Informationally Efficient Climate Policy: Designing Markets to Measure and Price Externalities*

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I study how policymakers can access and act on the information about climate change damages that is dispersed throughout the economy. I develop and analyze a new dynamic deposit-refund instrument (called "carbon shares") that can: i) efficiently price emissions conditional on information, ii) efficiently incentivize removal of past emissions conditional on information, and iii) efficiently aggregate dispersed information about the social cost of emissions. Conventional emission taxes generally succeed at only the first of these objectives. Rather than projecting damages in all future periods and all possible states of the world in order to calculate the optimal tax, the regulator here estimates damages as they are realized and empowers markets to perform price discovery about future damages.

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Despite the validity in principle of the tax-subsidy approach in the Pigouvian tradition, in practice it suffers from serious difficulties. For we do not know how to estimate the magnitudes of the social costs, the data needed to implement the Pigouvian tax-subsidy proposals.

– Baumol (1972, p. 316)

The practical problem, however, arises precisely because these facts are never so given to a single mind, and because, in consequence, it is necessary that in the solution of the problem knowledge should be used that is dispersed among many people.

– Hayek (1945, p. 530)

1 Introduction

Economists have long emphasized the informational advantages of market-based policies for controlling pollution. Market-based policies only require the regulator to measure the social cost of pollution (i.e., the externality), whereas command-and-control policies also require the regulator to measure each firm or agent's cost of reducing pollution. Because actual firms and agents know more about their own options to reduce pollution than does the regulator, market-based policies increase efficiency by empowering them to employ their most cost-effective options. However, even market-based policies may demand a lot of information from a regulator: it is often quite challenging to measure the social cost of pollution, as Baumol (1972) recognizes in the opening quotation. In particular, economists have struggled to measure the social cost of the carbon emissions that drive global climate change.[1] As a result, economists have not converged on an emission price to recommend to policymakers interested in market-based solutions.[2]

To date, measurement of social costs has been centralized among academics and regulators, but much information about the cost of climate change is dispersed throughout society.

[1]Nordhaus (2019, p. 1998) acknowledges, "In reality, projecting impacts is the most difficult task and has the greatest uncertainties of all the processes associated with global warming." Some economists even criticize the social cost estimates underlying the most prominent climate-economy models as "completely made up, with no theoretical or empirical foundation" (Pindyck, 2013, p. 868). Recent work estimates the effects of weather shocks and uses these effects to project the consequences of future climate change (Deschênes and Greenstone, 2007; Carleton and Hsiang, 2016), but despite many advances in this literature, fundamental questions remain about how to map consequences of weather shocks to consequences of climate change (Dell et al., 2014; Lemoine, 2021b).

[2]In a survey of 289 economists with expertise on climate change (Holladay et al., 2009), estimates of the social cost of carbon had a standard deviation of \$339 per tCO_2, around three times larger than the average estimate. The social costs of carbon calculated in Pindyck (2019) from another survey of 113 economists also shows substantial dispersion, with the standard deviation again of comparable magnitude to the mean.

Every person and firm on this planet is exposed to climate change. Each may have some information about their own particular exposure and about their own particular ability to adapt. Since at least Hayek (1945), a rich tradition in economics views markets as an algorithm that aggregates dispersed information about the costs and benefits of the many goods produced in society. Yet environmental economists have not studied how to design markets to aggregate information about externalities, and thereby to perform price discovery for social costs.

I design and analyze a new market-based policy instrument that simultaneously measures and controls externalities. In my setting, as is customary, firms in different sectors of the economy trade off the benefit of emitting carbon against the cost of complying with current policies. Carbon emissions generate warming that impacts each sector of the economy in an uncertain, stochastic, and potentially correlated fashion. I introduce two novel features to this environment. First, firms can pay to remove old emissions from the atmosphere.[3] Second, information about climate change damages is heterogeneous. Agents measure climate impacts in their own sectors, and a regulator measures the aggregate effect of climate change from data on final good production and temperature. Both types of measurements may be arbitrarily imperfect. Agents' measurements are private information, whereas the regulator reports its measurement to all actors in the economy and can use it to set policy.

I show that a regulator who uses an emission tax policy is unable to optimally use new information about social costs or to observe all of the information about social costs that is dispersed throughout society.[4] First, if the regulator did have access to all agents' information about social costs, then it could use an emission tax to efficiently control new emissions (as has long been known). However, an emission tax on its own cannot incentivize the removal of past emissions other than as means to offset ongoing emissions: once an emitter has paid the penalty, subsequent information is irrelevant even if it dramatically alters the estimated social cost of previously emitted carbon and warrants paying for its removal. An emission tax ultimately places the risk of needing to fund large-scale carbon removal on taxpayers, which could impose an impractical fiscal burden (Bednar et al., 2019).[5] Second, the regulator will not actually have access to all agents' information. I show that this regulator's emission tax is informationally inefficient in the plausible case that the regulator measures aggregate

[3]Carbon dioxide removal, or negative emission, strategies include chemically separating carbon dioxide from air ("direct air capture"), capturing emissions from power plants that burn biomass ("bioenergy with carbon capture and storage"), accelerating the weathering of rocks, enhancing uptake of carbon by forests or oceans, and more. See National Research Council (2015), Fuss et al. (2018), and National Academies of Sciences, Engineering, and Medicine (2018) for recent reviews. Recently, Microsoft and Stripe each received bids to undertake carbon removal for around $150 per t$CO_2$ on average (Joppa et al., 2021).

[4]Similar critiques apply to cap-and-trade programs, the quantity version of an emission tax.

[5]A forward-thinking regulator could save the revenue collected by an emission tax for the procurement of future carbon removal, but as analyzed below, even that revenue would be insufficient if carbon removal becomes desirable because the regulator learns climate change is more damaging than they had believed at the time of emission.

consequences only imperfectly and also in the plausible case that sectors have heterogeneous value shares and correlated exposure to climate change. In either case, an all-seeing regulator with access to all of the information dispersed throughout society would choose a different emission tax than would a more realistic, information-constrained regulator.

I propose a new instrument that I show can incentivize optimal carbon removal and can efficiently aggregate dispersed information. The new instrument is a dynamic deposit-refund scheme. The regulator requires that emitters pay a deposit at the time of emission and in exchange gives emitters a tradeable asset that I call a "carbon share". In each period, the regulator refunds part of the deposit to current shareholders based on whether its measure of aggregate realized damages was as bad as implied by the deposit. The equilibrium value of the carbon share reflects expected refunds, which are by construction smaller than the value of the deposit. Emitters have an incentive to reduce emissions in order to avoid giving up the deposit for the less valuable carbon share. In later periods, a carbon shareholder may decide to remove the underlying unit of carbon from the atmosphere in order to recover the share's deposit. And in each period, the equilibrium price of the carbon share reflects market participants' beliefs about the regulator's future measurements and thus about future damages from climate change.

I show that there exists a fully revealing rational expectations equilibrium in which the price of a carbon share perfectly aggregates the information dispersed throughout society and in which the incentives to reduce emissions and remove carbon are the same as in the welfare-maximizing, informationally efficient benchmark.[6] The two critical features of the policy are that the deposit be sufficiently large and that the regulator make a good-faith (albeit potentially imperfect) effort to measure and report aggregate recent damages. A large deposit is critical because the private cost of emitting carbon and the private benefit of removing carbon are both defined by traders' expectation of the difference between the deposit and expected refunds: I call this difference the expected stream of "damage charges" that correspond to the regulator's future measurements of aggregate damages. If the initial deposit is small, then some periods' damage charges are likely to be constrained by the deposit and thus be smaller than the regulator's measured damages. As the deposit becomes large, traders' expectations of damage charges converge to the expected future measurements of damages and thus to their current estimate of the marginal damage from carbon emissions, which makes the private cost of emitting converge to the social cost of carbon emissions. Numerical simulations suggest that a deposit around 2–3 times as large as the estimated social cost of emissions approximates optimal emission and removal incentives.

I also study the Bayesian Nash equilibrium of a game in demand functions in order to

[6] Following the convention in finance and information economics, I use "rational expectations" to indicate that Bayesian traders use prices to learn about others' information and trade optimally conditional on their posterior beliefs. The usage common in macroeconomics is slightly different, as that literature does not typically model asymmetric information. See Vives (2008, Chapter 3) and Campbell (2017, Chapter 12) for discussions.

assess equilibria that are "implementable" via a specific trading mechanism (see, among others, Vives, 2014; Rostek and Yoon, 2020). Each trader submits a demand schedule that accounts for their private information and for what they would infer about other traders' information from any given carbon share price they might observe. I show that carbon share prices in implementable equilibria use traders' private information to learn about correlated impacts in other sectors and to reduce the impact of measurement error in aggregate data, similar to how an all-seeing, informationally efficient regulator would update beliefs. Although implementable equilibria do not achieve full informational efficiency, they do aggregate information in ways similar to informationally efficient beliefs but not achievable by a regulator with access only to aggregate data.

This new policy shifts much of the work of projecting possible climate change damages from the regulator to markets. However, the regulator does still play a critical role in measuring realized damages from climate change (after all, climate change is still an externality). Traders use their private information about climate change damages and the observed carbon share price to forecast the regulator's future measurements because those determine future damage charges and refunds. Whatever the regulator attempts to measure therefore determines the market incentives to reduce emissions and to remove old emissions.[7] The regulator's measurements should be unbiased on average, as is also critical to setting the proper emission tax. However, here the regulator's measurements can also be arbitrarily noisy without undercutting the policy too severely, because carbon emission and removal incentives ultimately depend on the dispersed information about future measurements that is aggregated by the carbon share market.

This paper intersects with several distinct literatures. As mentioned above, a central theme throughout environmental economics is the importance of using market-based instruments to control pollution, whether in the form of emission taxes or cap-and-trade programs (see, among others, Metcalf, 2009; Stavins, 2022).[8] Instead of focusing exclusively on the role of emission prices in determining the budget sets of firms and households, I also consider how to design markets so that emission prices convey information about damages from climate change.[9] Where dispersed information about damages is relatively unimportant and

[7]The regulator could choose to measure whatever it cares about. For instance, it could apply equity weighting and/or value nonmarket impacts.

[8]The recommendation to address climate change by taxing emissions dates to at least Nordhaus (1977), and attempts to empirically estimate the consequences of climate change date to at least Mendelsohn et al. (1994). Weitzman (1974) shows that asymmetric information about abatement costs can break the equivalence between an emission tax and cap. I emphasize asymmetric information about the externality, which I show can make an emission tax or cap informationally inefficient.

[9]Other work focuses on the revelation of beliefs about the magnitude of climate change: Schlenker and Taylor (2021) show that weather derivatives are sensitive to climate model projections, and Hsu (2011) proposes futures on emission taxes set, following McKitrick (2011), according to a predefined function of temperature (see also Aliakbari and McKitrick, 2018). Here the focus is on predicting the aggregation of local damages rather than on predicting the level of global temperature, here price discovery determines emission

cleanup of past emissions is irrelevant (as may be true of particulate matter or lead pollution), the proposed policy performs like an emission tax. But where information about damages is dispersed, the proposed policy acts like improving the information underlying an emission tax, and where cleanup of past emissions is potentially relevant, the present policy can incentivize such cleanup without requiring the regulator to directly fund it. Climate change clearly demonstrates dispersed information about impacts and the possibility of ex post cleanup, and many other externalities will too.[10]

My proposed instrument is a dynamic deposit-refund instrument. Static deposit-refund schemes resolve difficulties monitoring—and thus taxing—improper waste disposal (e.g., Bohm, 1981; Russell, 1987; Fullerton and Kinnaman, 1995; Torsello and Vercelli, 1998).[11] I posit no problem monitoring either the act of emission or the act of carbon removal, but I do resolve difficulties in incentivizing carbon removal that arise from the regulator's imperfect ability to tax past emitters for emerging climate damages.[12] My dynamic deposit-refund scheme resolves the difficulties of taxing past emitters by imposing all costs upfront and offering rewards for subsequently claiming ownership of pollution. From the perspective of emitters, my policy combines a tax (in the form of the deposit) and a subsidy (in the value of the carbon share received), with emission incentives determined by the difference between the deposit and the value of the carbon share. This type of emission incentive is familiar from the static generalization of deposit-refund schemes in Fullerton and Wolverton (2000). Here, however, the level of the subsidy is not fixed by the regulator but is instead determined in equilibrium (in the form of the share prices) by private actors' information about climate

and carbon removal decisions rather than merely providing a signal useful for long-run investment, and the carbon share market proposed here is likely to have more liquidity than a prediction market or a futures market because the traded asset is a property right created by the (still commonplace) act of emitting.

[10]For instance, consider the externalities produced by orbital debris in space. Satellite owners could post a bond to fund an "orbital-use share" that would incentivize both optimal debris creation and optimal debris cleanup. Fees for launching satellites are the analogue of an emission tax. They fail to incentivize either active measures to avoid creating debris post-launch or cleanup of debris post-impact. Rao et al. (2020) propose orbital-use fees that are the analogue of taxing the stock of pollution, a policy option discussed in footnote 12. Such a policy is vulnerable to judgment-proofness problems induced by market churn. It also does not aggregate private information like the carbon share price does.

[11]Deposit-refund schemes have also been understood as means to avoid the fiscal costs of subsidies and the distributional costs of taxes (Bohm, 1981). Here one of the motivations is to avoid the fiscal costs of using the public purse to directly fund carbon removal.

[12]If the regulator could tax the stock of carbon in the atmosphere (rather than the flow of emissions into the atmosphere), then the incentive to remove carbon would respond to new information as the stock tax was updated, analogous to how incentives under carbon shares respond to updated refunds. Emitters' incentives would depend on their expectations of future stock taxes (see Appendix B), analogous to how incentives under carbon shares depend on expected future damage charges. However, the stock tax would be evaded as emitters go out of business over the many decades that carbon persists in the atmosphere, whereas carbon shares are instruments with positive payoffs that investors are willing to pay for. Stock taxes have been proposed in the context of climate change (Lemoine, 2007), mine remediation (White et al., 2012; Yang and Davis, 2018), and space orbits (Rao et al., 2020).

change impacts.[13]

The possibility of removing enough carbon from the atmosphere to make aggregate emissions "net negative" has become a prominent part of the climate policy discourse. The Intergovernmental Panel on Climate Change projects that limiting warming to 1.5°C (2°C) would require up to 700 (250) Gt CO_2 of net negative emissions over this century (IPCC, 2022), the 2021 U.S. Infrastructure Investment and Jobs Act provided $3.5 billion to establish carbon removal hubs, and the 2022 U.S. Inflation Reduction Act increased tax credits for capturing and storing carbon from the air from $50 to $180 per ton of CO_2. Despite the increasingly prominent discussion and promotion of carbon removal, I know of no work on market-based approaches to incentivizing optimal use of these technologies.[14] In the absence of alternative policy instruments, many assume that governments would directly subsidize carbon removal, despite concerns about the fiscal burden such subsidies would impose (see Bednar et al., 2019; Edenhofer et al., 2021).[15] I here propose a policy instrument that is revenue-positive for the government and efficiently adapts the scale of carbon removal to new information about the cost of climate change.

