# Staying at Zero with Affine Processes

## An Application to Term Structure Modelling

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**Bank of Canada and SF Fed**

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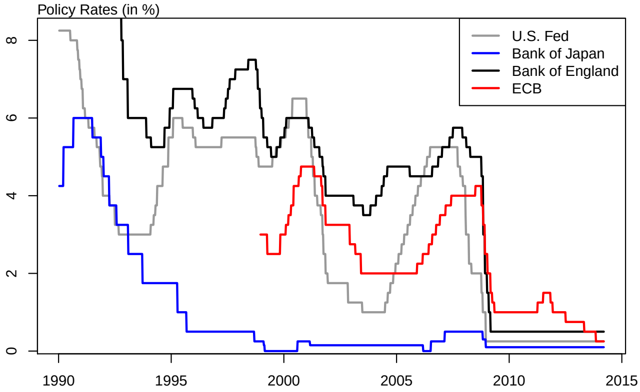
# Contents

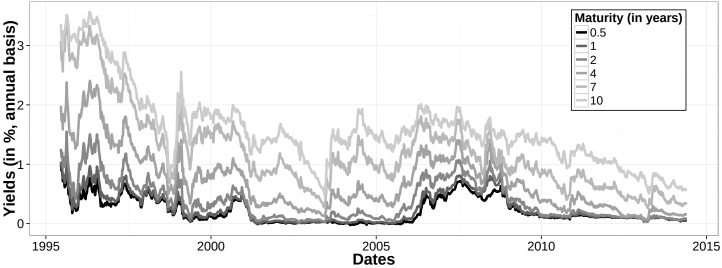
# Zero lower bound (ZLB)

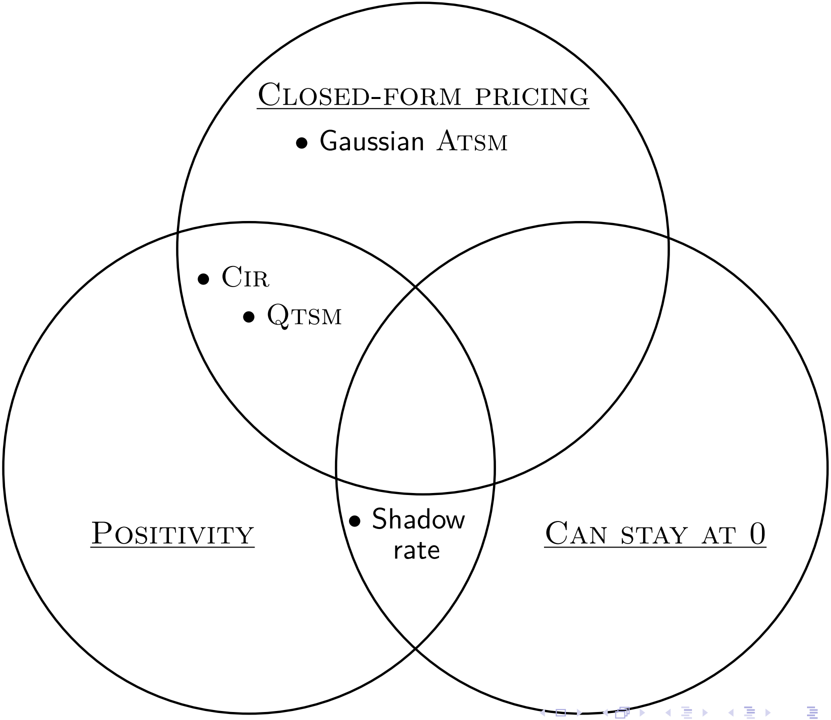
Several of the major central banks now face the ZLB

