimate affine and shadow-rate models with one or two yield factors in addition to the two macro factors, and denote these models by MA(1), MA(2), MZ(1), and MZ(2). We denote by $L = 1$ or 2 and $M = 2$ the number of yield and macro factors, respectively. The yield factors are the first L principal components of yields, again measured with error. Notably, the yield factors are latent and the macro factors are observable.[11] The affine macro-finance models correspond to the TS^f specification in JLS.

In our macro-finance models, the macroeconomic variables are spanned by the yield curve.[12] An alternative would have unspanned macro risks as in Joslin et al. (2012) and Wright (2011).

[10]In other words, all yields are measured with error, the yield factors are linear combinations of the true model-implied yields, and the weightings for these linear combinations correspond to the loadings of the first N principal components of the yield data.

[11]We do not allow for measurement errors on the macro factors, because in that case "the likelihood function largely gives up on fitting the observed macro factors in favor of more accurate pricing of bonds" (Joslin et al., 2013).

[12]A subtlety is that, for the shadow-rate models, the macro factors are spanned by the unobservable "shadow yields" and not by the model-implied yields.

In such models, current interest rates are independent of current macroeconomic information, and the short-term interest rate only depends on the yield factors. Here we maintain the assumption that macroeconomic conditions affect the short-term interest rate. This seems consistent with the view of the FOMC (as in the epigraph) that the short rate will be based on the unemployment and inflation rates and with much of the literature on macro-finance models (JLS).[13]

2.4 Data, filtering, and estimation

Our data consist of monthly observations of interest rates and macroeconomic variables from January 1985 to December 2012. For the short end of the yield curve, we use three-month and six-month T-bill rates.[14] The remaining rates are smoothed zero-coupon Treasury yields with maturities of one, two, three, five, seven, and ten years from Gürkaynak et al. (2007).[15] We measure economic activity by the unemployment gap, using the estimate of the natural rate of unemployment from the Congressional Budget Office. Inflation is measured by the year-over-year percent change in the consumer price index (CPI) for all items excluding food and energy, i.e., by core CPI inflation. We include the inflation and gap measures because these are closely linked to the target federal funds rate, the policy instrument of the Federal Reserve (Rudebusch, 2006, 2009). Figure 1 gives a view of our data. The top panel shows the yields with maturities of three months, two years, and ten years. The bottom panel shows the macroeconomic variables.

All $J = 8$ yields are measured with error. Denote the model-implied vector of yields by Y_t and the actual, observed yields by Y_t^o. The observation equation for yields is

$$Y_t^o = Y_t + e_t, \quad e_t \overset{iid}{\sim} N(0, \sigma_e^2 I_J). \tag{6}$$

This measurement error specification implies that the yield factors are latent. As mentioned above, the macroeconomic variables in the macro-finance models are assumed to be measured without error.

For the affine models, we have $Y_t = A + BX_t$, with J-vector A and $J \times N$-matrix B containing the usual affine loadings. In this case, the state-space system is linear, and the Kalman filter can be used for inferring the latent factors and calculating the likelihood. We can

[13] One consequence of this assumption is that the model implies a simple policy rule for the short rate, which during normal times can be used to assess the stance of monetary policy.

[14] T-bill rates are obtained from the Federal Reserve's H.15 release, see http://www.federalreserve.gov/releases/h15/data.htm.

[15] These yields are available at http://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html.

carry out fast and reliable maximum likelihood (ML) estimation using the following approach: First, we obtain accurate starting values by estimating versions of the models in which yield factors are priced perfectly. In that setting, estimates of μ and ϕ can be obtained using least squares due to the particular normalization used. The remaining parameters are easily obtained by maximizing the log-likelihood function for given VAR parameters. Second, we estimate the models with measurement error using the Kalman filter—the optimization quickly converges to the global maximum because of the good starting values. JLS make the point that this procedure delivers excellent starting values and works well for their yields-only model, and we find that this is also the case for the macro-finance models.

For the shadow-rate models, we have $Y_t = g(X_t)$, where the function $g(\cdot)$ is nonlinear and not known in closed form, but can be approximated using Monte Carlo simulation (see Appendix B). Since the observation equation becomes nonlinear, a different filtering method is needed. We use the Extended Kalman Filter (EKF) in this case, which is computationally efficient since it requires only the calculation of the Jacobian of $g(\cdot)$. We approximate the Jacobian numerically. For a given set of parameters, it takes less than one minute to run the EKF and obtain risk factors and yields for our chosen level of accuracy of the Monte Carlo simulations.

Estimation of the shadow-rate models using maximum likelihood and the EKF is computationally costly. We use a simple work-around which dramatically reduces computational cost and is satisfactory in practice: We obtain the parameters by estimating the affine models over a subsample during which the ZLB was largely irrelevant, specifically, the sample period ending in December 2007. We use these parameters for both the affine model and for the corresponding shadow-rate model. There are two assumptions underlying the validity of this approach. The first assumption is that the affine model and the shadow-rate model have close to identical implications on the estimation subsample, so that ML estimates of the model parameters are interchangeable between the two models. Appendix C provides evidence that this is a reasonable assumption by comparing affine and shadow-rate models over this subsample. The second assumption is that ending the sample before 2008 does not materially change the parameter estimates. We have performed the full estimation for the yields-only models $YZ(2)$ and $YZ(3)$—these have fewer parameters than the macro-finance models and hence are time-consuming but manageable to estimate. We found that the parameter estimates barely change and that the economic implications remain qualitatively and quantitatively the same.

3 Model evaluation

From a theoretical perspective, shadow-rate models have a fundamental advantage over affine models in that they impose the nonnegativity of nominal interest rates. But how relevant is this in practice? In this section, we evaluate the affine and shadow-rate models and describe problems when affine models are used during a period of near-zero interest rates.

3.1 Cross-sectional fit

The first dimension of model evaluation for any DTSM is the accuracy of its cross-sectional fit, because a low-dimensional factor model cannot fit all yields perfectly. Table 1 shows the root mean-squared pricing errors (RMSEs) across models for the whole cross section of yields and for each yield maturity separately. The top panel reports RMSEs for the whole sample, while the bottom panel reports the fit for the ZLB subsample, here taken to be the period from January 2008 to December 2012. [16]

The shadow-rate models fit the cross section of yields more accurately than their affine counterparts, and the improvements are larger for those models with worse fit. For example, the macro-finance model MA(1), which was the focus of JLS, has trouble fitting yields, with a total RMSE for the full sample of 60 basis points. The ZLB counterpart to this model, MZ(1), cuts the RMSE roughly in half. For affine models that fit the cross section well—notably the canonical three-factor yields-only model, YA(3)—the improvements from introducing a shadow-rate specification are more modest.

The increased accuracy of the shadow-rate models in the cross section is explained by the better fit of yields during the ZLB period from 2008 to 2012. The bottom panel of Table 1 shows that improvements in RMSEs are very substantial for this subsample, most dramatically for the MA(1) and MZ(1) models. In contrast, for the pre-2008 period the affine and shadow-rate models have essentially identical cross-sectional fit, as shown in Appendix C. Intuitively, shadow-rate models have an additional degree of freedom to fit yields, the shadow rate. This, however, only comes into play when yields are close to zero and the policy rate is likely to remain at the ZLB, because only then does the shadow rate differ from the short rate. During ZLB periods, the added flexibility of shadow-rate models can substantially improve cross-sectional fit.

[16] On December 16, 2008, the FOMC lowered the target for the federal funds rate to a range from 0 to 25 basis points, hence we choose December 2008 as the first month of the ZLB subsample.

3.2 Violations of the ZLB by affine models

To understand the relevance of the ZLB for term structure modeling in recent U.S. data, we now assess the extent to which affine models violate this constraint. Violations of the ZLB by a DTSM can occur when model-implied paths of future short rates drop below zero at some horizons. This can happen for either forward rates, i.e., risk-neutral ($\mathbb{Q}$-measure) expected future short rates, or for real-world ($\mathbb{P}$-measure) short-rate expectations. To systematically evaluate this type of violation, Table 2 shows the number of months for each affine model where forward rates or expected future short rates drop below zero. Also shown is the average length of horizon that the paths stay in negative territory. All of the affine models imply future short rates that violate the ZLB during the period of near-zero policy rates from 2008 to 2012. For yields-only models, the short-rate paths typically stay below zero for several months. For the macro-finance models, short-rate paths are often negative for over a year or more. The greater degree of ZLB violations by the macro-finance models is due to the high unemployment and subdued inflation during and after the Great Recession, which drag down expected policy rates.

For an affine model, even when the expectation for the short rate at some future point in time is positive, its implied probability distribution may put nonnegligible mass on negative outcomes. The time-t conditional probability, under the $\mathbb{P}$-measure, of negative future short rates at forecast horizon h is given by:

$$\alpha_{t,h} = P(r_{t+h} < 0|X_t) = N(-\tilde{\mu}_t^h/\tilde{\sigma}^h). \tag{7}$$

When the conditional mean is positive and large relative to $\tilde{\sigma}^h$, the probability α_t^h is close to zero. However, when interest rates are near zero and the forecast horizon is moderate, $\tilde{\mu}_t^h$ will also be near zero, and α_t^h can become quite large.

Figure 2 plots these conditional probabilities. The four panels in the figure show α_t^h for each affine model at horizons h = 6, 12, and 24 months. Note that during the extended period of monetary easing after the 2001 recession, the probability of negative future short rates was nonnegligible. For the more recent period of near-zero short rates from 2008 to 2012, all models imply that these probabilities are very high. The macro-finance affine models lead to larger probabilities over this period than the corresponding yields-only models. The high unemployment and low inflation toward the end of the sample produce very low estimated future short rates. Consequently, the implied α_t^h are close to one for this period.

3.3 Short-rate forecast accuracy at the ZLB

As we have shown, estimated affine models produce clear ZLB violations in the United States in recent years. We now turn to a forecast accuracy comparison of affine and shadow-rate models that also demonstrates the importance of explicitly incorporating the ZLB into a DTSM.

The federal funds rate and other short-term interest rates have recently been at the effective ZLB for several years. This extended period of near-zero short rates is difficult if not impossible for an affine DTSM to capture. The affine DTSM will project the short rate to steadily revert to its unconditional mean, and this smooth forecast trajectory means that the policy path will remain at zero for no longer than one period. Shadow-rate models, however, are able to represent a situation with short rates pegged near zero for some time.

