Tractable Term Structure Models–A New Approach

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Motivation

1. Interest rates are close to or have reached their lower bound across several markets globally.
2. Bounded positive interest rates imply large tractability or flexibility costs within the existing DTSM framework.
3. These costs are especially acute when exploring the volatility of yields over the cycle. As the level and slope of the yield curve evolves,
   - How does the volatility of bond yields evolve throughout the cycle?
   - How does the (hump-shaped) term structure of yield volatility evolve throughout the cycle?
   - How does volatility of the expectation and risk premium components evolve throughout the cycle?
     (Cieslak and Povala, 2015)
4. Contribution: we introduce Tractable Term Struture Models (TTSMs) to answer these questions.

Examples

Models with positive yields are restrictive:

1. Positive affine DTSM models
   - Restrictions on the correlation structure (only positive).
   - Restrictions to accommodate macro variables that changes signs.
   - Restrictions on the risk premium (Dai and Singleton, 2002; Joslin and Le, 2013).
2. Quadratic DTSM models or Black's DTMS
   - Tractable?
   - Limited to simple Gaussian state dynamics.

Motivation

- DTSMs are based on the fundamental theorems of asset pricing to ensure the Absence of Arbitrage.
- The focus is on the subset of "realistic" SDFs $M_t > 0$ such that:

$$\begin{aligned} P_{1,t} &= E_t[M_{t+1}] \text{ is closed form,} \\ P_{2,t} &= E_t[M_{t+1}M_{t+2}] \text{ is closed form,} \\ &..., \\ P_{n,t} &= E_t[M_{t+1}M_{t+2}...M_{t+n}] \text{ is closed form} \end{aligned}$$

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\end{aligned}
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- **Question: Can we bypass specifying the SDF to retain tractability and flexibility yet producing bond prices that are "close" to AOA?**

1. Our construction of bond prices

Assumption (1)

- *The n-period bond price P_n is given recursively by*

$$P_0(X_t) \equiv 1, \quad \forall X_t \tag{1}$$
$$P_n(X_t) = P_{n-1}(g(X_t)) \times exp(-m(X_t)), \tag{2}$$

- *given some state X_t with support $\underline{X}$,*
- *and some functions $m(\cdot)$, $g(\cdot)$ where $g(X_t) \in \underline{X}$ for every $X_t \in \underline{X}$.*

Assumption 1 guarantees pricing tractability.

1. Our construction of bond prices

- Example n=1:

$$P_1(X_t) = P_0(g(X_t)) \times exp(-m(X_t)) = exp(-m(X_t)) \qquad (3)$$

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- Example n=2:

$$\begin{aligned} P_2(X_t) &= P_1(g(X_t)) \times exp(-m(X_t)) \\ &= exp(-m(g(X_t))) \times exp(-m(X_t)) \end{aligned} \quad (4)$$

  - $g(\cdot)$ lets us price $P_n(\cdot)$ given $P_{n-1}(\cdot)$.

1. Properties of bond prices

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P1 — Positivity $P_n(X_t) \leq 1 \; \forall X \in \underline{X}$ *or equivalently* $y_{n,t} \geq 0$;

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The following choices of functions $m(\cdot)$, $g(\cdot)$ guarantee Properties **P1-P3**:

1. $m(\cdot)$ is continuous and monotonic with $m(X) \geq 0 \; \forall X \in \underline{X}$,
2. $g(X)$ is a contraction with unique fixed-point $g(X^*) = X^*$,
3. $g(X) = KX$.

1. Time series dynamics

Assumption (3)

The time series dynamics of X_t admits $\underline{X}$ as support and is such that yields for all maturities $y_{n,t} \equiv -log(P_n(X_t))/n$ have a joint distribution that is stationary and ergodic.

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- Virtually any time series dynamics is acceptable in our framework and will not affect any of our earlier results.
- This means that our framework is flexible enough to accommodate:
  - GARCH-like or stochastic volatility
  - DCC-like or stochastic correlation
  - Unspanned macro variables
  - Long or infinite lag structure
  - Shifting endpoints and unit roots.
  - ...