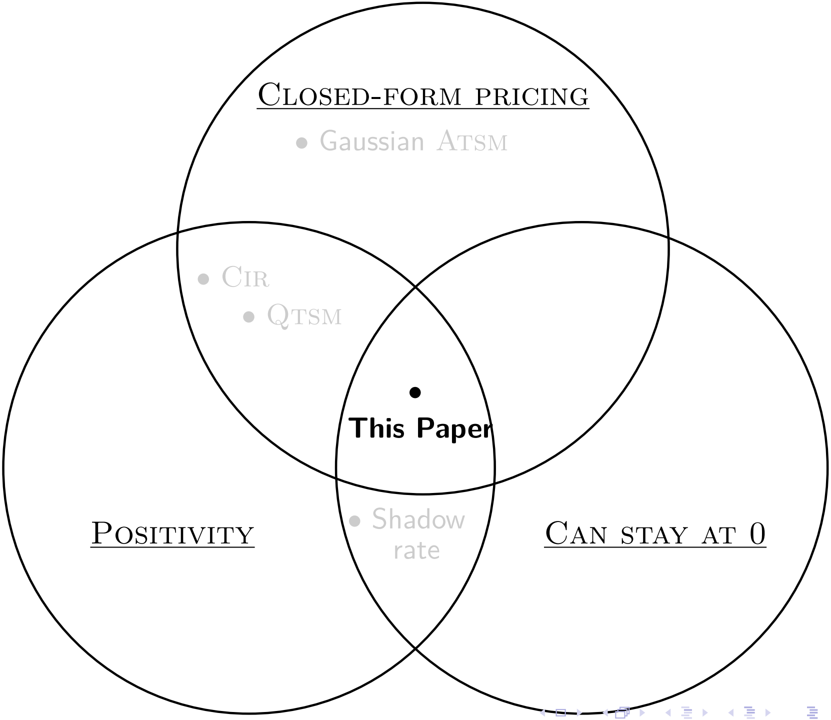


# Stylized facts to match

- The short-term nominal rate can stay at the ZLB for several periods...
- and in the meantime, longer-term yields can show substantial fluctuations [JGB yields from June 1995 to May 2014]




CLOSED-FORM PRICING
• Gaussian ATSM
• CIR
• QTSM
POSITIVITY
• Shadow rate
CAN STAY AT 0


Closed-form pricing
Gaussian Atsm
Cir
Qtsm
This Paper
Positivity
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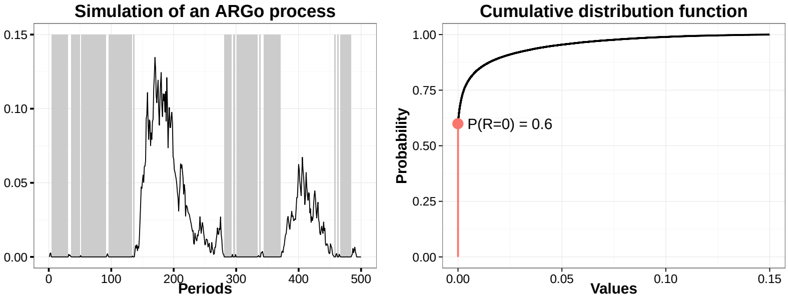
Can stay at 0

# Our ZLB model: a primer

$\longrightarrow$ We introduce a new affine process:



# What we do in this paper

- **We derive** Non-negative Affine processes staying at 0 ($ARG_0$ processes) to build a Term Structure Model which is:
    - providing positive yields for all maturities;
    - consistent with the ZLB (a short-rate experiencing prolonged periods at 0) *WHILE* long-term rates still fluctuates;
    - affine: thus closed-form formulas for bond-pricing and lift-off probabilities are available.
- Empirical assessment on JGB yields (June 1995 to May 2014). **Good performance** of our model in terms of:
    - fitting yield levels and conditional variances;
    - calculating Risk-Neutral *and* Historical lift-off probabilities.

# Related literature

- Term structure models at the ZLB: Black (1995), Ichiue & Ueno (2007), Kim & Singleton (2012), Krippner (2012), Renne (2012), Kim & Priebsch (2013), Wu & Xia (2013), Bauer & Rudebusch (2013), Christensen & Rudebusch (2013).
- Conditional volatilities of yields: Almeida *et al.* (2011), Bikbov & Chernov (2011), Filipovic, Larsson & Trolle (2013), Creal & Wu (2014), Christensen *et al.* (2014).
- Affine and Autoregressive Gamma processes: Darolles *et al.* (2006), Gourieroux & Jasiak (2006), Dai, Le & Singleton (2010), Creal & Wu (2013)
- Lift-off probabilities: Bauer & Rudebusch (2013), Swanson & Williams (2013)

# Contents

# Defining the **Gamma-Zero distribution**

We construct a new distribution in two steps:

- $Z \sim \mathcal{P}(\lambda) \Longrightarrow Z(\omega) \in \{\mathbf{0}, 1, 2, \ldots\}$ and $\mathbb{P}(Z = 0) = \exp(-\lambda)$.
- We define $X|Z \sim \gamma_Z(\mu)$, which implies:
  1. If $Z = 0$, $X$ is a Dirac point mass at 0.
  2. If $Z > 0$, $X$ is Gamma-distributed (continuous on $\mathbb{R}^+$).