To empirically assess the importance of this issue, we evaluate the accuracy of model-based short-rate forecasts during the ZLB sample. The forecast target is the three-month T-bill rate, the shortest interest rate in our data set, which has remained very close to zero during this subsample. For each month from December 2008 to June 2011, we calculate model-based forecasts of this rate for horizons of 3, 6, 9, 12, 15, and 18 months, and construct forecast errors as the difference between future observed values and model-based forecasts. Table 3 shows the RMSEs in percentage points for each horizon across models. The shadow-rate models predict the short rate more accurately than the affine models, and the differences in forecast accuracy are large. For the macro-finance models, the improvements are particularly striking. Comparing the MA(2) and MZ(2) models, the improvements of the shadow-rate model over the affine model are between 54% and 79%, depending on the forecast horizon. Clearly, the ability of shadow-rate models to forecast near-zero short rates is empirically important and leads to substantially more accurate short-rate forecasts than for affine models.

Taken together, our results on forecast accuracy, cross-sectional fit, and ZLB violations highlight the danger of seriously distorted inference when using affine DTSMs during a period of near-zero short-term interest rates. While an affine model with a sufficient number of risk factors would be able to accurately fit the cross section of interest rates, any type of economic inference is prone to be misleading. In particular, affine model estimates of near-term forward curves and policy paths, of risk-neutral rates and term premia estimates, as well as point and interval forecasts of future interest rates are likely to be seriously distorted when interest rates are near zero. Hence, it is advisable for researchers to instead use models that incorporate the ZLB constraint such as shadow-rate models.

Another interesting result that emerges from Table 3 is the clear benefit of incorporating macroeconomic information into DTSMs. Comparing the models MZ(2) and YZ(2), both of which have two yield factors and exhibit similar cross-sectional fit, we see that for horizons longer than six months the macro-finance model substantially outperforms the yields-only model. The inclusion of macroeconomic factors leads to improvements in forecast accuracy that range from 29% at the nine-month horizon to 73% at the 18-month horizon. More generally, both macro-finance shadow-rate models produce much more accurate forecasts than either of the yields-only models. This improvement illustrates that for forecasts and economic inference at the ZLB, it is particularly useful to augment the information set with macroeconomic factors. When the yield curve is constrained by the zero bound, it necessarily carries less information about the future path of monetary policy than during normal times. At these times, there are substantial benefits to including macroeconomic variables, which are unconstrained by a ZLB.

4 Shadow rates

This section first discusses the limited usefulness of estimated *current* shadow short rates. We then argue that the distribution of *future* shadow short rates, as reflected in shadow forward rates and shadow yields, and in particular the extent of its asymmetry, is more revealing about the impact of the ZLB on the term structure of interest rates.

4.1 Shadow short rates

Black (1995) described the shadow short rate as the short-term interest rate that would prevail in the absence of the option of holding physical currency. The obvious question arises what can we can learn from estimates of the shadow short rate at the ZLB. Figure 3 shows the time series of the shadow short rate implied by our four models, together with the three-month T-bill rate. When the short rate is well above zero, the various estimated shadow short rates generally match the observed short rate. During the recent ZLB period, however, the models disagree substantially about the value of the shadow short rate. The MA(1) model implies by far the most negative shadow short rate, on average around -3.6% over the period from December 2008 to December 2012. The MA(2) and YA(2) models lead to shadow short rates that are slightly negative, whereas the YA(3) model produces a shadow short rate that is mostly positive and very close to zero.

Evidently, the estimates of the shadow rate near the ZLB are highly model-dependent. Our estimates as well as those in other studies suggest that more flexible DTSMs, i.e., those with a larger number of risk factors and more flexibility to obtain a good cross-sectional fit, appear to produce shadow short rates that are closer to zero during ZLB periods.[17] On the other hand, models with poor cross-sectional fit, such as MA(1), appear to produce more negative shadow rates during ZLB periods.[18] While Ichiue and Ueno (2013) see a near-zero shadow rate during the ZLB period as an undesirable feature of shadow-rate model, this in fact appears to be a natural result for a sufficiently flexible model. Intuitively, good-fitting models do not need the additional degree of freedom that the shadow rate provides in fitting the cross section of yields, while worse-fitting models use the shadow rate to tweak the current yield curve closer to actual yields. Consequently, a flexible model will typically produce estimates of the shadow short rate that are fairly close to the actual short rate.

Some have interpreted the shadow short rate as a measure of the stance of monetary policy at the ZLB. Krippner (2012) and Ichiue and Ueno (2013) take take this interpretation and view the shadow short rate as one of the main outputs of their shadow-rate models. Bullard (2012) has taken Krippner's estimates of a very negative shadow rate in the United States as evidence of a very easy stance of monetary policy. However, our estimates and those in KS and Christensen and Rudebusch (2013) suggest that it is difficult to draw robust empirical conclusions from estimated current shadow short rates. Shadow-rate estimates are model-dependent, and in a ZLB situation, the short end of the term structure does not convey much information about the stance of monetary policy.

4.2 The distribution of future short rates

Although the current value of the shadow short rate is largely uninteresting, distribution of future shadow short rates—especially relative to the distribution of future short rates—reveals how strongly the ZLB constraint affects the term structure of interest rates. In this subsection, we examine this distribution and explain how the ZLB introduces an asymmetry into the distribution of future short rates.

To build intuition, we first consider a specific date and a specific future horizon, and graph the probability density of the future shadow short rate and of the future short rate under the $\mathbb{Q}$-measure for model MZ(2). The date is December 31, 2012, and the horizon is $h = 48$ months. Figure 4 shows the densities. For the future shadow rate, this density is

[17] Christensen and Rudebusch (2013) show that shadow-rate DTSMs estimated on Japanese bond yields differ substantially in their implications about the level of the current shadow short rate depending on the number of factors used. KS similarly find strong model-dependence of their shadow rate estimates.

[18] Another example is the model in Krippner (2012), who estimates a significantly negative shadow rate for the recent U.S. period that falls to around -8%, employing a highly restrictive two-factor model. Kripnner does not estimate his model but instead calibrates the parameters and the shadow rate at each point in time, which also might distort the shadow-rate estimates.

Gaussian, centered around $E^{\mathbb{Q}}(s_{t+h}|X_t)$. The density of the future short rate equals the dirac-delta at zero, indicated in the graph with a vertical line, and corresponds to the shadow rate density for positive values. As noted by KS, this density can have either one mode at zero, if $E^{\mathbb{Q}}(s_{t+h}|X_t) \leq 0$, or two modes, at zero and $E^{\mathbb{Q}}(s_{t+h}|X_t) > 0$. In the latter case, which is the one visualized in the figure, it is useful to define the larger mode of the two as the unique mode.

Due to the fact that the short rate cannot be negative, its distribution has a point mass at zero and is asymmetric. Therefore, its mode is smaller than its mean $E^{\mathbb{Q}}(r_{t+h}|X_t)$. The wedge between the two, which is determined by the probability of a negative future shadow rate, captures how important the ZLB is for forward rates at a certain horizon. During normal times, this probability is negligibly small, so that the mean and the mode will approximately coincide. The more relevant the ZLB becomes, the larger the asymmetry of the distribution of future short rates, and the larger the wedge will become. Importantly, the wedge and the relevance of the ZLB does not depend only on how close yields are to zero, but also on second moments.

In the subsequent analysis, we will continue to focus exclusively on risk-neutral distributions and its moments. The main advantage is that the parameters $\mu^{\mathbb{Q}}$ and $\phi^{\mathbb{Q}}$ are estimated very accurately, due to the large amount of cross-sectional information in the yield curve (Cochrane and Piazzesi, 2008; Kim and Orphanides, 2012). In contrast, inference about the VAR parameters μ and ϕ and about the real-world distribution of future short rates is fraught with severe statistical problems due to the high persistence of interest rates (Bauer et al., 2012; Duffee and Stanton, 2012). Importantly, our methodology and all the points we make below are equally applicable to the case when the analysis is carried over to $\mathbb{P}$-measure distributions and their moments. The reader should keep in mind that the term "expectations" will be understood to include a term premium component that adjusts for risk. In practice, forward rates over short and medium horizons are typically taken to closely correspond to real-world expectations, based on the reasonable assumption that risk premia are likely small at such maturities.[19]

[19] Another common practice is to derive risk-neutral probability densities from option prices and to assess variance, asymmetries, probabilities of large moves, and other aspects of these distributions—see for example the monthly Minneapolis Options Report produced by the Federal Reserve Bank of Minneapolis at http://www.minneapolisfed.org/banking/assetvalues/ (accessed 7/10/2013).

4.3 Shadow forward rates

To assess the relevance of the ZLB constraint for the term structure of interest rates at a certain point in time, it is particularly instructive to consider the moments discussed above across horizons. Forward rates correspond to $\mathbb{Q}$-expected future short rates.[20] We will speak of *shadow forward rates* to denote $\mathbb{Q}$-expected future shadow short rates. The *modal path* corresponds to the mode of the future short rate distribution across horizons, and at each horizon h, it equals the larger of zero and the shadow forward rate, i.e., $\max[0, E_t^{\mathbb{Q}} s_{t+h}]$.[21] This corresponds to the most likely value of the short-term interest rate at each future horizon.

Figure 5 displays these paths for two dates, June 30, 2011 and December 31, 2012, for models YZ(3) and MZ(2). (Here and in the following sections, we will focus on these two flexible models for parsimony.) For the earlier date, both models imply that there is a slight difference between forward rates and modal path at short horizons, but that this difference becomes small for horizons beyond two years. In contrast, 18 months later, the difference between the two curves has become very pronounced, and even for more distant horizons there is a substantial wedge between expectations of future short rate and future shadow short rates.

The figures also demonstrate the very limited amount of information of the shadow short rate at a given point in time. Its value is close to zero on both dates, which is clearly not representative of the whole curve or of the tightness of the ZLB constraint.

4.4 Shadow yields

Another useful comparison can be made between fitted yields and shadow yields. Shadow yields represent the yields that would prevail in the absence of the ZLB. They are approximately equal to average $\mathbb{Q}$-expected future shadow rates, just as actual yields reflect $\mathbb{Q}$-expected future short rates.[22] Therefore, the difference between shadow yields and fitted yields also captures how tight the ZLB constraint is at any given point in time.

Figure 6 shows actual yields together with the fitted yield curves and shadow yield curves implied by the shadow-rate models YZ(3) and MZ(2) on June 30, 2011, and on December

[20]The common definition of forward rates corresponds to a rate that can be contracted at t for a future investment by buying and selling bonds of different maturities, and such a rate will reflect not only $\mathbb{Q}$-expectations but also a convexity/Jensen-inequality term. We do not include this in our definition of forward rates, because we are interested exclusively in conditional means of future short rates.

[21]Notably, such a modal path can also be constructed from prices of derivatives such as Eurodollar options or interest rate caps.

[22]Shadow yields are calculated from shadow bond prices, where the short rate used for discounting is replaced by the shadow rate, i.e., in equation (4), r is replaced by s. Hence, calculation of shadow yields is carried out with the usual affine loadings.