2. How close are we to AOA?

Theorem 1: Nelson-Siegel Yield Curve

Bond prices generated using

$$m(X_t) = \begin{bmatrix} 1 & \frac{1-e^{-\lambda}}{\lambda} & \frac{1-e^{-\lambda}}{\lambda} - e^{-\lambda} \end{bmatrix} X_t, \tag{5}$$

$$g(X_t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-\lambda} & \lambda e^{-\lambda} \\ 0 & 0 & e^{-\lambda} \end{bmatrix} X_t, \tag{6}$$

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1. Implementations of the Nelson-Siegel model are not *strictly* free of arbitrage (Bjork and Christensen; Filipovic) and *the same applies here*.
2. Nevertheless, the empirical literature has long concluded that not much distinguishes NS from a fully-fledged DTSM implementation. (Diebold and Li; Christensen, Diebold and Rudebusch).
3. We also clarify how close TTSM are to strict AOA.

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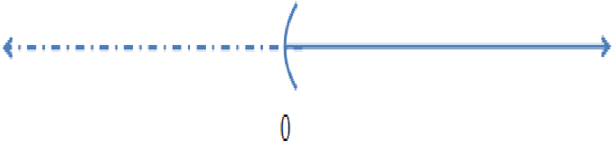

Figure: Prices of portfolios with strictly positive payoffs.

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Theorem 3: Self-Financing Arbitrage

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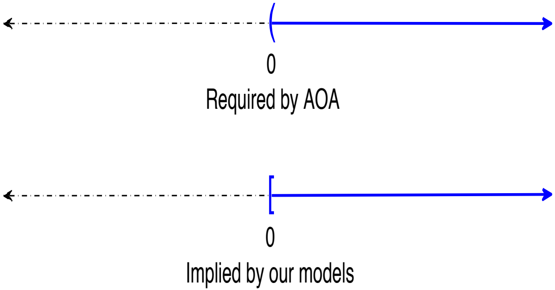

Figure: No Arbitrage Strategies: prices of portfolios with non-negative payoffs.

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Theorem 4: Transaction Costs
Our bond price construction allows no arbitrage opportunities in presence of transaction costs (however small)

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- How reasonable/important for us to think about transaction costs?
- Strictly speaking, we only need to invoke the transaction costs for self-financing portfolios. These must involve costly short positions.
    - See evidence in e.g., Duffie (1996); Krishnamurthy (2002); Vayanos and Weill (2008); and Banerjee and Graveline (2012)

3. Specification—f and g functions

- Choose $g(X) = KX$ and $m(X) = u(\theta, X)$ such that
  1. limit (i): Black's $max(0, \delta_0 + \delta_1' X_t)$ with $\theta_1 \to 0$,
  2. limit (ii): linear $\delta_0 + \delta_1' X$ with $\theta_1 \to \infty$.
- Guarantees positivity in spirit of $max(0, \delta_0 + \delta_1' X_t)$ but remains invertible:

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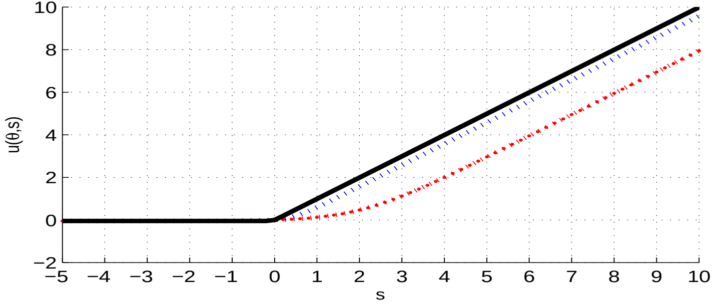


Figure: The max function and different shapes of the short-rate function $u(\theta, s)$

3. Specification—f and g functions

- Analytical yields/ forwards:

$$f_{n,t} = u(\theta, \delta_0 + \delta_1' K^n X_t) \tag{7}$$

- We can work with transformed forwards $\tilde{f}_{n,t}$,

$$\tilde{f}_{n,t} \equiv u^{-1}(\theta, f_{n,t}) = \delta_0 + \delta_1' K^n X_t \tag{8}$$

- We are back to the linear space: restate the model in terms of portfolios $\mathcal{P}_t = W\tilde{f}_{n,t}$ and proceed with preferred estimation method.

