Imperfect Knowledge about Asset Prices and Credit Cycles

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Abstract

I develop an equilibrium model with collateral constraints in which rational agents are uncertain and learn about the equilibrium mapping between fundamentals and collateral prices. Bayesian updating of beliefs by agents can endogenously generate booms and busts in collateral prices and largely strengthen the role of collateral constraints as an amplification mechanism through the interaction of agents' beliefs, collateral prices and credit limits. Over-optimism or pessimism is fueled when a surprise in price expectations is interpreted partially by the agents as a permanent change in the parameters governing the collateral price process and is validated by subsequently realized prices. I show that the model can quantitatively account for the recent US boom-bust cycle in house prices, household debt and aggregate consumption dynamics during 2001-2008. I also demonstrate that the leveraged economy with a higher steady state leverage ratio is more prone to self-reinforcing learning dynamics.

Keywords: Booms and Busts, Collateral Constraints, Learning, Leverage, Housing

JEL classifications: D83, D84, E32, E44

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"At some point, both lenders and borrowers became convinced that house prices would only go up. Borrowers chose, and were extended, mortgages that they could not be expected to service in the longer term. They were provided these loans on the expectation that accumulating home equity would soon allow refinancing into more sustainable mortgages. For a time, rising house prices became a self-fulfilling prophecy, but ultimately, further appreciation could not be sustained and house prices collapsed."

Bernanke, Speech, Monetary Policy and the Housing Bubble, at the Annual Meeting of the American Economic Association, Atlanta, Georgia, January 3, 2010

1 Introduction

The recent decade has witnessed a massive run-up and subsequent crash of house prices, as well as the remarkable role of the interaction of housing markets and credit markets in aggregate fluctuations in the US economy. Real house prices increased considerably in the decade before the recent financial crisis, as seen in the upper panel of figure 1.[1] They displayed relatively smaller variability before the year 2000 and increased by 35.9% from 2001 to 2006 in which house prices peaked. Associated with the price boom was a sharp increase in the household credit market debt/GDP ratio[2] and a consumption boom. As can be seen from the lower panel of figure 1, the household credit market debt/GDP ratio changed moderately before the year 2000 but increased from 45% in 2001 to 70% in 2006. Aggregate consumption[3] grew at about 3% per annum between 2001 and 2006, while its growth dropped sharply after house prices started to revert, as shown in figure 2.

Building upon the model of Kiyotaki and Moore (1997, henceforth KM), I develop a dynamic general equilibrium model with housing collateral constraints that can quantitatively account for the recent US boom-bust in house prices, and associated household credit market debt and aggregate consumption dynamics during 2001-2008 following the strong fall in real interest rates after the year 2000.

Much of recent research attempting to understand the recent house price dynamics include a housing collateral constraint. Examples are Boz and Mendoza (2010), Ferrero (2011) and Hoffmann, Krause and Laubach (2012). Despite the critical role in the recent financial and macroeconomic turmoil, the massive run-up of house prices is

[1] The data is taken from the OECD. Its definition is national wide single family house price index. The real house price index is the nominal house price index deflated by CPI price index. It is normalized to a value of 100 in 2000. The price-to-rent ratio and price-to-income ratio display a similar pattern.

[2] The household credit market debt/GDP ratio is measured by the absolute value of the ratio of net credit market assets of US household and non-profit organizations to GDP. The data is from the Flow of Funds Accounts of the U.S. provided by the Board of Governors of the Federal Reserve System.

[3] The data is from Federal Reserve System. It is the Real Personal Consumption Expenditures (series ID: PCECC96).

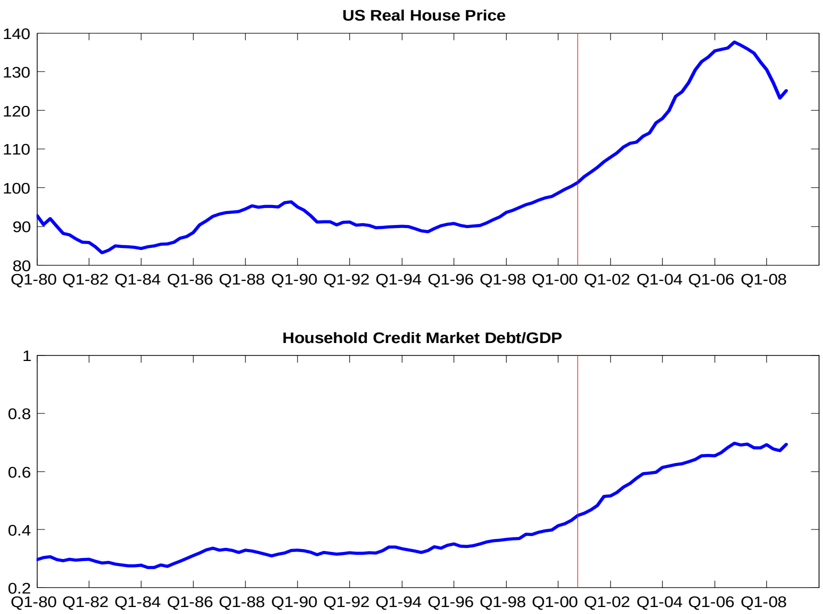

Figure 1: US Real House Prices and Household Credit Market Debt/GDP

extremely difficult to generate in most existing optimizing-agent DSGE models with housing collateral constraints. These models typically assume that agents could rationally foresee future collateral prices associated with any possible contingency. Therefore, the link between collateral prices and fundamentals is relatively tight, while the latter has relatively smaller variability.

In contrast to the previous literature with housing collateral constraints, I assume that agents have an incomplete model of the macroeconomy, knowing their own objective, constraints and beliefs but not the equilibrium mapping between fundamentals (e.g. preference shocks, collateral holdings) of the economy and collateral prices. Instead, agents have a completely specified subjective belief system about the collateral price process and make optimal decisions. By extrapolating from historical patterns in observed data they approximate this mapping to forecast future collateral prices.

The dynamic interaction of agents' price beliefs, credit limits and price realizations largely amplifies the effect of interest rate reductions and could give rise to expectation-driven house price booms. In addition, the role of collateral constraints as an amplification mechanism in aggregate fluctuations is largely strengthened due to more variability of collateral prices and to larger transfers of collateral between agents with different productivity relative to a RE version of the model.

An unexpected i.i.d negative shock to the interest rate is considered to illustrate the different dynamics of the learning model from the RE version of the model. In response to the shock, realized prices become higher than previously expected, inducing agents' belief revision and more optimistic expectation about future collateral prices

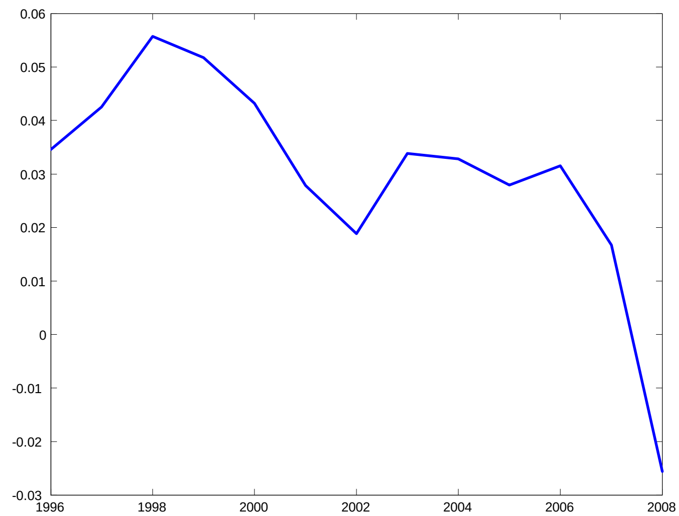

Figure 2: US Real Consumption Growth

than under RE. Credit limits are relaxed and larger loans are granted by lenders based on the optimism. With a larger borrowing capacity, borrowers can afford more and temporarily increase their collateral holdings. Realized prices partially validate agents' optimism, which leads to further optimism and persistent increases in actual prices. Associated with prolonged periods of increases in borrowers' collateral holdings, aggregate output and consumption expand due to shifts of collateral to more productive households.

Further rises in collateral prices will come to an end due to endogenous model dynamics. When the capital gain of collateral holding falls short of the down-payment (in the benchmark model and similarly in the extended model), the borrowers start to reduce their collateral holding and collateral prices revert subsequently. When the collateral prices fall short of agents' expectations, they revise their beliefs downward and become pessimistic. Credit limits are tightened due to pessimism about the future liquidation value of collateral and to shifts of collateral back to the lenders. The realized prices reinforce agents' pessimism, inducing periods of persistent downward adjustments of beliefs and actual prices. Realized prices and quantities decline faster toward and eventually converge to the steady state.

The learning model explains the US house prices boom and bust following the strong fall in real interest rates after the year 2000 and their staying at a low level for a long period. Responses of prices and quantities of the learning model are largely amplified relative to the RE version of the model due to the dynamic interaction of agents' beliefs, credit limits and price realizations. The model also generates a widening household credit market debt/GDP due to both the house price boom and rising amounts of collateral holdings by households. Aggregate output and consumption amplification arise from shifts of collateral to more productive borrowers.

The role of adaptive learning in asset pricing has been found limited in an endowment economy studied by Timmermann (1996) and in a production economy without collateral constraints in Carceles-Poveda and Giannitsarou (2008). In these models agents' perceived law of motion (PLM) has the same functional form as the REE and they learn about the parameters linking asset prices and fundamentals. The asset pricing equation in the credit-constrained economy with learning differs critically from them. It has an intrinsic property that collateral prices are influenced by the change of agents' price beliefs regardless of the belief specifications. Past beliefs come into play because they determine the inherited debt repayment of borrowers, which in turn offsets their net worth in the current period. This opens the possibility for the learning model to generate strong persistence in belief changes and hence in price changes, even though agents learn the parameters linking prices and fundamentals. I find that a leveraged economy with a higher steady state leverage ratio is more prone to self-reinforcing learning dynamics.

The transmission mechanism is consistent with the findings of Iacoviello and Neri (2010), which estimate a DSGE model with a housing collateral constraint via Bayesian methods using data from 1965 to 2006. They find an important role of monetary factors in housing cycles over the whole sample and an increasing role during the recent housing cycle. In addition, they also find nonnegligible spillover effect from housing markets to consumption over the whole sample and increasing importance of the effect in the recent housing cycle.

The rest of the paper is structured as follows. The next section discusses the related literature. Section 3 presents the benchmark model, agents' optimality conditions and the RE equilibrium. In section 4, I discuss the equilibrium with imperfect knowledge, the belief specification and the optimal learning behavior of agents. The mechanism of the learning model is inspected in section 5. I examine an extension of the model and a modification of agents' belief system in section 6. Quantitative results are presented in section 7. Section 8 concludes.

2 Related Literature

Collateral constraints have been studied as an amplification mechanism under RE transforming relative small shocks to the economy into large output fluctuations. Examples are KM, Kiyotaki (1998), Kocherlakota (2000), Krishnamurthy (2003), Cordoba and Ripoll (2004), and Liu, Wang and Zha (2011). More recently, Ferrero (2012) accounts for a sizable portion of the US house price boom and the current account deficit by combining a progressive relaxation of credit standards and departures of nominal interest rates from a standard monetary policy rule in a model with a housing collateral constraint. Allowing agents to be uncertain about the link between prices and fundamentals, the learning model generates additional non-fundamental fluctuations in collateral prices and strengthens the role of collateral constraints as an amplification mechanism.

Other models with imperfect information and learning have been developed to understand the recent house/land price dynamics given that it is difficult to reconcile the latter with relatively smaller variability of fundamentals in full information rational expectation models. For example, Boz and Mendoza (2010) study the role of learning about the riskiness of a new financial environment in a model with collateral constraints. Another example is Hoffmann, Krause and Laubach (2012) in which agents face uncertainty and learn about the long-run productivity growth. The interaction of the learning frictions and the collateral constraint helps to generate additional amplification of fundamental shocks. Agents in these models are endowed with knowledge about the equilibrium mapping from fundamentals to collateral prices and hence do not learn from equilibrium outcomes. My learning model differs by having feedback from equilibrium prices to agents' beliefs and possibly generates larger amplifications.

The paper is related to the literature which explores the role of shifting expectations in business cycle fluctuations, or asset pricing, or asset booms and busts, in particular based on learning dynamics. For example, Huang, Liu and Zha (2009) study implications of adaptive expectations in a standard growth model and find their model seems promising in generating plausible labor market dynamics. Another example of an application to the business cycle analysis is Eusepi and Preston (2011), which find learning friction amplifies technology shocks, improves the internal propagation and generates forecast errors that are consistent with business cycle properties of forecast errors for many variables from survey data. Milani (2011) estimates a New Keynesian Model with adaptive learning incorporating survey data on expectations and finds a crucial role of expectational shocks as a major driving force of the U.S. business cycle. Timmermann (1996) examines the role of learning about stock prices in an endowment economy. Carceles-Poveda and Giannitsarou (2008) study an asset pricing model with learning in a production economy with capital accumulation. Adam, Marcet, and Nicolini (2009) and Adam and Marcet (2010) develop learning models which can quantitatively replicate major stock pricing facts, generating booms and busts in stock prices and matching agents' return expectations as in survey data. Lansing (2010) examines a near-rational solution to Lucas-type asset pricing model and learning to generate intermittent stock bubbles and to match many quantitative features observed in the long-run US stock market data. The paper differs by incorporating a collateral constraint and studying the role of the interaction of shifting expectations and credit limits in asset pricing and macroeconomic fluctuations.

Adam, Kuang, and Marcet (2011, henceforth AKM) develop an open economy asset pricing model with housing collateral constraints and learning, which quantitatively accounts for the heterogeneous G7 house prices and the current account dynamics over 2001-2008. This paper differs from AKM along several important dimensions. Both models generate significant quantitative differences from the RE version of the models. A critical property is the dependence of collateral prices on the belief changes and hence the possibility of endogenously persistent belief and price changes. In the former this is due to the intrinsic property of the credit-constrained economy regardless of the belief specification, while in the latter this is due to learning about the persistent component of price growth and the use of price growth data to update beliefs by agents. The paper also examines the stability condition of the REE under learning, the dependence of learning dynamics on the leverage ratio, as well as dynamics of several different variables, such as household debt and aggregate consumption dynamics.

3 The Benchmark Model

In this section I present the benchmark model, which adopts the basic version of the KM model but has two differences. The first difference is a shock to lenders' preferences and hence to interest rates is added. More importantly, the belief specification and expectation formation in my model are different.

3.1 The Model Setup

There are two types of goods in the economy, durable assets, i.e., houses, and non-durable consumption goods, which are produced using houses but cannot be stored. The durable assets play a dual role: they are not only factors of production but also serve as collateral for getting loans. There are two types of infinitely lived risk-neutral agents, households and financial intermediaries, each of which has unit mass. Both produce and eat consumption goods. At each date t, there are two markets. One is a competitive spot market in which houses are exchanged for consumption at a price of q_t, while the other is a one-period credit market in which one unit of consumption at date t is exchanged for a claim to R_t units of consumption at date $t+1$.

The expected utility of a household i is

$$E_0^{\mathcal{P}^i} \sum_{t=0}^{\infty} (\beta^B(i))^t c_t^B(i) \tag{1}$$

where $\beta^B(i)$ is his subjective discount factor and $c_t^B(i)$ is his consumption in period t. The operator $E_0^{\mathcal{P}^i}$ denotes household $i's$ expectation in some probability space $(\Omega, S, \mathcal{P}^i)$, where Ω is the space of payoff relevant outcomes that the household takes as given in its optimization problem. The probability measure $\mathcal{P}^i$ assigns probabilities to all Borel subsets S of Ω. It may or may not coincide with objective probabilities emerged in the equilibrium. Further details about the Ω and $\mathcal{P}^i$ will be provided in the next section.