31, 2012. The left panels of Figure 6 show that in June 2011, shadow yields were, for longer maturities, about 20 to 30 basis points below actual yields. Thus, there was a noticeable effect of the ZLB on the entire yield curve. Although forward rates on this date were only noticeably affected at short maturities, yields at all maturities are constrained to some extent by the ZLB, simply because long-term yields reflect the behavior of average forward rates up to the specific maturity. The right panels show that for the later date, the differences between fitted and shadow yields at long maturities is much larger, around 60 to 70 basis points. The ZLB clearly was constraining yields more tightly at the end of 2012, in line with what we saw for forward rates.

These results show the tightness of the ZLB constraint for individual forward rates and yields, at a given point in time. The natural next step is to assess how this evolved over the entire ZLB period, and we will do so in the next section. A related issue is that it is desirable to have a univariate summary statistic that measures how tightly the *entire yield curve* is restricted by the ZLB. One intuitive possibility would be to calculate the area between the forward curve and the modal path, or between fitted yields and shadow yields, since this area reflects in some way the size of the wedge due to the ZLB asymmetry. However, it would be difficult to interpret such a measure. Instead, the estimated time until monetary policy liftoff from the ZLB, which we discuss in the next section, can provide a meaningful and intuitive summary of the impact of the ZLB on the entire term structure.

5 Monetary policy liftoff

During the recent period of near-zero interest rates in the United States, the timing of the future liftoff of the policy short rate from zero has been a constant focus of market commentary, Wall Street analysis, and policy discussion. The FOMC has provided explicit forward guidance about the likely duration of the period of near-zero policy rates, at first based on calendar dates and later based on economic outcomes. In December 2012, it stated that it expected policy liftoff to occur only after the unemployment rate had fallen to at least 6.5% in a context of price stability (as quoted in the epigraph). Of course, the yield curve incorporates, among other things, the expectations of market participants about the duration of the period of near-zero policy rates. How can yield curve information be used to estimate the timing of the future policy liftoff? The academic literature does not provide satisfactory answers to this question, partly because this is a new situation in the United States. Here, we discuss how to appropriately estimate the time until future policy liftoff, compare alternative approaches to this problem, and argue in favor of our preferred approach which leads to approximately optimal forecasts.

5.1 The distribution of the liftoff horizon

The future horizon where monetary policy liftoff will occur is a random variable, determined by the stochastic process for the shadow rate. The distribution of this random variable is the natural starting point for the forecasting problem, because it provides everything needed to construct optimal point and interval forecasts. Theoretically, the liftoff horizon closely corresponds to the hitting time of a given threshold for a diffusion process. For univariate diffusion processes, the probability density for this hitting time can in some cases be derived analytically. In an application to the problem at hand, Linetsky (2004) uses analytical results for hitting times of an Ornstein-Uhlenbeck process to calculate the distribution of the liftoff horizon in a shadow-rate Vasicek model. These results are also used in Ueno et al. (2006) and Ichiue and Ueno (2012), who consider one-factor shadow-rate DTSMs.

In a multi-factor shadow-rate DTSM, the liftoff distribution has to be obtained using simulation.[23] Our approach to do so is the following: Starting from the current values of the risk factors at time t, we simulate 10,000 sample paths using the risk-neutral dynamics in equation (2). For each simulation, we determine the horizon of liftoff using a threshold of 25 basis points. Because of the erratic nature of the shadow rate sample paths, we require that it stays above the threshold for 12 months for a crossing of the threshold to be denoted as policy liftoff.[24]

Figure 7 shows smoothed Kernel densities of the empirical liftoff distribution on December 31, 2012, based on simulations from models YZ(3) and MZ(2). The distribution is strongly skewed to the right—even very large outliers are not uncommon—so the mean is not a very useful summary of the central tendency. The median (mode) based on these simulations is 29 (11) months for model YZ(3) and 33 (26) months for model MZ(2). Notably, the mode is substantially lower than the median on both dates for both models, which we shall find is a general pattern. The figure also reports the interquartile range, and alternative liftoff estimates, based on the forward curve and the modal path, which will be described below.

It should be noted that the median is the optimal forecast under absolute-error loss. This loss function appears reasonable in this context, given the fat right tail of the target distri-

[23] Ichiue and Ueno (2007) use Monte Carlo simulations to obtain the distribution of the hitting time in their two-factor shadow-rate model.

[24] Because the sample path is erratic, it sometimes briefly jumps above the threshold but then returns into substantially negative territory. Furthermore, in some cases, the shadow rate starts out at t above the threshold but then turns negative for an extended period. Our definition of liftoff leads to better-behaved liftoff distribution than simply taking the first horizon with $s_{t+h} > 25$bps as the liftoff.

bution, since outliers do not have an unduly large effect. The mean, being optimal under squared-error loss, is strongly affected by the large positive outliers, and therefore results in unreasonably large estimates. At the other extreme is the mode, which is an optimal forecast for the case of step loss (also called 0–1 loss). This loss function is unappealing for the problem at hand, since the magnitude of the error is not taken into account, and the skewness is effectively ignored. Notably, in the studies cited above, the mode of this distribution is taken as the forecast of policy liftoff, without any reference to forecast optimality.[25]

5.2 Estimating liftoff using the term structure

Instead of using simulations, one might want to simply base liftoff estimates on the current term structure of interest rates. This method is very common in practice among financial market participants and analysts. One way to do so is to use the horizon when forward rates or futures rates rise above a certain threshold near zero, typically 25 basis points.[26] Since the forward rate curve reflects expectations for the future short-term interest rate, it appears natural to take the horizon at which these expectations lift off from zero as an estimate of the duration of the ZLB period. But a more nuanced interpretation of the yield curve is required, since this ignores the asymmetry of the distribution of the future short-term interest rate induced by the ZLB, which has a point mass at zero. Therefore, forward rates do not reflect the most likely value of the policy rate at a future point in time, as we saw in Section 4.2. To see that estimates based on forward rates can be misleading, consider a mean-preserving spread in the future shadow rate distribution. This would raise forward rates and push the liftoff estimate earlier, although the modal path was unaffected and arguably the liftoff estimate should not have changed.

Intuitively, the most likely path of future policy rates, the modal path, should form the basis for liftoff estimates. Many professional forecasters and financial market participants appear to do exactly this: They first construct a path for the future policy rate that they consider most likely, and then they take the first increase in this path beyond the threshold as an estimate of policy liftoff. An alternative way to view this is that analysts simultaneously determine their liftoff estimate and their modal path simultaneously.[27] Since our aim is to

[25] Ichiue and Ueno (2012) argue in favor of the mode over the mean based on what they view as better comparability with survey forecasts. They do not discuss the median or mean of the hitting-time distribution.

[26] Ueno et al. (2006) take the horizon where Euroyen futures rates exceed a given threshold as an estimate of future policy liftoff by the Bank of Japan. There are also examples in financial market commentary that appear to base liftoff estimates on this approach, including “Fed Likely to Push Back on Market Expectations of Rate Increase,” Wall Street Journal from 6/13/2013.

[27] The answers to the Primary Dealer Survey are consistent with the view that respondents base their liftoff estimate on the modal path. Other examples of analysis in line with this approach include “Reading the Tea

derive sensible estimates of policy liftoff from the term structure with the goal of matching alternative estimates, it appears logical to use a similar approach. Beyond its intuitive appeal and consistency with industry practice, a key advantage of using the modal path to estimate liftoff is that it is a very simple calculation for which no simulation or approximation is necessary—expected shadow rates are affine functions of current risk factors.[28] We will show below that it appears to coincide closely with the optimal forecast under absolute loss, and that it works very well empirically in comparison to outside estimates.

Figure 5, which was introduced in Section 4.2, illustrates the alternative liftoff estimates based on the term structure. Horizontal lines at 25 basis points indicates the threshold for liftoff. As shadow forward rates and the modal path are always below forward rates, the liftoff implied by the modal short rate path is always later than the liftoff estimated from forward rates. In December 2012, on which we will focus here, the differences are particularly pronounced, due to the stronger asymmetry of the future short rate distribution.

For model YZ(3), the modal path crosses this threshold from below at a horizon of 33 months, corresponding to liftoff in September 2015. This forecast is close to the median of the liftoff distribution (29 months). The forward curve, in contrast, crosses much earlier, at a horizon of 14 months, which would put liftoff at February 2014, which is close to the mode of the liftoff distribution. For model MZ(2), the modal-path forecast is also 33 months, which for this model corresponds exactly to the median of the liftoff distribution. The estimate based on the forward curve is 21 months, which would imply policy liftoff in September 2014, again close to the mode of the liftoff distribution. We will see below that the pairwise correspondence between modal-path forecast and median on the one hand, and forward-curve forecast and mode on the other hand, appears to be a general pattern.

It is useful to put these estimates in perspective based on FOMC communication and survey forecasts. In December 2012, the FOMC announced that it expected the policy rate to stay near zero at least until the unemployment rate fell below 6.5%. Out of 19 FOMC participants, 13 expected policy liftoff to occur in 2015, according to the Summary of Economic Projections (SEP), which also indicated that the central tendency of the participants' unemployment rate forecast for 2015 was 6.0-6.6%. In the January 2013 Blue Chip Financial Forecasts survey, responses for which are collected near the end of December, 80% of the respondents expected the time of a fall of the unemployment rate below 6.5% to be in 2015 or later. And of the respondents to the Federal Reserve Bank of New York's Primary Dealer survey in January 2013, 84% expected liftoff to occur in 2015 or later. In light of these observations, it appears

Leaves of Rate Expectations," Goldman Sachs US Economic Analyst from 7/3/2013.

[28]One can also use option prices and derive the modal path from the estimated risk-neutral probability densities. This provides a model-free estimate of policy liftoff.

that a sensible liftoff forecast on December 31, 2012, would have to be for a horizon of at least two years, more likely about 2.5 years or longer. The forecasts based on the modal path and those based on the median of the liftoff distribution appear reasonable from this perspective. In contrast, liftoff forecasts based on the forward curve or the mode of the liftoff distribution are substantially too low.

5.3 Alternative liftoff estimates over the recent ZLB period

The shadow-rate models allow us to estimate the liftoff horizon for each date in the sample. Figure 8 compares the estimates based on forward curve and modal path to the mode, median, and interquartile range from the simulated liftoff distribution. The top panel shows the estimates for the yields-only model YZ(3), while the bottom panel displays them for the macro-finance model MZ(2). The period shown in the graphs is from January 2008 to December 2012.