3. Specification

1. Joint VAR dynamics for yield portfolios $\mathcal{P}_t$ and unspanned macro variables U_t:

$$\begin{pmatrix} \mathcal{P}_{t+1} \\ U_{t+1} \end{pmatrix} = K_0^{\mathbb{P}} + K_1^{\mathbb{P}} \begin{pmatrix} \mathcal{P}_t \\ U_t \end{pmatrix} + \sqrt{\Sigma_t} \begin{pmatrix} \varepsilon_{\mathcal{P},t+1} \\ \varepsilon_{U,t+1} \end{pmatrix}, \qquad (9)$$

2. The innovations $\varepsilon_t \equiv (\varepsilon_{\mathcal{P},t+1}, \varepsilon_{U,t+1})' \sim N(0, \Sigma_t)$
3. Σ_t combines EGARCH(1,1) and DCC dynamics.
4. Yields: GSW forward rates from GSW; 1990 and 2015; quarterly maturities between 3 months and 10 years.
5. Macro: Survey forecasts of inflation and gdp 1-year ahead (Blue Chips Financials).

4. Results—Model nomenclature

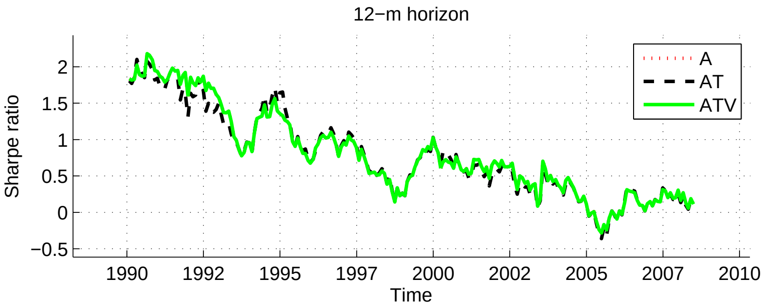
1. $A_0(3)$ Gaussian DTSM $\rightarrow$ A
2. Affine TTSM $\rightarrow$ AT
3. Affine TTSM with Volatility dynamics $\rightarrow$ ATV
4. Positive TTSM $\rightarrow$ PT
5. Positive TTSM with Volatility dynamics $\rightarrow$ PTV

Here: focus on cyclical volatility variations

In the paper: also check that pricing errors, forecasts, liftoff time, risk premium and Sharpe ratios are identical between models

4. Results—Sharpe Ratios

Figure: 2-year bond annual Sharpe ratio; 1990-2008



Essentially no differences between model-implied Sharpe ratios.

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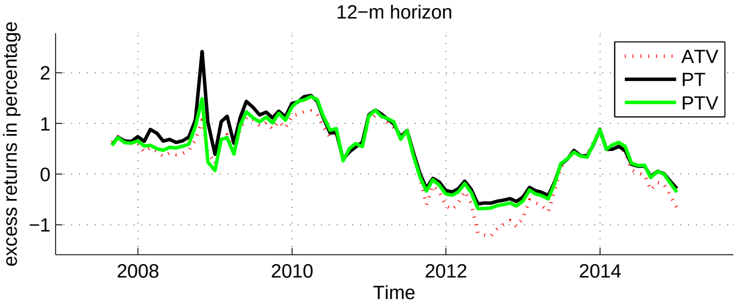

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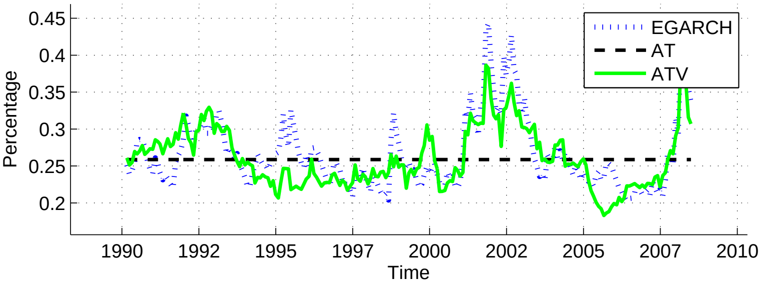
4. Results—Conditional volatility

Figure: 1-Year Yield Conditional Volatility 1990-2008



Volatility peaks in recession, adding to risk.