The household i produces with a constant return to scale technology. Only the $aH_t^B(i)$ component of the output is tradable in the market, while $\overline{e}H_t^B(i)$ is perishable and nontradable. His production function is

$$y_{t+1}^B(i) = (a + \overline{e})H_t^B(i) \tag{2}$$

where $H_t^B(i)$ is the amount of used houses. The introduction of nontradable output is to avoid continually postponement of consumption by households.

The household's production technology is assumed to be idiosyncratic in the sense that it requires his specific labor input. He always has the freedom to withdraw his labor, or in the language of Hart and Moore (1994), the household's human capital is inalienable. The households are potentially credit-constrained. The financial intermediaries protect themselves against risks of default by collateralizing the households' houses. The household i can at most pledge collateral $E_t^{\mathcal{P}^j} q_{t+1} H_t^B(i)$. Thus his borrowing constraint is

$$b_t^B(i) \leq \frac{E_t^{\mathcal{P}^j} q_{t+1}}{R_t} H_t^B(i) \tag{3}$$

where $b_t^B(i)$ is the amount of loans borrowed, $E_t^{\mathcal{P}^j} q_{t+1}$ the financial intermediary j's expectation about the collateral price in period $t+1$, and R_t gross interest rate between t and $t+1$. The borrowing constraint says that a household can get a maximum loan which is equal to the discounted expected liquidation value of his house holdings at $t+1$.

The household faces a flow-of-fund constraint

$$q_t(H_t^B(i) - H_{t-1}^B(i)) + R_{t-1} b_{t-1}^B(i) + c_t^B(i) \leq y_t^B(i) + b_t^B(i) \tag{4}$$

He produces consumption goods using houses and borrows from the credit market. He spends on consuming, repaying the debt, and investing in houses.

A financial intermediary j's preferences are specified by a linear utility function. She maximizes the following expected utility

$$E_0^{\mathcal{P}^j} \sum_{t=0}^{\infty} (\beta^L(j))^t A_t c_t^L(j) \tag{5}$$

where $\mathcal{P}^j$ is her subjective probability measure and $\beta^L(j)$ is her subjective discount factor. A_t is an i.i.d innovation to the financial intermediary's patience factor with $E[logA_t] = 0$ and $E[(\log A_t)^2] = \sigma_A^2$. She faces the following budget constraint:

$$q_t(H_t^L(j) - H_{t-1}^L(j)) + b_t^L(j) + c_t^L(j) \leq y_t^L(j) + R_{t-1} b_{t-1}^L(j) \tag{6}$$

where $H_t^L(j) - H_{t-1}^L(j)$ is her investment in collateral holdings. She uses a decreasing return to scale technology to produce, i.e., $y_{t+1}^L(j) = G^j(H_t^L(j))$, where $G^{j'} > 0$, $G^{j''} < 0$.

A few assumptions are made following the KM paper. The aggregate supply of the collateral is assumed to be fixed at $\overline{H}$. Later I will assume that all households (financial intermediaries) have the same subjective discount factor $\beta^B = \beta^B(i)$ for $\forall i$ ($\beta^L = \beta^L(j)$ for $\forall j$) and households are less patient than financial intermediaries, i.e., $\beta^B < \beta^L$. In addition, an assumption, i.e., $\overline{e} > (\frac{1}{\beta^B} - 1)a$, is made to ensure that in equilibrium households will not want to consume more than the perishable consumption goods.[4]

[4] The implication of this assumption is elaborated in the original KM paper and also briefly reviewed later.

3.2 Optimality and Market Clearing Conditions

Recall how individual household i makes his optimal decisions with respect to consumption, borrowing and collateral demand in the original KM paper. Since return to investment in collateral holding is sufficiently high as shown in KM,[5] he prefers to borrow up to the maximum, consume only the nontradable part of his output and invest the rest in collateral holding. His optimal consumption is

$$c_t^B(i) = \bar{e} H_{t-1}^B(i) \tag{7}$$

and optimal borrowing

$$b_t^B(i) = \frac{E_t^{\mathcal{P}^j} q_{t+1}}{R_t} H_t^B(i) \tag{8}$$

The household uses both his own resources and external borrowing to finance collateral holdings. Given that the household consumes only the nontradable output, his net worth at the beginning of date t contains the value of his tradable output $aH_{t-1}^B(i)$, and the value of the collateral held from the previous period $q_t H_{t-1}^B(i)$, net of the debt payment, $R_{t-1} b_{t-1}^B(i)$. The household i's demand on collateral could be derived from (2), (4), (7),and (8)

$$H_t^B(i) = \frac{1}{q_t - \frac{1}{R_t} E_t^{\mathcal{P}^j} q_{t+1}} [(a + q_t) H_{t-1}^B(i) - R_{t-1} b_{t-1}^B(i)] \tag{9}$$

where $q_t - \frac{1}{R_t} E_t^{\mathcal{P}^j} q_{t+1}$ is the down-payment required to buy a unit of house.

Except for the initial period, every period the household i inherits debt $b_{t-1}^B(i)$ from the previous period[6] where

$$b_{t-1}^B(i) = \frac{E_{t-1}^{\mathcal{P}^j} q_t}{R_{t-1}} H_{t-1}^B(i) \tag{10}$$

His debt repayment $R_{t-1} b_{t-1}^B(i)$ is influenced by the expectation of collateral price at period t formed at period $t-1$, i.e., $E_{t-1}^{\mathcal{P}^j} q_t$. After plugging (10) into (9), the collateral

[5] Recall the calculation in the original KM paper. Consider a marginal unit of tradable consumption at date t. The borrower could consume it and get utility 1. Alternatively he could invest it in collateral holding and produce consumption goods. In the next period, he will consume the nontradable part of production and invest further the tradable part, and so forth. KM show that the discounted sum of utility of investing it at date t will exceed the utility of immediately consuming it, which is 1. Similarly, the return to investment will also be larger than the other choice, saving it for one period and then investing. Hence the collateral constraint will always be binding.

[6] I assume for the initial period (10) also holds.

demand[7] of the household i is derived as following:

$$H_t^B(i) = \frac{1}{q_t - \frac{1}{R_t} E_t^{\mathcal{P}^j} q_{t+1}} (a + q_t - E_{t-1}^{\mathcal{P}^j} q_t) H_{t-1}^B(i) \tag{11}$$

Note borrowers' collateral demand are influenced by expectations at two successive periods, $E_{t-1}^{\mathcal{P}^j} q_t$ and $E_t^{\mathcal{P}^j} q_{t+1}$.The former comes from the inherited debt repayment. The dependence may give rise to interesting dynamics under learning, as analyzed later.

Users' cost of collateral is defined as the opportunity cost of holding collateral for one more period, which is

$$u_t^e = q_t - \frac{1}{R_t} E_t^{\mathcal{P}^j} q_{t+1}$$

A financial intermediary j is not credit constrained and her demand for collateral is determined by the point at which the present value of the marginal product of collateral is equal to the user cost of holding collateral

$$\frac{1}{R_t} G^{j'}\left(H_t^L(j)\right) = q_t - \frac{1}{R_t} E_t^{\mathcal{P}^j} q_{t+1} \tag{12}$$

Aggregation yields $H_t^B = \int_0^1 H_t^B(i)$, $H_t^L = \int_0^1 H_t^L(j)$, $b_t^B = \int_0^1 b_t^B(i)$, and $b_t^L = \int_0^1 b_t^L(j)$. Denote by y_t the aggregate output in period t, which is the sum of the production by borrowers and lenders

$$\begin{aligned} y_t &= \int_0^1 y_t^B(i) + \int_0^1 y_t^L(j) &(13)\\ &= (a + \overline{e}) H_{t-1}^B + G(H_{t-1}^L) &(14) \end{aligned}$$

Given that households are less patient than financial intermediaries, in equilibrium the former will borrow from the latter and the rate of interest rate will always be equal to the financial intermediaries' rate of time preference; that is

$$R_t = \frac{A_t}{\beta^L e^{\frac{1}{2}\sigma_A^2}} \tag{15}$$

Market clearing implies $H_t^B + H_t^L = \overline{H}$ and $b_t^B = b_t^L$. Due to zero net supply of loans and collateral assets, aggregate consumption c_t will be equal to aggregate output

[7] A related paper by Assenza and Beradi (2009, JEDC, henceforth AB) enriches the KM model with adaptive learning focusing on voluntary default of borrowers. The borrowers' collateral demand equation in their paper, the counterpart of equation (11), does not include the capital gains/losses of collateral holdings $(q_t - E_{t-1}^{\mathcal{P}^j} q_t) H_{t-1}^B(i)$. Kuang (2012) shows that the "optimality" conditions in AB imply agents' "optimal" choices are either suboptimal or infeasible. It also discusses whether this may affect the E-stability condition of the RE equilibrium, propagation of productivity shocks, and the timing of default of borrowers under heterogenous learning rules.

$y_{t.}$ Since aggregate investment is automatically zero in the model, I introduce a fixed, exogenous amount of autonomous investment I.[8] This captures the investment and government absorption in the data. So the GDP in the model is the sum of aggregate consumption and investment

$$GDP_t = c_t + I \tag{16}$$

Denote $(Debt/GDP)_t$ the household credit market debt/GDP ratio, which is calculated by

$$(Debt/GDP)_t = b_t^B / GDP_t \tag{17}$$

3.3 The Steady State and the MSV Rational Expectation Equilibrium

Assuming homogeneity among all borrowers and all lenders, symmetric equilibrium requires $H_t^B = H_t^B(i)$, $H_t^L = H_t^L(j)$, $b_t^B = b_t^B(i)$, and $b_t^L = b_t^L(j)$. There exists a unique non-stochastic steady state. The steady state value of the interest rate, the collateral price, the users' cost, lenders' collateral holding, borrowers' collateral holding, borrowing, and borrowers' consumption are $R = \frac{1}{\beta^L}$, $q = \frac{aR}{R-1}$, $u = a$, $H^L = G'^{-1}(Ra)$, $H^B = \overline{H} - H^L$, $b^B = qH^B/R$ and $c^B = \overline{e}H^B$, respectively.

Suppress indices of agents here and denote by $\frac{1}{\eta}$ the steady state elasticity of the users' cost of collateral with respect to borrowers' collateral holdings

$$\frac{1}{\eta} \equiv \frac{d \log u^e(H_t^B)}{d \log H_t^B}|_{H_t^B = H^B} = -\frac{d \log G'(H_t^L)}{d \log H_t^L}|_{H_t^L = H^L} \times \frac{H^B}{\overline{H} - H^B}$$

The elasticity is the product of the financial intermediaries' marginal product of houses and the ratio of the households' collateral holdings to the financial intermediaries' at the steady state.

Define $\widehat{A}_t = A_t - 1$. Appendix A shows that log-linearizing the borrowers' collateral demand equation (11) yields

$$\widehat{H}_t^B = \frac{R}{R-1}[(\widehat{q}_t - E_{t-1}^{\mathcal{P}}\widehat{q}_t) - (\widehat{q}_t - \frac{1}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1})] + \widehat{H}_{t-1}^B - \frac{1}{R-1}\widehat{A}_t \tag{18}$$

In combination with the assumption of fixed supply of collateral, log-linearizing the lenders' collateral demand equation (12) leads to the following equation

$$\widehat{q}_t = \frac{1}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1} + \frac{1}{\eta}\frac{R-1}{R}\widehat{H}_t^B - \widehat{A}_t \tag{19}$$

Plugging equation (18) into (19), I obtain

$$\widehat{q}_t = \gamma_1 E_t^{\mathcal{P}}\widehat{q}_{t+1} - \gamma_2 E_{t-1}^{\mathcal{P}}\widehat{q}_t + \gamma_3 \widehat{H}_{t-1}^B + \gamma_4 \widehat{A}_t \tag{20}$$

[8] This assumption is also made in Boz and Mendoza (2010).

where $\gamma_1 = \frac{1}{R}(1+\frac{1}{\eta})$, $\gamma_2 = \frac{1}{\eta}$, $\gamma_3 = \frac{R-1}{R}\frac{1}{\eta}$, and $\gamma_4 = -(1+\frac{1}{\eta}\frac{1}{R})$.

Denote by parameters with a "*bar*" the value that appears in the rational expectations solution. Using the method of undermined coefficients, I derive the Minimum State Variables (MSV) RE solution for collateral prices and borrowers' collateral holdings in the benchmark economy

$$\widehat{q}_t = \bar{\phi}^m + \bar{\phi}^p \widehat{H}^B_{t-1} + \bar{\phi}^s \widehat{A}_t \tag{21}$$

$$\widehat{H}^B_t = \overline{\varkappa}^m + \overline{\varkappa}^p \widehat{H}^B_{t-1} + \overline{\varkappa}^s \widehat{A}_t \tag{22}$$

where $\bar{\phi}^m = 0$, $\bar{\phi}^p = \frac{R-1}{R}\frac{1}{\eta+1-\frac{\eta}{R}}$, $\bar{\phi}^s = -(\frac{\frac{1}{\eta}}{R-1}+1)$, $\overline{\varkappa}^m = 0$, $\overline{\varkappa}^p = \frac{\eta}{1+\eta}$, and $\overline{\varkappa}^s = \frac{\eta(1-\frac{1}{R})+1}{(\eta+1)(1-\frac{1}{R})(R-1)}$. Note the RE solution for borrowers' collateral holdings is an AR(1) process and collateral prices ARMA(1,1) process.

4 Equilibrium with Imperfect Knowledge

In the rational expectations equilibrium, agents are endowed with knowledge about the equilibrium mapping from the history of collateral holdings and lenders' preference shocks to collateral prices. Below I assume homogeneous expectations among all agents but relax the assumption that the homogeneity of agents is common knowledge, in particular, agents do not know other agents' discount factors and beliefs about future collateral prices. Relaxation of the informational assumption leads to agents in the model being uncertain about the equilibrium mapping between collateral prices and fundamentals. I discuss the underlying probability space conditional on which agents form their expectation and the equilibrium concept of the model. Afterwards agents' near-REE beliefs are specified and their optimal learning behavior is studied given their belief system and information set.

4.1 The Underlying Probability Space and the Internally Rational Expectation Equilibrium

I now describe the probability space $(\Omega, S, \mathcal{P})$. Following Adam and Marcet (2011), I extend the state space of outcomes to contain not only the sequence of fundamentals, i.e., borrowers' collateral holdings and the shock to lenders' patience factor, but also other pay-off relevant variables, collateral prices. Both borrowers and lenders view the process for q_t, A_t and H^B_t as external to their decision problem and the probability space over which they condition their choices is given by $\Omega = \Omega_q \times \Omega_A \times \Omega_{H^B}$ where $\Omega_X = \Pi_{t=0}^{\infty} R_+$ and $X \in \{q, A, H^B\}$. The probability spaces contain all possible sequences of prices, lenders' preference shocks and borrowers' collateral holdings. I denote the set of all possible histories up to period t by $\Omega^t = \Omega^t_q \times \Omega^t_A \times \Omega^t_{H^B}$ and its typical element is denoted by $\omega^t \in \Omega^t$. The RE belief is nested as a special case in which the probability measure $\mathcal{P}$ features a singularity in the joint density of prices and fundamentals. Since equilibrium pricing functions are assumed to be known to agents under RE, conditioning their choices on the collateral price process is redundant.