To evaluate the model-based estimates, we compare them to two alternative calculations of future liftoff dates by the private sector. The first is from the Primary Dealer Survey of the Federal Reserve Bank of New York, which is publicly available going back to January 2011.[29] In particular, we report the median of the Primary Dealers' modal forecasts for the time of policy liftoff.[30] The second alternative source of liftoff estimates is from the projections of the future path of the federal funds rate by Macroeconomic Advisers, which are also modal forecasts, i.e., the most likely scenario for Fed policy in their view.

The modal-path estimate is closely in line with the median of the liftoff distribution. For model MZ(2), the two are essentially identical from 2010 onward. Evidently, the horizon where the modal path crosses the ZLB threshold is nearly optimal as a liftoff forecast in terms of absolute loss.For practical purposes, it is satisfactory to see that forecasting liftoff based on our suggested straightforward approach using the modal path is justified based on more thorough analysis of the forecasting problem.

While we do not present a formal proof, the close correspondence of the modal-path forecast and the median of the liftoff distribution is quite intuitive: Denote by h_{mp} the horizon where the modal path crosses the threshold. Due to the symmetry of the shadow-rate distribution, it is equally likely for the shadow-rate path to be above or below the threshold at $t + h_{mp}$. Therefore, the probability mass for the event of liftoff between t and $t + h_{mp}$ will be close

[29]See http://www.newyorkfed.org/markets/primarydealer_survey_questions.html (accessed February 8, 2013) for the questions and answers for each survey.

[30]In the survey, the respondents are asked to provide the "estimate for [the] most likely quarter and year of [the] first target rate increase."

to 0.5, as will the probability of liftoff after $t + h_{mp}$.[31] This would place the median of the liftoff distribution close to h_{mp}, so the modal-path forecast of liftoff would be approximately absolute-loss optimal.

Our preferred liftoff estimate is generally close to the outside estimates from the Primary Dealer survey and Macroeconomic Advisers, and therefore appear to accurately measure what is considered the most likely duration of the ZLB period. In particular, the models capture the substantial increase in the most likely ZLB duration between mid 2011 and late 2011. The FOMC announced on August 9, 2011, that it expected a near-zero policy rate until at least mid-2013, which caused the expected duration to jump up (see also Swanson and Williams, 2012). This information is reflected in the outside estimates as well as in our model-based estimates. At the end of our sample, in December 2012, the model-path estimates are 33 months for both models, implying liftoff in the third quarter of 2015. This exactly corresponds with the forecasts of both the Primary Dealer survey and Macroeconomic Advisers.

The mode of the liftoff distribution is typically similar to the estimate based on the forward-curve estimate, particularly for the model MZ(2) for the period after 2010. While previous papers analyzing policy liftoff have to our knowledge exclusively focused on the mode as the liftoff forecast, our results show clearly that this estimate delivers unsatisfactory forecasts. The liftoff estimates are too early compared to absolute-loss optimal forecasts and outside estimates. The difference is particularly pronounced late in the ZLB period.[32] The increasing downward bias reflects the greater constraining effect of the ZLB and the consequent greater asymmetry in the future short-rate distribution. In December 2012, estimates based on forward rates imply liftoff in 14 months for model YZ(3) model, and in 21 months for model MZ(2). That is, compared to the absolute-loss optimal forecasts and the outside estimates, liftoff is estimated to occur at least a year too early.

The uncertainty underlying liftoff forecasts is visualized in the graphs by shaded confidence bands based on the interquartile range of the liftoff distribution.[33] Naturally, uncertainty has increased with the increasing expected duration of the ZLB period, and it is quite substantial in December 2012. Judging by the quantiles of the distribution from model MZ(2) at that time, which were also reported in Figure 7, liftoff could plausibly occur between 22 and

[31] The liftoff event can be defined in different ways, but it appears desirable to largely precludes cases where liftoff occurs before $t + h_{mp}$ and the modal shadow rate at that point is below the threshold. The smaller the likelihood of this occurrence, the closer the probability of liftoff before h_{mp} will be to 0.5.

[32] The mode of the simulated distribution also has the undesirable property of being close to zero during some months when reasonable liftoff estimates are substantially positive, due to the extreme skew of the true distribution and the nature of the shadow-rate simulations.

[33] These confidence bands only reflect uncertainty based on shocks to the term structure and not parameter uncertainty. To incorporate such estimation uncertainty, one would need an extensive bootstrap analysis or, ideally, a fully Bayesian estimation framework.

54 months in the future. Another important point to note is that these quantiles sensibly capture the uncertainty around the median, but the mean or mode are often near or outside their boundaries. Even with more conservative quantiles, the strong skewness of the liftoff distribution would argue for reporting them together with the median as the appropriate point estimate.

Comparing model-based estimates of the time until policy liftoff using models YZ(3) and MZ(2) also shows the benefit of incorporating macroeconomic information. Figure 8 indicates that model MZ(2) is more accurate than model YZ(2) in the sense that the estimated durations are closer, on average, to the outside estimates. To make this concrete, we calculate the root-mean-squared difference of model-based vs. alternative estimates. For the Primary Dealer estimates, this difference is 4.6 months for model MZ(2), compared to 5.5 months for YZ(3). For the estimates from Macro Advisers, the distance is 4.2 months for model YZ(3), and 3.5 months for model MZ(2). Measured in this way, the macro-finance model MZ(2) estimates the duration of the ZLB period more accurately than the yields-only model YZ(3). Furthermore, the incorporation of macroeconomic information also lowers the uncertainty around the estimated liftoff horizon. Figure 8 shows the narrower confidence bands for model MZ(2) compared with model YZ(3). Evidently, using macroeconomic information in addition to yield curve information lowers the dispersion of the liftoff distribution, hence improving the precision of the liftoff estimates.

In sum, the modal path from a shadow-rate model provides an estimate of policy liftoff that is simple to calculate and performs well empirically. Simulating the distribution of the liftoff horizon provides an assessment of the uncertainty around this point forecast, and demonstrates its near-optimality under absolute-error-loss.

5.4 Further interpretation of the liftoff horizon

The estimated liftoff horizon can provide a simple measure of the constraining effects of the ZLB and even the stance of monetary policy. Regarding the former, the lengthening estimated liftoff horizon in Figure 8 indicates the ZLB is increasingly constraining the yield curve from 2009 to 2012. This finding is consistent with Swanson and Williams (2012), who measure the tightness of the ZLB using the sensitivity of different interest rates to macroeconomic news. They argue that the two-year yield became significantly constrained in 2011. Our liftoff estimates are in line with these results, reflecting the switch to more explicit forward guidance by the FOMC in fall 2011, which pushed out the expected duration of the period with near-zero policy rates. While the empirical approach of Swanson and Williams (2012) answers the question whether a particular interest rate was or was not constrained by the ZLB at a given point in time, our approach provides a more continuous summary statistic of the tightness of the ZLB constraint on the yield curve.

The estimated liftoff horizon also provides a summary of the stance of monetary policy. It captures one of the most important dimensions of monetary policy at the ZLB, namely, how long the policy rate can be expected to stay near zero. For example, based on the liftoff estimates in Figure 8, the stance of policy became increasingly accommodative from 2009 to 2012, while at the ZLB, the observed short-term interest rate does not convey any information about the policy stance. However, the expected duration of the ZLB period provides an incomplete measure of monetary policy because it will not necessarily capture shifts in the slope of the yield curve.[34] In particular, focusing on the time to liftoff ignores the anticipated pace of tightening after liftoff, which is another important dimension of monetary policy expectations at the ZLB. Still, as a practical matter, the anticipated pace of tightening following liftoff appears to have held fairly steady during the recent episode.[35] Another aspect of monetary policy that the liftoff horizon will not capture are the effects of bond purchases conducted by the Federal Reserve, which appear to have significantly lowered longer-term interest rates, as documented by Gagnon et al. (2011) and others. To the extent that these purchases resulted in lower interest rates at intermediate maturities without corresponding changes in the likely duration of the period of near-zero policy rates, our liftoff estimates would lengthen without any change in the expected policy path. However, in practice, these purchases often went hand in hand with corresponding changes in policy expectations and a longer expected liftoff horizon (Bauer and Rudebusch, 2011). In any case, our approach could be extended to take explicitly into account changes in risk premia due to Federal Reserve balance sheet policies.

6 Conclusion

Using U.S. data, we estimate Gaussian affine and shadow-rate DTSMs with a variety of risk factors and elucidate some important issues about U.S. monetary policy at the zero bound. We estimate mean and modal paths for future short rates, taking into account the asymmetric probability distribution of future short rates at a range of projection horizons, and assess the associated dates for monetary policy liftoff from the ZLB. We argue that forecasts of policy liftoff using the term structure should be based on the modal path of future short rates, which

[34] That is, the yield curve can become steeper or flatter—changing the importance of the ZLB and altering the stance of policy—without any change in the liftoff date.

[35] The median of the Primary Dealer survey's modal forecasts has fairly consistently indicated about a 2 percentage point increase in the fed funds rate over the two years following liftoff.

is a near-optimal forecast and performs well empirically. We find that the increasing model-implied expectations of liftoff from 2009 to 2012 are very closely matched by private-sector and survey forecasts. Furthermore, unlike the current shadow short rate, the expected duration of the ZLB period can provide a useful measure of the stance of monetary policy and the tightness of the ZLB. Finally, we document the benefits of including macroeconomic information in shadow-rate models, which improves inference at the ZLB about future monetary policy.

Our work can be extended in several promising directions. A natural application of our framework would be the recent period of near-zero interest rates in euro countries and the United Kingdom, as well as the long period of low interest rates in Japan over the past 20 years. Regarding our modeling framework, one could evaluate and impose restrictions on the risk pricing, which would tighten the link between the cross-sectional and time series dynamics of the risk factors (Joslin et al., 2012; Bauer, 2011). This holds some appeal because it would improve parsimony, overcome some of the statistical issues related to the highly persistent nature of interest rates, and lead to more precise inference about short-rate expectations and policy liftoff under the real-world $\mathbb{P}$-measure (also see Bauer et al., 2012). An alternative technique to pin down interest rate expectations would augment the information set to include survey forecasts of future interest rates, as in Kim and Orphanides (2012). Finally, the use of a Bayesian inference framework holds some promise for estimation of shadow-rate models: On a practical level, the use of modern Markov chain Monte Carlo methods could improve the computational efficiency of the estimation. More fundamentally, such a framework would allow for an accurate description of the uncertainty around current and future shadow rates, and around liftoff estimates, taking into account not only shocks to the dynamic system but also parameter uncertainty and possibly model uncertainty.