**Definition**

The non-negative r.v. $X \sim \gamma_{\mathbf{0}}(\lambda, \mu)$, $\lambda > 0$ and $\mu > 0$, if

$$X \,|\, Z \sim \gamma_Z(\mu) \quad \text{with} \quad Z \sim \mathcal{P}(\lambda)$$

$$\Rightarrow \qquad \mathbb{P}(X = 0) = \mathbb{P}(Z = 0) = \exp(-\lambda)\,.$$

# A mixture distribution

In other words, $X \sim \gamma_0(\lambda, \mu)$ if its (complicated) p.d.f. is:

$$f_X(x\,;\,\lambda,\mu) = \sum_{z=1}^{+\infty} \left[ \frac{\exp(-x/\mu)\, x^{z-1}}{(z-1)!\,\mu^z} \times \frac{\exp(-\lambda)\lambda^z}{z\,!} \right] \mathbb{1}_{\{x>0\}} + \exp(-\lambda)\mathbb{1}_{\{x=0\}}$$

However, simple Laplace transform:

$$\varphi_X(u\,;\,\lambda,\mu) := \mathbb{E}\left[\exp(uX)\right] = \exp\left[\lambda \frac{u\mu}{(1-u\mu)}\right] \quad \text{for} \quad u < \frac{1}{\mu}\,.$$

$\Longrightarrow$ Exponential-affine in $\lambda$.

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$\Longrightarrow$ Exponential-affine in $\lambda$.

# Introducing dynamics: the ARG$_0$ process

Main goal: Build a dynamic affine process with zero point mass.

## Definition

$(X_t)$ is a ARG$_0(\alpha, \beta, \mu)$ if $(X_{t+1}|\underline{X_t})$ is Gamma-zero distributed:

$$(X_{t+1}|\underline{X_t}) \sim \gamma_0(\alpha + \beta X_t, \mu) \quad \text{for} \quad \alpha \geq 0,\ \mu > 0,\ \beta > 0\,.$$

Again, simple conditional LT, exponential-affine in $X_t$:

$$\begin{aligned} \varphi_{X,t}(u\,;\,\alpha,\beta,\mu) &:= \mathbb{E}_t\left[\exp(uX_{t+1})\right] \\ &= \exp\left[\frac{u\mu}{1-u\mu}(\alpha+\beta\,X_t)\right]\,, \quad \text{for} \quad u < \frac{1}{\mu}\,. \end{aligned}$$

# Interesting features and properties

Key properties:

- Non-negative process.
- Affine process: the conditional Laplace transform is exp-affine.
  $\varphi_{X,t}(u\,;\,\alpha,\beta,\mu) := \mathbb{E}_t\left[\exp(uX_{t+1})\right] = \exp\left[a(u)X_t + b(u)\right]$
- Staying at zero with probability:
  $$\mathbb{P}(X_{t+1} = 0|X_t = 0) = \exp(-\alpha) \neq 0.$$
  - $\alpha \neq 0 \implies$ zero is not absorbing.
  - in our multivariate yield curve model this probability will be time-varying, function of all date-$t$ factors;
- Closed-form moments (affine conditional cumulants).

# Relation to the original ARG process

- We extend the $ARG_0(\alpha, \beta, \mu)$ process to the more general $ARG_\nu(\alpha, \beta, \mu)$ case:

**$ARG_\nu(\alpha, \beta, \mu)$ process**

$X_t$ follows an $ARG_\nu(\alpha, \beta, \mu)$ process if:

$X_{t+1} \,|\, Z_{t+1} \sim \gamma_{\nu+Z_{t+1}}(\mu)$ with $Z_{t+1} \,|\, X_t \sim \mathcal{P}(\alpha + \beta X_t)$

- $\nu = 0 \implies ARG_0$ process.
- $\nu > 0,\ \alpha = 0 \implies$ ARG process of Gouriéroux and Jasiak (2006).

| | $\nu = 0$ | $\nu > 0$ |
|---|---|---|
| Positivity | Yes | Yes |
| Affine | Yes | Yes |
| Zero point mass | Yes | No |

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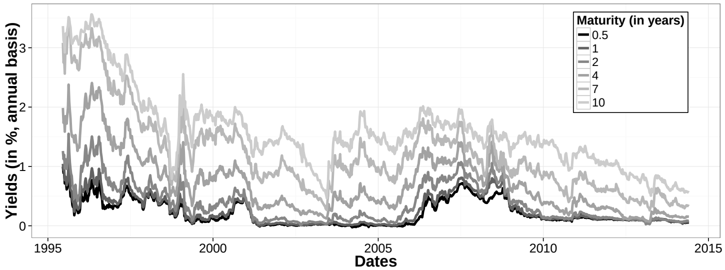
# Stylized facts to match (1)

- short-term nominal rate at the ZLB for several periods
- longer-term yields showing substantial fluctuations [JGB yields from June 1995 to May 2014]



# Risk-neutral dynamics

- The state of the economy is defined by a $n$-dimensional vector $X_t$. These factors follow a $VARG_\nu$ process under $\mathbb{Q}$ (the same under $\mathbb{P}$).

**$VARG_\nu$ processes**

$X_t$ follows a $\text{VARG}_\nu(\alpha, \beta, \mu)$ if, $\forall t$, $\forall i$:

- $Z_{i,t+1}|X_t \sim \mathcal{P}(\alpha_i + \beta_i' X_t)$.
- $X_{i,t+1}|Z_{i,t+1} \sim \gamma_{Z_{i,t+1}+\nu_i}(\mu_i)$ cond. indep across $i$.