The agents are assumed to be "Internally Rational"[9] as defined below, i.e., maximizing their expected utility under uncertainty, taking into account their constraints, and conditioning their choice variables over the history of all external variables. Their expectations about future external variables are evaluated based on their consistent set of subjective beliefs specified in the subsequent subsection, which is endowed to them at the outset.

Definition 1 *Internal Rationality*

a) A household i is "Internally Rational" if he chooses $(b_t^B(i), H_t^B(i), c_t^B(i)) : \Omega^t \to R^3$ to maximize the expected utility (1) subject to the flow-of-fund constraint (4), the collateral constraint (3) and his production function, taking as given the probability measure $\mathcal{P}^i$.

b) A financial intermediary j is "Internally Rational" if she chooses $(b_t^L(j), H_t^L(j), c_t^L(j)) : \Omega^t \to R^3$ to maximize the expected utility (5) subject to the flow-of-fund constraint (6) and her production function, taking as given the probability measure $\mathcal{P}^j$.

Note the internal rationality of agents is tied neither to any specific belief system nor to the learning behavior of agents. However, the belief system is usually specified with some near-rationality concept and it is natural to introduce learning behavior of agents.

In the following I specify the equilibrium of the economy. Let (Ω_A, P_A) be a probability space over the space of histories of preference shocks Ω_A. Denote P_A the 'objective' probability measure for lenders' preference shocks. Let $\omega_A \in \Omega_A$ denote a typical infinite history of lenders' preference shocks.

Definition 2 *Internally Rational Expectations Equilibrium*

The Internally Rational Expectation Equilibrium (IREE) consists of a sequence of equilibrium price functions $\{q_t\}_{t=0}^{\infty}$ where $q_t : \Omega_A^t \to R_+$ for each t, contingent choices $(c_t^B(i), c_t^L(j), b_t^B(i),\ b_t^L(j), H_t^B(i), H_t^L(j)) : \Omega^t \to R^6$ and probability beliefs $\mathcal{P}^i$ for each household i and $\mathcal{P}^j$ for each financial intermediary j, such that
(1) all agents are internally rational, and
(2) when agents evaluate $(c_t^B(i), c_t^L(j), b_t^B(i), b_t^L(j), H_t^B(i), H_t^L(j))$ at equilibrium prices, markets clear for all t and all $\omega_A \in \Omega_A$ almost surely in P_A.

In the Internally Rational Expectations Equilibrium, expectations about collateral prices are formed based on agents' subjective belief system, which are not necessarily equal to the 'objective' density. Collateral prices and borrowers' collateral holdings are determined by equations (18) and (19) after agents' probability measures $\mathcal{P}$ are specified.

[9] This follows Adam and Marcet (2011).

4.2 Agents' Belief System and Optimal Learning Behavior

I now describe agents' probability measure $\mathcal{P}$ and derive their optimal learning algorithm. Agents' belief system is assumed to have the same functional form as the RE solution for collateral prices (21) and for borrowers' collateral holdings (22). Agents believe collateral prices and borrowers' collateral holdings depend on past aggregate borrowers' collateral holdings.[10] It can be represented as following:[11]

$$\begin{aligned} \widehat{q}_t &= \phi^m + \phi^p \widehat{H}^B_{t-1} + \epsilon_t && (23) \\ \widehat{H}^B_t &= \varkappa^m + \varkappa^p \widehat{H}^B_{t-1} + \varrho_t && (24) \end{aligned}$$

given $\widehat{H}^B_0$, where

$$\begin{pmatrix} \epsilon_t \\ \varrho_t \end{pmatrix} \sim iiN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2_\epsilon & 0 \\ 0 & \sigma^2_\varrho \end{pmatrix} \right) \tag{25}$$

Unlike under rational expectations, they are assumed to be uncertain about the parameters and the shock precisions $(\phi^m, \phi^p, \frac{1}{\sigma^2_\epsilon}, \varkappa^m, \varkappa^p, \frac{1}{\sigma^2_\varrho})$, which is a natural assumption given that internally rational agents cannot derive the equilibrium distribution of collateral prices. Note agents' beliefs about $(\varkappa^m, \varkappa^p, \frac{1}{\sigma^2_\varrho})$ do not matter for equilibrium outcomes because only one-step ahead expectations enter the equilibrium under internal rationality in the model. So I omit belief updating equations for $(\varkappa^m, \varkappa^p, \frac{1}{\sigma^2_\varrho})$ for the rest of the paper.

Denote K the precision of the innovation ϵ_t, i.e., $K \equiv \frac{1}{\sigma^2_\epsilon}$. Agents' uncertainty at time zero are summarized by a distribution

$$(\phi^m, \phi^p, K) \sim f$$

The prior distribution of unknown parameters is assumed to be a Normal-Gamma distribution as following

$$\begin{aligned} K &\sim G(\gamma_0, d_0^{-2}) && (26) \\ (\phi^m, \phi^p)' \mid K = k &\sim N((\theta^m_0, \theta^p_0)', (\nu_0 k)^{-1}) && (27) \end{aligned}$$

The residual precision K is distributed as a Gamma distribution, and conditional on the residual precision K unknown parameters (ϕ^m, ϕ^p) are jointly normally distributed. The deviation of this prior from the REE prior will vanish assuming agents' initial

[10] The shock to lenders' preference is observable but not included in agents' regression. Including it will generate a singularity in the regression if initial beliefs coincide with the rational expectations equilibrium given it is the only shock in the model.

[11] This is analogous to learning the parameter linking prices and dividend in stock pricing models. Note the dividend here is the marginal product of lenders and a function of borrowers' collateral holding. After log-linearization, the (percentage deviation of) dividend is just a constant multiple of (percentage deviation of) borrowers' collateral holding.

beliefs are at the RE value $\theta = \bar{\theta} = (\bar{\theta}^m, \bar{\theta}^p)'$, and they have infinite confidence in their beliefs about the parameters, i.e., $\gamma_0 \to \infty$, and $\nu_0 \to \infty$.

For the sake of notational compactness, for the rest of this section I denote y_t and x_t the collateral price $\widehat{q}_t$ and $(1, \widehat{H}^B_{t-1})$, respectively. $\theta_t \equiv (\theta^m_t, \theta^p_t)$ stands for the posterior mean of (ϕ^m, ϕ^p).

Given agents' prior beliefs (26) and (27), optimal behavior implies that agents' beliefs are updated by applying Bayes' law to market outcomes. Appendix B shows that the posterior distribution of unknown parameters is given by

$$\begin{aligned} K|\omega^t &\sim G(\gamma_t, d_t^{-2}) \qquad (28) \\ (\phi^m, \phi^p)'|K = k, \omega^t &\sim N((\theta^m_t, \theta^p_t)', (\nu_t k)^{-1}) \qquad (29) \end{aligned}$$

where the parameters $(\theta^m_t, \theta^p_t, \nu_t, \gamma_t, d_t^{-2})$ evolve recursively as following

$$\begin{aligned} \theta_t &= \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t (y_t - x_t'\theta_{t-1}) \qquad (30) \\ \nu_t &= \nu_{t-1} + x_t x_t' \qquad (31) \\ \gamma_t &= \gamma_{t-1} + \frac{1}{2} \qquad (32) \\ d_t^{-2} &= d_{t-1}^{-2} + \frac{1}{2}(y_t - x_t'\theta_{t-1})'(x_t x_t' + \nu_{t-1})^{-1}\nu_{t-1}(y_t - x_t'\theta_{t-1}) \qquad (33) \end{aligned}$$

To avoid simultaneity between agents' beliefs and actual outcomes, I assume information on the data, i.e., prices and collateral holdings, is introduced with a delay in θ_t. So I actually use

$$\begin{aligned} \theta_t &= \theta_{t-1} + (x_{t-1} x_{t-1}' + \nu_{t-1})^{-1} x_{t-1} (y_{t-1} - x_{t-1}'\theta_{t-1}) \qquad (34) \\ \nu_t &= \nu_{t-1} + x_{t-1} x_{t-1}' \qquad (35) \end{aligned}$$

A micro-founded belief system justifying this delay could be provided following Adam and Marcet (2010).

Equations (34) and (35) are equivalent to the following Recursive Least Square (RLS) learning algorithm

$$\begin{aligned} \theta_t &= \theta_{t-1} + \frac{1}{t+N} S_t^{-1} x_{t-1}(y_{t-1} - x_{t-1}'\theta_{t-1}) \qquad (36) \\ S_t &= S_{t-1} + \frac{1}{t+N}(x_{t-1}x_{t-1}' - S_{t-1}) \qquad (37) \end{aligned}$$

when the initial parameter is set to $\nu_0 = NS_0$. Then it can be shown that for subsequent periods we have $\nu_t = (t+N)S_t$, for $\forall t \geq 1$. Therefore, N in the above equations measures the precision of initial beliefs.

The term $y_{t-1} - x_{t-1}'\theta_{t-1}$ in equation (36) is agents' price expectational error at period t. According to (36) and (37), a surprise in agents' price expectation will induce a revision of their beliefs or the parameters linking prices and fundamentals.

5 Understanding the Learning Model

In this section some preliminary views are firstly provided on why the learning model can generate additional propagation of a shock relative to a RE version of the model. The learning dynamics is then analyzed more formally. I investigate the E-stability of the RE equilibrium, i.e., whether and when the learning process converges to the REE. In addition, I examine a deterministic version of the model to study the transitional learning dynamics.

5.1 Preliminary Views on the Mechanism

Reproducing the log-linearized borrowers' collateral demand equation (18)

$$\widehat{H}_t^B = \frac{R}{R-1}[(\widehat{q}_t - E_{t-1}^{\mathcal{P}}\widehat{q}_t) - (\widehat{q}_t - \frac{1}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1})] + \widehat{H}_{t-1}^B - \frac{1}{R-1}\widehat{A}_t \tag{38}$$

and the collateral pricing equation (19)

$$\widehat{q}_t = \frac{1}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1} + \frac{1}{\eta}\frac{R-1}{R}\widehat{H}_t^B - \widehat{A}_t \tag{39}$$

To illustrate the different dynamics of the learning model, I consider a one-time unanticipated i.i.d. negative shock to borrowers' patience factor and hence an unexpected reduction in the interest rate.[12] The economy is assumed to start at its non-stochastic steady state and initially agents' beliefs about unknown parameters are at the RE level.

The RE solution for prices and collateral holdings are summarized in (21) and (22). Under rational expectations, borrowers' demand on collateral increases following an unexpected interest rate reduction. In the impact period, collateral is transferred from lenders to borrowers. Due to the fixed supply of collateral and the decreasing return to scale technology of lenders to produce, users' cost of collateral rises above the steady state value. Since borrowers' current investment in collateral holding raises their ability to borrow in the next period, there will be persistence in their collateral holdings. The users' cost of collateral stays above the steady state for many periods. Under RE, the collateral price is the discounted sum of current and future users' costs. The persistence in the users' cost reinforces the effect on collateral prices and collateral values, which leads to a larger effect on collateral transfers and aggregate activities.

After the shock disappears, expectations about future collateral prices realize themselves and there will be no capital gains or losses in borrowers' collateral holdings. The higher-than-steady-state users' cost chokes off further rise in borrowers' demand on collateral. Collateral prices and borrowers' collateral holdings will revert immediately

[12] Recall due to the risk-neutrality of lenders, the equilibrium interest rate in the model, i.e. the interest rate, is determined by their subjective discount factor and not affected by other endogenous variables, see equation (15). The original KM model considered an unexpected shock to both borrowers' and lenders' productivity to illustrate the RE equilibrium dynamics.

toward the steady state. Prices and quantities converge persistently and monotonically to the steady state.

Unlike under RE, capital gains or losses in agents' collateral holdings, in the form of expectational errors, may still arise in the learning model even in the absence of shocks, which generates additional variations in borrowers' net worth. An intrinsic property of the credit-constrained economy with learning is that borrowers' collateral demand is influenced not only by current beliefs about future collateral prices but also by past beliefs about current collateral prices. On the one hand, the past beliefs affect borrowers' inherited debt holdings, which in turn offset their current net worth, as can be seen more clearly from equation (11). On the other hand, the down-payment is affected by agents' current beliefs about future prices. The change of borrowers' collateral holdings will depend on the change of agents' beliefs about collateral prices. Equation (38) says that without the shock borrowers' collateral holdings will increase when the capital gain (of holding one unit of collateral by borrowers) outweighs the downpayment to buy one unit of collateral.

From equation (39), collateral prices in the learning model depend on both one-step ahead forecasts of collateral prices $E_t^{\mathcal{P}} \widehat{q}_{t+1}$ and the current users' cost of collateral. The price expectations are determined by both agents' beliefs (parameter estimates) and borrowers' collateral holdings as under RE. As borrowers' collateral holdings depend on the change of agents' beliefs and lenders use the former to forecast future collateral prices, actual collateral prices will depend on the change of agents' beliefs.

The impact responses of all variables in the learning model are the same as those under RE, because the learning agents have correct forecast functions initially. Nevertheless, the learning model generates additional propagation due to belief revisions and the interaction of beliefs and price realizations. After the shock disappears, a positive surprise in the collateral price induces an upward belief revision. Agents partially interpret the price expectational errors due to the temporary shock, as a permanent change in the parameters governing the collateral price process. They become more optimistic about future collateral prices due to both more optimistic beliefs and rising amount of collateral holdings by borrowers. The credit limit is relaxed based on lenders' optimistic expectations about the liquidation value of collateral. With larger borrowing capacity, borrowers can afford more and temporarily increase their collateral holdings when the capital gain outweighs the down-payment to buy one unit of collateral, as can be seen from equation (38).

After the shock disappears, collateral prices may rise further due to more optimistic price expectations and rising users' cost of collateral. The realized prices may reinforce agents' optimism and leads to further optimism when using price realizations to update their belief. Learning about collateral prices can give rise to dynamic feedback between agents' beliefs and actual prices through the relaxation of credit limits, which generates additional propagation of the shock as well as prolonged periods of expansion of prices and quantities. As can be seen from the quantitative results later, collateral price amplifications are driven mainly by the expectation about future collateral prices, while the variation of users' costs due to shifts of collateral between borrowers and lenders has a smaller effect.

Collateral price increases will be choked off for a number of reasons. For example, adverse fundamental shocks such as shocks to the interest rate, or endogenous model dynamics may lead to lower capital gain than the users' cost. Borrowers will then start to reduce their demand for collateral, and collateral prices will revert subsequently. When collateral prices fall short of agents' expectations, according to (36) and (37), their beliefs will be updated downward and they become pessimistic. The realization of collateral prices implied by the actual law of motion will be low, which leads to further pessimism. The prices and quantities decline faster toward the steady state. A more formal analysis of the learning dynamics is presented in the next subsection.

Denote by Y the steady state value of aggregate output. Log-linearizing aggregate output (14) yields

$$\widehat{y}_t = \frac{(a+\bar{e}) - G'}{(a+\bar{e})} \frac{(a+\bar{e})H^B}{Y} \widehat{H}^B_{t-1}$$

Aggregate output is equal to the product of the productivity gap $\frac{(a+\bar{e})-G'}{(a+\bar{e})}$ between borrowers and lenders, the production share of borrowers $\frac{(a+\bar{e})H^B}{Y}$ and the redistribution of collateral. Aggregate consumption $\widehat{c}_t$ will be the same as aggregate output because of zero net investment in housing. The learning model generates larger shifts of collateral to more productive households and hence output and consumption amplification relative to a RE version of the model.