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A Affine bond pricing

Under the assumptions of Section 2.1, bond prices are exponentially affine functions of the pricing factors:

$$P_t^m = e^{\mathcal{A}_m + \mathcal{B}_m X_t},$$

and the loadings $\mathcal{A}_m = \mathcal{A}_m(\mu^{\mathbb{Q}}, \phi^{\mathbb{Q}}, \delta_0, \delta_1, \Sigma)$ and $\mathcal{B}_m = \mathcal{B}_m(\phi^{\mathbb{Q}}, \delta_1)$ follow the recursions

$$\begin{aligned}
\mathcal{A}_{m+1} &= \mathcal{A}_m + (\mu^{\mathbb{Q}})'\mathcal{B}_m + \frac{1}{2}\mathcal{B}_m'\Sigma\Sigma'\mathcal{B}_m - \delta_0 \\
\mathcal{B}_{m+1} &= (\phi^{\mathbb{Q}})'\mathcal{B}_m - \delta_1
\end{aligned}$$

with starting values $\mathcal{A}_0 = 0$ and $\mathcal{B}_0 = 0$. Model-implied yields are determined by $y_t^m = -m^{-1}\log P_t^m = A_m + B_m X_t$, with $A_m = -m^{-1}\mathcal{A}_m$ and $B_m = -m^{-1}\mathcal{B}_m$. Risk-neutral yields, the yields that would prevail if investors were risk-neutral, can be calculated using

$$\tilde{y}_t^m = \tilde{A}_m + \tilde{B}_m X_t, \quad \tilde{A}_m = -m^{-1}\mathcal{A}_m(\mu, \phi, \delta_0, \delta_1, \Sigma), \quad \tilde{B}_m = -m^{-1}\mathcal{B}_m(\phi, \delta_1).$$

Risk-neutral yields reflect policy expectations over the life of the bond, $m^{-1}\sum_{h=0}^{m-1} E_t r_{t+h}$, plus a convexity term. The yield term premium is defined as the difference between actual and risk-neutral yields, $ytp_t^m = y_t^m - \tilde{y}_t^m$.

B Monte Carlo bond pricing

To obtain bond prices and yields for the shadow-rate model, we resort to Monte Carlo simulation. For given values of the risk factors, the price of a bond with maturity m is

$$P_t^m = E_t^{\mathbb{Q}}\left[\exp\left(-\sum_{i=0}^{m-1} r_{t+i}\right)\right].$$

Since this expectation cannot be found analytically, we approximate it by simulating $M = 500$ paths of the risk factors of length m, where each sample path is obtained using the $\mathbb{Q}$-measure VAR in equation (2), starting from the given initial value X_t. Using the ZLB short-rate equation (5) we obtain the sampled paths for the short rate. Denote the value of the short rate in simulation j at time $t+i$ by $r_{t+i}^{(j)}$. The approximate bond price is given by

$$\tilde{P}_t^m = M^{-1}\sum_{j=1}^{M} \exp\left(-\sum_{i=0}^{m-1} r_{t+i}^{(j)}\right).$$

We use antithetic sampling to obtain the shock sequences, by taking the shock sequence for replication j as the negative of the shock sequence for replication $j-1$. This improves the accuracy of the approximation for any given M, because it introduces negative dependence between pairs of replications.

To assess the accuracy of our Monte Carlo simulation for bond prices, we compare Monte

Table B.1: Accuracy of Monte Carlo yields

Model	Total	3m	6m	1y	2y	3y	5y	7y	10y
YA(2)	0.14	0.00	0.00	0.01	0.04	0.08	0.06	0.11	0.35
YA(3)	0.85	0.00	0.00	0.01	0.04	0.20	0.72	1.17	1.96
MA(1)	0.47	0.00	0.00	0.01	0.04	0.11	0.42	0.75	1.01
MA(2)	0.38	0.00	0.00	0.01	0.03	0.07	0.48	0.78	0.56

Notes: Root mean-squared difference, in basis points, between analytical and Monte Carlo yields for affine models. Sample period: January 1985 to December 2012.

Carlo bond prices and analytical bond prices for the affine model. The Monte Carlo bond prices are obtained as described above, with the only difference being that we use the affine short-rate equation (1) instead of the ZLB short-rate equation (5). We use our estimated parameters, together with the risk factors filtered using the Kalman filter in the affine model, and consider the accuracy of Monte Carlo yields over the full sample from 1985 to 2012.

Table B.1 shows the approximation error, measured as the root mean-squared difference between exact affine yields and approximate Monte Carlo yields, in basis points. The approximation error is miniscule at short and medium maturities, and rarely exceeds one basis point. Evidently, the approximation of model-implied yields using our Monte Carlo method, even for a moderate amount of replications ($M = 500$), is very accurate.

C Affine vs. shadow-rate models over the 1985–2007 subsample

Here we provide evidence that the implications of the affine and shadow-rate models on the estimation subsample 1985 to 2007 are very similar. This is part of the justification for using in our shadow-rate models those parameters that are estimated for the affine models.

We use the parameters estimated for the affine model and calculate model-implied yields using both the affine and shadow-rate models over the pre-2008 sample period. For each model we first filter the latent risk factors appropriately from the observed yields, using the Kalman filter for the affine models, and the Extended Kalman filter for the shadow-rate models.

Table C.1 compares the model fit of the affine and shadow-rate models over the estimation subsample. In the first panel, it shows the RMSEs in basis points, to assess whether affine and shadow-rate models differ in terms of cross-sectional fit over this subsample. The RMSEs are very similar for each pair of affine and shadow-rate models, which demonstrates that the cross-sectional fit is essentially identical. The second panel of the table shows the discrepancy between yields implied by the affine and shadow-rate models, again measured as a root mean-squared difference. We see that the discrepancy is very small—model-implied yields from affine and shadow-rate models typically differ by less than one basis point.

These results show that the affine model and the shadow-rate model have close to identical implications on the estimation subsample. Therefore, it appears that the assumption

Table C.1: Comparison of affine and shadow-rate model from 1985 to 2007

Model	Total	3m	6m	1y	2y	3y	5y	7y	10y
Cross-sectional fit									
YA(2)	13.98	17.73	11.22	19.60	15.98	11.25	9.06	8.91	13.97
YZ(2)	13.96	17.77	11.27	19.59	16.03	11.23	8.85	8.88	13.86
YA(3)	6.07	6.36	9.25	9.29	2.89	4.44	4.17	2.76	5.46
YZ(3)	6.10	6.33	9.22	9.39	2.86	4.56	4.25	2.72	5.45
MA(1)	28.77	40.85	33.80	22.86	12.08	13.55	22.93	30.78	38.51
MZ(1)	28.75	40.85	33.71	22.90	12.06	13.56	22.95	30.76	38.52
MA(2)	9.56	13.74	10.26	10.31	9.98	8.45	4.15	5.15	10.80
MZ(2)	9.58	13.64	10.39	10.36	9.98	8.44	4.15	5.19	10.88
Discrepancy between affine and ZLB fitted yields									
YA(2)/YZ(2)	0.41	0.26	0.19	0.08	0.20	0.37	0.45	0.13	0.93
YA(3)/YZ(3)	0.28	0.14	0.13	0.25	0.24	0.37	0.42	0.24	0.31
MA(1)/MZ(1)	0.47	0.34	0.36	0.36	0.32	0.23	0.21	0.61	0.89
MA(2)/MZ(2)	0.58	0.58	0.32	0.12	0.57	0.70	0.34	0.34	1.10

Notes: Cross-sectional fit, measured by root mean-squared errors of model-implied yields, and discrepancy between affine and ZLB fitted yields, measured by root mean-squared differences, all measured in basis points. Sample period: January 1985 to December 2007.

is reasonable that maximum likelihood estimates of the model parameters are interchangeable between each affine model and its ZLB counterpart, if estimation is constrained to the pre-2008 subsample.

Table 1: Cross-sectional fit

Model	Total	3m	6m	1y	2y	3y	5y	7y	10y
Full sample									
YA(2)	14.0	17.1	10.8	18.1	16.2	13.1	9.0	8.8	15.4
YZ(2)	13.5	16.8	11.0	18.3	15.3	12.0	8.7	8.5	14.2
YA(3)	6.8	7.2	8.7	10.2	3.0	5.9	5.7	2.8	7.1
YZ(3)	6.7	7.0	8.8	10.1	3.4	6.0	6.0	2.9	6.6
MA(1)	60.2	78.6	71.8	54.9	19.0	20.0	52.2	68.3	80.3
MZ(1)	29.1	37.7	31.7	25.0	22.0	20.4	22.2	29.5	38.3
MA(2)	12.0	17.1	9.9	14.9	12.8	9.8	8.0	7.4	12.3
MZ(2)	10.3	13.7	10.6	12.4	10.7	9.2	6.3	5.6	11.2
ZLB subsample 2008-2012									
YA(2)	14.0	14.0	8.7	7.8	16.9	19.3	8.7	8.1	20.7
YZ(2)	11.4	10.9	10.0	10.1	11.6	14.9	8.1	6.5	15.7
YA(3)	9.3	10.3	5.8	13.6	3.3	10.2	10.1	3.0	11.9
YZ(3)	9.1	9.7	6.7	12.7	5.0	10.1	10.8	3.7	10.4
MA(1)	128.3	164.0	153.7	120.3	36.8	37.3	113.3	147.7	171.2
MZ(1)	30.7	16.5	20.3	33.0	45.2	38.4	18.1	22.5	37.3
MA(2)	19.5	27.6	8.0	27.4	21.4	14.5	16.7	13.7	17.8
MZ(2)	13.2	14.1	11.3	19.3	13.6	12.2	12.1	7.2	12.6

Notes: Root mean-squared errors of model-implied yields in basis points. Full sample: January 1985 to December 2012. ZLB subsample: January 2008 to December 2012.