This paper constitutes a novel link between environmental economics and the literature on asymmetric information in financial markets. Since Grossman (1976, 1978), much literature studies financial markets' ability to efficiently aggregate dispersed information. I here study an asset tied to an externality. I consider a setting that mixes elements of large- and small-market models: there are many small traders, but they are attached to a finite number of sectors and thus have a finite number of distinct information sets. Each small trader does not account for how its own trades affect the price (i.e., is nonstrategic) but does account for how the aggregate decisions of similarly informed traders move the price (i.e., uses information rationally). The setting and the conclusion that implementable equilibria are partially revealing share features of the monopolistic competition environment in Kyle (1989), but Kyle (1989) assumes that traders' signals are independent of each other. My focus on correlated information is similar to studies of "fundamental value" (Vives, 2011; Rostek and Weretka, 2012; Vives, 2014), but here payoffs are common across traders because they depend on damage charges that are applied uniformly to all shareholders (the externality is

[13]I show that the ideal deposit would equal the worst-case social cost of carbon. Others have previously proposed that fees on materials or products be set to their most harmful possible environmental fate, with fees refunded in accord with the harmfulness of actual outcomes (e.g., Solow, 1971; Mills, 1972; Bohm and Russell, 1985; Costanza and Perrings, 1990; Boyd, 2002). These informal proposals employ arguments based on ambiguity aversion, difficulties monitoring pollution, or difficulties posed by judgment-proofness.

[14]Conventional emission pricing policies could incentivize use of carbon dioxide removal technologies up to the point at which net emissions are zero. However, the European Union's flagship cap-and-trade program historically has not provided the credits for carbon dioxide removal that could sustain even this limited incentive (Scott and Geden, 2018; Rickels et al., 2020). Bednar et al. (2021) propose "carbon removal obligations" that would extend standard cap-and-trade schemes to allow temporarily overshooting longer-run carbon targets.

[15]Bednar et al. (2019) calculate that the subsidies required for carbon removal could exceed even the share of output that the U.S. spends on defense.

common to shareholders). My environment therefore mixes payoffs that are a pure common value with an information structure reminiscent of studies of fundamental value.

Dynamics are also central to my analysis. My traders aim to predict the regulator's future measurements, both because they directly determine a carbon share's refunds and because they determine capital gains as future traders use those signals to update their own beliefs about subsequent refunds. Much of the literature on asymmetric information in financial markets analyzes static models. I analytically solve a dynamic equilibrium model of information aggregation by assuming that each generation of traders has access to all public information from previous periods (i.e., to all previous prices and regulator reports) but not to any private information from previous periods. This assumption is similar to the critical assumption in Vayanos (1999, 2001) because traders begin each period in a symmetric informational position.[16]

The next section describes the economic and informational environment. Section 3 derives outcomes in the informationally efficient, welfare-maximizing benchmark. Section 4 analyzes emission taxes, focusing on the constraints they face in using new information and in observing all available information. Section 5 formally defines carbon shares and establishes the conditions under which this policy resolves the issues demonstrated with emission taxes. The final section concludes. The appendix contains numerical details, analysis of stock taxes, and proofs.

2 Setting

2.1 Production, Consumption, and Emissions

Let there be a unit mass of households and $N > 1$ intermediate-good sectors, each of which is perfectly competitive. Output from sector i in period t is

$$Y_{it} = \exp[-\zeta_{it} T_t]\, L_{it}\, Y^{it}(e_{it}).$$

$L_{it} \in [0, 1]$ is labor offered by households to sector i in exchange for wage w_{it}. The representative firm in sector i has gross production function $Y^{it}(e_{it})$, with $e_{it} \geq 0$ indicating emissions. $Y^{it}(\cdot)$ is strictly increasing and concave. Temperature T_t imposes damages ζ_{it} in sector i at time t.[17] The multiplicative effect of climate damages follows the DICE model (Nordhaus, 1992, 2013), among others, and the exponential form for damages follows Golosov et al.

[16]Other dynamic models assume that traders are risk-neutral (e.g., Kyle, 1985) or myopic (e.g., Singleton, 1987), or they assume ordered information structures (e.g., Wang, 1993) that do not make sense in the present paper's multisector economy. In Vives (1995), the asset is liquidated after a finite number of periods, at which time its true payoff is revealed. Here carbon shares can endure forever and the true nature of climate impacts is never revealed.

[17]The index i could equivalently be interpreted as indicating either regions or sector-region pairs.

(2014) and Lemoine (2021a), among others. Firms can condition their emission decisions on ζ_{it}.

The representative firm in sector i can fund the removal of quantity $R_{it} \geq 0$ of emissions from the atmosphere. It purchases this emission removal from a competitive industry whose costs $c(R_t)$ as a share of total output (see below) depend on aggregate removal $R_t \triangleq \sum_{i=1}^{N} R_{it}$, with $c(\cdot)$ strictly positive, strictly increasing, and strictly convex.[18]

Cumulative emissions up to time t are $M_t = M_0 + \sum_{s=0}^{t-1} \left[\sum_{i=1}^{N} e_{is} - R_s \right]$, with pre-policy cumulative emissions $M_0 \geq 0$ given. Time t warming is $T_t = \alpha\, M_t$, with $\alpha > 0$. This representation recognizes that carbon dioxide is a globally mixed pollutant and follows recent scientific findings that global temperature is approximately a linear function of cumulative emissions (see Dietz and Venmans, 2019, among others). Firms are small, so they ignore the effects of their own emissions on temperature.

Total output is Cobb-Douglas:

$$Y_t = \prod_{i=1}^{N} (Y_{it})^{\kappa_i} ,$$

with each $\kappa_i > 0$ and $\sum_{i=1}^{N} \kappa_i = 1$. Aggregate consumption C_t is no greater than net output:

$$C_t \leq (1 - c_t(R_t))Y_t.$$

The representative household has logarithmic utility:

$$u(C_t) = \ln(C_t).$$

Time t welfare is the present value of expected utility:

$$\sum_{s=t}^{\infty} \frac{1}{(1+r)^{s-t}} E_t \left[u(C_s) \right] ,$$

with per-period discount rate r and with the information set defined in each application below.

In equilibrium, firms maximize the expected present value of profits subject to prices, households maximize utility subject to budget constraints, and all markets clear. Time 0 consumption is the numeraire.

[18]Convexity in removal costs reflects both the cost of removing carbon from the atmosphere and the potential scarcity of sites for storing carbon after removal.

2.2 Informational Environment

I now describe the informational environment. All agents are Bayesian.

Agents affiliated with sector i observe $\zeta_{it}+\lambda_{it}$, where $\zeta_{it}=\zeta_i+\bar{\zeta}_i+\epsilon_{it}$. The ζ_i are unknown and unobserved. The $\bar{\zeta}_i$ are public knowledge and represent prior expected damages in each sector. Assume only that $\sum_{k=1}^{N}\kappa_k\bar{\zeta}_k<(r/\alpha)\kappa_i Y^{i0\prime}(0)/Y^{i0}(0)$ for some $i\in\{1,...,N\}$, which will ensure that welfare-maximizing aggregate emissions are strictly positive in the initial period. The ϵ_{it} and λ_{it} are random variables that are each normally distributed, mean-zero, unobserved, and uncorrelated either across sectors or over time. The variance of each ϵ_{it} is $\sigma^2>0$, and the variance of each λ_{it} is $\omega^2\geq 0$. The ϵ_{it} represent random exposure to global temperature. That randomness could result from randomness in the mapping from global temperature to temperatures in locations relevant to sector i and/or from randomness in sector i's exposure to its locations' temperatures. The λ_{it} represent agents' potentially imperfect ability to measure the effect of temperature on sectoral production.

The regulator and firms have a common jointly normal prior over the ζ_i at time 0. Each ζ_i has a prior mean of zero and has prior variance $\tau_0^2>0$.[19] The correlation between any pair ζ_i and ζ_j (for $i\neq j$) is $\Gamma\in[0,1]$, a known parameter. This correlation determines how signals of damages in one sector provide information about damages in another sector. If $\Gamma=0$, then the unknown component of damages is independent across sectors. If $\Gamma>0$, then the unknown component of damages has a common component across sectors, as when impacts in one sector affect other sectors or as when vulnerability to weather is correlated across sectors.

The regulator does not observe sectoral production or input choices. Instead, at the end of time t, the regulator uses observed total output to measure $\tilde{\zeta}_t+\tilde{\lambda}_t$, where $\tilde{\zeta}_t\triangleq\sum_{i=1}^{N}\kappa_i\zeta_{it}$.[20] The $\tilde{\lambda}_t$ are random variables that are normally distributed, mean-zero, and serially uncorrelated. Their variance is $\tilde{\omega}^2\geq 0$. They reflect the possibility of measurement error in aggregate data and of additional imprecision due to having to estimate $\tilde{\zeta}_t$ from aggregate data. The regulator shares the measured $\tilde{\zeta}_t+\tilde{\lambda}_t$ with all agents in the economy.

The timing within a period t is that intermediate-good firms make emission decisions, markets clear based on realized production, agents observe $\zeta_{it}+\lambda_{it}$, and finally the regulator observes $\tilde{\zeta}_t+\tilde{\lambda}_t$.

[19] Assuming a prior mean of zero is not restrictive, as nonzero means can be absorbed into the $\bar{\zeta}_i$.

[20] Firms' equilibrium production choices are independent of T_t (see (A-12) through (A-14)), so the regulator can estimate $\tilde{\zeta}_t$ from a time series of Y_t and T_t.

3 Informationally Efficient, Welfare-Maximizing Benchmark

Begin by considering welfare-maximizing emissions and carbon removal. Define $\hat{E}_t$ as the expectation operator based on all information available up to time t, $\hat{\boldsymbol{\mu}}_t$ and $\hat{\boldsymbol{\Omega}}_t$ as the vector of posterior means and the posterior covariance matrix for the ζ_i based on information up to time t, and $\hat{\mu}_t$ and $\hat{\Omega}_t$ as the posterior mean and variance of $\sum_{i=1}^N \kappa_i \zeta_i$ based on information up to time t.[21]

Welfare-maximizing outcomes solve the following Bellman equation:

$$\hat{W}(T_t, \hat{\boldsymbol{\mu}}_t, \hat{\boldsymbol{\Omega}}_t) = \max_{\boldsymbol{L}_t, \boldsymbol{e}_t, R_t \geq 0} \hat{E}_t \left[u(C_t) + \frac{1}{1+r} \hat{W}(T_{t+1}, \hat{\boldsymbol{\mu}}_{t+1}, \hat{\boldsymbol{\Omega}}_{t+1}) \right],$$

where $\boldsymbol{L}_t$ and $\boldsymbol{e}_t$ indicate vectors of labor and emissions in each sector. Taking first-order conditions and then recursively substituting from the envelope theorem yields the following conditions that must hold for all i:

$$\frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})} \begin{cases} = \frac{1}{r}\alpha \left[\sum_{k=1}^N \kappa_k \bar{\zeta}_k + \hat{\mu}_t \right] & \text{if } e_{it} > 0 \\ \leq \frac{1}{r}\alpha \left[\sum_{k=1}^N \kappa_k \bar{\zeta}_k + \hat{\mu}_t \right] & \text{if } e_{it} = 0 \end{cases}, \tag{1}$$

$$\frac{c_t'(R_t)}{1 - c_t(R_t)} \begin{cases} = \frac{1}{r}\alpha \left[\sum_{k=1}^N \kappa_k \bar{\zeta}_k + \hat{\mu}_t \right] & \text{if } R_t > 0 \\ \geq \frac{1}{r}\alpha \left[\sum_{k=1}^N \kappa_k \bar{\zeta}_k + \hat{\mu}_t \right] & \text{if } R_t = 0 \end{cases}. \tag{2}$$

On the right-hand side, the terms in brackets yield per-period expected damages per unit of warming, the α converts to units of emissions, and the $1/r$ converts to present value. The first condition equates the marginal benefit of emissions to the marginal social cost of emissions when emissions are strictly positive. If $Y^{it\prime}(0)$ is sufficiently small, then $e_{it} = 0$. The second condition equates the marginal cost of carbon removal to the marginal social cost of emissions when carbon removal is strictly positive. If $c_t'(0)$ is sufficiently large, then $R_t = 0$. As $\hat{\mu}_t$ increases, e_{it} falls and R_t eventually begins increasing. Negative emissions, in which $R_t > \sum_{i=1}^N e_{it}$, become optimal when $\sum_{k=1}^N \kappa_k \bar{\zeta}_k + \hat{\mu}_t$ is sufficiently large.

Now consider beliefs. As in other literature, the benchmark of informational efficiency updates beliefs from the available signals $\zeta_{it} + \lambda_{it}$ and $\tilde{\zeta}_t + \tilde{\lambda}_t$.