Denote by $\overline{C}$ and $\overline{GDP}$ aggregate consumption and GDP at the steady state, respectively. Log-linearizing (16) yields

$$\widehat{GDP}_t = \frac{\overline{C}}{\overline{GDP}} \widehat{c}_t \tag{40}$$

and (17) yields

$$\begin{aligned} (\widehat{Debt/GDP})_t &= \widehat{b}^B_t - \widehat{GDP}_t && (41) \\ &= E^{\mathcal{P}}_t \widehat{q}_{t+1} + H^B_t - \widehat{R}_t - \frac{\overline{C}}{\overline{GDP}} \widehat{y}_t && (42) \end{aligned}$$

where $\widehat{b}^B_t$ can be calculated by log-linearizing equation (8) and imposing the symmetry of the equilibrium.

In response to the real interest rate reduction, the household credit market Debt/GDP ratio in the learning model increases by more than under RE due to both a further rise in collateral holdings held by households and in rising house prices.

5.2 Belief Dynamics

The belief dynamics is now analyzed more formally. I investigate the Expectational-Stability (E-stability) of the REE (21)-(22), in particular whether and under which conditions agents' beliefs will converge (locally) to the REE beliefs. This can be analyzed by applying the standard stochastic recursive algorithm (SRA) techniques elaborated in Evans and Honkapohja (2001). Furthermore, I examine a deterministic version of the learning model to study the transitional dynamics.

Recall agents perceive prices and borrowers' collateral holdings to evolve according to (23)− (24), while their beliefs are updated following (36) and (37). The state variables of the learning algorithm are $x_t = (1 \quad \widehat{H}^B_{t-1})'$. Agents' conditional expectations are $E^{\mathcal{P}}_{t-1}\widehat{q}_t = \phi'_{t-1}x_{t-1}$ and $E^{\mathcal{P}}_t\widehat{q}_{t+1} = \phi'_t x_t$ where $\phi_t \equiv (\phi^m_t \quad \phi^p_t)'$. Substituting the conditional expectations into model equations (19) and (20), I get the actual law of motion (ALM) for collateral prices under learning

$$\widehat{q}_t = T_1(\phi^m_{t-1}, \phi^m_t, \phi^p_t) + T_2(\phi^p_{t-1}, \phi^p_t)\widehat{H}^B_{t-1} + T_3(\phi^p_t)\widehat{A}_t \tag{43}$$

where $T_1(\phi^m_{t-1}, \phi^m_t, \phi^p_t) = \frac{(\gamma_1\phi^m_t - \gamma_2\phi^m_{t-1})(1+\frac{\phi^p_t}{\gamma_3 R}) - \frac{\gamma_1}{\gamma_3}\frac{\phi^m_t\phi^p_t}{R}}{1+\frac{\phi^p_t}{\gamma_3 R} - \gamma_1\phi^p_t\frac{1}{\gamma_3}}$, $T_2(\phi^p_{t-1}, \phi^p_t) = \frac{\gamma_3 - \gamma_2\phi^p_{t-1}}{1-\frac{\gamma_1\phi^p_t}{\gamma_3+\phi^p_t\frac{1}{R}}}$ and $T_3(\phi^p_t) = \frac{\gamma_3}{1-\frac{\gamma_1\phi^p_t}{\gamma_3+\phi^p_t\frac{1}{R}}}$. Combining (19) and (43), I obtain borrowers' collateral holding

$$\widehat{H}^B_t = \frac{\widehat{q}_t - \frac{\phi^m_t}{R}}{\gamma_3 + \frac{1}{R}\phi^p_t} \tag{44}$$

Below Θ_0 is defined as the set of admissible parameters in the benchmark learning model.

Definition 3 *The Set* Θ_0

The set of admissible parameters $\Theta_0 \equiv \{(\eta, R)|\eta > 0, R > 1\}$.

The T-map mapping agents' subjective beliefs to actual parameters in the ALM is $T(\phi^m, \phi^p) \equiv (T_1, T_2)(\phi^m, \phi^p) \equiv (\frac{(\gamma_1\phi^m - \gamma_2\phi^m)(1+\frac{\phi^p}{\gamma_3 R}) - \frac{\gamma_1}{\gamma_3}\frac{\phi^m\phi^p}{R}}{1+\frac{\phi^p}{\gamma_3 R} - \gamma_1\phi^p\frac{1}{\gamma_3}}, \frac{\gamma_3 - \gamma_2\phi^p}{1-\frac{\gamma_1\phi^p}{\gamma_3+\phi^p\frac{1}{R}}})$. Local stability of the MSV REE is determined by the stability of the following associated ODEs

$$\begin{aligned}
\frac{d\phi^m}{d\tau} &= T_1(\phi^m, \phi^p) - \phi^m \\
\frac{d\phi^p}{d\tau} &= T_2(\phi^p, \phi^p) - \phi^p
\end{aligned}$$

The following condition establishes a sufficient condition for the E-stability of the MSV equilibrium (21).

Proposition 4

The MSV equilibrium (21) and (22) for the model economy represented by equations (18) and (19) is E-stable for any admissible parameters in Θ_0.
Proof. See Appendix C. ■

The users' cost of collateral plays an important role in stabilizing collateral holdings and prices around the neighborhood of the REE equilibrium. This can be seen more clearly after reformulating equation (38) and dropping the innovation term

$$\widehat{H}_t^B = \frac{R}{R-1}[\frac{1}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1} - E_{t-1}^{\mathcal{P}}\widehat{q}_t] + \widehat{H}_{t-1}^B \tag{45}$$

The following illustration may help to understand the E-stability condition. Fixing agents' beliefs ϕ^m at the RE value 0 and ϕ^p above the RE value, which implies that there is a deviation of collateral price expectations above and away from the RE level. Agents' conditional expectations are $E_{t-1}^{\mathcal{P}}\widehat{q}_t = \phi^p \widehat{H}_{t-1}^B$ and $E_t^{\mathcal{P}}\widehat{q}_{t+1} = \phi^p \widehat{H}_t^B$. Using equations (43) and (44), I obtain $E_t^{\mathcal{P}}\widehat{q}_{t+1} = \phi^p \frac{\widehat{q}_t}{\gamma_3 + \frac{1}{R}\phi^p} = \frac{\phi^p}{\gamma_3 + \frac{1}{R}\phi^p} T_2(\phi^p, \phi^p)\widehat{H}_{t-1}^B$. It can be shown for all admissible parameters in Θ_0 that the actual elasticity of collateral prices with respect to collateral holdings $T_2(\phi^p, \phi^p)$ is low enough such that $\frac{\phi^p}{\gamma_3 + \frac{1}{R}\phi^p} T_2 < \phi^p$. This implies further that the users' cost of collateral outweighs the capital gain, i.e., $\frac{1}{R}E_t^{\mathcal{P}}\widehat{q}_{t+1} < E_{t-1}^{\mathcal{P}}\widehat{q}_t$. Borrowers' collateral holding will be reduced and so do collateral prices subsequently. Therefore, the asymptotic local stability of the REE is achieved. Roughly speaking, given that the E-stability condition is satisfied and estimates are around the neighborhood of the steady state, we have $\beta_t \to \bar{\beta}$ and $\nu_t \to \infty$ almost surely.[13]

Although eventually agents' belief will converge to the REE belief under the learning rule (34)-(35), the learning model may display strong persistence in belief and price changes during the transition to the REE. This is interesting given that house price changes display strong positive serial correlation at short time horizon, such as one year, as shown by Case and Shiller (1989), and Glaeser and Gyourko (2006).

A deterministic version of the learning model is examined to study the transitional learning dynamics by assuming $\widehat{A}_t = 0$ for all t. I further consider a simplified PLM without learning about $\bar{\phi}^m$ or the steady state, that is, $\widehat{q}_t = \phi_{t-1}^p \widehat{H}_{t-1}^B + \omega_t$. I focus on the T-map mapping from agents' beliefs about the slope coefficient to the parameter in the ALM, $T_2(\phi_{t-1}^p, \phi_t^p) = \frac{\gamma_3 - \gamma_2 \phi_{t-1}^p}{1 - \frac{\gamma_1 \phi_t^p}{\gamma_3 + \phi_t^p \frac{1}{R}}}$, which also determines critically the dynamics of the model with learning about $\bar{\phi}^m$.

As I analyzed previously, the economy with endogenous credit constraints has the property that borrowers' collateral holdings and hence collateral prices depend not only on current beliefs but also on past beliefs. The T-map T_2 contains both ϕ_t^p and ϕ_{t-1}^p. The latter come into play because they affect the inherited debt repayment, which in

[13]Once convergence of agents' estimates in the collateral price process is achieved, agents' belief about the parameter estimates in borrowers' collateral holding equation will also converge to the RE value.

turn offset borrowers' net worth in the current period. This opens the possibility of persistent belief changes in the learning model.

Below momentum[14] in agents' beliefs is defined as one way to capture the persistence in the change of agents' beliefs. Denote b_t agents' belief (parameter estimate) at period t, and $\bar{b}$ the corresponding value at the RE level.

Definition 5 *Momentum*

Momentum is defined as:
(1) if $b_t \leq \bar{b}$ and $b_t > b_{t-1}$, then $b_{t+1} > b_t$.
(2) if $b_t \geq \bar{b}$ and $b_t < b_{t-1}$, then $b_{t+1} < b_t$.

Note b_{t-1}, b_t, and $\bar{b}$ correspond to ϕ_{t-1}^p, ϕ_t^p and $\overline{\phi}^p$ in the learning model, respectively. Suppose agents' belief or parameter estimate is adjusted upward (downward) but still not exceed (not below) the RE level, this will be followed by further upward (downward) belief adjustment.

The following result shows that momentum in beliefs arises more easily in the learning economy with a higher elasticity of the users' costs of collateral with respect to borrowers' collateral holdings, i.e., $\frac{1}{\eta}$, or a higher steady state leverage ratio.

Proposition 6

A sufficient condition ensuring that the benchmark learning economy displays momentum in agents' belief (around the neighborhood of REE beliefs[15]) is either
(1)when $\frac{1}{\eta} > \frac{1}{3}$,
or
(2)when $\frac{1}{\eta} \leq \frac{1}{3}$ and the steady state leverage ratio $1/R > \frac{1}{R(\eta)}$ with $R(\eta) \equiv \frac{\eta}{2}[1 - \sqrt{1 - \frac{4}{\eta+1}}]$.

Proof. See Appendix D. ■

When agents' belief arrives at the RE level from below (above), that is, $\phi_{t-1}^p < \phi_t^p \leq \overline{\phi}^p$ ($\phi_{t-1}^p > \phi_t^p \geq \overline{\phi}^p$), the realization of the parameter in the actual law of motion $T_2(\phi_{t-1}^p, \phi_t^p)$ will be higher (lower) than the RE value if the above conditions hold. Agents' belief updating equations (36)-(37) implies

$$\begin{aligned} \phi_{t+1}^p &= \phi_t^p + \frac{1}{t+N} S_{t+1}^{-1} \widehat{H}_t^B (\widehat{q}_t - \widehat{H}_t^B \phi_t^p) \\ &= \phi_t^p + \frac{1}{t+N} S_{t+1}^{-1} \left(\widehat{H}_t^B\right)^2 (T_2(\phi_{t-1}^p, \phi_t^p) - \phi_t^p) \end{aligned}$$

Using realized collateral prices, agents will update their belief further upward (downward).

[14] This follows Adam, Marcet and Nicolini (2009).

[15] Due to the denominator of the T_2 mapping is nonlinear in current belief ϕ_t^p, a first-order Taylor expansion of the denominator around the REE belief is done for deriving this proposition, as can be seen in Appendix D.

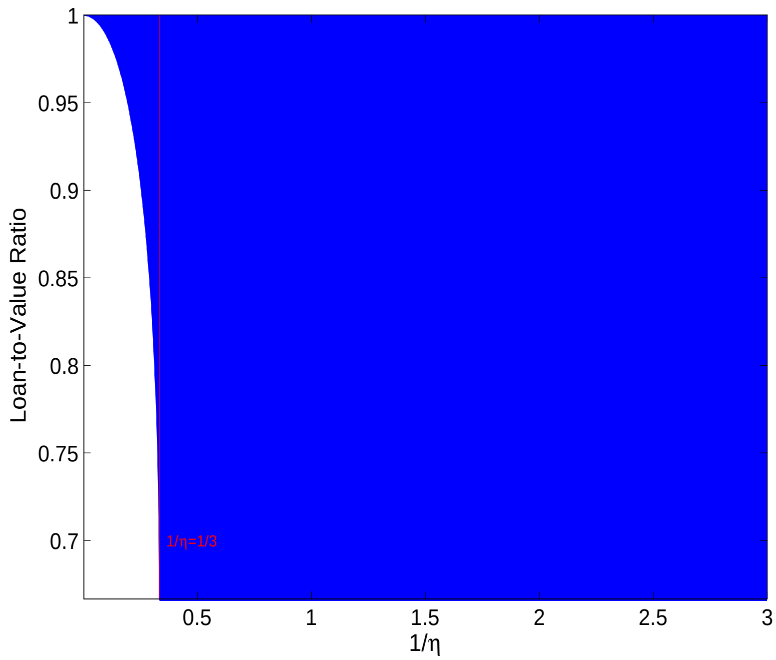

Figure 3: Threshold function $\frac{1}{R}(\frac{1}{\eta})$ and parameter combinations generating momentum

In figure 3, I plot the threshold leverage ratio (or loan-to-value ratio) $\frac{1}{R}$ as a function of $\frac{1}{\eta}$, i.e., $\frac{1}{R}(\frac{1}{\eta})$, and summarize parameter combinations that generate momentum in beliefs in the shaded area. The threshold leverage ratio is a decreasing function in $\frac{1}{\eta}$ if $\frac{1}{\eta} < \frac{1}{3}$. This proposition says that regardless of the steady state leverage ratio, the learning economy exhibits momentum in beliefs if the elasticity of the users' costs with respect to borrowers' collateral holdings are larger than $\frac{1}{3}$. When the elasticity $\frac{1}{\eta}$ is relatively small, the model exhibits momentum in beliefs only when the leverage ratio is sufficiently high. It can be seen from figure 3 that for relatively small $\frac{1}{\eta}$, momentum in beliefs can be present in a learning economy with higher leverage ratios but not with lower leverage ratios.

The existing literature also use an alternative specification of the collateral constraint $b_t^B(i) \leq q_t H_t^B(i)$, which implies that the maximum loans an individual household can get is (possibly a fraction of) the current collateral market values instead of the expected liquidation value of collateral, for example, in Boz and Mendoza (2010). The transitional learning dynamics analyzed here may be robust to this alternative specification of collateral constraints. Borrowers' collateral holdings will critically depend on past price beliefs, which affect the collateral price at $t-1$, i.e., q_{t-1}, and hence the inherited debt holdings $b_{t-1}^B(i)$. This generates the possibility of momentum in belief and price changes.

6 Model Extension and Modification of Agents' Belief

In the benchmark model, the steady state leverage ratio $\frac{1}{R}$ is unrealistically high and also determined by the interest rate. To reduce the leverage and separate it from the interest rate, the benchmark model is extended to include a proportional transaction cost τ,[16] so that the maximum loan borrowers can get is a certain fraction of the expected present liquidation value of collateral. In addition, agents' belief system is modified such that they may perceive parameters in their subjective model to drift over time or follow a random walk process. The quantitative results in section 7 are based on the modified version of the model discussed in this section.