Table 2: Violations of the ZLB

	Forward rates		Short-rate expectations	
Model	frequency	avg. length	frequency	avg. length
YA(2)	25	1.4	30	3.3
YA(3)	11	4.4	16	9.2
MA(1)	50	10.5	50	27.7
MA(2)	34	10.5	49	22.3

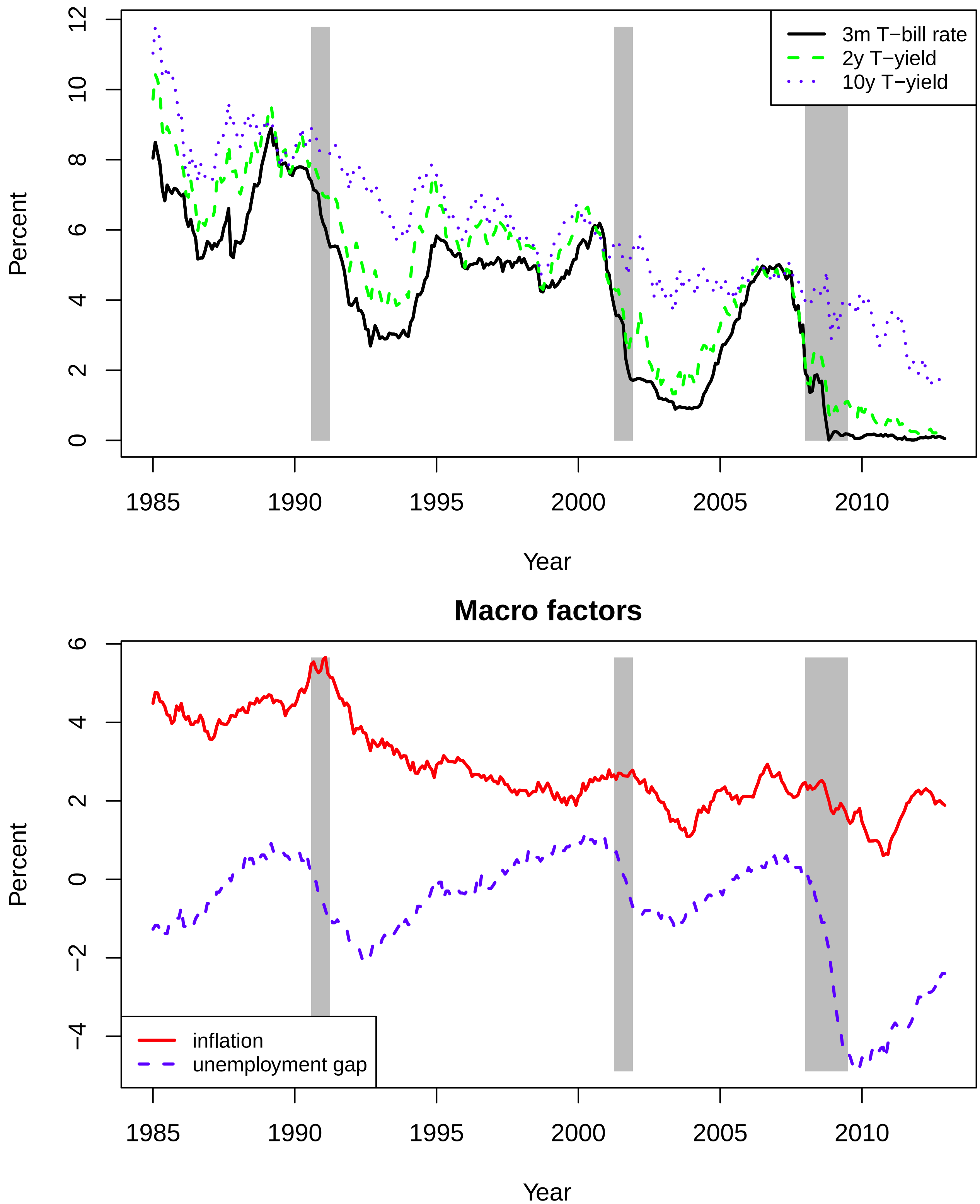
Notes: Number of months, between January 2008 and December 2012, in which some forward rates (column two) or short-rate expectations (column four) drop below zero, and average length of horizon with rates in negative territory.

Table 3: Forecast accuracy at the ZLB

Model	3m	6m	9m	12m	15m	18m
YA(2)	8.6	13.4	23.3	32.3	41.6	51.9
YZ(2)	10.2	11.9	15.0	17.8	22.9	30.7
YA(3)	11.0	14.9	16.9	22.3	33.6	49.3
YZ(3)	9.4	9.6	10.4	14.3	22.8	35.3
MA(1)	171.9	149.6	126.6	103.5	80.2	56.6
MZ(1)	13.1	12.0	11.0	10.9	11.0	10.1
MA(2)	25.2	44.4	51.4	49.6	41.4	29.2
MZ(2)	11.5	11.4	10.6	10.4	9.9	8.4

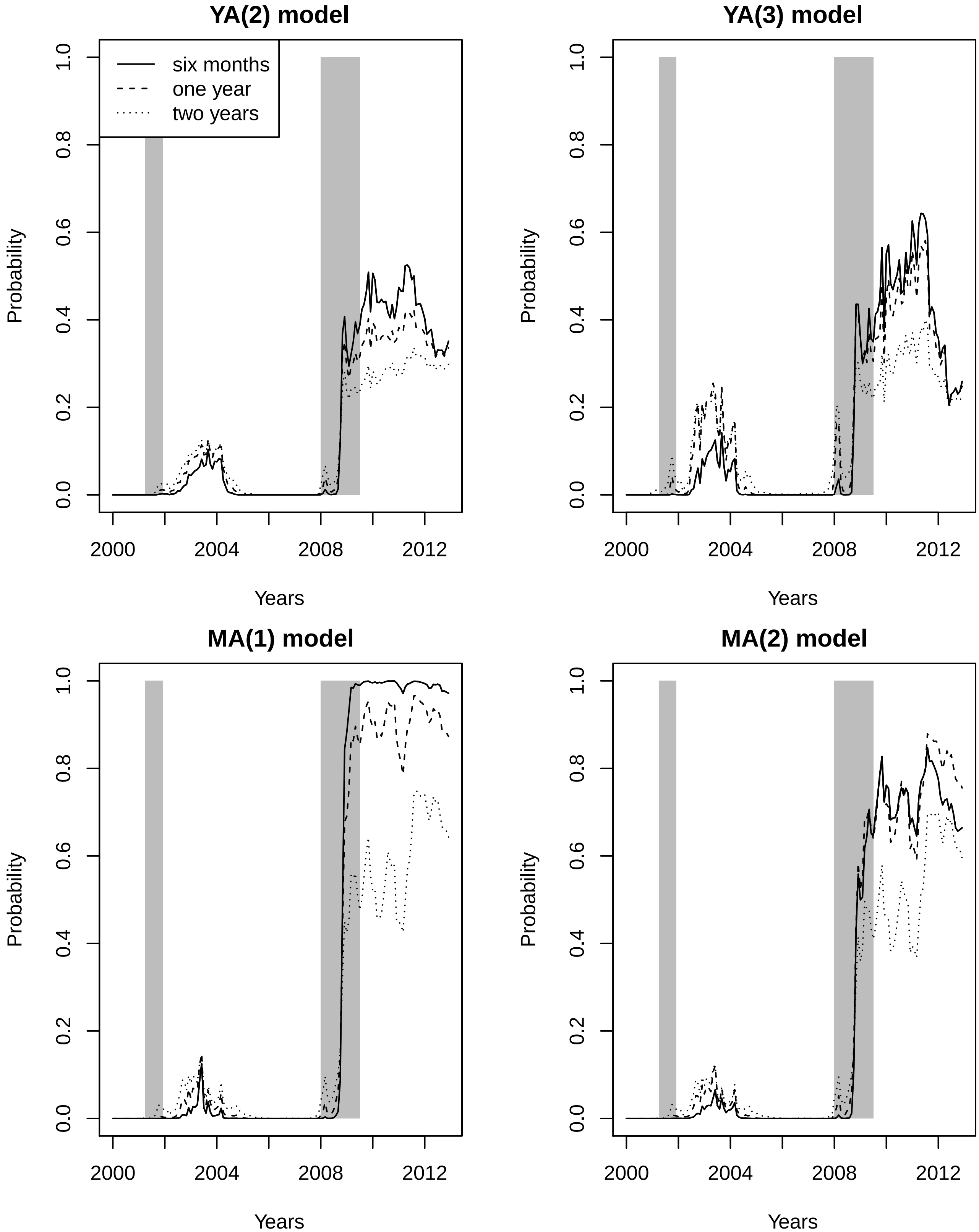
Notes: RMSEs in basis points for model-based forecasts (under $\mathbb{P}$-measure) of the future three-month T-bill rate at various forecast horizons. Forecast period: December 2008 to June 2011.

Figure 1: Yields and macroeconomic data



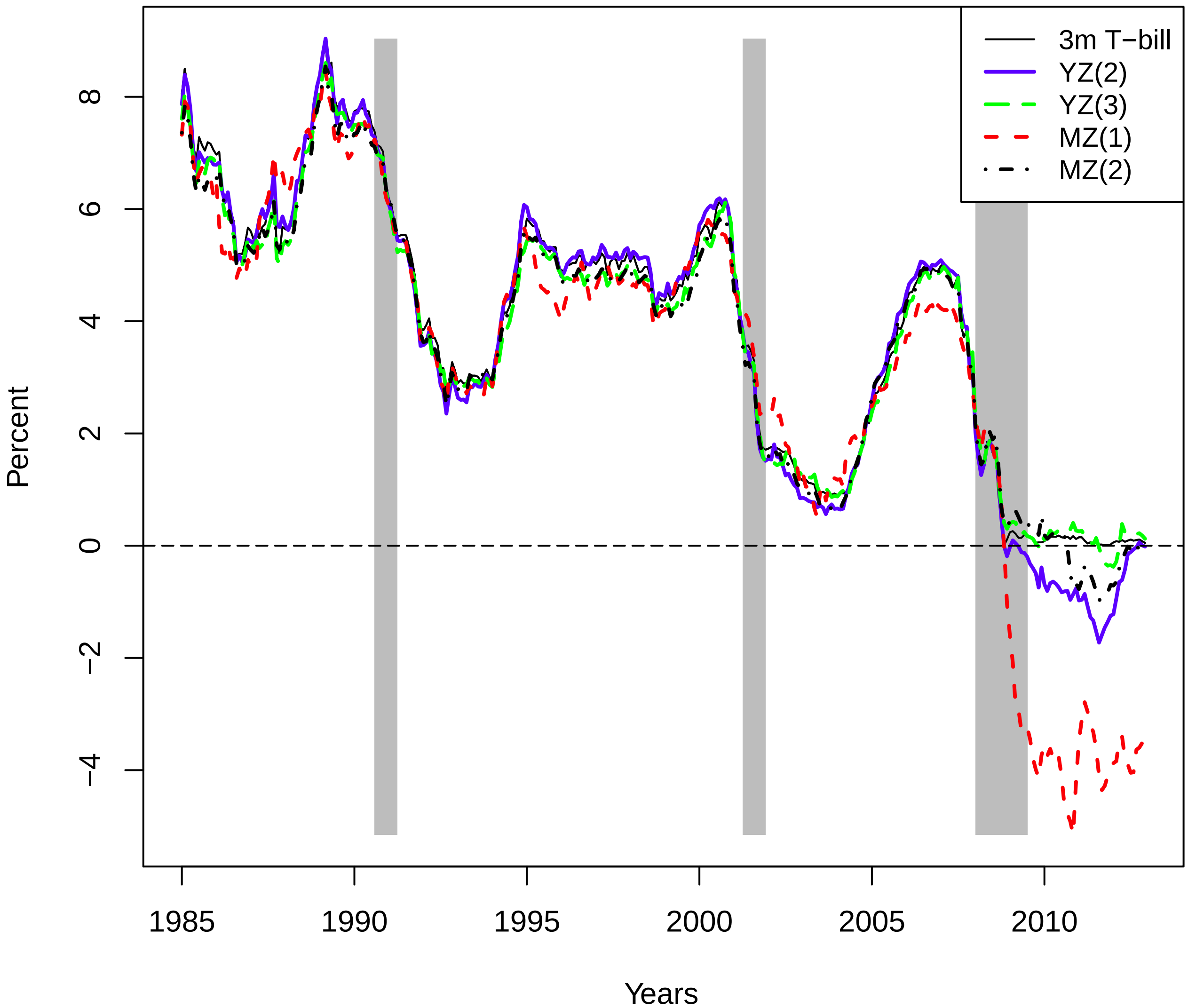
Notes: Top panel shows three selected yields, for the three-month, two-year, and ten-year maturities. Bottom panel shows the macroeconomic data. Sample period: January 1985 to December 2012