- Conditional $\mathbb{Q}$-moments (same formulas under $\mathbb{P}$):

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}(X_{t+1}) &= \mu^{\mathbb{Q}} \odot (\alpha^{\mathbb{Q}} + \beta^{\mathbb{Q}\prime} X_t + \nu) \\ \mathbb{V}_t^{\mathbb{Q}}(X_{t+1}) &= \text{diag}\left[\mu^{\mathbb{Q}} \odot \mu^{\mathbb{Q}} \odot \left(\nu + 2\alpha^{\mathbb{Q}} + 2\beta^{\mathbb{Q}\prime} X_t\right)\right]\end{aligned}$$

*Note*: Conditional correlations can be allowed.

# Short-rate specification

- The vector of factors $X_t$ is split into two: $X_t = (X_t^{(1)'}, X_t^{(2)'})'$ where:
  - *(i)* All components of $X_t^{(1)}$ have $\nu_j = 0$ (point mass at 0).
  - *(ii)* All components of $X_t^{(2)}$ have $\nu_j > 0$ (no point mass).
  - *(iii)* $\mu_j^{\mathbb{P}} = 1$, $\beta^{\mathbb{P}}$ and $\beta^{\mathbb{Q}}$ lower-triangular (identification).
- The short-term rate $r_t$ is given by:

$$\boxed{r_t = \delta' X_t^{(1)}} \quad (= r_{min} + \delta' X_t^{(1)}\,,\ \textit{if } LB \neq 0) \tag{1}$$

**Key Property**

$\{$Eq.(1) $+$ *(i)*$\} \Rightarrow r_t$ has a zero point mass.

# Other Properties:

{Eq.(1) + *(iii)*}:

$$r_t = \delta' X_t^{(1)}$$
$$\begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix} = \text{constant} + \begin{pmatrix} \beta_{11}^{\mathbb{Q}} & \beta_{12}^{\mathbb{Q}} \\ 0 & \beta_{22}^{\mathbb{Q}} \end{pmatrix} \begin{pmatrix} X_{t-1}^{(1)} \\ X_{t-1}^{(2)} \end{pmatrix} + \xi_t^{\mathbb{Q}}$$

- We have $X^{(2)} \xrightarrow{\text{G.C.}} X^{(1)}$
- and thus $X_t^{(2)}$ appears in the short rate conditional $\mathbb{Q}$-expectations (hence in long rates).

$\Longrightarrow$ long-term yields can move during the ZLB.

# Pricing Formulas

The model belongs to the class of ATSM:

- Explicit closed-form bond-pricing
- Yields are affine in the factors for all maturities:

$$R_t(h) = -\frac{1}{h}({A_h}' X_t + B_h) = \overline{A}_h' X_t + \overline{B}_h.$$

- Recursive pricing formulas:

$$\begin{aligned} A_h &= -\delta + \beta^{\mathbb{Q}} \left( \frac{A_{h-1} \odot \mu^{\mathbb{Q}}}{1 - A_{h-1} \odot \mu^{\mathbb{Q}}} \right) \\ B_h &= B_{h-1} + \alpha^{\mathbb{Q}\prime} \left( \frac{A_{h-1} \odot \mu^{\mathbb{Q}}}{1 - A_{h-1} \odot \mu^{\mathbb{Q}}} \right) - \nu' \log\left(1 - A_{h-1} \odot \mu^{\mathbb{Q}}\right) \end{aligned}$$

# The historical dynamics

- The SDF is exp-affine with market price of risk vector $\theta$:

$$\frac{d\mathbb{P}_{t,t+1}}{d\mathbb{Q}_{t,t+1}} = \exp\left[\theta' X_{t+1} - \psi_t^{\mathbb{Q}}(\theta)\right]$$

## Change of measure property

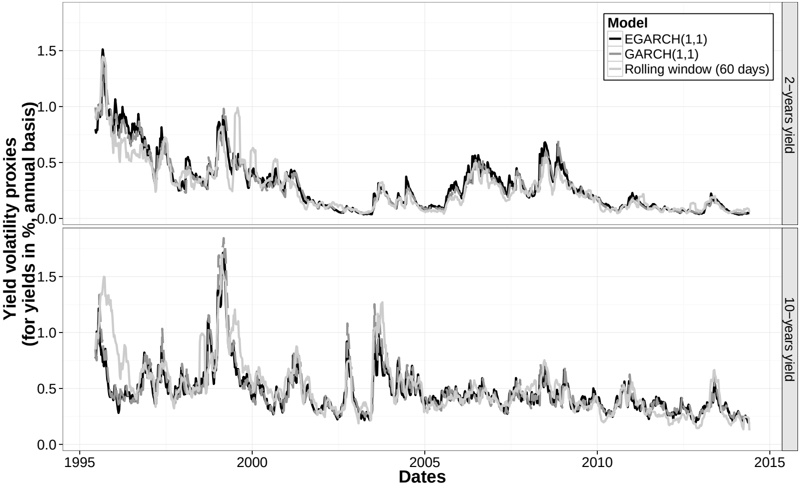
$X_t$ follows a $\text{VARG}_\nu(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}}, \mu^{\mathbb{P}})$ process under the historical measure $\mathbb{P}$.