Proposition 1 (Informationally Efficient Beliefs)**.** *There exists* $\hat{Z}_t \in [0, 1)$ *such that* $\hat{Z}_t \to 0$

[21] Throughout, I use a hat (ˆ) to indicate outcomes under the informationally efficient welfare-maximizing benchmark, a tilde (˜) to indicate outcomes under an emission tax policy, and a breve (˘) to indicate outcomes under the carbon share policy.

as $\tilde{\omega}^2/\omega^2 \to \infty$ *and*

$$\begin{aligned}\hat{\mu}_t =& \hat{Z}_t \left[\frac{1}{t}\sum_{j=0}^{t-1}[\tilde{\zeta}_j + \tilde{\lambda}_j] - \sum_{k=1}^{N}\kappa_k\bar{\zeta}_k\right] \\ &+ \frac{(1-\hat{Z}_t)(1-\Gamma)\tau_0^2 - \hat{Z}_t\sigma^2/t}{(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t}\sum_{k=1}^{N}\kappa_k\left[\frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{kj} + \lambda_{kj}] - \bar{\zeta}_k\right] \\ &+ \frac{\sigma^2/t + (1-\hat{Z}_t)\omega^2/t}{(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t}\frac{N\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2}\frac{1}{N}\sum_{k=1}^{N}\left[\frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{kj} + \lambda_{kj}] - \bar{\zeta}_k\right].\end{aligned} \tag{3}$$

Proof. Apply the projection theorem to a random vector formed from $\sum_{k=1}^{N}\kappa_k\zeta_k$, the N sectoral signals, and the aggregate signal. See Appendix C. □

Corollary 1 (Special Cases for Informationally Efficient Beliefs)**.**

i If $\omega^2 = 0$*, then*

$$\begin{aligned}\hat{\mu}_t =& \frac{(1-\Gamma)\tau_0^2}{(1-\Gamma)\tau_0^2 + \sigma^2/t}\overbrace{\sum_{k=1}^{N}\kappa_k\left[\frac{1}{t}\sum_{j=0}^{t-1}\zeta_{kj} - \bar{\zeta}_k\right]}^{=\lim_{\tilde{\omega}^2\to 0}\frac{1}{t}\sum_{j=0}^{t-1}\tilde{\zeta}_j - \sum_{k=1}^{N}\kappa_k\bar{\zeta}_k} \\ &+ \frac{\sigma^2/t}{(1-\Gamma)\tau_0^2 + \sigma^2/t}\frac{N\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2 + \sigma^2/t + N\Gamma\tau_0^2}\frac{1}{N}\sum_{k=1}^{N}\left[\frac{1}{t}\sum_{j=0}^{t-1}\zeta_{kj} - \bar{\zeta}_k\right].\end{aligned}$$

ii If $\tilde{\omega}^2 = 0$ *and* $\Gamma = 0$*, then*

$$\hat{\mu}_t = \frac{\tau_0^2}{\tau_0^2 + \sigma^2/t}\left[\frac{1}{t}\sum_{j=0}^{t-1}\tilde{\zeta}_j - \sum_{k=1}^{N}\kappa_k\bar{\zeta}_k\right].$$

iii If $\tilde{\omega}^2 = 0$ *and each* $\kappa_i = 1/N$*, then*

$$\hat{\mu}_t = \frac{(1-\Gamma)\tau_0^2 + N\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2 + \sigma^2/t + N\Gamma\tau_0^2}\left[\frac{1}{t}\sum_{j=0}^{t-1}\tilde{\zeta}_j - \sum_{k=1}^{N}\kappa_k\bar{\zeta}_k\right].$$

Proof. See Appendix D. □

The informationally efficient benchmark aggregates information from the private signals and the public signals. The first part of the corollary describes mean beliefs when dispersed agents do not suffer measurement error ($\omega^2 = 0$). In this case, the informationally efficient benchmark has no use for the aggregate signals $\tilde{\zeta}_j + \tilde{\lambda}_j$. Instead, it weights the sectoral signals as if a perfectly measured version of the aggregate signal were available (first line) and, when $\Gamma > 0$, it also uses the disentangled (i.e., unweighted) sectoral signals directly because signals in relatively unimportant sectors with small κ_i provide information about damages in all other sectors (second line). In fact, if $\Gamma = 1$, informationally efficient beliefs do not weight the sectoral signals by the κ_i at all (the first line vanishes) because each sector's signal provides the same information about aggregate damages as any other sector's signal, whether or not the signal comes from a sector that receives only a small weight in aggregate output.

The second and third parts of Corollary 1 describe mean beliefs when there is no measurement error in the aggregate signal ($\tilde{\omega}^2 = 0$). The second part shows that the informationally efficient benchmark has no use for the disaggregated sectoral signals $\zeta_{kj} + \lambda_{kj}$ when sectoral signals are independent ($\Gamma = 0$). And the third part shows that the informationally efficient benchmark has no use for the disaggregated sectoral signals $\zeta_{kj} + \lambda_{kj}$ when sectors have identical value shares of output and thus receive identical weights in the aggregate signal (i.e., each $\kappa_i = 1/N$). In either case, mean beliefs are what one would expect from simple applications of the familiar univariate normal-normal updating rule.

Proposition 1 shows that informationally efficient beliefs in general use both types of information. The aggregate signal (first line in (3)) provides information that can mitigate the consequences of measurement error in private signals, and the private signals (second line) provide information used to construct an alternate version of the aggregate signal that mitigates the consequences of measurement error in the aggregate signal. Finally, efficient updating also leverages correlation across sectoral effects (third line) to learn from sectors whose small κ_i mean they do not directly matter much for aggregate outcomes.

4 Regulation by Emission Taxes

Now consider a regulator who maximizes welfare by taxing firms' period t net emissions at rate ν_t. Firms can avoid the tax either by reducing emissions or by simultaneously contracting for removal to offset ongoing emissions. The regulator returns any tax revenue to households as lump-sum transfers. I initially assume that the regulator's tax revenue must be weakly positive in each period before weakening that assumption in Section 4.1.

The regulator sets the time t tax at the beginning of the period so as to maximize welfare conditional on its time t beliefs and subject to market equilibrium. The regulator's chosen time t tax is therefore a function of the aggregate measurements from times 0 through $t-1$. Denote the regulator's mean belief about $\sum_{k=1}^{N} \kappa_k \zeta_k$ at the time t information set as $\tilde{\mu}_t$.

The following proposition gives the optimal emission tax:

Proposition 2 (Emission Tax)**.** *There exists* $\bar{\nu}_t > 0$ *such that* $\sum_{i=1}^{N} e_{it} - R_t = 0$ *if and only if* $\nu_t \geq \bar{\nu}_t$*. The regulator maximizes welfare with a tax of*

$$\nu_t = \min\left\{\bar{\nu}_t,\ C_0 \frac{\alpha}{r}\left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \tilde{\mu}_t\right]\right\}.$$

Proof. See Appendix E. □

When $\nu_t < \bar{\nu}_t$, the tax is determined by the present value of expected aggregate damages.[22] Using that tax, firms' first-order conditions (A-13) and (A-14) become

$$\frac{\alpha}{r}\left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \tilde{\mu}_t\right] = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})},$$

$$\frac{\alpha}{r}\left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \tilde{\mu}_t\right] = \frac{c_t'(R_t)}{1 - c_t(R_t)}.$$

Once we account for the possibility of corner solutions, these are the conditions for welfare-maximization given in (1) and (2) when $\tilde{\mu}_t = \hat{\mu}_t$.

We see two possible reasons why a tax may not attain the informationally efficient welfare-maximizing benchmark. First, it could be that $C_0 \frac{\alpha}{r}\left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \tilde{\mu}_t\right] > \bar{\nu}_t$. In this case, the benchmark from Section 3 would have negative net emissions whereas the regulator's feasible equilibrium has zero net emissions. I refer to this possibility as inducing a loss due to inefficiency in using information, as it indicates a failure to implement the information collected from the economy. Second, it could be that $\tilde{\mu}_t \neq \hat{\mu}_t$. In this case, the benchmark from Section 3 would be based on beliefs that are different from the regulator's beliefs. I refer to this possibility as inducing a loss from observing information, as it indicates a failure to collect all of the information in the economy. I explore each in turn.

4.1 Loss Due to Inefficiency in Using Information

Temporarily assume that the regulator observes all signals $\zeta_{it} + \lambda_{it}$, as when firms truthfully communicate to the regulator or the regulator collects data as precise as what firms have access to. In this case, $\tilde{\mu}_t = \hat{\mu}_t$, so the regulator can set the informationally efficient tax and we have no loss from inefficiency in observing information.

[22] The combination of logarithmic utility and the damage specification means that uncertainty is not priced directly, as in Golosov et al. (2014). For a more general constant relative risk aversion utility function, the optimal tax would be sensitive to future consumption and would include a risk premium (see Lemoine, 2021a).

It is immediately obvious from the foregoing analysis that the policy described in Proposition 2 imposes losses relative to the welfare-maximizing benchmark when there is some chance that the constraint $\bar{\nu}_t$ binds (i.e., that negative net emissions become optimal). That chance is driven by the possibility of observing information that makes $\tilde{\mu}_t$ large and by the possibility that technological progress in carbon removal makes R_t large for any given ν_t (i.e., makes $\bar{\nu}_t$ small).

One might object that the modeled revenue constraint is too stringent. A forward-thinking regulator could save the revenue collected from emission taxes and dedicate it to funding carbon removal. In effect, such a policy establishes a lockbox for emission tax revenue that allows the regulator to procure some level of negative net emissions without needing to raise money from taxpayers. It changes the revenue constraint from a static one that must hold in each period to a dynamic one that must hold across periods.

Consider the implications of a dynamic revenue constraint in a world in which the regulator does not learn about damages but in which technological progress in carbon removal can eventually make negative net emissions optimal even under the prior belief. Proposition 2 establishes that the regulator's unconstrained-optimal tax would be

$$\nu_t = C_0 \frac{\alpha}{r} \left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \tilde{\mu}_t \right].$$

This tax is unaffected by the possibility of technological progress in carbon removal technologies and is constant over time in the absence of learning about damages (i.e., when $\tilde{\mu}_t$ is constant over time). Because this tax is also the subsidy a regulator would like to offer for carbon removal in a negative net emission scenario, the tax that the regulator collects at the time of emission is exactly equal to the subsidy the regulator would subsequently offer to remove that same unit of emission from the atmosphere. The dynamic revenue constraint would therefore never bind unless it became optimal to bring future carbon stocks below their initial level M_0.

Now let the regulator learn about damages. In that case negative net emissions could become optimal because the regulator observed unfavorable information about climate damages. Such information could increase $\tilde{\mu}_t$ by enough to make negative net emissions optimal, whether or not there is progress in carbon removal technology. The tax the regulator collects at the time of emission is then strictly less than the subsidy the regulator would subsequently offer to remove that same unit of emission from the atmosphere. The regulator has sufficient tax revenue in its lockbox to bring carbon only part of the way back to M_0. The more pessimistic damage estimates become, the more likely this constraint on the regulator's ability to fund negative net emissions becomes binding. And if carbon removal technology simultaneously progresses quickly, then this constraint becomes even more likely to bind. The regulator might again be unable to procure the optimal level of negative net emissions without raising funds from taxpayers.

The following corollary shows that the regulator also distorts emission decisions in anticipation that the constraint might bind:

Corollary 2 (Emission Tax With A Lockbox)**.**

1. *Consider a time t in which net emissions are strictly positive. The tax with a lockbox is strictly greater (strictly less) and emissions are strictly less (strictly greater) than given in Proposition 2 if marginally raising that tax increases (decreases) tax revenue.*

2. *Consider a time $t+s$ in which net emissions are weakly (strictly) negative. The optimal price that the regulator offers for carbon removal is weakly (strictly) less, and net emissions are weakly (strictly) greater, than the welfare-maximizing benchmark.*

Proof. See Appendix F. □

The marginal value of a higher emission tax is comprised of its marginal value in a setting without revenue constraints and its effect on future negative emission constraints via its effect on the revenue that will be stored in a lockbox. The latter effect distorts the emission tax away from its unconstrained-optimal level in order to prepare for the possibility that sufficiently negative net emissions become optimal. When emissions are strictly positive, raising an emission tax increases revenue by charging more per unit of net emissions but reduces revenue by reducing net emissions. The optimal tax in the presence of a lockbox is higher (lower) when the former (latter) dominates. Small distortions in the emission tax do not impose first-order costs today, but they can raise extra revenue that provides first-order benefits by weakening a potential future constraint on negative net emissions.[23]

Once the regulator is already paying for negative emissions out of the lockbox, reducing the emission tax clearly leaves more revenue in the lockbox: a lower price requires the regulator to pay less per unit of carbon removal and also procures less carbon removal. A regulator in such a condition prepares for the possibility of future binding constraints by reducing the price offered for carbon removal.

In summary, we have seen that

1. A regulator who must obey a period-by-period revenue constraint cannot procure negative emissions using an emission tax. An emission tax therefore cannot attain first-best should negative emissions eventually become optimal, even if the regulator has perfect information.

2. A regulator who stores emission tax revenue in a lockbox for funding future carbon removal can fund the optimal level of carbon removal if the regulator does not learn about damages over time. An emission tax-plus-lockbox therefore can attain first-best in the presence of technological progress in carbon removal.

[23]The possibility of hitting the constraint at some future time did not distort the optimal tax in Proposition 2 because the combination of logarithmic utility and multiplicative-exponential damages makes the optimal tax independent of future emission and removal trajectories.

3. A regulator who stores emission tax revenue in a lockbox for funding future carbon removal might not be able to fund the optimal level of carbon removal if the regulator learns about damages over time. An emission tax-plus-lockbox therefore may not attain first-best in the presence of new information about climate damages. And the optimal use of the lockbox distorts emissions and removal decisions in all other periods so as to increase funds in the lockbox.

That final point shows that information is critical to inefficiencies in procuring negative emissions: an emission tax may not be able to optimally use new information about the social cost of emissions, should that new information be sufficiently pessimistic and the regulator not be able to costlessly offer arbitrarily large subsidies from taxpayer funds.

4.2 Loss Due to Inefficiency in Observing Information

Now allow asymmetric information but assume that carbon removal is infeasible, as when $c_t'(0)$ is so large that carbon removal is irrelevant under plausible beliefs. We have no loss from inefficiency in using information and instead analyze a loss from inefficiencies in observing information.

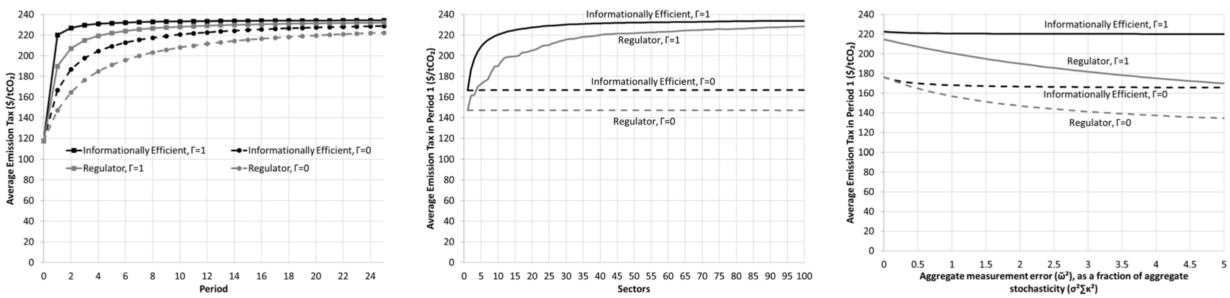
The following result describes the regulator's time t posterior estimate of $\sum_{k=1}^{N} \kappa_k \zeta_k$.