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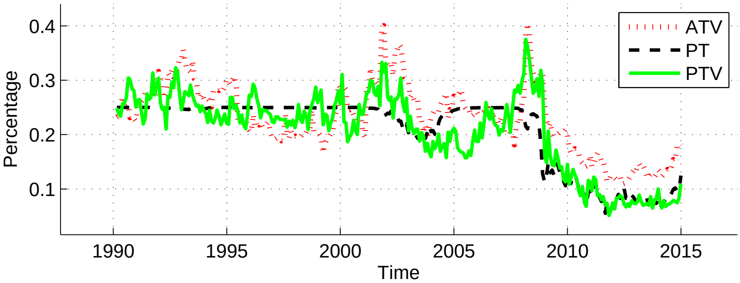
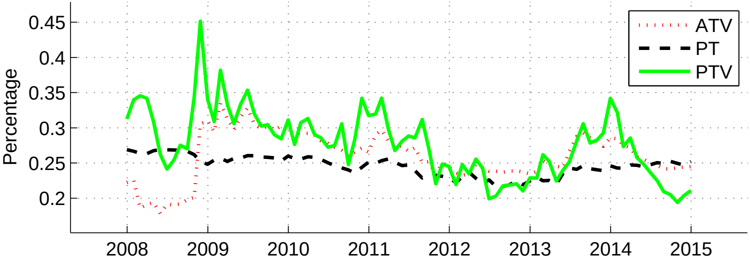


Need volatility compression to capture short-term volatility near the lower bound.

4. Results—Conditional volatility

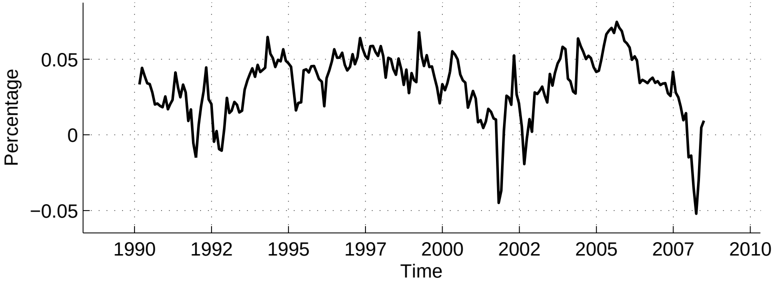
Figure: 10-year Yield Conditional Volatility (2008-2015)



Still need time-varying factor volatility to match volatility of long-term yields near the lower bound

4. Results—Conditional volatility

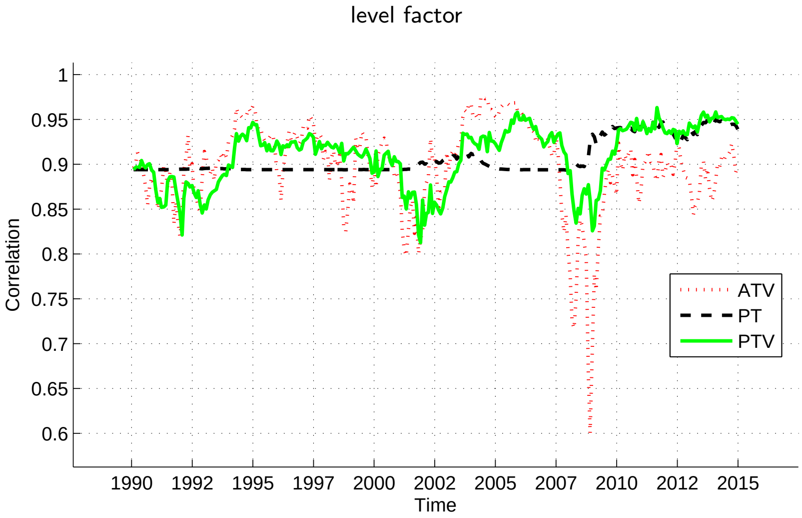
Figure: Volatility hump:Difference between 12-month and 1-month ahead volatility.