$$\alpha_j^{\mathbb{P}} = \frac{\alpha_j^{\mathbb{Q}}}{1 - \theta_j\, \mu_j^{\mathbb{Q}}}, \qquad \beta_j^{\mathbb{P}} = \frac{1}{1 - \theta_j\, \mu_j^{\mathbb{Q}}}\beta_j^{\mathbb{Q}}, \qquad \mu_j^{\mathbb{P}} = \frac{\mu_j^{\mathbb{Q}}}{1 - \theta_j\, \mu_j^{\mathbb{Q}}}.$$

*Rk: $\nu$ is the same under both measures.*

# Stylized facts to match (2)

Conditional volatilities: time-varying and maturity-dependent.



# How to treat it

- Conditional variance of yields:

$$\begin{aligned} & \mathbb{V}_t^{\mathbb{P}}\left[R_{t+1}(h)\right] \\ = \; & \overline{A}_h' \mathbb{V}_t^{\mathbb{P}}(X_{t+1}) \overline{A}_h \\ = \; & \overline{A}_h' \left\{ \operatorname{diag}\left[ \mu^{\mathbb{P}} \odot \mu^{\mathbb{P}} \odot \left( \nu + 2\alpha^{\mathbb{P}} + 2\beta^{\mathbb{P}\prime} X_t \right) \right] \right\} \overline{A}_h \end{aligned}$$

- Time-varying and maturity-dependent.

# Advantages of an affine framework

## NATSM properties

- Yields $R_t(h)$ are non-negative;
- Long-term yields can move while $r_t = 0$ for several periods;
- Unconditional first two moments are available in closed-form;
- Conditional first two moments of yields are affine in $X_t$ (available in closed-form);
- Yields forecasts are explicitly affine in $X_t$;

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# Estimation technique

State vector $Y_t = (R_t', V_t', S_t')'$ affine in $X_t$:

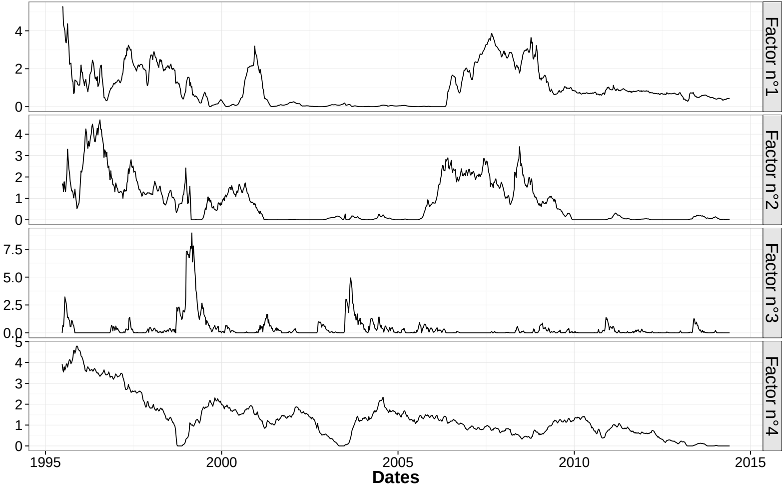
- $R_t$ = yield levels (6 maturities);
- $V_t$ = 2- and 10-y yield conditional (EGARCH) variance;
- $S_t$ = SPF for 3-m and 1-y ahead 10-y yield;
- prelim. estimations have suggested $\dim(X_t^{(1)}) = 1$, $\dim(X_t^{(2)}) = 3$ and $\nu = 0$;

## Estimation technique

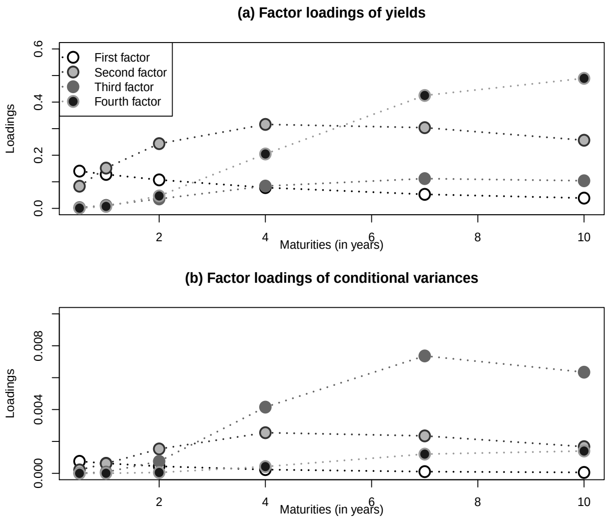
Linear Kalman-filter-based QML:

$$\begin{cases} X_{t+1} = m + MX_t + \Sigma_t^{1/2} \varepsilon_{t+1} \\ Y_t = \Gamma_0 + \Gamma_1 X_t + \Omega \, \eta_t \end{cases},$$