Proposition 3 (Regulator's Beliefs)**.**

$$\tilde{\mu}_t = \frac{(1-\Gamma)\tau_0^2 \sum_{i=1}^{N} \kappa_i^2 + \Gamma\tau_0^2}{(1-\Gamma)\tau_0^2 \sum_{i=1}^{N} \kappa_i^2 + \Gamma\tau_0^2 + \frac{1}{t}[\tilde{\omega}^2 + \sigma^2 \sum_{i=1}^{N} \kappa_i^2]} \frac{1}{t} \sum_{j=0}^{t-1} \left(\tilde{\zeta}_j + \tilde{\lambda}_j - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right). \tag{4}$$

Proof. Follows from application of the conventional univariate normal-normal Bayesian updating formula, observing that the prior variance is $\tau_0^2 \sum_{i=1}^{N} \kappa_i^2 + 2\Gamma\tau_0^2 \sum_{i=1}^{N} \sum_{k=i+1}^{N} \kappa_i \kappa_k$ and using $\sum_{i=1}^{N} \kappa_i = 1$. □

The weight placed on the aggregate measurement in (4) increases in Γ: positive correlation increases the regulator's prior uncertainty about aggregate damages and thus mechanically increases the weight placed on the aggregate measurement. Posterior beliefs are exactly the same as those formed by a counterfactual regulator in a world with variance $\check{\tau}_0^2 = (1-\Gamma)\tau_0^2 + \Gamma\tau_0^2/[\sum_{i=1}^{N} \kappa_i^2]$ and correlation $\check{\Gamma} = 0$. In contrast, Proposition 1 showed that positive correlation among the unknown sector-specific effects ζ_i increases the weight that informationally efficient beliefs place on the disentangled sectoral measurements. Positive correlation enables informationally efficient beliefs to use information from one sector to update beliefs about the others: when $\Gamma > 0$, informationally efficient updating will not be identical to some other world that lacked correlation but had a redefined variance. Correlation affects the regulator differently because it does not have access to the disentangled signals.



(a) Evolution of Beliefs (b) Influence of N in period 1 (c) Influence of $\tilde{\omega}^2$ in period 1

Figure 1: An example of how informationally efficient beliefs and the regulator's beliefs would each evolve on average (i.e., of $E_0[\hat{\mu}_t|\boldsymbol{\zeta}]$ and $E_0[\tilde{\mu}_t|\boldsymbol{\zeta}]$). Appendix A details the calibration. The emission tax optimal at prior beliefs is \$118 per tCO_2, and simulations assume that the emission tax conditional on true knowledge of the ζ_i would be twice as large (\$236 per tCO_2). Left: Evolution of average beliefs over the first 25 periods, with $N = 10$ and $\tilde{\omega}^2/[\sigma^2 \sum_{i=1}^{N} \kappa_i^2] = 2$. Middle: The effect of the number of sectors (N) on period 1 beliefs. Right: The effect of aggregate measurement error ($\tilde{\omega}^2$) on period 1 beliefs.

The following corollary delineates conditions under which the regulator's beliefs are informationally efficient.

Corollary 3 (Informationally Efficient Regulator)**.** *For $t > 0$, $\tilde{\mu}_t = \hat{\mu}_t$ with probability 1 if (i) $\tilde{\omega}^2 = 0$ and either (iia) $\Gamma = 0$ or (iib) each $\kappa_i = 1/N$.*

Proof. Follows straightforwardly from Corollary 1 and Proposition 3. □

The regulator's beliefs are informationally efficient if there is no measurement error at the aggregate level ($\tilde{\omega}^2 = 0$) and either there is no correlation among sectoral effects ($\Gamma = 0$) or sectors have identical weights in production (each $\kappa_i = 1/N$). Otherwise Proposition 1 showed that informationally beliefs generally use disaggregated signals unavailable to the regulator. Because all three of the conditions in Corollary 3 are plausibly violated in reality, an actual regulator's beliefs are likely to be informationally inefficient.

Figure 1 provides an example that illustrates how the regulator's beliefs (gray) differ from informationally efficient beliefs (black), for the extreme cases when unknown sectoral effects are independent of each other ($\Gamma = 0$, dashed) and are perfectly correlated with each other ($\Gamma = 1$, solid). Based on the calibration described in Appendix A, the emission tax that would be optimal at time 0 beliefs is \$118 per tCO_2. The depicted simulations assume that the initial tax that would be optimal with perfect information about the ζ_i would be twice as large. Each curve averages over 1 million trajectories for $\hat{\mu}_t$ and $\tilde{\mu}_t$.

The left panel assesses how beliefs converge to the truth.[24] In these cases, informationally efficient beliefs converge to the truth faster than do the regulator's beliefs. Both types of beliefs converge faster when sectors are perfectly correlated with each other than when sectors are independent of each other: informationally efficient beliefs infer more from each observation in the presence of correlation, and the regulator places more weight on the data when correlation increases the prior variance.

The middle and right panels plot the emission tax chosen, on average, after observing signals from time 0. The middle panel shows that the number of sectors N does not affect beliefs when sectoral effects are independent (the plot scales σ^2 and ω^2 so that aggregate variance is independent of N). However, correlation makes the average speed of learning increase in the number of sectors N, and informationally efficient beliefs in particular update much faster when the economy has multiple sectors that provide information about each other. The right panel shows that aggregate measurement error $\tilde{\omega}^2$ (which increases to the right) markedly slows learning by the regulator. In contrast, informationally efficient beliefs are less sensitive to aggregate measurement error because they can use the disentangled sectoral signals directly and thereby mitigate that source of error.[25]

Whereas the drawback in Section 4.1 was inefficiency in using the available information should that information warrant negative emissions, the drawback here is the inefficiency in observing all available information. The best emission tax that a regulator can implement will generally differ from the emission tax that the regulator would choose based on all of the information in the economy.

5 A Policy Framework that Dominates Conventional Emission Pricing

We have seen that conventional emission pricing does not perform ideally at either collecting or using information about the social cost of greenhouse gas emissions. As a result, there is space for a policy to do better than conventional market-based instruments. I now describe such a policy, in the form of a novel dynamic deposit-refund instrument.

This new type of policy requires each emitter to post a deposit $D \geq 0$ per unit of

[24] The average speed of convergence is used here for illustration, but it is not a measure of the quality of beliefs. Such a measure would also account for the standard deviation of beliefs. For instance, when $\Gamma = 0.5$, the regulator's beliefs converge towards the true value on average faster than do informationally efficient beliefs, but they are more sensitive to randomness in any particular trajectory.

[25] It is hard to detect visually in the figure, but aggregate measurement error does slow learning even for informationally efficient beliefs with $\Gamma = 1$. Also, the convergence of informationally efficient beliefs and the regulator's beliefs when $\Gamma = 0$ and $\tilde{\omega}^2 = 0$ illustrates Corollary 3.

emissions. We can express D as

$$D = \frac{1}{r} C_0 \, \alpha \left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \bar{\mu} \right]. \tag{5}$$

The parameter $\bar{\mu} \geq -\sum_{i=1}^{N} \kappa_i \bar{\zeta}_k$ defines beliefs about per-period climate damages implied by the deposit.[26] In exchange for the deposit, the emitter receives a transferable asset that is attached to the unit of carbon emitted. I refer to the asset as a "carbon share" because it reflects a claim on a part of the carbon in the atmosphere.

At the end of each period, the policymaker applies a damage charge to each outstanding carbon share. This charge is set equal to the lesser of the period t measured marginal damage from carbon emissions and the per-period damages implied by the deposit:

$$\Delta_t = C_0 \, \alpha \, \min \left\{ \tilde{\zeta}_t + \tilde{\lambda}_t, \; \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \bar{\mu} \right\}. \tag{6}$$

The damage charge is returned lump sum to consumers.[27] The policymaker refunds to carbon shareholders the difference between the damage charge and the per-period damages implied by the deposit:

$$\begin{aligned} d_t =& r \, D - \Delta_t \\ =& C_0 \, \alpha \max \left\{ 0, \; \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \bar{\mu} - (\tilde{\zeta}_t + \tilde{\lambda}_t) \right\}. \end{aligned} \tag{7}$$

The refunds d_t are weakly positive. No refund is paid in the period of emission. The deposit acts like principal, some of which is returned to agents over time in the form of refunds and some of which is reclaimed by the regulator in the form of damage charges. Over the lifetime of a carbon share, the present value of total refunds and damage charges recovers the deposit:

$$\begin{aligned} \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [d_{t+s} + \Delta_{t+s}] =& \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} r D \\ =& D. \end{aligned}$$

In each period subsequent to emission, a carbon share's owner decides whether to leave its attached unit of carbon in the atmosphere. If the owner removes the carbon from the

[26] The deposit would equal the optimal emission tax in Proposition 2 if $\bar{\mu} = \tilde{\mu}_t$.

[27] In a second-best setting, revenue from damage charges could be used to offset revenue from distortionary taxes. A full analysis of such a setting should consider how to adapt both the damage charges and the deposit (see Fullerton and Wolverton, 2000, 2005).

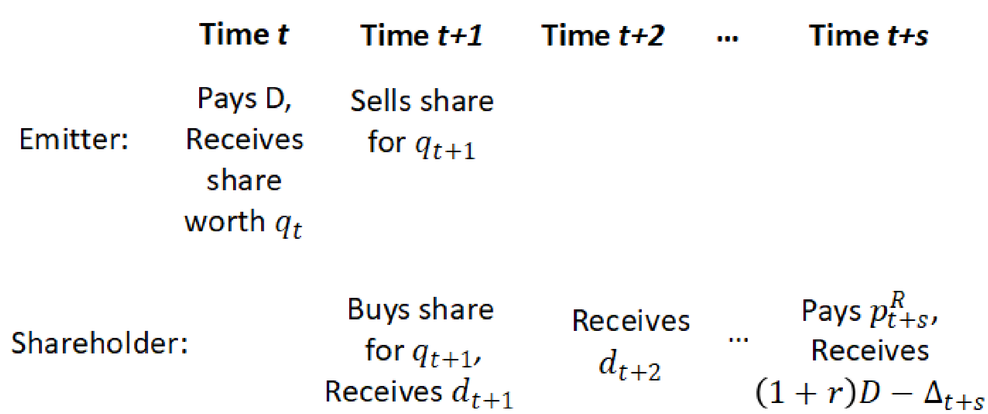


Figure 2: Example of the life of a carbon share. Here the share is attached to a unit of time t emissions, the emitter decides to sell the share at time $t+1$, and the new shareholder decides to remove the underlying unit of carbon from the atmosphere at time $t+s$.

atmosphere in time t, they receive $(1+r)D - \Delta_t$ and the share is retired; otherwise they receive refund d_t and can keep or sell the share. The carbon share is therefore an option to claim the deposit by spending on carbon removal. The shareholder receives the refunds d_t whether exercising or holding the option, but the shareholder loses the charges Δ_t as long as the option is unexercised. Shares are clearly valuable, because the worst they do is pay zero refunds. If the owner of a carbon share were to declare bankruptcy or otherwise liquidate, its creditors would want the carbon share so they could receive its refunds and have the option to eventually reclaim the full deposit.

Figure 2 provides an example of payoffs over time under the carbon share policy. At time t, an emitter posts the deposit D and in return receives a carbon share whose market value is q_t (to be analyzed below). At time $t+1$, the emitter in this example decides to sell the share to a third party for the market price q_{t+1}. That third party claims the time $t+1$ refunds d_{t+1} and continues to do so until either selling the share or removing the underlying unit of carbon from the atmosphere. At time $t+s$, the third party in this example does decide to remove the underlying unit of carbon from the atmosphere, which costs p^R_{t+s}. At that point, the regulator retires the carbon share and pays the third party $(1+r)D - \Delta_{t+s}$.

I assume all agents discount at rate r. Use $\check{W}$ to denote welfare along a realized trajectory under the carbon share policy defined above and use $\hat{W}$ to denote welfare along a realized trajectory under the welfare-maximizing, informationally efficient benchmark:

$$\breve{W} = \sum_{s=0}^{\infty} \frac{1}{(1+r)^s} u(\breve{C}_s),$$
$$\hat{W} = \sum_{s=0}^{\infty} \frac{1}{(1+r)^s} u(\hat{C}_s).$$

The full-information expected loss from using the carbon share policy is:

$$\breve{L} = E_0 \left[\hat{W} - \breve{W} \middle| \boldsymbol{\zeta} \right],$$

where $\boldsymbol{\zeta}$ is a vector of the ζ_i.

5.1 Improved Efficiency in Using Information

First assume informational efficiency, so that all actors in the economy see all of the $\zeta_{it} + \lambda_{it}$ at each time t. We saw in Section 4.1 that an emission tax may fail to use information efficiently when information justifies negative emissions. I will show that carbon shares can improve outcomes if the deposit D is sufficiently large. Section 5.2 will analyze carbon shares when agents have private information.

Define $\hat{q}_t$ as the carbon share's value in period t prior to observing the $\zeta_{it} + \lambda_{it}$, $\tilde{\zeta}_t + \tilde{\lambda}_t$, Δ_t, or d_t, where the hat notation reflects that the carbon share price in this section uses informationally efficient beliefs (as opposed to the share price to be defined in Section 5.2). The following lemma establishes the equilibrium value of the carbon share:

Lemma 1 (Carbon Share Value)**.**

$$\hat{q}_t = \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} \hat{E}_t[d_{t+j}]. \tag{8}$$

Proof. See Appendix G.