Figure 2: Affine model probabilities of negative future short rates



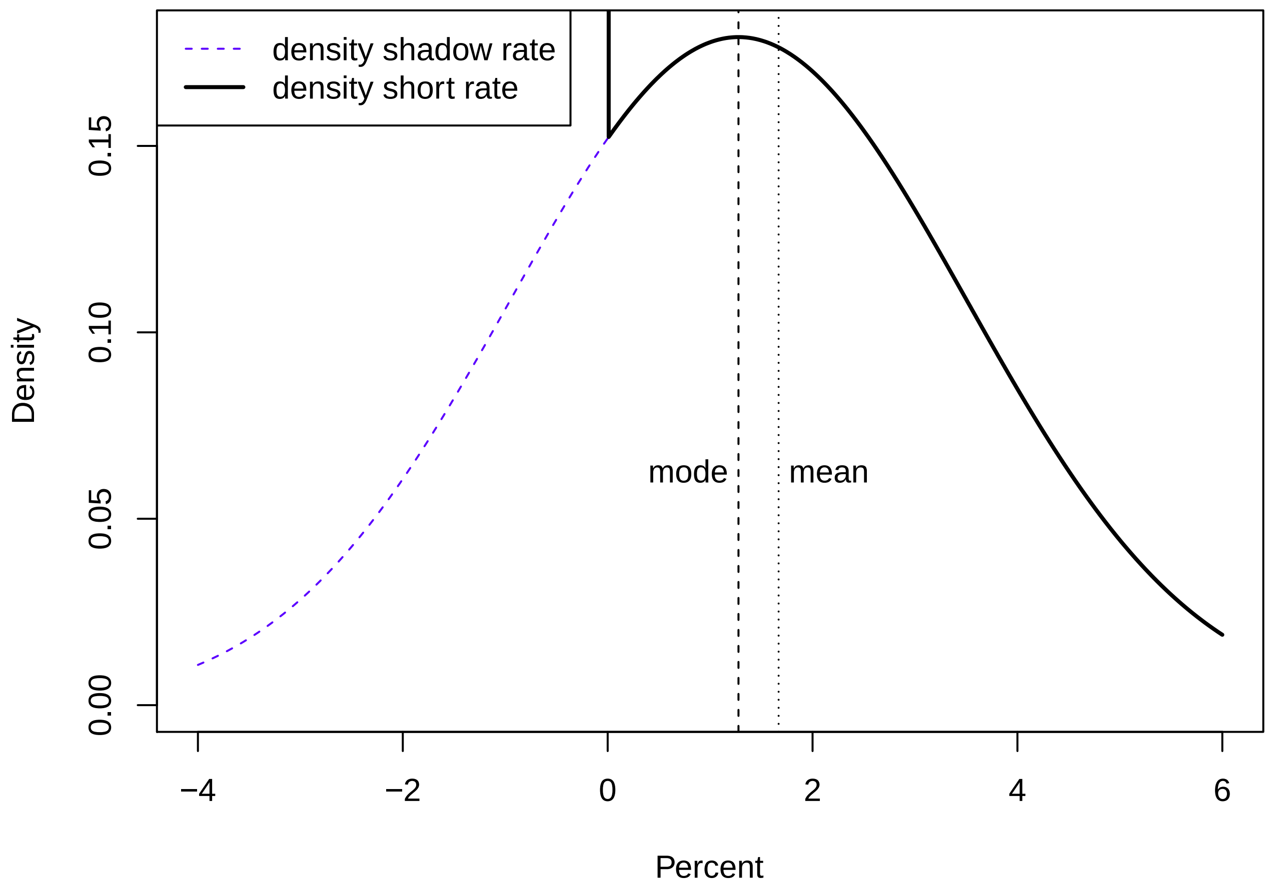
Notes: Model-implied real-world ($\mathbb{P}$) probabilities of negative future short-term interest rates at horizons of six months, one year, and two years. Shaded areas correspond to NBER recessions. Sample period: January 2000 to December 2012

Figure 3: Shadow short rates



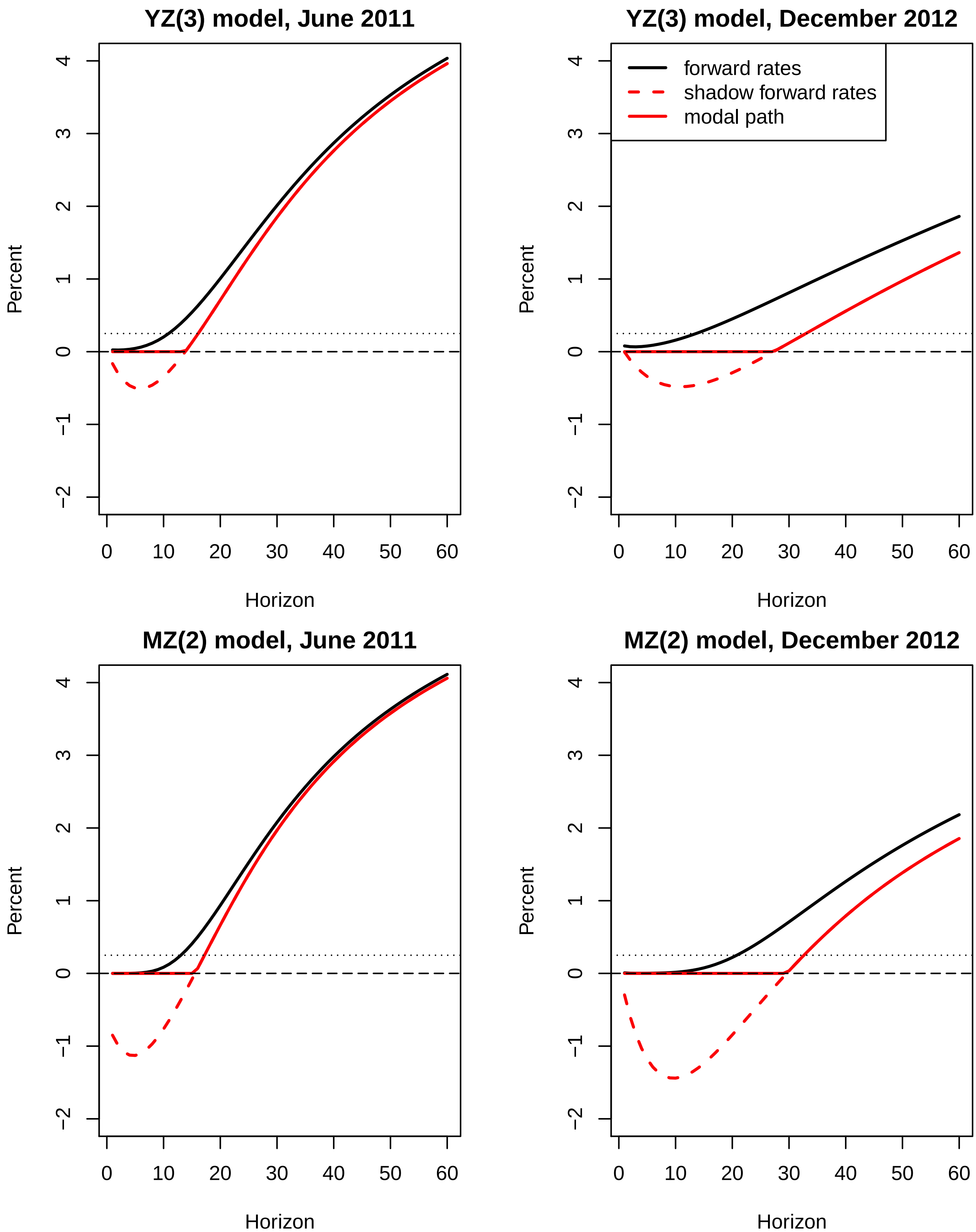
Notes: Shadow short rates implied by shadow-rate models and the three-month T-bill rate. Shaded areas correspond to NBER recessions. Sample period: January 1985 to December 2012.