6.1 Extended Model with Proportional Transaction Cost

I assume that if borrowers repudiate their debt obligations, lenders can repossess borrowers' collateral by paying a proportional transaction cost $\tau E_t^{\mathcal{P}} q_{t+1} H_t^B$. Now the borrower's collateral constraint becomes

$$b_t^B \leq (1-\tau)\frac{E_t^{\mathcal{P}} q_{t+1}}{R_t} H_t^B \tag{46}$$

The maximum loan borrowers can get is $(1-\tau)E_t^{\mathcal{P}} q_{t+1} H_t^B / R_t$. The steady state leverage ratio is now $\frac{1-\tau}{R}$. Note the benchmark model is nested in the extended one as a special case when $\tau = 0$.

Optimal behavior implies that borrowers still consume only the nontradable part of the production and borrow up to the limit.[17] The system of equations representing the dynamics for the extended model economy are

$$H_t^B = \frac{a + q_t - (1-\tau)E_{t-1}^{\mathcal{P}} q_t}{q_t - \frac{1}{R_t}(1-\tau)E_t^{\mathcal{P}} q_{t+1}} H_{t-1}^B \tag{47}$$

$$H_t^L = G'^{-1}(R_t u_t^e) \tag{48}$$

where $u_t^e = q_t - \frac{1}{R_t} E_t^{\mathcal{P}} q_{t+1}$. The down-payment for buying one unit of collateral, i.e., $q_t - \frac{1}{R_t}(1-\tau)E_t^{\mathcal{P}} q_{t+1}$, differs from the users' cost and the former is larger.

The steady state of the extended model differs from that of the benchmark model, which has $u = \frac{a}{(1-\tau)}$, $q = \frac{aR}{(R-1)(1-\tau)}$, $H^L = G'^{-1}(\frac{aR}{1-\tau})$ and $H^B = \overline{H} - H^L$.

[16] One explanation is that debt enforcement procedures in real world are significantly inefficient and some value is lost during such procedure, as documented by Djankov, Hart, Mcliesh and Shleifer (2008).

[17] To achieve this in the extended model, I need to assume $\frac{\overline{e}}{a} > [\frac{1}{\beta^B(1-\tau)} - 1]$ to ensure the return to investing a marginal unit of tradable consumption good in collateral holding is larger than consuming or saving it at the steady state. Note this assumption holds for the parameterization in the quantitative exercise later.

Log-linearizing the borrowers' collateral demand equation yields

$$\widehat{q}_t = \xi_1 E_t^{\mathcal{P}} \widehat{q}_{t+1} - \xi_2 E_{t-1}^{\mathcal{P}} \widehat{q}_t + \xi_3 \widehat{H}_{t-1}^B + \xi_4 \widehat{A}_t \tag{49}$$

where $\xi_1 = \frac{1}{R} + \frac{1}{\eta}\frac{R-1}{R}\frac{1}{1-\frac{1}{R}(1-\tau)}\frac{1-\tau}{R}$, $\xi_2 = (1-\tau)\frac{1}{\eta}\frac{R-1}{R}\frac{1}{1-\frac{1}{R}(1-\tau)}$, $\xi_3 = \frac{1}{\eta}\frac{R-1}{R}$ and $\xi_4 = -(1 + \frac{\frac{1-\tau}{R}}{1-\frac{1}{R}(1-\tau)}\frac{1}{\eta}\frac{R-1}{R})$. The lenders' collateral demand equation (19) is unchanged in the case with transaction cost. The system governing the extended model economy consists of equations (19) and (49).

The RE solution for collateral prices is following

$$\widehat{q}_t = \bar{\zeta}^m + \bar{\zeta}^p \widehat{H}_{t-1}^B + \bar{\zeta}^s \widehat{A}_t \tag{50}$$

where $\bar{\zeta}^m = 0$, $\bar{\zeta}^p = \frac{(1-\frac{1}{R}(1-\tau))\frac{1}{\eta}}{1+(1-\tau)(\frac{1}{\eta}-\frac{1}{R})}$ and $\bar{\zeta}^s = \frac{\xi_4(\frac{1}{R}\bar{\zeta}^p+\xi_3)+\xi_1\bar{\zeta}^p}{\frac{1}{R}\bar{\zeta}^p+\xi_3-\xi_1\bar{\zeta}^p}$.

Learning agents use borrowers' collateral holdings to forecast collateral prices, so their conditional expectations are $E_{t-1}\widehat{q}_t = \zeta_{t-1}^m + \zeta_{t-1}^p \widehat{H}_{t-1}^B$ and $E_t \widehat{q}_{t+1} = \zeta_t^m + \zeta_t^p \widehat{H}_t^B$. Plugging these expectations into equations (19) and (49), I obtain the actual law of motion for collateral prices under learning

$$\widehat{q}_t = T_1(\zeta_{t-1}^m, \zeta_t^m, \zeta_t^p) + T_2(\zeta_{t-1}^p, \zeta_t^p)\widehat{H}_{t-1}^B + T_3(\zeta_t^p)\widehat{A}_t \tag{51}$$

where $T_1(\zeta_{t-1}^m, \zeta_t^m, \zeta_t^p) = \frac{(\xi_1\zeta_t^m - \xi_2\zeta_{t-1}^m)(1+\frac{\zeta_t^p}{\xi_3 R}) - \frac{\xi_1}{\xi_3}\frac{\zeta_t^m\zeta_t^p}{R}}{1+\frac{\zeta_t^p}{\xi_3 R}-\xi_1\zeta_t^p\frac{1}{\xi_3}}$, $T_2(\zeta_{t-1}^p, \zeta_t^p) = \frac{\xi_3-\xi_2\zeta_{t-1}^p}{1-\frac{\xi_1\zeta_t^p}{\xi_3+\zeta_t^p\frac{1}{R}}}$ and $T_3(\zeta_t^p) = \frac{(\xi_4+\frac{\xi_1\zeta_t^p}{\xi_3+\frac{1}{R}\zeta_t^p})}{1-\frac{\xi_1\zeta_t^p}{\xi_3+\frac{1}{R}\zeta_t^p}}$.

Define the admissible parameter space Θ_1 as following.

Definition 7

The admissible parameter space $\Theta_1 \equiv \{(\eta, R, \tau) | \eta > 0, R > 1, 0 \le \tau < 1\}$.

The following proposition examines the E-stability condition of the MSV equilibrium (50).

Proposition 8

The MSV equilibrium (50) for the economy represented by (19) and (49) is E-stable for all admissible parameters in Θ_1.
Proof. see appendix E. ■

The deterministic dynamics of the learning model are examined by assuming $\widehat{A}_t = 0$ for all t. Again a simplified PLM without learning about $\bar{\zeta}^m$ or the steady state, that is, $\widehat{q}_t = \zeta_{t-1}^p \widehat{H}_{t-1}^B + \omega_t$, is considered. The T-map, mapping agents' belief to the actual

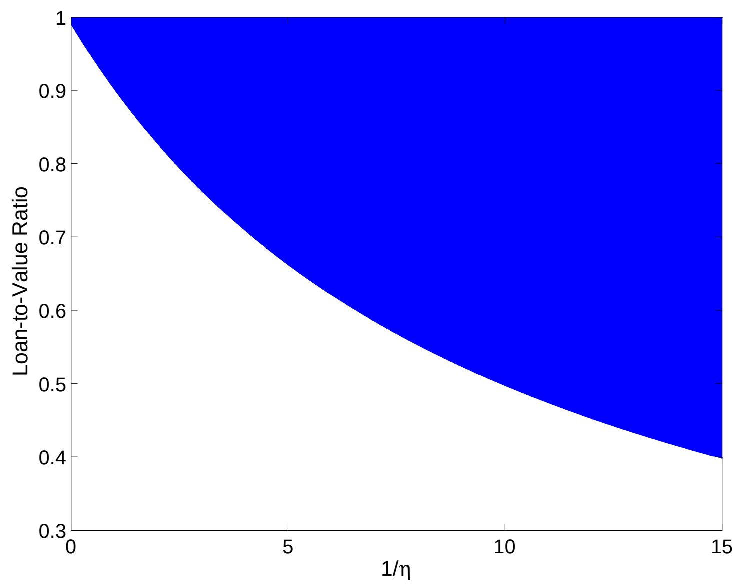

Figure 4: Threshold function $\frac{1-\tau}{R}(\frac{1}{\eta})$ and parameter combinations generating momentum

slope coefficient, is $T_2(\zeta^p_{t-1}, \zeta^p_t) = \frac{\xi_3 - \xi_2 \zeta^p_{t-1}}{1 - \frac{\xi_1 \zeta^p_t}{\xi_3 + \zeta^p_t \frac{1}{R}}}$. The property that the learning model displays momentum in beliefs may emerge given that current collateral prices depend not only on current beliefs but also past beliefs about current prices. The following condition summarizes the dependence of this property on key parameters of the model.

Proposition 9

A sufficient condition[18] guaranteeing momentum in beliefs (around the neighborhood of REE beliefs) in the extended learning model is that parameter combinations of (η, R, τ) satisfy $\frac{1-\tau}{R} > \frac{1}{g(R)\frac{1}{\eta}+1}$ where $g(R) = R(\sqrt{(R-1) + \frac{(R-1)^2}{4}} + \frac{R-1}{2})$.

Proof. see appendix F. ∎

As an example, I set the gross quarterly interest rate R to 1.0088, which is the steady state value of the interest rate I choose in the quantitative exercise later. The shaded area of figure 4 summarizes the parameter combinations $(\frac{1-\tau}{R}, \frac{1}{\eta})$ under which there is momentum in beliefs in the extended learning model.[19] The threshold steady state loan-to-value ratio as a function of $\frac{1}{\eta}$, i.e, $\frac{1-\tau}{R} = \frac{1}{g(R)\frac{1}{\eta}+1}$, is also plotted, which is

[18] It can be shown that when $\tau = 0$, this condition will collapse to the condition in proposition 6.

[19] The parameter combinations generating momentum in beliefs are not sensitive to a wide range of the steady state value of the interest rate R chosen here.

decreasing in the elasticity $\frac{1}{\eta}$. As can be seen from this figure, momentum[20] will arise in the extended learning model when the elasticity of the users' costs with respect to borrowers' collateral holdings is relatively large or the steady state leverage ratio is relatively large.

6.2 Modification of Agents' Belief System

The belief system I assumed in section 4.2 implies that agents' beliefs converge over time to the REE beliefs and the volatility of prices decreases over time. Below the agents' belief system is modified such that they perceive that the fundamental parameter ϕ_t keeps changing over time. Specifically, agents perceive the following random walk model of coefficient variation[21]

$$\phi_t = \phi_{t-1} + \iota_t \qquad E\iota_t\iota_t' = R_{1t} \tag{52}$$

$$y_t = \phi_t' x_t + \varsigma_t \qquad E\varsigma_t\varsigma_t' = R_{2t} \tag{53}$$

where y_t, x_t denote the collateral price $\widehat{q}_t$ in agents' regression and $(1, \widehat{H}_{t-1}^B)$, respectively. Define $P_t = E[(\theta_t - \phi_t)(\theta_t - \phi_t)']$, where θ_t stands for agents' estimates of ϕ_t. The prior distribution of ϕ_0 is assumed to be normal, i.e., $N(\theta_0, P_{0|0})$.

Agents learn unknown parameters ϕ_t via Bayes' law. The posterior mean θ_t can be represented by the following basic Kalman filter recursions[22]

$$\theta_t = \theta_{t-1} + L_t[y_t - x_t'\theta_{t-1}] \tag{54}$$

$$L_t = \frac{P_{t|t-1}x_t}{R_{2t} + x_t'P_{t|t-1}x_t} \tag{55}$$

$$P_{t+1|t} = P_{t|t-1} - \frac{P_{t|t-1}x_t x_t' P_{t|t-1}}{R_{2t} + x_t'P_{t|t-1}x_t} + R_{1t+1} \tag{56}$$

Assume further that agents perceive $R_{1t} = \frac{g}{1-g}P_{t-1|t-1}$ and $R_{2t} = \frac{1}{g}$. The above Kalman filter recursions lead to

$$\theta_t = \theta_{t-1} + L_t[y_t - x_t'\theta_{t-1}]$$

where

$$L_t = \frac{P_{t-1|t-1}x_t}{\frac{1-g}{g} + x_t'P_{t-1|t-1}x_t}$$

$$P_{t|t} = \frac{1}{1-g}\left[P_{t-1|t-1} - \frac{P_{t-1|t-1}x_t x_t' P_{t-1|t-1}}{x_t'P_{t-1|t-1}x_t + \frac{1-g}{g}}\right]$$

[20] Though the parameterizations in the quantitative exercise later do not fall in the shaded area here, persistence in agents' beliefs and in collateral price changes could still arise when the learning friction interacts with real interest rate reductions.

[21] The equation governing the evolution of borrowers' collateral holdings is omitted because learning about the parameters in the equation does not matter for the equilibrium as explained earlier.

[22] The derivations of the Kalman filter recursions can be found in, e.g., Harvey (1989). Note for the model considered here the prior distribution about ϕ_t is the same as the posterior about ϕ_{t-1}, i.e., $\theta_{t|t-1} = \theta_{t-1|t-1}$, so I suppress the conditioned information set and use θ_t for both.

With $S_t^{-1} = P_{t|t}$, appendix G shows the above updating equations are equivalent to the following constant gain learning algorithm

$$\theta_t = \theta_{t-1} + gR_t^{-1}x_t(y_t - x_t'\theta_{t-1}) \tag{57}$$
$$S_t = S_{t-1} + g(x_t x_t' - S_{t-1}) \tag{58}$$

Again to avoid simultaneity between agents' beliefs and actual outcomes, I assume information on the data is introduced with a delay in θ_t. So I actually use

$$\theta_t = \theta_{t-1} + gR_t^{-1}x_{t-1}(y_{t-1} - x_{t-1}'\theta_{t-1}) \tag{59}$$
$$S_t = S_{t-1} + g(x_{t-1}x_{t-1}' - S_{t-1}) \tag{60}$$

Agents discount past observations and give relatively more importance to new data, keeping track of the structural changes in the economy. Unlike the learning algorithm with a decreasing gain, parameter estimates coming from a constant gain learning algorithm can not point-converge to a single value even in a time-invariant economy, but they could still converge in distribution around the true value as long as the gain parameter is sufficiently small.[23]

7 Quantitative Results

I now estimate the learning model to the U.S. economy and show that the learning model can quantitatively account for the recent house prices boom and bust and the associated household debt and aggregate consumption dynamics. Around the year 2001, the US real interest rate considerably dropped and stayed low for an extended period of time, before rising again around the year 2006. The average of 1-year ahead ex-ante real mortgage interest rates[24] from 1997Q1 to 2000Q4 was 3.51%, while the average of real interest rates between 2001Q1-2005Q4 was 2.28%.

I conduct the following experiment. Initially the economy is assumed to be at the steady state and agents' beliefs at 2000Q4 are set to the RE value. The low real interest rates after 2000Q4 and subsequent increases are captured in the following stylized way. The annualized real interest rate at the steady state is set to 3.51%. I let the interest rate fall from 2001Q1, stay unchanged at 2.28% until 2005Q4, and then go back to the steady state. The model is used to predict real house prices, consumption and debt/GDP ratio during 2001Q1-2008Q4. The model predictions below do not use any data after 2000Q4, except for the stylized path information about the real interest rate. Following Campbell (1994), I set the steady state consumption-GDP ratio to 0.745. Then $\widehat{GDP}_t$ is calculated via (40).

[23] The convergence properties of learning models under constant gain learning algorithm are discussed in details in Evans and Honkapohja (2001).