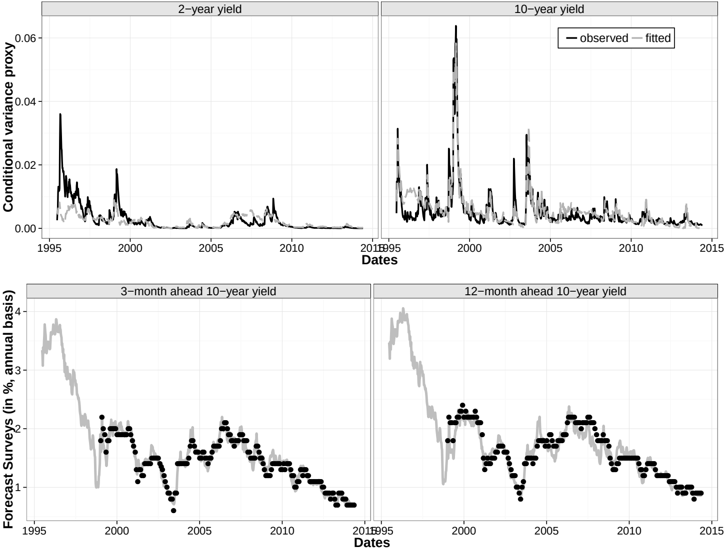
# Filtered factors



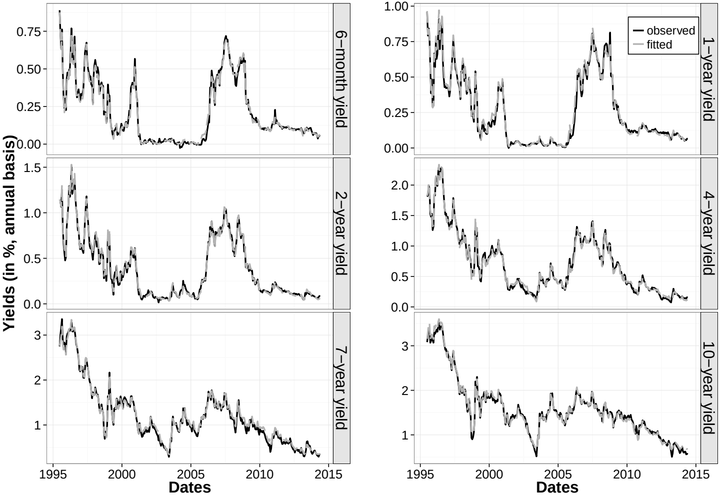
# Factor loadings of yields and conditional variances



# Fit of Conditional Variances and SPFs



# Fit of Yields



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# Lift-off probability dates under $\mathbb{P}$ *and* $\mathbb{Q}$

We calculate the following probabilities:

- $\mathbb{P}(r_{t+k} = 0 \,|\, \underline{X_t})$ and $\mathbb{Q}(r_{t+k} = 0 \,|\, \underline{X_t})$;
- $\mathbb{P}(r_{t+k} < 25\, bps. \,|\, \underline{X_t})$ and $\mathbb{Q}(r_{t+k} < 25\, bps. \,|\, \underline{X_t})$.

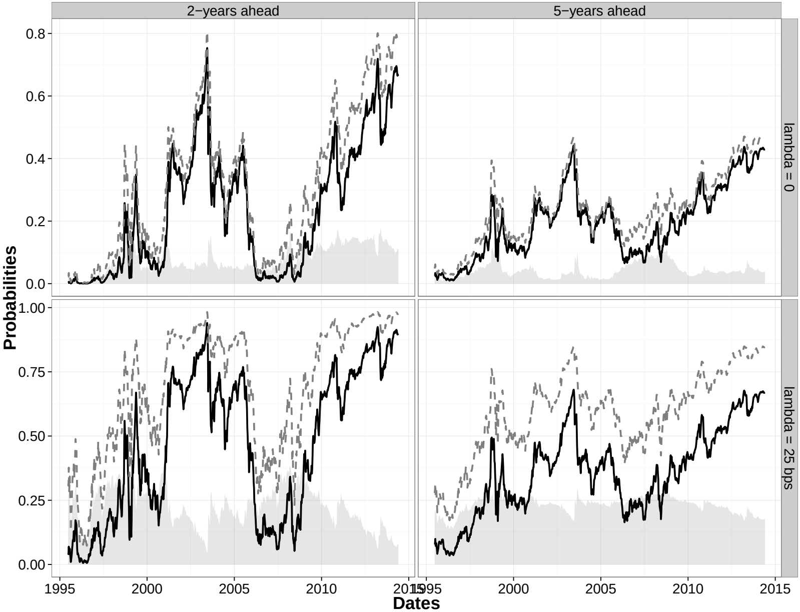
**Useful formula**

If $z \in \mathbb{R}^+$ and $\varphi_z(u)$ its Laplace transform.