Volatility term structure downward-sloping in recession.

4. Results—The changing role of level and slope

Figure: Principal components R^2s from yields' conditional correlation matrix.



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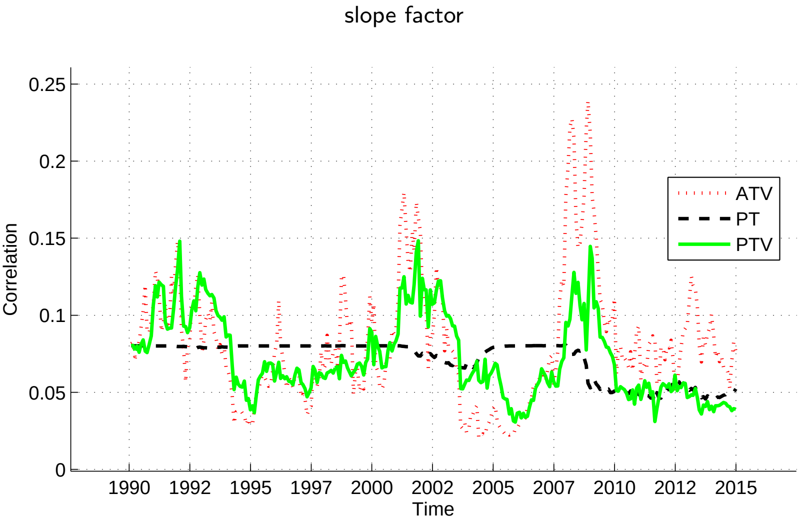

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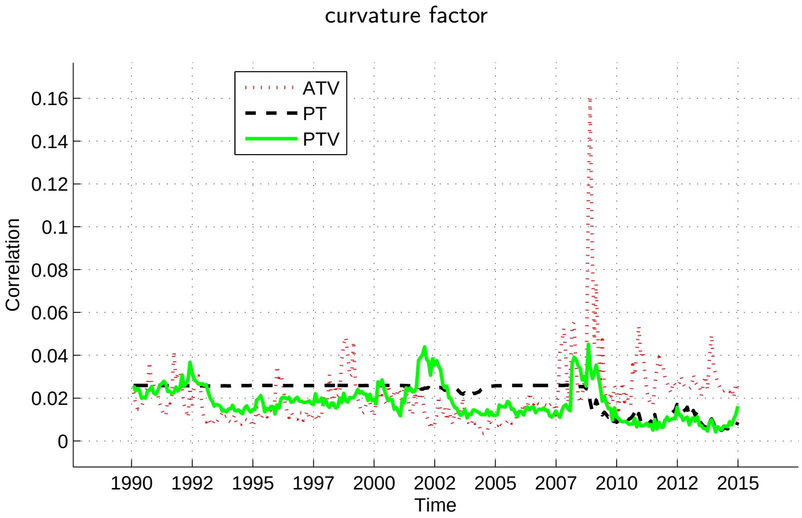

Conclusion

1. Propose Tractable Term Structure Models (TTSMs).
   - We specify bond prices directly without imposing a parametric SDF.
   - Like Nelson-Siegle curves, bond prices are nearly but not strictly AOA.
   - Imposition of lower bound is straightforward without giving away flexibility, tractability and ease of implementation.

2. Empirically:
   - DTSM and TTSM risk premium and Sharpe ratios are essentially the same away from the lower bound.
   - TTSM can match volatility dynamics both near and away from the lower bound.
   - The relative importance of level risk and slope risk changes plays a key role.