[24] The mortgage rate I use is the "one-year adjustable rate mortgage average in the United States" from Freddie Mac (seriesID: MORTGAGE1US). The ex-ante real interest rate is calculated as mortgage rate minus the median expected 1 year ahead CPI inflation rate from the survey of professional forecasters.

Denote by ck the product of the productivity gap $\frac{(a+\bar{e})-G'}{(a+\bar{e})}$ and borrowers' production share $\frac{(a+\bar{e})H^B}{Y}$ in aggregate output. The gain parameter g, the elasticity $\frac{1}{\eta}$, the parameter τ, and the parameter ck, are chosen to minimize the absolute distance between the learning model predicted and actual house prices, consumption and debt/GDP ratio as following

$$\sum_{t=2001Q1}^{2008Q4}\left(\frac{|\widehat{q}_t-\widehat{\mathbf{q}}_t|}{std(\widehat{\mathbf{q}}_t)}+\frac{|\widehat{c}_t-\widehat{\mathbf{c}}_t|}{std(\widehat{\mathbf{c}}_t)}+\frac{|\widehat{Debt/GDP}_t-\widehat{\mathbf{Debt/GDP}}_t|}{std(\widehat{\mathbf{Debt/GDP}}_t)}\right)$$

where boldface letters denote actual data and std stands for standard deviation.

The minimization yields that $g=0.065$, $\frac{1}{\eta}=2.46$,[25] $\tau=0.45$, and $ck=0.43$. This choice of τ implies that the steady state loan-to-value ratio is 0.54.[26] The value of ck implies roughly, say, both the productivity gap and borrowers' production share are $\frac{2}{3}$.[27]

Recall the interest rate at period t in my model is $R_t=\frac{A_t}{\beta^L e^{\frac{1}{2}\sigma_A^2}}$. To get low interest rates during 2001-2005, two alternatives are assumed. The first one corresponds to choosing a sequence of 20 quarters lower-than-steady-state realizations of lenders' patience factor A_t to match the low interest rate, i.e., 2.28%. Alternatively I assume lenders' discount factors β^L shift upward exogenously during 2001-2005 and back to their old value during 2006-2008, meanwhile A_t is assumed to be at its steady state value 1 throughout 2001-2008.[28]

Model predictions under RE for these two alternatives are provided below. The parameterizations of the RE models are the same as the estimated learning model. "RE-I" model is used to denote the RE model with repeated unexpected shocks, while "RE-II" model stands for the RE model with exogenous shifts in lenders' discount factors β^L. Note the performance of the learning model will be the same under either of the two assumptions because expectations about future interest rates do not enter the system of equations governing the model economy. The assumption of a sequence of transitory shocks is only used for the purpose of illustrating the different dynamics under RE and learning. The quantitative results of the learning model presented in figure 6 do not rely on this assumption because they can be alternatively obtained in the second scenario.

[25] The KM considered a larger $\frac{1}{\eta}$. A larger $\frac{1}{\eta}$ improves the amplification of both the RE and the learning model. Nevertheless, the improvement for the RE model is very limited. Further discussions of this paramter can be found in footnote 32.

[26] This is consistent with the estimate of household loan-to-value ratio by Iacoviello (2005) with mean 0.55 and standard deviaton 0.09.

[27] The productivity gap of $\frac{2}{3}$ is also considered by Cordoba and Ripoll (2004) in their figure 5.

[28] Admittedly, these are short-cuts, but necessary ways, to model interest rate movement in my context.

7.1 Response to 1% i.i.d. shock to interest rate

Figure 5 depicts the responses to an unexpected negative shock to lenders' patience factor A_t and hence to the interest rate. In the impact period, the real house price under RE rises by about 1.2%, while consumption and debt/GDP ratio rise by about 0.6% and 2%,[29] respectively. However, they do not rise further after the shock disappears. Consumption decays exponentially, while the house price drops substantially and then converges persistently to the steady state. The learning model generates additional propagation of the shock due to belief revisions and the dynamic interaction between beliefs and price realizations. The peak responses of house prices, consumption and debt/GDP ratio are 1.2%, 0.73%, and 2.3%, respectively. The learning model also generates positive persistence in forecast errors,[30] as can be seen from the lower right panel.

The impulse response functions here is roughly consistent with the estimation of Iacoviello and Neri (2010). They find that following 1% negative i.i.d. monetary policy shock real house prices increase by about 0.65%, while the response of consumption is hump-shaped and its peak is 0.5% below the steady state.[31]

7.2 Boom and bust in house prices, debt and aggregate consumption dynamics

Figure 6 contrasts model predictions of the learning model and of the "RE-I model" with actual data. Under RE, prices and quantities jump upward following real interest rate reduction. House prices continue to increase due to the persistence in the users' costs and repeated unexpected negative shocks to the interest rate. They peak at about 14.4% above the steady state. After the shocks to interest rate disappear, house price starts to revert. The RE model under-predicts largely the levels of prices and quantities.[32]

The learning model predicts house prices, debt/GDP ratio and consumption rather well, in particular during the price boom years. Following the real interest rate reduction, real house prices under the learning model increase at a faster pace than under RE. The learning model generates large additional amplification of prices and quantities

[29] Note debt/GDP ratio here is percentage changes from the steady state value.

[30] The forecast error is defined as $\widehat{q}_t - E_{t-1}\widehat{q}_t$.

[31] They considered 1% *positive* i.i.d. monetary policy shock, which will lead to a decrease of house prices by about 0.65% and hump-shaped response of consumption with the trough 0.5% below the steady state. Since their model is linearized RE model, the impulse response functions are symmetric with respect to positive and negative shocks.

[32] Given the pattern of the interest rates I consider, the response of house prices in the "RE-I" model will be larger if the elasticity $\frac{1}{\eta}$ is larger. But this improvement is very limited. Similarly, the improvement of the performance of the RE model with larger leverage ratio is very limited as long as the leverage ratio is not too close to 1, otherwise the transfer of collateral holding will be dramatically large. Regardless of the value of these two parameters, the REE house prices will revert when the interest rate starts to revert. So the RE model cannot match the turning point of house prices.

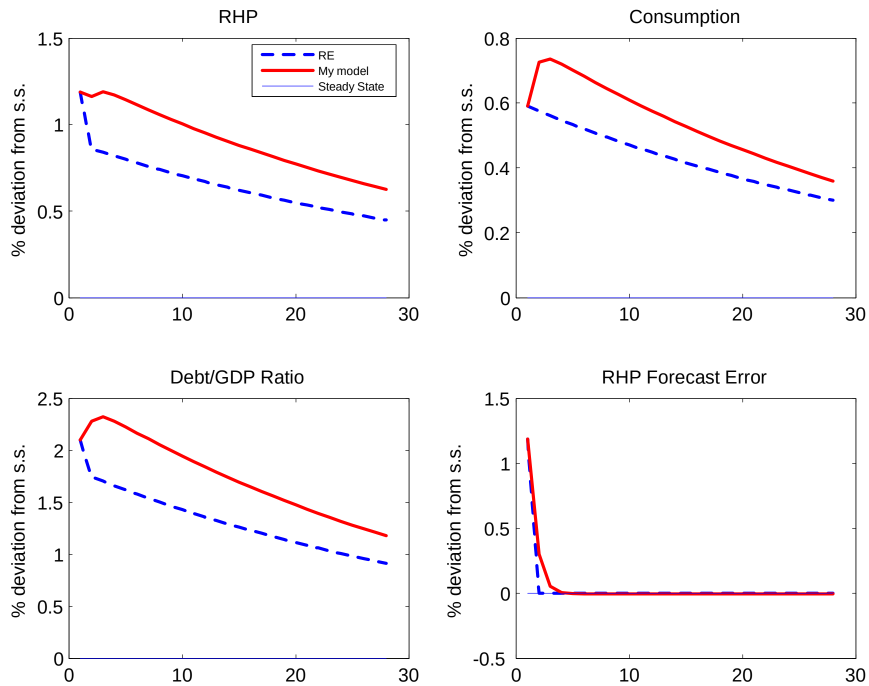

Figure 5: Response to 1% unexpected negative shock to interest rate

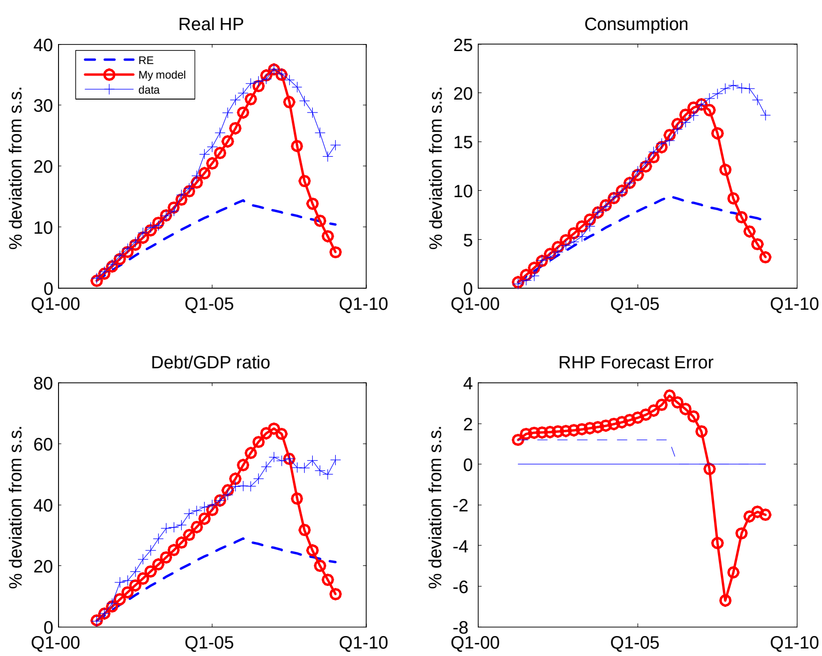

Figure 6: Model Predictions of the RE-I Model, Learning Model and Actual Data

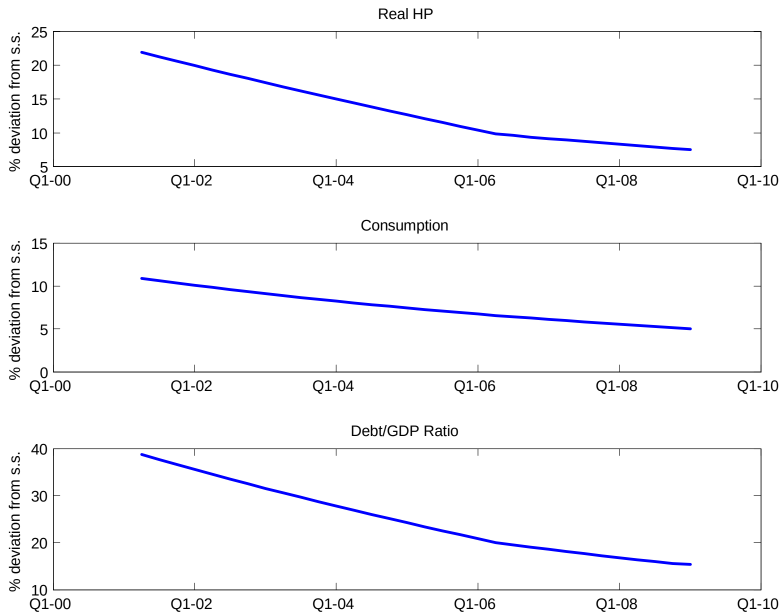

Figure 7: Predictions of the RE-II Model with Anticipated Interest Rate Movement

relative to the RE version of the model. The peak of predicted house prices under the learning model is about 35.9% at 2006Q4, which is about 2.5 times the peak response of the RE model. House price booms arise mainly from more optimistic expectation about future prices due to both more optimistic beliefs and the rising amount of collateral held by households. The rising household credit market debt/GDP ratio is due to both the house price boom and the rising amount of collateral held by households. The learning model also generates a consumption boom due to shifts of collateral to more productive households. The peak response of consumption in the learning model is 18.8%, which is twice as large as that in the RE model.

House prices in the "RE-I model" start to revert once the shocks disappear, while the learning model matches rather well the turning point of house prices in the data. House prices in the learning model rise further for a few quarters as in the data even after the shocks disappear. This arises from belief revisions and the interaction of beliefs and price realizations.

The forecast errors of the RE model are constant during 2001Q1-2005Q4 and then become zero afterwards. They are completely driven by the pattern of exogenous shocks. In contrast, the learning model generates internal and positive persistence in forecast errors.

7.3 RE price dynamics with anticipated interest rate movement

Figure 7 displays the "RE-II" model dynamics,[33] i.e., when the low interest rates during 2001Q1-2005Q4 are interpreted as a result of an exogenous upward shift of lenders' discount factor β^L. Except for the initial period, agents understand the effects of such structural change and could perfectly foresee the entire path of prices and quantities given that there is no remaining uncertainty after the initial real rate reduction. The real house prices jump immediately upward and then converge to the steady state. This is inconsistent with the pattern of prices and quantities observed in the data. In particular, the model does not generate persistent increases in house prices due to the lack of capital gains in borrowers' collateral holdings after the initial period.

8 Conclusions

The paper presents a general equilibrium model with housing collateral constraints in which agents have imperfect knowledge and learn about the parameters linking prices and fundamentals. An intrinsic property of the credit-constrained economy with learning is that collateral holdings and collateral prices depend not only on current price beliefs but also on past price beliefs regardless of the specification of agents' beliefs. I also find a leveraged economy with a higher leverage ratio is more prone to self-reinforcing learning dynamics when agents' subjective beliefs are allowed to have small deviations from REE beliefs. Estimated to the US economy, the learning model can quantitatively account for the recent US boom and bust in house prices, as well as the household debt and aggregate consumption dynamics following the persistent fall in the level of the real interest rate after the year 2000.

The nonlinear dependence of economic volatilities on the leverage ratio may help to understand economic volatilities of aggregate variables across regimes with different leverages or in cross-country comparisons. For example, in studying the behavior of money, credit and macroeconomic indicators for 14 countries over the year 1870-2008, Schularick and Taylor (2011) find output losses today are as large as Pre-WW2 despite more activist policies and the presence of deposit insurance and allude to the important role of increased leverage in the financial sector. Second, the model provides additional rationale for reasonable capital requirement regulation to avoid extremely high leverage ratio regime.

Asset prices/values play a large role in aggregate fluctuations through many channels such as households, corporate balance sheets, bank capital channels, etc. It would be interesting to study and quantify further the role of the interaction of agents' uncertainty in financial markets considered here with other kind of credit market frictions

[33] For simulating the model in such senario, I firstly solve the law of motion for prices and quantities during 2006Q1-2008Q4. Then with them I recursively solve backward the policy function until 2001Q1.

in aggregate fluctuations. It would also be interesting to look into how the uncertainty in financial markets interacts with economic agents' decisions in other markets, such as the labor market. Finally, the model facilitates the discussion of how monetary policies can affect whether a bubble occurs in the first place and how they can affect the speed at which it deflates, as well as the appropriate design of policies to stabilize the economy and the financial system.

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A Log-linearization of the Benchmark Model

The following system of equations represents the dynamics of the benchmark economy

$$H_t^B(i) = \frac{(a + q_t - E_{t-1}^{\mathcal{P}} q_t)}{q_t - \frac{1}{R_t} E_t^{\mathcal{P}^j} q_{t+1}} H_{t-1}^B(i) \tag{61}$$

$$u_t^e = q_t - \frac{1}{R_t} E_t^{\mathcal{P}} q_{t+1} = \frac{G'(H_t^L)}{R_t} \tag{62}$$

The interest rate $R_t = \frac{A_t}{\beta^L}$ with steady state value $R = \frac{1}{\beta^L}$. The steady state value of endogenous variables are $q = \frac{aR}{R-1}$, $u = a$, $H^G = G'^{-1}(Ra)$, $H^B = \overline{H} - H^G$, $b^B = qH^B/R$ and $c^B = \overline{e}H^B$.