Figure 4: Distribution of future short rate



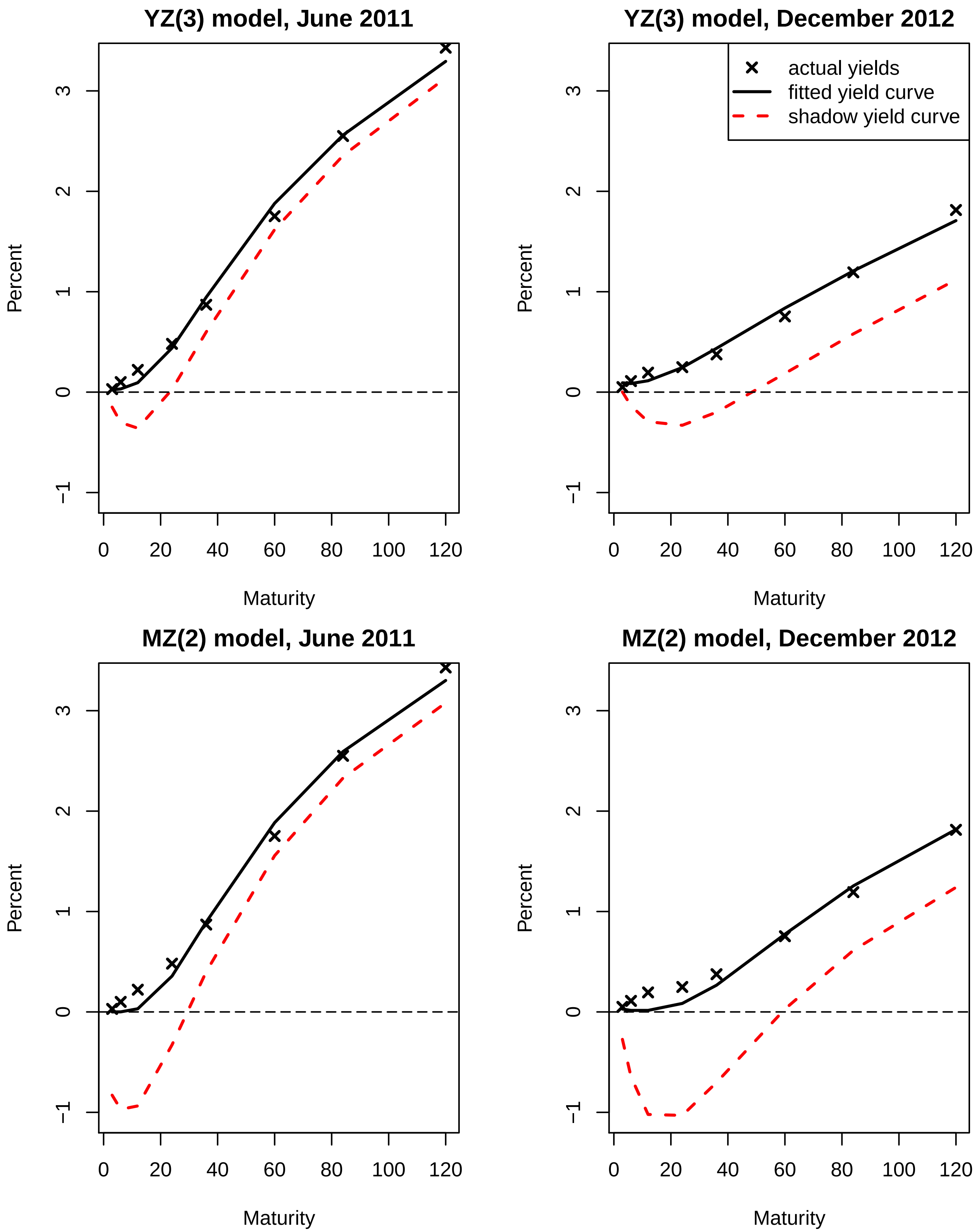
Notes: Densities of future shadow rate and of future short rate, based on model MZ(2), at horizon of 48 months, on December 31, 2012. Vertical lines show the mode and mean of the distribution of the future short rate.

Figure 5: Forward rates, shadow forward rates, and modal paths



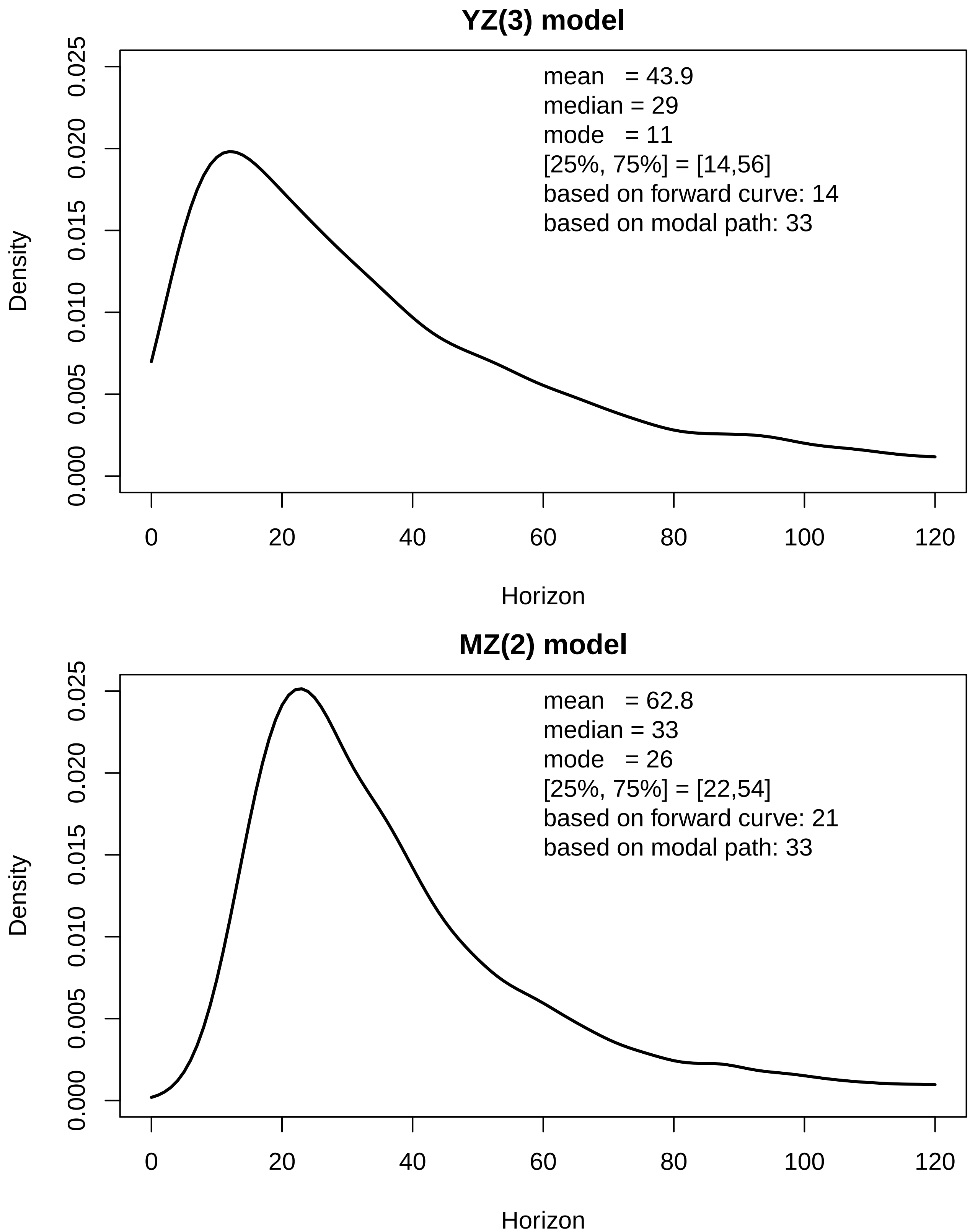
Notes: Forward rates, shadow forward rates, and modal path on June 30, 2011, and on December 31, 2012.

Figure 6: Fitted and shadow yield curves



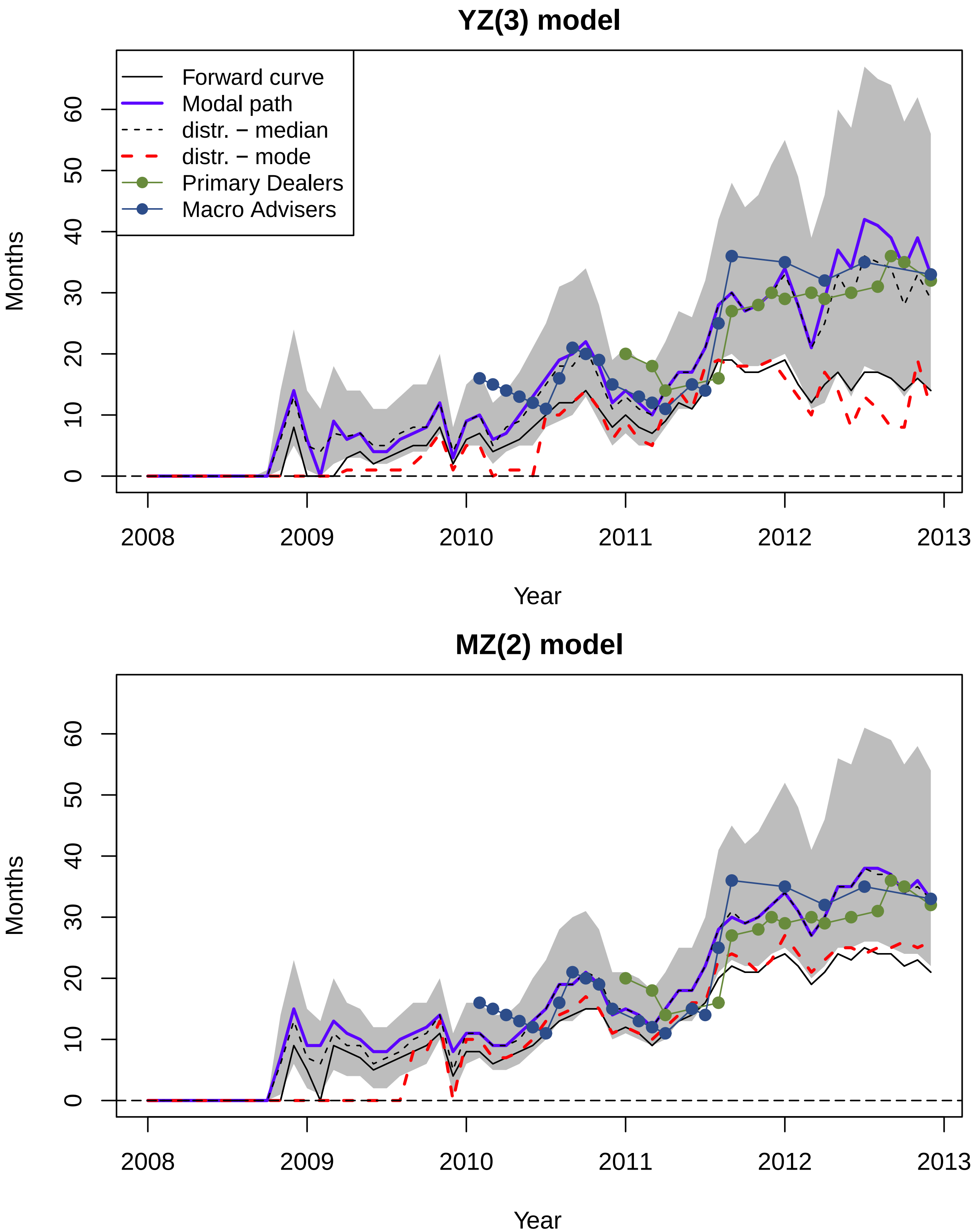
Notes: Actual, fitted, and shadow yield curves on June 30, 2011, and on December 31, 2012.

Figure 7: Distribution of liftoff horizon



Notes: Approximate densities for the distribution of the future liftoff horizon, for December 31, 2012

Figure 8: Alternative liftoff estimates



Notes: Estimated horizon (in months) until policy liftoff from the ZLB, based on the forward curve, the modal path, the median and mode of the liftoff distribution, and interquartile range (gray-shaded). Also shown is the predicted duration of the ZLB period from the Primary Dealer survey (median response) and from Macroeconomic Advisers. Period: January 2008 to December 2012.