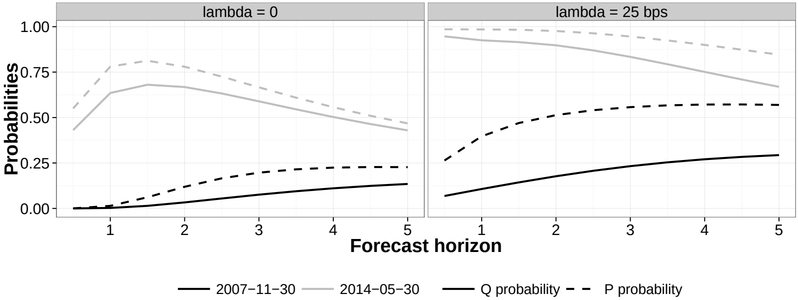
$$\mathbb{P}(z = 0) = \lim_{u \to -\infty} \varphi_z(u) \,.$$

Next two plots ($\mathbb{Q}$ is the black solid line):

- *Time-series dimension*: $t$ varies ($k = 2$yrs and 5yrs).
- *Horizon dimension*: $k$ varies ($t = 11/30/07$ and $05/30/14$).


2−years ahead
5−years ahead
lambda = 0
lambda = 25 bps
Probabilities
Dates
0.0
0.2
0.4
0.6
0.8
0.00
0.25
0.50
0.75
1.00
1995
2000
2005
2010
2015

# Horizon dimension of probabilities



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# Summary and further research

We have derived **affine non-negative processes staying at 0** and built an affine term-structure model (**NATSM**) gathering:

- a **short-rate consistent with the ZLB** experiencing periods at 0 while **long-run rates still fluctuates**;
- **closed-form formulas** for bond-pricing and lift-off probabilities.

An empirical assessment showed performance of our model for:

- **fitting yield levels and conditional variances**;
- calculating risk-neutral *and* historical **lift-off probabilities**.

**Further research:** Empirical comparison of NATSMs, derivatives pricing.

Thank you for your attention.

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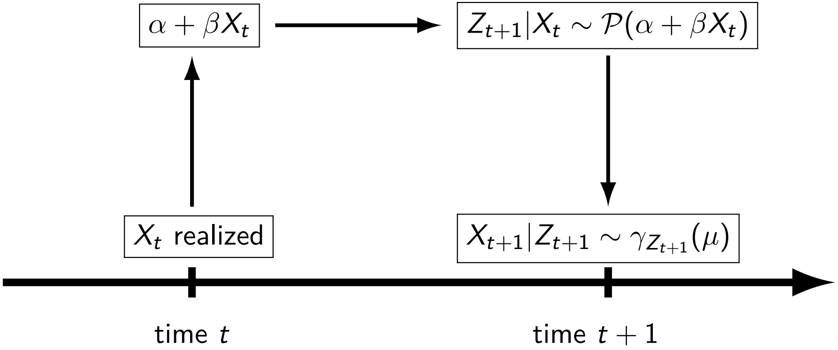
Table : Parameter estimates

| | $\mathbb{P}$-parameters | | | $\mathbb{Q}$-parameters | |
|---|---|---|---|---|---|
| | Estimates | Std. | | Estimates | Std. |
| $\alpha_4$ | 3.2455 | 0.1118 | | 3.2347 | 0.1113 |
| $\beta_{1,1}$ | 0.9663 | 0.0078 | | 0.9794 | 0.0042 |
| $\beta_{2,2}$ | 0.9978 | 0.0005 | | 0.9957 | 0.0006 |
| $\beta_{3,3}$ | 0.9486 | 0.0044 | | 0.9705 | 0.0023 |
| $\beta_{4,4}$ | 0.9967 | 0.0005 | | 0.9933 | 0.0003 |
| $\beta_{2,1}$ | 0.0308 | 0.0041 | | 0.0308 | 0.0041 |
| $\beta_{3,2}$ | 0.1094 | 0.0059 | | 0.1120 | 0.0061 |
| $\beta_{4,3}$ | $3.88{\cdot}10^{-4}$ | $2.28{\cdot}10^{-5}$ | | $3.87{\cdot}10^{-4}$ | $2.27{\cdot}10^{-5}$ |
| $\mu_1$ | 1 | – | | 1.0135 | 0.0040 |
| $\mu_2$ | 1 | – | | 0.9980 | 0.0005 |
| $\mu_3$ | 1 | – | | 1.0231 | 0.0023 |
| $\mu_4$ | 1 | – | | 0.9967 | 0.0003 |
| Other Parameters | | | | | |
| $\delta_1$ | 0.0030 | 0.0003 | | | |
| $\theta_1$ | -0.0133 | 0.0039 | $\theta_2$ | 0.0020 | 0.0005 |
| $\theta_3$ | -0.0226 | 0.0022 | $\theta_4$ | 0.0033 | 0.0003 |
| $\sigma_R$ | 0.0407 | 0.0003 | | | |
| $\sigma_V$ | $3 \cdot 10^{-3}$ | – | $\sigma_S$ | 0.15 | – |