Define $\widehat{A}_t \equiv A_t - 1$. Log-linearizing equation (62) yields

$$\widehat{u}_t^e = \frac{1}{\eta} \widehat{H}_t^B = \frac{R}{R-1}(\widehat{q}_t - \frac{1}{R_t} E_t^{\mathcal{P}} \widehat{q}_{t+1} + \widehat{A}_t) \tag{63}$$

Log-linearizing equation (61) leads to

$$\widehat{H}_t^B = \frac{R}{R-1}[(\widehat{q}_t - E_{t-1}^{\mathcal{P}} \widehat{q}_t) - (\widehat{q}_t - \frac{1}{R} E_t^{\mathcal{P}} \widehat{q}_{t+1})] + \widehat{H}_{t-1}^B + \widehat{A}_t \tag{64}$$

Plugging equation (64) into (63), I obtain

$$\widehat{q}_t = \gamma_1 E_t^{\mathcal{P}} \widehat{q}_{t+1} - \gamma_2 E_{t-1}^{\mathcal{P}} \widehat{q}_t + \gamma_3 \widehat{H}_{t-1}^B + \gamma_4 \widehat{A}_t \tag{65}$$

where $\gamma_1 = \frac{1}{R}(1 + \frac{1}{\eta})$, $\gamma_2 = \frac{1}{\eta}$, $\gamma_3 = \frac{R-1}{R}\frac{1}{\eta}$, and $\gamma_4 = -(1 + \frac{1}{\eta}\frac{1}{R})$.

B Derivation of the Bayesian Posterior Mean[34]

I assume the prior distribution of unknown parameters, i.e., the parameters linking prices and fundamentals (ϕ^m, ϕ^p) and the residual precision $K \equiv \frac{1}{\sigma_\varepsilon^2}$, is a Normal-Gamma distribution as following

$$\begin{aligned} K &\sim G(\gamma_0, d_0^{-2}) \\ (\phi^m, \phi^p)' \mid K &= k \sim N((\theta_0^m, \theta_0^p)', (\nu_0 k)^{-1}) \end{aligned}$$

The prior distribution of K is a gamma distribution and the conditional prior of $\phi \equiv (\phi^m, \phi^p)$ given K is a multivariate normal distribution.

I drop the terms which do not involve (θ, k) by using the proportionality symbol. The conditional probability of the collateral price is a normal distribution with following conditional probability density function

$$p(y_t|\theta, k) \propto k^{\frac{1}{2}} exp\{-\frac{k}{2}(y_t - x_t'\theta)'(y_t - x_t'\theta)\}$$

The prior density of the parameters is following

[34] The derivation follows DeGroot (1974).

$$p(\theta, h) \propto k^{\gamma_{t-1}-1} exp\{-d_{t-1}^{-2} k\} k^{\frac{1}{2}} exp\{-\frac{k}{2}(\theta - \theta_{t-1})'\nu_{t-1}(\theta - \theta_{t-1})\}$$

I show the posterior distribution of the parameters are following

$$\begin{aligned}
\theta | K &= k \sim N(\theta_t, (\nu_t k)^{-1}) \\
K &\sim G(\gamma_t, d_t^{-2})
\end{aligned}$$

with probability density function

$$p(\theta, k|y_t) \propto k^{\gamma_t - 1} exp\{-d_t^{-2} k\} k^{\frac{1}{2}} exp\{-\frac{k}{2}(\theta - \theta_t)'\nu_t(\theta - \theta_t)\}$$

where

$$\begin{aligned}
\theta_t &= \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t (y_t - x_t' \theta_{t-1}) \\
\nu_t &= \nu_{t-1} + x_t x_t' \\
\gamma_t &= \gamma_{t-1} + \frac{1}{2} \\
d_t^{-2} &= d_{t-1}^{-2} + \frac{1}{2}(y_t - x_t'\theta_{t-1})'(x_t x_t' + \nu_{t-1})^{-1} \nu_{t-1} (y_t - x_t' \theta_{t-1})
\end{aligned}$$

The above equations can be derived using Baye's law. The critical intermediate steps are presented here. The posterior density

$$p(\theta, k|y_t) \propto p(y_t|\theta, k) p(\theta, k)$$

It can be derived from the right hand side that the posterior mean of the parameters is

$$\begin{aligned}
&\theta_t \\
= \; & (x_t x_t' + \nu_{t-1})^{-1} (\nu_{t-1}\theta_{t-1} + x_t y_t) \\
= \; & (x_t x_t' + \nu_{t-1})^{-1} \nu_{t-1}\theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t y_t \\
= \; & \theta_{t-1} - (x_t x_t' + \nu_{t-1})^{-1} x_t x_t' \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t y_t \\
= \; & \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t (y_t - x_t'\theta_{t-1})
\end{aligned}$$

Note

$$\begin{aligned}
& (y_t - x_t'\theta)'(y_t - x_t'\theta) + (\theta - \theta_{t-1})'\nu_{t-1}(\theta - \theta_{t-1}) \\
= \; & y_t' y_t - 2\theta' x_t y_t + \theta' x_t x_t' \theta + \theta' \nu_{t-1} \theta - 2\theta' \nu_{t-1} \theta_{t-1} + \theta_{t-1}' \nu_t \theta_{t-1} \\
= \; & \theta'(x_t x_t' + \nu_{t-1})\theta - 2\theta'(x_t y_t + \nu_{t-1}\theta_{t-1}) + y_t' y_t + \theta_{t-1}' \nu_{t-1} \theta_{t-1} \\
= \; & \left(\theta - (x_t x_t' + \nu_{t-1})^{-1}(\nu_{t-1}\theta_{t-1} + x_t y_t)\right)' (x_t x_t' + \nu_{t-1}) \left(\theta - (x_t x_t' + \nu_{t-1})^{-1}(\nu_{t-1}\theta_{t-1} + x_t y_t)\right) \\
& \left[y_t' y_t + \theta_{t-1}' \nu_{t-1}\theta_{t-1} - (\nu_{t-1}\theta_{t-1} + x_t y_t)'(x_t x_t' + \nu_{t-1})^{-1}(\nu_{t-1}\theta_{t-1} + x_t y_t)\right]
\end{aligned}$$

C Proof of Proposition 4

Note agents' belief about the parameters $(\varkappa^m, \varkappa^p, \frac{1}{\sigma_\varrho^2})$ in the borrowers' collateral holding process (24) do not affect equilibrium outcome. So it suffices to examine only the associated ODEs for (ϕ^m, ϕ^p). Local stability of the MSV RE is determined by the stability of the following associated ODEs

$$\begin{aligned}\frac{d\phi^m}{d\tau} &= T_1(\phi^m, \phi^p) - \phi^m \\ \frac{d\phi^p}{d\tau} &= T_2(\phi^p, \phi^p) - \phi^p\end{aligned}$$

where $T_1(\phi^m_{t-1}, \phi^m_t, \phi^p_t) = \frac{(\gamma_1\phi^m_t - \gamma_2\phi^m_{t-1})(1+\frac{\phi^p_t}{\gamma_3 R}) - \frac{\gamma_1}{\gamma_3}\frac{\phi^m_t\phi^p_t}{R}}{1+\frac{\phi^p_t}{\gamma_3 R} - \gamma_1\phi^p_t\frac{1}{\gamma_3}}$, $T_2(\phi^p_{t-1}, \phi^p_t) = \frac{\gamma_3 - \gamma_2\phi^p_{t-1}}{1-\frac{\gamma_1\phi^p_t}{\gamma_3+\phi^p_t\frac{1}{R}}}$, $\gamma_1 = \frac{1}{R}(1+\frac{1}{\eta})$, $\gamma_2 = \frac{1}{\eta}$, and $\gamma_3 = \frac{R-1}{R}\frac{1}{\eta}$. The rational expectations solution for collateral prices is

$$\widehat{q}_t = \bar{\phi}^m + \bar{\phi}^p \widehat{K}^F_{t-1} + \bar{\phi}^s \widehat{A}_t \tag{66}$$

where $\bar{\phi}^m = 0$, $\bar{\phi}^p = \frac{R-1}{R}\frac{1}{\eta+1-\frac{\eta}{R}}$ and $\bar{\phi}^s = -(\frac{\frac{1}{\eta}}{R-1}+1)$.

The E-stability for the above MSV equilibrium requires the eigenvalues of the Jacobian of the right hand side of the ODEs is negative. Since ϕ^m does not show up in the ODE for β^p, the eigenvalues will be on the diagonal of the Jacobian and only two partial derivatives, i.e.,$\frac{\partial T_1(\phi^m,\phi^p)}{\partial \phi^m}|_{\phi^m=\bar{\phi}^m, \phi^p=\bar{\phi}^p}$ and $\frac{\partial T_2(\phi^p,\phi^p)}{\partial \phi^p}|_{\phi^p=\bar{\phi}^p}$, matter for the E-stability.

Plugging the parameters $\gamma' s$, I obtain

$$\begin{aligned}T_2(\phi^p, \phi^p) &= \frac{\frac{1}{\eta}(\phi^p - \frac{R-1}{R})}{\frac{(\phi^p-(R-1))}{\eta\phi^p+(R-1)}} \\ &= \frac{(\phi^p - \frac{R-1}{R})\frac{1}{\eta}(\eta\phi^p + (R-1))}{(\phi^p - (R-1))}\end{aligned}$$

The derivative with respect to ϕ^p is

$$\begin{aligned}\frac{\partial T_2(\phi^p, \phi^p)}{\partial \phi^p}|_{\phi^p=\bar{\phi}^p} &= \frac{\phi^p - \frac{R-1}{R} + \frac{R-1}{\eta}}{\phi^p - (R-1)} \\ &= \frac{\frac{\eta}{R} - \eta + R(1+\frac{1}{\eta} - \frac{1}{R})}{1 - R(\eta + 1 - \frac{\eta}{R})}\end{aligned} \tag{67}$$

Note the denominator of (67) is negative if $R > 1$. Below I show the numerator is positive. The numerator is positive is equivalent to

$$\begin{aligned}\frac{\eta}{R} - \eta + R + \frac{R}{\eta} - 1 &> 0 \\ (\eta - R)^2 + R^2\eta &> 0\end{aligned}$$

which holds for all admissible parameters. Note

$$T_1(\phi^m, \phi^p) = \frac{(\gamma_1 - \gamma_2(1 + \frac{\phi^p}{\gamma_3 R}))\phi^m}{1 + \frac{\phi^p}{\gamma_3 R} - \gamma_1 \phi^p \frac{1}{\gamma_3}}$$

Given that $T_1(\overline{\phi}^m, \overline{\phi}^p) = \overline{\phi}^m = 0$, the first derivative

$$\begin{aligned}\frac{\partial T_1(\phi^m, \phi^p)}{\partial \beta^m}|_{\beta^m = \bar{\beta}^m, \beta^p = \bar{\beta}^p} &= \frac{(\gamma_1 - \gamma_2(1 + \frac{\phi^p}{\gamma_3 R}))}{1 + \frac{\phi^p}{\gamma_3 R} - \gamma_1 \phi^p \frac{1}{\gamma_3}}|_{\beta^m = \bar{\beta}^m, \beta^p = \bar{\beta}^p} \\ &= \frac{(\gamma_1 - \gamma_2)\gamma_3 R - \gamma_2 \overline{\phi}^p}{\gamma_3 R + \overline{\phi}^p (1 - R\gamma_1)} \\ &= \frac{(1 + \frac{1}{\eta} - \frac{R}{\eta})(\eta + 1 - \frac{R}{\eta}) - 1}{(\eta + 1 - \frac{R}{\eta})R - 1}\end{aligned}$$

Since $1 + \frac{1}{\eta} - \frac{R}{\eta} < R$ holds for all admissible parameters, I have $\frac{\partial T_1(\phi^m, \phi^p)}{\partial \beta^m}|_{\beta^m = \bar{\beta}^m, \beta^p = \bar{\beta}^p} < 1$. To sum up, the MSV equilibrium is E-stable for all admissible parameters.

D Proof of Proposition 6

Recall the T-map on agents' perceived slope coefficients $T_2(\phi^p_{t-1}, \phi^p_t)$.

$$T_2(\phi^p_{t-1}, \phi^p_t) = \frac{\gamma_3 - \gamma_2 \phi^p_{t-1}}{1 - \frac{\gamma_1 \phi^p_t}{\gamma_3 + \phi^p_t \frac{1}{R}}} \tag{68}$$

Substituting γ_1, γ_2 and γ_3 into (68) and simplifying the latter yields

$$T_2 = \frac{\frac{R-1}{R} - \phi^p_{t-1}}{1 - f(\phi^p_t)} \tag{69}$$

where $f(\phi_t^p) \equiv 2 - \frac{(\eta+1)(R-1)}{\eta\phi_t^p + R-1}$. Further algebra yields

$$T_2(\phi_{t-1}^p, \phi_t^p) = \bar{\phi}^p + \frac{-\bar{\phi}^p(1-f(\phi_t^p)) + (\frac{R-1}{R} - \phi_{t-1}^p)}{1-f(\phi_t^p)} \tag{70}$$

$$= \bar{\phi}^p + \frac{-\bar{\phi}^p(1-f(\phi_t^p)) + \bar{\phi}^p(1-f(\bar{\phi}^p)) - (\frac{R-1}{R} - \bar{\phi}^p) + (\frac{R-1}{R} - \phi_{t-1}^p)}{1-f(\phi_t^p)} \tag{71}$$

$$= \bar{\phi}^p + \frac{(\bar{\phi}^p - \phi_{t-1}^p) - \bar{\phi}^p(f(\bar{\phi}^p) - f(\phi_t^p))}{1-f(\phi_t^p))} \tag{72}$$

$$\simeq \bar{\phi}^p + \frac{(\bar{\phi}^p - \phi_{t-1}^p) - \bar{\phi}^p f'(\bar{\phi}^p)(\phi^p - \phi_t^p)}{1-f(\phi_t^p))} \tag{73}$$

In equation (71), I use the REE belief $\bar{\phi}^p$ is a fixed point of the T_2 map, i.e., $\bar{\phi}_p = \frac{\frac{R-1}{R} - \bar{\phi}_p}{1-f(\bar{\phi}_p)}$. In the last step I do a first-order Taylor approximation of the function f at the RE belief. A sufficient condition to guarantee momentum is $1 > \bar{\phi}^p f'(\bar{\phi}^p)$. Then given $\phi_{t-1}^p < \phi_t^p < \bar{\phi}^p$, I have $T_2(\phi_{t-1}^p, \phi_t^p) > \bar{\phi}^p$. Belief updating equation (36) implies

$$\begin{aligned} \phi_{t+1}^p &= \phi_t^p + \frac{1}{t+N} S_{t+1}^{-1} \widehat{H}_t^B (\widehat{q}_t - \widehat{H}_t^B \phi_t^p) \\ &= \phi_t^p + \frac{1}{t+N} S_{t+1}^{-1} \left(\widehat{H}_t^B\right)^2 (T_2(\phi_{t-1}^p, \phi_t^p) - \phi_t^p) \end{aligned}$$

This implies further that agents will update their belief further upward, i.e., $\phi_{t+1}^p > \phi_t^p$. Similarly, given $\phi_{t-1}^p > \phi_t^p > \bar{\phi}$, I have $T_2(\phi_{t-1}^p, \phi_t^p) < \bar{\phi}$. Then belief updating implies $\phi_{t+1}^p < \phi_t^p$.