□

The equilibrium value of the carbon share is the expected present value of the refunds that it claims. The value of holding a carbon share therefore derives from the possibility that damages will not be as bad as implied by $\bar{\mu}$. At the time of emission, the firm's net outlays per unit of non-abated emissions are $D - (\hat{q}_t - \hat{E}[d_t]) \in [0, D]$.[28]

The following assumption ensures that it would never be optimal to remove enough carbon to bring atmospheric carbon and temperature below their initial levels:

[28] If future damages were guaranteed to be zero in every period, then the present value of the stream of refunds at the time of emission would be D, and if future damages were guaranteed to exceed the per-period value implied by D in every period, then the present value of the stream of refunds at the time of emission would be zero. Therefore $\hat{q}_t - \hat{E}[d_t] \in [0, D]$.

Assumption 1 (Current Carbon Will Not Be Removed). *$\hat{R}_t \leq M_t - M_0$ for all $t \geq 0$.*

Even the highest-removal scenarios for the coming century do not project bringing carbon or temperature below current levels (IPCC, 2022), so this assumption is likely to be met by any carbon share policy begun in the next few years. The following proposition relates the period t loss to the deposits required at earlier times:

Proposition 4 (Efficiency Conditional on Information). *Let Assumption 1 hold. Then $\breve{L} \to 0$ as $D \to \infty$.*

Proof. See Appendix H. □

The proposition establishes that the carbon share policy achieves the welfare-maximizing benchmark as the deposit becomes large. The proof shows that the time t private values for reducing emissions and removing carbon are each equal to

$$\sum_{j=1}^{\infty} \frac{1}{(1+r)^j} \hat{E}_t[\Delta_{t+j}].$$

Emitters lose the difference between the initial deposit and the initial value of the share they receive, and that difference is the present value of expected damage charges. Carbon removal benefits shareholders by preventing the loss of future damage charges. From (6), the damage charges are the current period's realized marginal damage when $\bar{\mu}$ (and thus D) is large. Therefore the present value of expected future damage charges under large D is just the present value of expected marginal damage from emissions, which is the social cost of carbon familiar from much work on the economics of climate change.[29]

As the deposit becomes large, the carbon share policy maintains the emission reduction incentives of an emission tax but approaches efficiency in using information even in the presence of carbon removal. Comparing to results in Section 4.1, the carbon share policy outperforms an emission tax with a static revenue constraint if net negative emissions might ever become optimal, and the carbon share policy outperforms an emission tax with a dynamic revenue constraint if optimal removal might exhaust the cumulative revenue collected from emission taxes. The carbon share policy's incentives can be understood by contrasting it with an emission tax policy that attaches a distinct lockbox to each unit of emissions, so that the most a regulator can spend to remove old emissions is what the regulator collected at the time of emission. In that case, increasing the emission tax at some time would increase

[29] I have used normal distributions for tractability and ease of exposition. If I instead assumed that the distribution of damages had finite support, then the carbon share policy would achieve the welfare-maximizing benchmark as each $\bar{\mu}$ (and thus each D) approaches some finite value from below. Under this interpretation, the carbon share policy approaches efficiency as the deposit approaches the worst-case social cost of carbon.

the amount the regulator could later spend on removal, but such a change in the emission tax would overincentivize emission reductions. In contrast, increasing the carbon share's deposit increases both the efficiency of its emission price and the efficiency of its removal incentives.

The optimal carbon share policy provides the same incentives as would the optimal tax on the stock of carbon previously emitted by a firm (as opposed to the conventional tax on the flow of carbon emissions studied in Section 4).[30] However, whereas firms could avoid a carbon stock tax by declaring bankruptcy, carbon shares are valuable assets that investors want to hold, financed at the time of emission by the deposit. Carbon shares therefore avoid judgment-proofness problems that can bedevil stock taxes. Moreover, a stock tax would lack the information aggregation benefits to be described in Section 5.2.

One might be concerned about whether the deposit would challenge firms' liquidity (see Shogren et al., 1993). Recall that firms receive a carbon share in return for their deposit and can immediately sell this valuable asset on. From (A-21), their net outlays per unit of emissions are the exact same outlays required by the traditional Pigouvian carbon tax. This is why an arbitrarily large deposit does not distort firms' emission incentives. If the market for carbon shares is decently thick, a carbon share policy need not be any more financially challenging than a conventional carbon emission tax.[31]

But one might still wonder about the scale of the deposit. If the deposit is not so large, then the highest potential damage charges are truncated by the constraints imposed by the deposits, which reduces the expected damage charges that firms use to guide emission and removal decisions. The possibility of hitting deposits' constraints therefore increases emissions and reduces removal. Ex post, options on carbon shares would reveal whether traders deemed it likely that the value of a carbon share would approach zero, as when damage charges are constrained by the deposit. Ex ante, a numerical exercise detailed in Appendix A provides some indication of how large a deposit may be necessary. This exercise takes damage estimates from the survey in Pindyck (2019) and considers the probability that any given deposit would be insufficient to cover the implied damage charges (i.e., that $\Delta_t < C_0\,\alpha\,[\tilde{\zeta}_t + \tilde{\lambda}_t]$ in equation (6)). In this calibration, expected damages imply a tax of \$118 per tCO_2. Figure 3 shows that a deposit roughly twice as large (\$250 per tCO_2) would suffice in all but the worst 10% of cases, and a deposit roughly three times as large (\$400 per tCO_2) would suffice in all but the worst 5% of cases. An adequate deposit may therefore be well within an order of magnitude of what the carbon tax would have been.

[30] Appendix B shows that the optimal time t stock tax would be $\lim_{\bar{\mu}\to\infty} \Delta_t$.

[31] Gross outlays are also capped because any firm could avoid posting the deposit by reducing its emissions. The growing number of firms making zero emission pledges and recent cost projections for removal technologies both suggest that even the maximum gross outlays are limited to a reasonable scale.

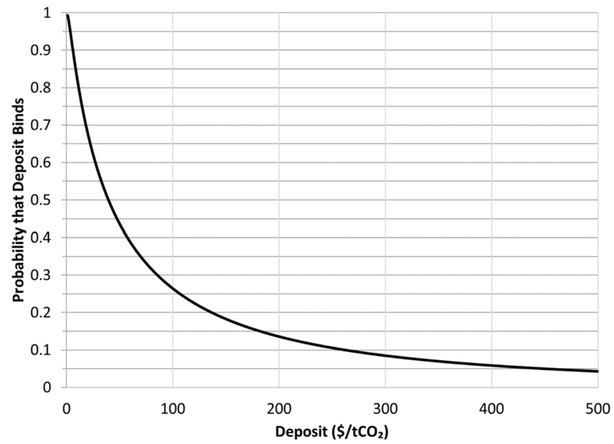


Figure 3: The probability that deposits of various sizes would bind, based on the calibration in Appendix A and using $\tilde{\omega}^2 = \sigma^2 = 0$.

5.2 Improved Efficiency in Observing Information

I now investigate whether the market for carbon shares aggregates dispersed information about climate change damages. I therefore allow asymmetric information, as in Section 4.2. I condition results on arbitrarily large D so as to highlight potential inefficiencies in observing information rather than in using information. From Section 5.1, emissions and carbon removal will be optimal conditional on information.

A continuum of traders normalized to unit mass is attached to sector i. At the beginning of time t, all agents have a symmetric, common prior over $\sum_{k=1}^{N} \kappa_k \zeta_k$, based on the regulator's measured aggregate damages in earlier periods and the prices of carbon shares in earlier periods. The time 0 prior is as described in Section 2.2, and the prior at the beginning of time t assigns variance τ_t^2 to each ζ_k. The price q_t of carbon shares at the beginning of time t reflects this information. Firms make emission and removal decisions based on this price and the population consumes accordingly. Subsequently, traders attached to sector i measure $\zeta_{it} + \lambda_{it}$.[32] They trade carbon shares based on this differentiated information. The market clears at price $\breve{q}_t^*$, but noise traders make the observed price $\breve{q}_t = \breve{q}_t^* + \theta_t$, where θ_t is a mean-zero, independently and identically distributed, normal random variable, with variance $\Theta^2 > 0$.[33] Between periods t and $t+1$, the regulator measures $\tilde{\zeta}_t + \tilde{\lambda}_t$ from its data

[32] Traders do not need to be only in emitting sectors; they could be in any sector with information about damages. Here, that possibility would be reflected by $Y^{it\prime}(0) = 0$, in which case sector i would have zero time t emissions but could be affected by damages.

[33] Noise has long been recognized as critical for the existence of partially revealing equilibria (e.g., Hellwig, 1980; Grossman and Stiglitz, 1980; Diamond and Verrecchia, 1981; Admati, 1985). Noise traders are often also interpreted as stochastic shocks to aggregate supply. Here noisy prices are critical for the existence of equilibrium because traders recognize the consequences of their continuum of companions all bidding with

on aggregate output, returns refunds d_t to shareholders based on shareholdings at the end of period t, and issues new shares to firms based on period t emissions.[34]

At the beginning of time t, traders of type i have share holdings y_{it} and wealth w_{it}. The y_{it} include shares issued in all previous periods that are still active (i.e., for which the underlying unit of carbon has not yet been removed). Traders have the ability to invest in a riskless asset with return r. After observing their private signals, traders choose their net demand X_{it} to maximize their expected utility of wealth at the beginning of period $t+1$:

$$\max_{X_{it}} E_t \left[-\exp\left[-A\left((1+r)(w_{it} + (X_{it} + y_{it})d_t - X_{it}\breve{q}_t) + (y_{it} + X_{it})q_{t+1} \right) \right] \middle| \, \zeta_{it} + \lambda_{it}, \breve{q}_t \right],$$

with $A > 0$ the coefficient of absolute risk aversion and E_t indicating expectations based on common information available at the beginning of time t. Traders have exponential utility, as opposed to the logarithmic utility function of the representative household. Exponential utility is critical to the analysis in this section because exponential utility yields linear asset demand functions that are independent of wealth and amenable to aggregation. For these reasons, exponential utility (including its implementation as quadratic payoffs) is used in nearly all literature on asymmetric information in asset markets.[35]

In a (noisy) rational expectations equilibrium, markets clear with traders inferring from prices whatever information they can and bidding to maximize utility conditional on that information. This equilibrium is defined as fully revealing if the carbon share price reveals the same information about $\sum_{k=1}^{N} \kappa_k \zeta_k$ as would observing all the disentangled ζ_{kt}. The following proposition establishes properties of this equilibrium:

Proposition 5 (Efficient Equilibrium)**.** *Let Assumption 1 hold. A fully revealing rational expectations equilibrium with $\breve{L} = 0$ exists in the limit as $\Theta^2 \to 0$ and $D \to \infty$.*

Proof. See Appendix J. □

As noise traders lose influence and the deposit becomes large, there exists a fully revealing rational expectations equilibrium, in which the price of a carbon share reflects informationally efficient beliefs and traders hold those same beliefs upon observing their private information and the carbon share price. By aggregating traders' private information about damages, the carbon share market improves on the regulator's ability to estimate damages, and by defining

the same information. In Vives (2014), the information structure has fundamental values and small traders, which precludes the need for noise traders, but here the payoffs themselves are pure common values and thus retain the rationale for noise traders despite traders being small (see footnote 36).

[34] These new shares can be handled by including time t emitters in the set of time $t+1$ traders.

[35] If we give the representative household in Section 2.1 exponential utility over consumption and also let damages be additive rather than multiplicative, then the learning dynamics are unchanged and, as $A \to 0$ (so as to eliminate risk premia, as with log utility), the regulator's tax in Section 4 and the damage charge in Section 5.1 are altered only by losing the C_0 normalization.

the marginal cost of emitting, the carbon share market simultaneously implements that information to control emissions and incentivize carbon removal. We have therefore designed a decentralized policy instrument that can implement the welfare-maximizing, informationally efficient benchmark that is generally unattainable via an emission tax instrument.

However, it is well-known that a fully revealing rational expectations equilibrium is not always implementable: it may be that no trading mechanism can actually deliver this equilibrium. In particular, if the carbon share price is a sufficient statistic for all information in the economy, then traders should ignore their private information, in which case it is unclear how their private information ends up being summarized by the equilibrium price. I therefore also study an equilibrium in demand functions (e.g., Kyle, 1989). Here traders submit demand functions that account for their observed sectoral signals $\zeta_{it} + \lambda_{it}$ and for the information they would infer from an observed price $\breve{q}_t$. Traders treat $\breve{q}_t$ as exogenous (i.e., they are price-takers) but do recognize how their observed signals influence that price through the beliefs of other traders in their sector.[36] Following much previous literature, I associate an implementable equilibrium with a Bayesian Nash equilibrium of this game and study linear equilibria.[37]

Define $\breve{\mu}_t$ as traders' posterior estimate of $\sum_{k=1}^{N} \kappa_k \zeta_k$ at the beginning of time t, after observing $\breve{q}_s$ and $\tilde{\zeta}_s + \tilde{\lambda}_s$ for all $s \in \{0, ..., t-1\}$. The following proposition characterizes emissions in an implementable equilibrium.

Proposition 6 (Implementable Equilibrium)**.** *In the limit as $D \to \infty$, there exists a linear equilibrium in demand functions in which the marginal cost of emissions and marginal benefit of carbon removal are each equal to*

$$\frac{1}{r} C_0 \alpha \left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \breve{\mu}_t \right]$$

[36] As in Vives (2014), the continuum of traders solves the "schizophrenic" problem of Hellwig (1980) because price-taking behavior is here individually optimal. However, I study an economy with a finite number of sectors and so the signals observed in some sector can affect the price.