# ARG$_0$ Summary



# Univariate case: lift-offs formulas

- $Z \in \mathbb{R}^+$ and $\varphi_Z(u)$ its Laplace transform.

$$\mathbb{P}_Z\{0\} = \lim_{u \to -\infty} \varphi_Z(u) \,.$$

- Lift-off probabilities: $(X_t) \sim \text{ARG}_0(\alpha, \beta, \mu)$ and $\varphi_{t,h}(u_1, \dots, u_h)$ its multi-horizon conditional Laplace transform.
  - $\mathbb{P}(X_{t+h} = 0 \,|\, X_t) = \lim_{u \to -\infty} \varphi_{t,h}(0, \dots, 0, u)$
  - $\mathbb{P}\big[X_{t+1} = 0, \dots, X_{t+h} = 0 \,\big|\, X_t\big] = \lim_{u \to -\infty} \varphi_{t,h}(u, \dots, u)$ $= \exp(-\alpha\, h - \beta\, X_t)\,,$
  - $\mathbb{P}\big[X_{t+1} = 0, \dots, X_{t+h} = 0, X_{t+h+1} > 0 \,\big|\, X_t)\big]$ $= \exp\left[-\alpha\, h - \beta\, X_t\right] \left[1 - \exp(-\alpha)\right]\,, \quad h > 1.$

# Multivariate Case

- $Z \in \mathbb{R}_+^n$ and $\varphi_Z(u_1, \ldots, u_n)$ its Laplace transform.

$$\mathbb{P}_Z\{0, \ldots, 0\} = \lim_{u \to -\infty} \varphi_Z(u, \ldots, u).$$

- Notations: Multi-horizon conditional LT.

$$\begin{aligned}
\varphi_{t,k}^{\mathbb{P}}(u_1, \ldots, u_k) &= \mathbb{E}^{\mathbb{P}}\left[\exp\left(u_1' X_{t+1} + \ldots + u_k' X_{t+k}\right) \middle| X_t\right] \\
&= \exp\left[\mathcal{A}_k' X_t + \mathcal{B}_k\right] \\
\varphi_{R,t,k}^{(h)\mathbb{P}}(v_1, \ldots, v_k) &= \mathbb{E}\left[\exp\left(v_1 R_{t+1}(h) + \ldots + v_k R_{t+k}(h)\right) | X_t\right]
\end{aligned}$$

# Lift-offs

- $\mathbb{P}\left[r_{t+k} = 0 \,|\, X_t\right] = \lim_{u \to -\infty} \varphi^{(1)\mathbb{P}}_{R,t,k}(0, \ldots, 0, u)$
- $\mathbb{P}\left[r_{t+1} = 0, \ldots, r_{t+k} = 0 \,|\, X_t\right]$
  $$= \lim_{u \to -\infty} \varphi^{(1)\mathbb{P}}_{R,t,k}(u, \ldots, u) = p_{r,t,k} \quad \text{(say)}$$
- $\mathbb{P}\left[r_{t+1} = 0, \ldots, r_{t+k-1} = 0, r_{t+k} > 0 \,|\, X_t\right] = p_{r,t,k-1} - p_{r,t,k}$
- $\mathbb{P}\left[v' R^{(t+k)}_{t+1}(h) > \lambda \,|\, X_t\right]$
  $$= \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \frac{Im\left[\varphi^{(h)\mathbb{P}}_{R,t,k}(i\, v\, x) \exp(-i\, \lambda\, x)\right]}{x}\, dx$$

# Useful remarks

**Remark 1**

Stationarity conditions are easily imposed:

$$X_t \quad \text{stationary} \iff \forall j \in \{1, \dots, n\}, \quad \rho_j := \mu_j \beta_{j,j} < 1\,.$$

**Remark 2**

The assumption of conditional independence can be relaxed keeping the affine structure of the multivariate process $X_t$.

$\implies$ Recursive discrete-time affine process (*mimeo*).