It can be shown that the condition $1 > \bar{\phi}^p f'(\bar{\phi}^p)$ is equivalent to

$$(R - \frac{\eta}{2})^2 > \eta^2(\frac{1}{4} - \frac{1}{1+\eta}) \tag{74}$$

Case 1: if $\frac{1}{4} < \frac{1}{1+\eta}$ or $\frac{1}{\eta} > \frac{1}{3}$, (74) is satisfied. Case 2: if $\frac{1}{4} > \frac{1}{1+\eta}$ or $\eta > 3$, then I have either $R < \frac{\eta}{2}(1 - \sqrt{1 - \frac{4}{1+\eta}})$ or $R > \frac{\eta}{2}(1 + \sqrt{1 - \frac{4}{1+\eta}})$. The latter is dropped because it will imply the gross real rate $R > 1.5$.[35]

[35] The other reason that the gross interest rate R should not be too large is the assumption, i.e., $\bar{c} > (\frac{1}{\beta^B} - 1)a$, is more likely violated the higher is R or the lower is β^L. This assumption guarantees that borrowers will not want to consume more than the bruised consumption good.

E Proof of Proposition 8

Local stability of the MSV RE is determined by the stability of the following associated ODEs

$$\begin{aligned}\frac{d\zeta^m}{d\tau} &= T_1(\zeta^m,\zeta^p)-\zeta^m \\ \frac{d\zeta^p}{d\tau} &= T_2(\zeta^p,\zeta^p)-\zeta^p\end{aligned}$$

where $T_1(\zeta^m,\zeta^p) = \frac{(\xi_1\zeta^m-\xi_2\zeta^m)(1+\frac{\zeta^p}{\xi_3 R})-\frac{\xi_1}{\xi_3}\frac{\zeta^m\zeta^p}{R}}{1+\frac{\zeta^p}{\xi_3 R}-\xi_1\zeta^p\frac{1}{\xi_3}}$, $T_2(\zeta^p,\zeta^p) = \frac{\xi_3-\xi_2\zeta^p}{1-\frac{\xi_1\zeta^p}{\xi_3+\zeta^p\frac{1}{R}}}$, $\xi_1 = \frac{1}{R}+\frac{1}{\eta}\frac{R-1}{R}\frac{1}{1-\frac{1}{R}(1-\tau)}\frac{1-\tau}{R}$, $\xi_2 = (1-\tau)\frac{1}{\eta}\frac{R-1}{R}\frac{1}{1-\frac{1}{R}(1-\tau)}$, $\xi_3 = \frac{1}{\eta}\frac{R-1}{R}$, $\overline{\zeta}^m = 0$, and $\overline{\zeta}^p = \frac{(1-\frac{1}{R}(1-\tau))\frac{1}{\eta}}{1+(1-\tau)(\frac{1}{\eta}-\frac{1}{R})}$.

The E-stability condition requires the eigenvalues of the Jacobian of the right hand side of the above ODEs are negative. Since ζ^m does not show up in the ODE for ζ^p, the eigenvalues will be on the diagonal of the Jacobian and only two partial derivatives, i.e., $\frac{\partial T_1(\zeta^m,\zeta^p)}{\partial\zeta^m}|_{\zeta^m=\overline{\zeta}^m,\zeta^p=\overline{\zeta}^p}$ and $\frac{\partial T_2(\zeta^p,\zeta^p)}{\partial\zeta^p}|_{\zeta^p=\overline{\zeta}^p}$, matter for the E-stability.

$$\begin{aligned}T_2(\zeta^p,\zeta^p) &= \frac{\xi_3-\xi_2\zeta^p}{1-\frac{\xi_1\zeta^p}{\xi_3+\zeta^p\frac{1}{R}}} \\ &= \frac{(\xi_3+\frac{\zeta^p}{R})(\xi_3-\xi_2\zeta^p)}{\xi_3+(\frac{1}{R}-\xi_1)\zeta^p}\end{aligned}$$

The derivative of T_2 with respect to ζ^p is

$$\frac{\partial T_2(\zeta^p,\zeta^p)}{\partial\zeta^p}|_{\zeta^p=\overline{\zeta}^p} = \frac{(\frac{1}{R}-\xi_2)\xi_3-\overline{\zeta}^p(\frac{2}{R}\xi_2+\frac{1}{R}-\xi_1)}{\xi_3+(\frac{1}{R}-\xi_1)\overline{\zeta}^p}$$

I proceed to show the above expression is smaller than 1.

$$\frac{\partial T_2(\zeta^p,\zeta^p)}{\partial\zeta^p}|_{\zeta^p=\overline{\zeta}^p} < 1$$

is equivalent to

$$(\frac{1}{R}-\xi_2)\xi_3-\overline{\zeta}^p(\frac{2}{R}\xi_2+\frac{1}{R}-\xi_1) < \xi_3+(\frac{1}{R}-\xi_1)\overline{\zeta}^p$$

Rearranging the above inequality yields

$$(\frac{1}{R}-\xi_2-1)\xi_3 < 2\overline{\zeta}^p(\frac{\xi_2}{R}+\frac{1}{R}-\xi_1)$$

Plugging the parameters, I can show the left hand side of the above inequality is negative and the right hand side is zero.

Now I turn to the first derivative. Recall $T_1(\zeta^m, \zeta^p) = \frac{\xi_1\zeta^m - \xi_2\zeta^m(1+\frac{\zeta^p}{\xi_3 R})}{1+\frac{\zeta^p}{\xi_3 R} - \xi_1\zeta^p\frac{1}{\xi_3}}$ and

$$\begin{aligned}\frac{\partial T_1(\zeta^m, \zeta^p)}{\partial \zeta^m}|_{\zeta^m=\overline{\zeta}^m, \zeta^p=\overline{\zeta}^p} &= \frac{\xi_1 - \xi_2(1+\frac{\overline{\zeta}^p}{\xi_3 R})}{1+\frac{\overline{\zeta}^p}{\xi_3 R} - \xi_1\overline{\zeta}^p\frac{1}{\xi_3}} \\ &= \frac{(\xi_1-\xi_2)\xi_3 R - \xi_2\overline{\zeta}^p}{\xi_3 R + \overline{\zeta}^p(1-\xi_1 R)}\end{aligned}$$

Note the denominator of the above derivative is positive. I then show the above derivative is smaller than 1, which is equivalent to show

$$(\xi_1-\xi_2)\xi_3 R - \xi_2\overline{\zeta}^p < \xi_3 R + \overline{\zeta}^p(1-\xi_1 R)$$

Rearranging the above inequality yields

$$(\xi_1-\xi_2-1)\xi_3 R < (\xi_2 + 1 - \xi_1 R)\overline{\zeta}^p$$

After I plug in the parameters, it could be shown the left hand side of the above inequality is negative and the right hand side is zero.

F Proof of Proposition 9

Define $s(\zeta_t^p) = \frac{\xi_1\zeta_t^p}{\xi_3+\zeta_t^p\frac{1}{R}}$. Recall the T-map is

$$\begin{aligned}T_2(\zeta_{t-1}^p, \zeta_t^p) &= \frac{\xi_3 - \xi_2\zeta_{t-1}^p}{1-\frac{\xi_1\zeta_t^p}{\xi_3+\zeta_t^p\frac{1}{R}}} \\ &= \frac{\xi_3 - \xi_2\zeta_{t-1}^p}{1-s(\zeta_t^p)} \\ &= \overline{\zeta}^p + \frac{-\overline{\zeta}^p(1-s(\zeta_t^p)) + \overline{\zeta}^p(1-s(\overline{\zeta}^p)) - (\xi_3 - \xi_2\overline{\zeta}^p) + \xi_3 - \xi_2\zeta_{t-1}^p}{1-s(\zeta_t^p)} \\ &= \overline{\zeta}^p + \frac{\xi_2(\overline{\zeta}^p - \zeta_{t-1}^p) - \overline{\zeta}^p(s(\overline{\zeta}^p) - s(\zeta_t^p))}{1-s(\zeta_t^p)} \\ &\simeq \overline{\zeta}^p + \frac{\xi_2(\overline{\zeta}^p - \zeta_{t-1}^p) - \overline{\zeta}^p s'(\overline{\zeta}^p)(\overline{\zeta}^p - \zeta_t^p)}{1-s(\zeta_t^p)}\end{aligned}$$

Following the proof of proposition (6), a sufficient condition to guarantee momentum in belief is $\xi_2 > \overline{\zeta}^p s'(\overline{\zeta}^p)$. Plugging the parameters, this condition is equivalent to

$$(R^2(R-1) - \eta R(R-1) - \eta^2)(1-\tau)^2 + R\eta(R(R-1)+2\eta)(1-\tau) - R^2\eta^2 > 0$$

It can be simplified to

$$(\eta(R-(1-\tau))-\frac{R(R-1)(1-\tau)}{2})^2<R^2(1-\tau)^2((R-1)+\frac{(R-1)^2}{4})$$

Case 1: assuming $\eta(R-(1-\tau))>\frac{R(R-1)(1-\tau)}{2}$, I get $\frac{2(R-(1-\tau))}{R(R-1)(1-\tau)}>\frac{1}{\eta}>\frac{\frac{R}{1-\tau}-1}{g(R)}$. Case 2: assuming $\eta(R-(1-\tau))<\frac{R(R-1)(1-\tau)}{2}$, I obtain $\frac{1}{\eta}>\max\{\frac{2(R-(1-\tau))}{R(R-1)(1-\tau)},\frac{\frac{R}{1-\tau}-1}{g(R)}\}$.

Given $\frac{2(R-(1-\tau))}{R(R-1)(1-\tau)}>\frac{\frac{R}{1-\tau}-1}{g(R)}$ for all admissible parameters, I arrive at $\frac{1}{\eta}>\frac{\frac{R}{1-\tau}-1}{g(R)}$, which is equivalent to $\frac{1-\tau}{R}>\frac{1}{g(R)\frac{1}{\eta}+1}$, where $g(R)=R(\sqrt{(R-1)+\frac{(R-1)^2}{4}}+\frac{R-1}{2})$. It can be shown that when $\tau=0$, the condition here will collapse to the condition in proposition 9.

G Deriving the Constant-Gain Learning Algorithm from Bayesian Updating[36]

Agents perceive the following random walk model of coefficient variation

$$\begin{aligned}\phi_t &= \phi_{t-1}+\iota_t \qquad E\iota_t\iota_t'=R_{1t} && (75)\\ y_t &= \bar{\theta}_{t-1}'x_t+\varsigma_t \qquad E\varsigma_t\varsigma_t'=R_{2t} && (76)\end{aligned}$$

Define $P_{t-1}=E[(\phi_{t-1}-\theta_{t-1})(\phi_{t-1}-\theta_{t-1})]$. The prior belief about $\bar{\theta}_0$ are $N(\theta_0,P_{0|0})$. The posterior of θ_t can be represented by the basic Kalman filter, which takes the form of following recursions[37]

$$\begin{aligned}\theta_t &= \theta_{t-1}+L_t[y_t-x_t'\theta_{t-1}] && (77)\\ L_t &= \frac{P_{t|t-1}x_t}{R_{2t}+x_t'P_{t|t-1}x_t} && (78)\\ P_{t+1|t} &= P_{t|t-1}-\frac{P_{t|t-1}x_tx_t'P_{t|t-1}}{R_{2t}+x_t'P_{t|t-1}x_t}+R_{1t+1} && (79)\end{aligned}$$

Furthermore, agents are assumed to perceive $R_{1t}=\frac{g}{1-g}P_{t-1|t-1}$ and $R_{2t}=\frac{1}{g}$. Note $P_{t|t-1}=P_{t-1|t-1}+R_{1t}=\frac{1}{1-g}P_{t-1|t-1}$. Equations $(78)-(79)$ become

$$\begin{aligned}\theta_t &= \theta_{t-1}+L_t[y_t-x_t'\hat{\theta}_{t-1}] && (80)\\ L_t &= \frac{P_{t-1|t-1}x_t}{\frac{1-g}{g}+x_t'P_{t-1|t-1}x_t} && (81)\end{aligned}$$

[36] The derivation follows Ljung (1991) and Sargent (1999).

[37] Note for the model considered here I have $\theta_{t|t-1}=\theta_{t-1|t-1}$, so I suppress the conditioned information set and use θ_t for both.

And equation (77) becomes

$$P_{t+1|t} - R_{1t+1} = P_{t|t-1} - \frac{P_{t|t-1} x_t x_t' P_{t|t-1}}{R_{2t} + x_t' P_{t|t-1} x_t} \tag{82}$$

$$P_{t|t} = \frac{1}{1-g} \left[P_{t-1|t-1} - \frac{P_{t-1|t-1} x_t x_t' P_{t-1|t-1}}{x_t' P_{t-1|t-1} x_t + \frac{1-g}{g}} \right] \tag{83}$$

The constant gain learning algorithm is following

$$\theta_t = \theta_{t-1} + g R_t^{-1} x_t (y_t - \theta_{t-1}' x_t) \tag{84}$$

$$R_t = R_{t-1} + g \left(x_t x_t' - R_{t-1} \right) \tag{85}$$

Below I show the above two formulations are equivalent. Use $R_t^{-1} \equiv P_{t|t}$, equation (85) yields

$$R_t^{-1} = \left((1-g) R_{t-1} + g x_t x_t'^{-1} \right) \tag{86}$$

$$= \frac{1}{1-g} R_{t-1}^{-1} - \frac{1}{1-g} R_{t-1}^{-1} x_t \left[x_t' \frac{1}{1-g} R_{t-1}^{-1} x_t + \frac{1}{g} \right]^{-1} x_t' \frac{1}{1-g} R_{t-1}^{-1} \tag{87}$$

$$= \frac{1}{1-g} \left[P_{t-1|t-1} - \frac{P_{t-1|t-1} x_t x_t' P_{t-1|t-1}}{x_t' P_{t-1|t-1} x_t + \frac{1-g}{g}} \right] \tag{88}$$

From equation (86) to equation (87), the matrix inversion formula is used and stated in lemma 1 below. Specifically, it is applied with $A = (1-g)R_{t-1}$, $B = x_t$, $C = g$, $D = x_t'$.

Now I proceed to match equation (80) and (84). It suffices to show that $g R_t^{-1} x_t = L_t$.

$$g R_t^{-1} x_t = g P_{t|t} x_t \tag{89}$$

$$= \frac{g}{1-g} \left[P_{t-1|t-1} - \frac{P_{t-1|t-1} x_t x_t' P_{t-1|t-1}}{\frac{1-g}{g} + x_t' P_{t-1|t-1} x_t} \right] x_t \tag{90}$$

$$= \frac{P_{t-1|t-1} x_t}{\frac{1-g}{g} + x_t' P_{t-1|t-1} x_t} \tag{91}$$

$$= L_t \tag{92}$$

From equation (89) to (90), equation (83) is used.

Lemma 1. Let A, B, C and D be matrices of compatible dimensions, so that the product BCD and the sum A+BCD exist. Then

$$[A + BCD]^{-1} = A^{-1} - A^{-1} B [D A^{-1} B + C^{-1}]^{-1} D A^{-1} \tag{93}$$

Proof: see Ljung and Soederstroem (1983) pp. 19. (Sketch: show the RHS of (93) multiplied by $A + BCD$ from the right is equal to identity matrix.)