[37] The equilibrium is symmetric in terms of strategies defined over ζ_{it} and $\breve{q}_t$. Of course, the actual demand schedules will not be symmetric as each will depend on the observed ζ_{it}.

and

$$\breve{\mu}_t = \sum_{k=0}^{t-1} \tilde{\pi}_k \left(\tilde{\zeta}_k + \tilde{\lambda}_k - \sum_{j=1}^{N} \kappa_j \bar{\zeta}_j \right)$$
$$+ \sum_{k=0}^{t-1} \frac{\breve{\pi}_k}{\chi_k} \underbrace{\left[\frac{(1-\Gamma)\tau_k^2 + \sigma^2}{\tau_k^2 + \sigma^2 + \omega^2} \sum_{i=1}^{N} \breve{\kappa}_{ik} \kappa_i \left(\zeta_{ik} + \lambda_{ik} - \bar{\zeta}_i \right) + \frac{N \Gamma \tau_k^2}{\tau_k^2 + \sigma^2 + \omega^2} \frac{1}{N} \sum_{i=1}^{N} \breve{\kappa}_{ik} (\zeta_{ik} + \lambda_{ik} - \bar{\zeta}_i) \right]}_{\text{from } \breve{q}_k}$$
$$- \sum_{k=0}^{t-1} \frac{\breve{\pi}_k}{\chi_k} \frac{r}{C_0 \alpha (\breve{\chi}_k + r)} \theta_k, \tag{9}$$

where $\breve{\kappa}_{kt}$, χ_k *and* $\breve{\chi}_k$ *are each* $\in (0,1)$. *If* Γ, ω^2, $\tilde{\omega}^2$, *and* Θ^2 *are sufficiently small, then* $\tilde{\pi}_k \in (0,1)$ *and* $\lim_{\Gamma, \tilde{\omega}^2 \to 0} \breve{\pi}_k$ *is arbitrarily close to zero.*

Proof. See Appendix K.
Sketch: The proof uses traders' first-order conditions to determine demand for carbon shares in each sector. The equilibrium price $\breve{q}_t^*$ sets aggregate net demand to zero given the beliefs traders form from their private signals, the observed carbon share price, and expectations of q_{t+1}. Because normal-normal Bayesian updating implies that $\breve{\mu}_{t+1}$ is a linear function of $\breve{\mu}_t$, $\breve{q}_t$, and $\tilde{\zeta}_t$, so too is expected q_{t+1}. The proof then constructs a signal $\tilde{q}_t$ of aggregate damages implied by $\breve{q}_t$. By normal-normal Bayesian updating, each sector i trader's posterior mean for $\tilde{\zeta}_t + \tilde{\lambda}_t$ is a linear function of $\breve{\mu}_t$, $\tilde{q}_t$, and ζ_{it}. We can thus express the price signal $\tilde{q}_t$ as an unknown linear function of sectoral signals. The projection theorem yields each type of trader's posterior mean for aggregate damages conditional on observed sectoral information and on the observed price signal. Matching coefficients and applying Brouwer's fixed-point theorem yields posterior beliefs that are self-fulfilling via the price and also yields the market-clearing price, both as functions of the unknown coefficients that determine $\breve{\mu}_{t+1}$. Beliefs $\breve{\mu}_{t+1}$ follow from multivariate normal-normal updating, matching coefficients to the conjectured form, and recursively substituting backwards for earlier $\breve{\mu}_k$ and earlier $\breve{q}_k$. □

Carbon shares act like imposing an emission tax based on beliefs $\breve{\mu}_t$. The first line in (9) determines the weight placed on previous periods' aggregate measurements of damages. The second line describes how agents learn from the past prices $\breve{q}_k$ of carbon shares. Those past prices embed two types of information: a first piece learns from a version of the aggregate signal constructed from sectoral signals, and a second piece takes advantage of the correlation among sectoral effects to learn from the disentangled signals. The ability to construct a version of the aggregate signal that is affected by sectoral measurement error but not by aggregate measurement error and the ability to use the correlation across sectors to learn more efficiently were critical to the informationally efficient beliefs but were missing from the regulator's beliefs. Both types of learning require access to the sectoral signals, and the carbon share price aggregates these sectoral signals so that later agents do not actually have to observe all past private information.[38]

The most notable differences with respect to the informationally efficient beliefs described in Proposition 1 are the randomness induced by noise traders (third line in (9)) and the presence of the $\breve{\kappa}_{ik}$. As the $\kappa_i \to 1/N$ (i.e., as sectors become symmetric), the $\breve{\kappa}_{it}$ approach a constant $\breve{\kappa}_t \in (0,1)$ and the second line approaches

$$\sum_{k=0}^{t-1} \frac{\breve{\pi}_k}{\chi_k} \breve{\kappa}_k \left[\frac{(1-\Gamma)\tau_k^2 + \sigma^2}{\tau_k^2 + \sigma^2 + \omega^2} \sum_{i=1}^{N} \kappa_i \left(\zeta_{ik} + \lambda_{ik} - \bar{\zeta}_i \right) + \frac{N\Gamma\tau_k^2}{\tau_k^2 + \sigma^2 + \omega^2} \frac{1}{N} \sum_{i=1}^{N} (\zeta_{ik} + \lambda_{ik} - \bar{\zeta}_i) \right].$$

The term in brackets is similar to a term in Proposition 1.[39] This expression illustrates that the carbon price does aggregate dispersed information in implementable equilibria and moreover aggregates that information in a fashion analogous to—albeit not identical to—informationally efficient beliefs.

The downward adjustments due to the $\breve{\kappa}$ reflect two forces: traders in each sector shade their bids to reflect their awareness of their own sector's signals leaking into the asset price (from equation (A-43)), and risk-averse traders' demand for carbon shares decreases in the variance of the returns they will earn (from equation (A-23)). The following corollary examines the $\breve{\kappa}$ in more detail.

Corollary 4 (Traders' Distortions)**.** *Consider the* $\{\breve{\kappa}_{1t}, ..., \breve{\kappa}_{Nt}\}$ *defined in Proposition 6.*

i $\lim_{\Theta^2 \to 0} \breve{\kappa}_{it} = 0$

ii If $\kappa_i = 1/N$ *for all* $i \in \{1, ..., N\}$*, then* $\breve{\kappa}_t$ *increases in* Θ^2 *and* $\lim_{\Theta^2 \to \infty} \breve{\kappa}_t = 1/N$.

iii Without loss of generality, order sectors by κ_i*. As* $\Theta^2 \to \infty$*, the sequence* $\{\breve{\kappa}_{1t}, ..., \breve{\kappa}_{Nt}\}$ *is monotone increasing, with* $\breve{\kappa}_{1t} \leq 1/N$ *and* $\breve{\kappa}_{Nt} \geq 1/N$*. The latter two inequalities are strict if* $\kappa_1 < 1/N$*.*

Proof. See Appendix L. □

[38] Under the conditions of the proposition, traders' posterior beliefs do not rely on past share prices ($\breve{\pi}_k \approx 0$) when sectoral effects are uncorrelated ($\Gamma = 0$) and the aggregate signal is perfectly measured ($\tilde{\omega}^2 = 0$). This result should be unsurprising given the analysis of the regulator in Section 4.2.

[39] One difference is the σ^2 in the numerator of the first term, which was missing from Proposition 1. Carbon share traders are trying to predict the aggregate measurement that will be released following the same period because this measurement determines immediate refunds and subsequent carbon share prices. This aggregate measurement will include their privately observed stochastic sectoral shocks, whose variance is σ^2. These contemporaneous shocks do not hinder learning about the coming aggregate measurement. In contrast, informationally efficient beliefs formed from time t signals in Proposition 1 predict aggregate measurements in all subsequent periods, in which case the contemporaneous stochastic sectoral shocks are pure noise that hinders learning.

The first result implies that carbon share prices fail to aggregate information as noise traders become irrelevant. This is a manifestation of the same force that prevents the fully revealing rational expectations equilibrium from being implementable: traders whose information is fully revealed by the equilibrium price do not trade on that information. In contrast, the proof shows that bid shading vanishes as $\Theta^2 \to \infty$. Thus there is a tension between minimizing the consequences of the third line in (9) and minimizing the consequences of bid shading for the $\breve{\kappa}$. And the effects of risk aversion remain in the $\breve{\kappa}$ as $\Theta^2 \to \infty$. The second part of the corollary shows that the effects of risk aversion are symmetric when the sectors are symmetric (i.e., with identical κ_i). And the third part of the corollary shows that the effects of risk aversion are more severe for sectors that have smaller value shares in final-good production, because traders in those sectors observe signals that are less informative about returns to carbon shares and thus perceive additional risk that makes them less willing to trade carbon shares.

In sum, a carbon share policy attains informational efficiency in a fully revealing rational expectations equilibrium and does aggregate information in implementable equilibria in a fashion that is imperfect but analogous to the aggregation performed within informationally efficient updating. By using markets to perform price discovery for social cost, a carbon share policy mitigates the difficulties that an emission tax's regulator faces in observing the information available in the economy.

6 Conclusions and Discussion

I have advanced a new perspective on environmental policymaking. In place of the traditional emphasis on asymmetric information about firms' costs of eliminating emissions, I have emphasized asymmetric information about the social cost of emissions. I have shown that conventional emission taxes neither aggregate dispersed information nor enable full use of potential information about the severity of climate change. Instead, I have shown that a new market-based instrument I call "carbon shares" aggregates dispersed information and enables full use of new information without losing the desirable properties of emission taxes and other conventional market-based instruments.

The proposed policy conceives of a different role for the regulator. Traditionally, the regulator must project the marginal harm from emissions in all possible states of the world and in all future time periods in order to settle on an emission price. Here, however, the regulator need only measure damage as it is realized, which is a more conventional econometric exercise and which could be institutionalized, as with the production of price indices, employment statistics, and national accounts data. This new instrument shifts the burden of projecting future damages from the regulator to market traders. These traders form their own damage estimates from information produced by the regulator, from the observed prices of carbon shares, and from their own private information. This type of belief updating is a common task in markets.

This proposal generates four immediate questions. First, how would a regulator actually estimate the realized aggregate impacts of climate change? This task is different from the task economists have traditionally undertaken in projecting future damages from climate change. It is closer to the attribution studies now regularly undertaken in which climate scientists test how climate change affected the likelihood of recent realized weather events. And governments already do regularly produce economic measures that are noisy yet are of critical importance for policymaking and directly determine monetary payments. As one example, the U.S. consumer price index determines social security benefits and other transfer payments and is a prominent input to monetary policy, but it is imperfectly estimated and there is disagreement even about what it should be estimating (National Research Council, 2002; Schultze, 2003). The present challenge may be no greater.

Second, would the regulator have credibility to estimate realized impacts faithfully? If an econometric framework could be developed that became widely accepted, then the estimation may here be institutionalized as with the production of other federal statistics—and to the extent this estimation relies on standard data, it may be less vulnerable to political influence than the U.S. government's estimates of the social cost of carbon have been (see Voosen, 2021). A real-world implementation of the policy might also constrain the change in damage charges from period to period, which would reduce the flexibility to respond to new information but also reduce vulnerability to transient political influence.

Third, how would this instrument affect incentives to coordinate policy internationally? I have followed a long tradition in analyzing the benchmark of a global regulator. However, climate policy is in practice fragmented among countries. Future work should compare international dimensions of this policy to carbon taxes, cap-and-trade programs, and other policy options. In particular, the ability to institutionalize the damage charge calculations and to explicitly adopt country weights in the damage charge calculations could each affect incentives to coordinate policy: these calculations may have more credibility than a global carbon tax would enjoy and countries may be incentivized to join a coalition in order to have their damages counted.

Finally, would traders have an incentive to collect additional information about climate impacts? To date, the development of better scientific monitoring and modeling systems has primarily been the task of governments and universities. However, such information should have a market value under the proposed policy, as I conjecture that the implementable equilibrium does not suffer the paradox of Grossman and Stiglitz (1976, 1980). One might thus expect traders to invest in information production, so that a carbon share policy may not just aggregate the information already dispersed throughout the economy but also improve that information. Such information might also have a market value under an emission tax (as agents want to understand their own exposure to climate change and to forecast future emission taxes), but this information is plausibly much more valuable under a carbon share policy because it determines the immediate payoffs from trading carbon shares. I leave the analysis of incentives to collect information in various policy environments to future work.

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Appendix to "Informationally Efficient Climate Policy"

This appendix contains numerical detals, proofs, and additional formal analysis.

A Numerical Details

There is no broad agreement on a distribution to use for climate change impacts. I calibrate the distribution for aggregate impacts $\sum_{i=1}^{N} \kappa_i[\bar{\zeta}_i + \zeta_i]$ to Pindyck (2019). In 2016, he asked around 1,000 climate scientists and economists to report their subjective percentiles for the percentage reduction in GDP that climate change will cause in fifty years, assuming that no additional emission controls are enacted before then. He fit four distributions to the results and found that a lognormal distribution produced the highest corrected R^2. The location parameter for his fitted lognormal distribution is -2.446 and the scale parameter is 1.476. His distribution describes a parameter (which he labels ϕ) that is equal to $T_{2065} \sum_{i=1}^{N} \kappa_i[\bar{\zeta}_i + \zeta_i]$. Using this estimate and treating T_{2065} as known, $\sum_{i=1}^{N} \kappa_i[\bar{\zeta}_i + \zeta_i]$ is lognormally distributed with location parameter $-2.446 - \ln(T_{2065})$ and scale parameter 1.476.

The value for T_{2065} should be the temperature that experts would have expected to hold based on their information in 2016. The IPCC's AR5 summarizes knowledge around that time. Hausfather and Peters (2020) suggest that a no-additional-emission-controls scenario is consistent with SSP4–6.0 from the IPCC's AR6. So I consider RCP 6.0 in the IPCC's AR5. There, the mean of the CMIP5 models is for 2.2 °C of warming in 2046–2065 relative to a 1986–2005 reference period, which in turn is on average 0.61 °C warmer than over 1850–1900 (Collins et al., 2013, Table 12.2). I therefore fix $T_{2065} = 2.2 + 0.61 = 2.81$ °C. This implies a location parameter for $\sum_{i=1}^{N} \kappa_i[\bar{\zeta}_i + \zeta_i]$ of -3.48.

In order to determine either the emission tax or damage charges, it remains to calibrate r, α, and C_0. I take r to be the policymaker's consumption discount rate. According to World Bank date, average growth in global output per capita was 1.85% per year over 2000-2019. Choosing an annual utility discount rate of 1.5% and a coefficient of relative risk aversion of 1 to match the log utility specification, the Ramsey rule implies that $r = 0.015 + 1 * 0.0185 = 0.0335$, or 3.4% per year.

The parameter α, or the "transient climate response to cumulative carbon emissions", is 1.6/1000 °C/Gt C, from the central value in Matthews et al. (2009). This value is consistent with Collins et al. (2013) and Dietz and Venmans (2019) and is the same as used in Rudik (2020).

I calibrate initial consumption C_0 to World Bank data. In 2021, global output was $86.7 trillion in year 2015 US dollars. Converting to year 2021 US dollars using World Bank deflators, I set $C_0 = 97,975$ billion dollars.

Now consider the calculations underlying Figure 3. Substituting (5) into (6) yields the probability that the damage charge Δ_t is constrained by the deposit D from reaching its first-best value of $C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t]$:

$$Pr(\Delta_t < C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t]) = 1 - F(rD/[C_0\alpha]),$$

where $F(\cdot)$ is the cumulative density function for $\tilde{\zeta}_t + \tilde{\lambda}_t$. In the absence of either measurement error in the aggregate signal or stochasticity in climate impacts (i.e., if $\tilde{\omega}^2 = 0$ and $\sigma^2 = 0$), $F(\cdot)$ is completely determined by the lognormal distribution of $\sum_{i=1}^N \kappa_i[\bar{\zeta}_i + \zeta_i]$ defined above.

A.1 Additional Parameters for the Evolution of Beliefs

To analyze the evolution of beliefs in Figure 1, I consider a normal distribution that has the same mean and, when $\Gamma = 0$, variance as the lognormal distribution for $\sum_{i=1}^N \kappa_i[\bar{\zeta}_i + \zeta_i]$ described above:

$$\sum_{i=1}^N \kappa_i \bar{\zeta}_i = \exp(-3.48 + 1.48^2/2) = 0.0916,$$

and

$$\tau_0^2 \sum_{i=1}^N \kappa_i^2 = [\exp(1.48^2) - 1] \exp(-2 * 3.48 + 1.48^2) = 0.0658.$$

I assume that $N = 10$ in the base specification and that the κ_i are drawn from a symmetric Dirichlet distribution with concentration parameter 1. The regulator and all agents know the values of the κ_i.

Now consider the sources of noise. First, I assume that $\sigma^2 \sum_{i=1}^N \kappa_i^2$ is the same as the prior variance $\tau_0^2 \sum_{i=1}^N \kappa_i^2$, so that

$$\sigma^2 = \frac{0.0658}{\sum_{i=1}^N \kappa_i^2}.$$

Second, in the base specification, I assume that aggregate measurement error has twice the variance as aggregate stochasticity:

$$\tilde{\omega}^2 = 2 * \sigma^2 \sum_{i=1}^N \kappa_i^2.$$

Third, I assume that sectoral measurement errors have half the variance as sectoral stochasticity:

$$\omega^2 = 0.5 * \sigma^2.$$

For the purposes of simulating beliefs, I assume that the true value of $\sum_{i=1}^N \kappa_i \zeta_i$ is the same as $\sum_{i=1}^N \kappa_i \bar{\zeta}_i$, so that the full-information optimal tax is twice the prior tax. I draw the ζ_i from a multivariate normal distribution with mean vector equal to $\sum_{i=1}^N \kappa_i \bar{\zeta}_i$ and I then adjust the draws ex post by adding a constant to ensure that $\sum_{i=1}^N \kappa_i \zeta_i = \sum_{i=1}^N \kappa_i \bar{\zeta}_i$. I draw

1 million trajectories for the random variables conditional on these ζ_i. These trajectories yield 1 million trajectories for $\hat{\mu}_t$ and $\tilde{\mu}_t$ from equations (3) and (4), respectively. Averaging over these trajectories yields the trajectories $E_0[\hat{\mu}_t|\boldsymbol{\zeta}]$ and $E_0[\tilde{\mu}_t|\boldsymbol{\zeta}]$ despicted in the figure.

B Taxing the Stock of Carbon

Assume informational efficiency, so that all agents observe all $\zeta_{it} + \lambda_{it}$. Denote firm i's cumulative emissions from time 0 to time $t-1$ as M_{it}. In period t, the regulator taxes M_{it} at rate $\tilde{\nu}_t$.

Final-good firms' problem is as in Appendix E. I here assume that all actors observe all signals in the economy and that firms discount the future at rate r.

The representative intermediate-good producer in sector i solves the following Bellman equation:

$$J(M_{it}, \hat{\mu}_t, \hat{\Omega}_t) = \max_{L_{it}, e_{it}, R_{it}} \Big\{ \hat{E}_t \left[p_{it} \exp[-\zeta_{it} T_t] \right] L_{it} Y^{it}(e_{it}) - w_{it} L_{it} - \tilde{\nu}_t M_{it} - p_t^R R_{it} + \frac{1}{1+r} \hat{E}_t [J(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})] \Big\},$$

with each control weakly positive. At an interior solution, the first-order condition for emissions is

$$\hat{E}_t \left[p_{it} \exp[-\zeta_{it} T_t] \right] L_{it} Y^{it\prime}(e_{it}) = -\frac{1}{1+r} \hat{E}_t [J_1(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})],$$

and the first order condition for carbon removal is

$$p_t^R = -\frac{1}{1+r} \hat{E}_t [J_1(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})].$$

Substitute for p_{it} and then C_t in the first-order conditions as in Appendix E:

$$-\frac{1}{1+r} \hat{E}_t [J_1(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})] = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})} C_0, \tag{A-1}$$

$$-\frac{1}{1+r} \hat{E}_t [J_1(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})] = \frac{c_t'(R_t)}{1 - c_t(R_t)} C_0. \tag{A-2}$$

The envelope theorem yields:

$$J_1(M_{it}, \hat{\mu}_t, \hat{\Omega}_t) = -\tilde{\nu}_t + \frac{1}{1+r} \hat{E}_t [J_1(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})].$$

Recursively substituting, we find:

$$J_1(M_{it}, \hat{\mu}_t, \hat{\Omega}_t) = -\sum_{s=0}^{\infty} \frac{1}{(1+r)^s} \hat{E}_t[\tilde{\nu}_{t+s}].$$

Advancing by one timestep, substituting into (A-1) and (A-2), and applying the law of iterated expectations, an interior solution must satisfy:

$$\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \hat{E}_t[\tilde{\nu}_{t+s}] = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})} C_0,$$
$$\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \hat{E}_t[\tilde{\nu}_{t+s}] = \frac{c_t'(R_t)}{1-c_t(R_t)} C_0.$$

Now set $\tilde{\nu}_t = C_0 \alpha [\tilde{\zeta}_t + \tilde{\lambda}_t]$. The foregoing conditions become:

$$\alpha \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \hat{E}_t[\tilde{\zeta}_{t+s} + \tilde{\lambda}_{t+s}] = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})},$$
$$\alpha \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \hat{E}_t[\tilde{\zeta}_{t+s} + \tilde{\lambda}_{t+s}] = \frac{c_t'(R_t)}{1-c_t(R_t)},$$

which in turn are equivalent to

$$\alpha \frac{1}{r} \left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \hat{\mu}_t \right] = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})},$$
$$\alpha \frac{1}{r} \left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \hat{\mu}_t \right] = \frac{c_t'(R_t)}{1-c_t(R_t)}.$$

Once we adjust for the possibility of corner solutions, these conditions are the same as the conditions for welfare maximization in (1) and (2). Therefore this choice of $\tilde{\nu}_t$ must be the optimal choice for a regulator who can commit to a choice rule at time 0.

C Proof of Proposition 1

Consider the zero-mean random vector

$$s_t \triangleq \begin{bmatrix} \sum_{k=1}^{N} \kappa_k \zeta_k \\ \frac{1}{t} \sum_{j=0}^{t-1} [\zeta_{1j} + \lambda_{1j}] - \bar{\zeta}_1 \\ \vdots \\ \frac{1}{t} \sum_{j=0}^{t-1} [\zeta_{Nj} + \lambda_{Nj}] - \bar{\zeta}_N \\ \frac{1}{t} \sum_{j=0}^{t-1} [\tilde{\zeta}_j + \tilde{\lambda}_j] - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \end{bmatrix}.$$

Observe that

$$\hat{\mu}_t \triangleq E\left[\sum_{k=1}^{N}\kappa_k\zeta_k \left| \frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{1j}+\lambda_{1j}]-\bar{\zeta}_1, ..., \frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{Nj}+\lambda_{Nj}]-\bar{\zeta}_N, \frac{1}{t}\sum_{j=0}^{t-1}[\tilde{\zeta}_j+\tilde{\lambda}_j]-\sum_{k=1}^{N}\kappa_k\bar{\zeta}_k\right.\right].$$

Let Ψ_t indicate the $(N+1)\times(N+1)$ covariance matrix of the final $(N+1)\times 1$ elements of s_t and Σ_t indicate the $1\times(N+1)$ vector of covariances between $\sum_{k=1}^{N}\kappa_k\zeta_k$ and the other elements of s_t, so that

$$\Sigma_t \triangleq \left[\kappa_1\tau_0^2+(1-\kappa_1)\Gamma\tau_0^2, \quad \dots, \quad \kappa_N\tau_0^2+(1-\kappa_N)\Gamma\tau_0^2, \quad \tau_0^2\sum_{k=1}^{N}\kappa_k^2+2\Gamma\tau_0^2\sum_{k=1}^{N}\sum_{j=k+1}^{N}\kappa_k\kappa_j\right].$$

From the projection theorem,

$$\hat{\mu}_t = \Sigma_t\Psi_t^{-1}\begin{bmatrix}\frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{1j}+\lambda_{1j}]-\bar{\zeta}_1\\ \vdots \\ \frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{Nj}+\lambda_{Nj}]-\bar{\zeta}_N\\ \frac{1}{t}\sum_{j=0}^{t-1}[\tilde{\zeta}_j+\tilde{\lambda}_j]-\sum_{k=1}^{N}\kappa_k\bar{\zeta}_k\end{bmatrix}. \tag{A-3}$$

Consider Ψ_t^{-1}. Label the $N\times N$ upper-left block of Ψ_t as Ψ_A, the $N\times 1$ upper right block as Ψ_B, the $1\times N$ lower left block as Ψ_C, and the 1×1 lower right block as Ψ_D. From familiar results for block matrix inversion,

$$\Psi_t^{-1} = \begin{bmatrix}\Psi_A^{-1}+\Psi_A^{-1}\Psi_B(\Psi_D-\Psi_C\Psi_A^{-1}\Psi_B)^{-1}\Psi_C\Psi_A^{-1} & -\Psi_A^{-1}\Psi_B(\Psi_D-\Psi_C\Psi_A^{-1}\Psi_B)^{-1}\\ -(\Psi_D-\Psi_C\Psi_A^{-1}\Psi_B)^{-1}\Psi_C\Psi_A^{-1} & (\Psi_D-\Psi_C\Psi_A^{-1}\Psi_B)^{-1}\end{bmatrix}. \tag{A-4}$$

Element (i,i) of Ψ_A is $\tau_0^2+\sigma^2/t+\omega^2/t$, element (i,j) of Ψ_A is $\Gamma\tau_0^2$ for $i\neq j$, the ith element of Ψ_B and Ψ_C is $\kappa_i(\tau_0^2+\sigma^2/t)+(1-\kappa_i)\Gamma\tau_0^2$, and Ψ_D equals $(\tau_0^2+\sigma^2/t)\sum_{k=1}^{N}\kappa_k^2+2\Gamma\tau_0^2\sum_{k=1}^{N}\sum_{j=k+1}^{N}\kappa_k\kappa_j+\tilde{\omega}^2/t$.

Conjecture that each diagonal of Ψ_A^{-1} is

$$\frac{\tau_0^2+\sigma^2/t+\omega^2/t+(N-2)\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}$$

and each off-diagonal of Ψ_A^{-1} is

$$\frac{-\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}.$$

Under the conjecture for Ψ_A^{-1}, each diagonal element in $\Psi_A\Psi_A^{-1}$ is

$$\begin{aligned}&[\tau_0^2+\sigma^2/t+\omega^2/t]\frac{\tau_0^2+\sigma^2/t+\omega^2/t+(N-2)\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}\\ &-\sum_{k=1}^{N-1}\Gamma\tau_0^2\frac{\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2-(N-1)[\Gamma\tau_0^2]^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2}\\ =&1,\end{aligned}$$

and each off-diagonal element in $\Psi_A \Psi_A^{-1}$ is

$$\Gamma\tau_0^2 \frac{\tau_0^2 + \sigma^2/t + \omega^2/t + (N-2)\Gamma\tau_0^2}{(\tau_0^2 + \sigma^2/t + \omega^2/t)^2 + (\tau_0^2 + \sigma^2/t + \omega^2/t)(N-2)\Gamma\tau_0^2 - (N-1)[\Gamma\tau_0^2]^2}$$
$$- [\tau_0^2 + \sigma^2/t + \omega^2/t] \frac{\Gamma\tau_0^2}{(\tau_0^2 + \sigma^2/t + \omega^2/t)^2 + (\tau_0^2 + \sigma^2/t + \omega^2/t)(N-2)\Gamma\tau_0^2 - (N-1)[\Gamma\tau_0^2]^2}$$
$$- \sum_{k=1}^{N-2} \Gamma\tau_0^2 \frac{\Gamma\tau_0^2}{(\tau_0^2 + \sigma^2/t + \omega^2/t)^2 - (N-1)[\Gamma\tau_0^2]^2 + (\tau_0^2 + \sigma^2/t + \omega^2/t)(N-2)\Gamma\tau_0^2}$$
$$=0.$$

We have shown that $\Psi_A \Psi_A^{-1}$ is the identity matrix under the conjecture for Ψ_A^{-1} and thus have confirmed that the conjecture is correct. Observe that the denominator of each element simplifies to

$$\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right].$$

With Ψ_A^{-1} in hand, we now calculate Ψ_t^{-1} from (A-4). Element i of $\Psi_C \Psi_A^{-1}$ and also of $\Psi_A^{-1}\Psi_B$ is

$$\frac{\tau_0^2 + \sigma^2/t + \omega^2/t + (N-2)\Gamma\tau_0^2}{(\tau_0^2 + \sigma^2/t + \omega^2/t)^2 + (\tau_0^2 + \sigma^2/t + \omega^2/t)(N-2)\Gamma\tau_0^2 - (N-1)[\Gamma\tau_0^2]^2}[\kappa_i(\tau_0^2 + \sigma^2/t) + (1-\kappa_i)\Gamma\tau_0^2]$$
$$+ \sum_{k=1,\neq i}^{N} \frac{-\Gamma\tau_0^2}{(\tau_0^2 + \sigma^2/t + \omega^2/t)^2 + (\tau_0^2 + \sigma^2/t + \omega^2/t)(N-2)\Gamma\tau_0^2 - (N-1)[\Gamma\tau_0^2]^2}$$
$$[\kappa_k(\tau_0^2 + \sigma^2/t) + (1-\kappa_k)\Gamma\tau_0^2]$$
$$= \frac{(1-\Gamma)\tau_0^2 + \sigma^2/t}{(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t}\kappa_i + \frac{\Gamma\tau_0^2\omega^2/t}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]}.$$

We then have:

$$\Psi_C \Psi_A^{-1} \Psi_B = \sum_{i=1}^{N} \frac{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\kappa_i + \Gamma\tau_0^2\omega^2/t}{(\tau_0^2 + \sigma^2/t + \omega^2/t)^2 + (\tau_0^2 + \sigma^2/t + \omega^2/t)(N-2)\Gamma\tau_0^2 - (N-1)[\Gamma\tau_0^2]^2}$$
$$\left[\kappa_i((1-\Gamma)\tau_0^2 + \sigma^2/t) + \Gamma\tau_0^2\right]$$
$$= \frac{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]^2 \sum_{i=1}^{N} \kappa_i^2}{(\tau_0^2 + \sigma^2/t + \omega^2/t)^2 + (\tau_0^2 + \sigma^2/t + \omega^2/t)(N-2)\Gamma\tau_0^2 - (N-1)[\Gamma\tau_0^2]^2}$$
$$+ \frac{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right] + \left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\omega^2/t}{(\tau_0^2 + \sigma^2/t + \omega^2/t)^2 + (\tau_0^2 + \sigma^2/t + \omega^2/t)(N-2)\Gamma\tau_0^2 - (N-1)[\Gamma\tau_0^2]^2}\Gamma\tau_0^2.$$

And

$$
\begin{aligned}
&(\Psi_D - \Psi_C \Psi_A^{-1} \Psi_B)^{-1}\\
=&\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\\
&\Bigg\{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\\
&\qquad\qquad\Big[(\tau_0^2 + \sigma^2/t)\sum_{k=1}^N \kappa_k^2 + 2\Gamma\tau_0^2 \sum_{k=1}^N \sum_{j=k+1}^N \kappa_k\kappa_j + \tilde{\omega}^2/t\Big]\\
&\qquad - \Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]^2 \sum_{i=1}^N \kappa_i^2\\
&\qquad - \Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Gamma\tau_0^2 - \Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\Gamma\tau_0^2\omega^2/t\Bigg\}^{-1}\\
=&\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\\
&\Bigg\{\omega^2/t\Bigg(\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\sum_{k=1}^N \kappa_k^2 + \Gamma\tau_0^2\omega^2/t\Bigg)\\
&\qquad + \Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\tilde{\omega}^2/t\Bigg\}^{-1},
\end{aligned}
$$

where the final line uses $\sum_{k=1}^N \kappa_k = 1$. Element i of $\Psi_A^{-1}\Psi_B(\Psi_D - \Psi_C\Psi_A^{-1}\Psi_B)^{-1}$ and also of $(\Psi_D - \Psi_C\Psi_A^{-1}\Psi_B)^{-1}\Psi_C\Psi_A^{-1}$ is

$$
\begin{aligned}
&\Bigg\{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\kappa_i + \Gamma\tau_0^2\omega^2/t\Bigg\}\\
&\Bigg\{\omega^2/t\Bigg(\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\sum_{k=1}^N \kappa_k^2 + \Gamma\tau_0^2\omega^2/t\Bigg)\\
&\qquad + \Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\tilde{\omega}^2/t\Bigg\}^{-1}.
\end{aligned}
$$

Element (i,i) of $\Psi_A^{-1} + \Psi_A^{-1}\Psi_B(\Psi_D - \Psi_C\Psi_A^{-1}\Psi_B)^{-1}\Psi_C\Psi_A^{-1}$ is

$$\frac{(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + (N-1)\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]}$$
$$+ \frac{\left\{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\kappa_i + \Gamma\tau_0^2\omega^2/t\right\}}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]}$$
$$\left\{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\kappa_i + \Gamma\tau_0^2\omega^2/t\right\}$$
$$\Bigg\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2 + \Gamma\tau_0^2\omega^2/t\right)$$
$$+ \left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\Bigg\}^{-1},$$

and element (m,n) of $\Psi_A^{-1} + \Psi_A^{-1}\Psi_B(\Psi_D - \Psi_C\Psi_A^{-1}\Psi_B)^{-1}\Psi_C\Psi_A^{-1}$ is, for $m \neq n$,

$$\frac{-\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]}$$
$$+ \frac{\left\{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\kappa_m + \Gamma\tau_0^2\omega^2/t\right\}}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]}$$
$$\left\{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\kappa_n + \Gamma\tau_0^2\omega^2/t\right\}$$
$$\Bigg\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2 + \Gamma\tau_0^2\omega^2/t\right)$$
$$+ \left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\Bigg\}^{-1}.$$

The foregoing pieces define Ψ_t^{-1} from (A-4). $\Sigma_t\Psi_t^{-1}$ is $1 \times N+1$ and, from (A-3), determines how $\hat{\mu}_t$ uses the signals in s_t. Element $k \in \{1, ..., N\}$ of $\Sigma_t \Psi_t^{-1}$ is

$$
\begin{aligned}
&[\kappa_k \tau_0^2 + (1-\kappa_k)\Gamma\tau_0^2] \frac{(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + (N-1)\Gamma\tau_0^2}{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]} \\
&+ \frac{\left\{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\kappa_k + \Gamma\tau_0^2\omega^2/t\right\}}{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]} \\
&\quad \sum_{i=1}^{N} [\kappa_i \tau_0^2 + (1-\kappa_i)\Gamma\tau_0^2] \left\{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\kappa_i + \Gamma\tau_0^2\omega^2/t\right\} \\
&\quad \left\{\omega^2/t\left(\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\sum_{k=1}^{N}\kappa_k^2 + \Gamma\tau_0^2\omega^2/t\right)\right. \\
&\qquad \left. + \Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\tilde{\omega}^2/t\right\}^{-1} \\
&- [(1-\kappa_k)(1-\Gamma)\tau_0^2 + (N-1)\Gamma\tau_0^2] \frac{\Gamma\tau_0^2}{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]} \\
&- \left[\tau_0^2 \sum_{k=1}^{N}\kappa_k^2 + 2\Gamma\tau_0^2 \sum_{k=1}^{N}\sum_{j=k+1}^{N}\kappa_k\kappa_j\right] \\
&\quad \left\{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\kappa_k + \Gamma\tau_0^2\omega^2/t\right\} \\
&\quad \left\{\omega^2/t\left(\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\sum_{k=1}^{N}\kappa_k^2 + \Gamma\tau_0^2\omega^2/t\right)\right. \\
&\qquad \left. + \Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\tilde{\omega}^2/t\right\}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{(1-\Gamma)\tau_0^2}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\kappa_k\\
&+\frac{\sigma^2/t+\omega^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\frac{\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2}\\
&-[\omega^2/t]\left(\frac{(1-\Gamma)\tau_0^2+\sigma^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\kappa_k+\frac{\omega^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\frac{\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2}\right)\\
&\quad\left\{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2\sum_{i=1}^N\kappa_i^2+\left[\sigma^2/t+\omega^2/t\right]\Gamma\tau_0^2\right\}\\
&\quad\left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\sum_{k=1}^N\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right.\\
&\qquad\left.+\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1}. \quad \text{(A-5)}
\end{aligned}
$$

Element $N+1$ of $\Sigma_t\Psi_t^{-1}$ is

$$
\begin{aligned}
&-\sum_{k=1}^N[\kappa_k\tau_0^2+(1-\kappa_k)\Gamma\tau_0^2]\left\{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\kappa_k+\Gamma\tau_0^2\omega^2/t\right\}\\
&\quad\left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\sum_{k=1}^N\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right.\\
&\qquad\left.+\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1}\\
&+\left[\tau_0^2\sum_{k=1}^N\kappa_k^2+2\Gamma\tau_0^2\sum_{k=1}^N\sum_{j=k+1}^N\kappa_k\kappa_j\right]\\
&\quad\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\\
&\quad\left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\sum_{k=1}^N\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right.\\
&\qquad\left.+\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
=&[\omega^2/t]\left\{\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big](1-\Gamma)\tau_0^2\sum_{k=1}^N\kappa_k^2+\Big[\sigma^2/t+\omega^2/t\Big]\Gamma\tau_0^2\right\}\\
&\left\{\omega^2/t\left(\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t\Big]\sum_{k=1}^N\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right.\\
&\left.+\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]\tilde{\omega}^2/t\right\}^{-1}. \quad \text{(A-6)}
\end{aligned}
$$

With these elements, we can determine $\hat{\mu}_t$ from (A-3).

Define

$$
\begin{aligned}
\hat{Z}_t \triangleq&[\omega^2/t]\left\{\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big](1-\Gamma)\tau_0^2\sum_{k=1}^N\kappa_k^2+\Big[\sigma^2/t+\omega^2/t\Big]\Gamma\tau_0^2\right\}\\
&\left\{[\omega^2/t]\left(\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t\Big]\sum_{k=1}^N\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right.\\
&\left.+[\tilde{\omega}^2/t]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]\right\}^{-1}. \quad \text{(A-7)}
\end{aligned}
$$

Clearly $\hat{Z} \geq 0$. Observe that $\sum_{k=1}^N \kappa_k^2$ is minimized when each $\kappa_k = 1/N$ and thus

$$
\begin{aligned}
\hat{Z}_t <&[\omega^2/t]\left\{\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big](1-\Gamma)\tau_0^2\sum_{k=1}^N\kappa_k^2+\Big[\sigma^2/t+\omega^2/t\Big]\Gamma\tau_0^2\right\}\\
&\left\{[\omega^2/t]\left(\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big](1-\Gamma)\tau_0^2\sum_{k=1}^N\kappa_k^2+\Gamma\tau_0^2[\sigma^2/t+\omega^2/t]\right)\right.\\
&\left.+[\tilde{\omega}^2/t]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]\right\}^{-1}\\
<&1.
\end{aligned}
$$

Therefore $\hat{Z}_t \in [0, 1)$. By inspection, $\hat{Z}_t \to 0$ as $\tilde{\omega}/\omega \to \infty$. The expression in the proposition follows from (A-3), (A-5), (A-6), and the definition (A-7).

D Proof of Corollary 1

From Proposition 1, $\hat{Z}_t \to 0$ as $\omega^2 \to 0$ if $\tilde{\omega}^2 > 0$. In that case, the expression in part (i) follows from (3), $\lambda_{kj} = 0$, and the definition of $\tilde{\zeta}_j$. Now consider the case in which $\tilde{\omega}^2 = 0$ and $\omega^2 \to 0$. From (A-7),

$$\hat{Z}_t = \frac{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2 \sum_{k=1}^N \kappa_k^2 + \left[\sigma^2/t + \omega^2/t\right]\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^N \kappa_k^2 + \Gamma\tau_0^2\omega^2/t}$$

if $\tilde{\omega}^2 = 0$, which goes to

$$\frac{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2 \sum_{k=1}^N \kappa_k^2 + [\sigma^2/t]\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^N \kappa_k^2}$$

as $\omega^2 \to 0$. Substituting into (3), using $\lambda_{kj} = \tilde{\lambda}_j = 0$, and applying the definition of $\tilde{\zeta}_j$ yields the expression in part (i). We have established the first part of the corollary.

From (A-7),

$$\lim_{\Gamma,\tilde{\omega}^2 \to 0} \hat{Z}_t = \frac{\left[\tau_0^2 + \sigma^2/t + \omega^2/t\right]\tau_0^2 \sum_{k=1}^N \kappa_k^2}{\left[\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^N \kappa_k^2} = \frac{\tau_0^2}{\tau_0^2 + \sigma^2/t}.$$

Substituting into (3) and setting $\tilde{\lambda}_{kj} = 0$ yields the expression in part (ii) of the corollary.

From (A-7),

$$\begin{aligned}\lim_{\tilde{\omega}^2 \to 0, \kappa_i \to 1/N\,\forall i} \hat{Z}_t =& \frac{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2 + \left[\sigma^2/t + \omega^2/t\right]N\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right] + N\Gamma\tau_0^2\omega^2/t} \\ =& \frac{(1-\Gamma)\tau_0^2 + N\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2 + \sigma^2/t + N\Gamma\tau_0^2}.\end{aligned}$$

Using this,

$$\lim_{\tilde{\omega}^2\to 0,\kappa_i\to 1/N\,\forall i}\frac{(1-\hat{Z}_t)(1-\Gamma)\tau_0^2-\hat{Z}_t\sigma^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}$$
$$=-\frac{N\Gamma\tau_0^2\sigma^2/t}{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]+N\Gamma\tau_0^2\omega^2/t},$$
$$\lim_{\tilde{\omega}^2\to 0,\kappa_i\to 1/N\,\forall i}\frac{\sigma^2/t+(1-\hat{Z}_t)\omega^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\frac{N\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2}$$
$$=\frac{N\Gamma\tau_0^2\sigma^2/t}{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]+N\Gamma\tau_0^2\omega^2/t}.$$

Substituting into (3) and setting $\tilde{\lambda}_{kj}=0$ yields the expression in part (iii) of the corollary.

E Proof of Proposition 2

I first solve for market equilibrium conditional on the choice of tax and then consider how the regulator would design the tax to maximize welfare.

Let p_t be the price of consumption, p_{it} be the time t price of intermediate i, and p_t^R be the price paid for removal. Final-good firms solve:

$$\max_{R_t,\{Y_{it}\}_{i=1}^N}\left\{p_t\left(1-c_t(R_t)\right)Y_t-\sum_{i=1}^N p_{it}Y_{it}+p_t^R R_t\right\}.$$

The first-order conditions are

$$\begin{aligned}p_{it}=&p_t\kappa_i\frac{C_t}{Y_{it}},\\ p_t^R=&p_t\frac{c_t'(R_t)}{1-c_t(R_t)}C_t.\end{aligned}\tag{A-8}$$

The latter condition becomes a $\leq$ when $R_t=0$.

The representative intermediate-good producer in sector i solves

$$\max_{L_{it},e_{it},R_{it}\geq 0}\left\{E_{it}\left[p_{it}\exp[-\zeta_{it}T_t]\right]L_{it}Y^{it}(e_{it})-w_{it}L_{it}-\nu_t\max\{e_{it}-R_{it},0\}-p_t^R R_{it}\right\}.$$

The $\max\{e_{it}-R_{it},0\}$ reflects that, unless the regulator could disciriminate subsidies by sectors, the regulator cannot pay firms in sector i for negative emissions or else it would violate its revenue constraint if emissions in other sectors net out to zero or less. A maximum clearly has $e_{it}-R_{it}\geq 0$.

The first-order condition for L_{it} is

$$w_{it} = E_{it}\left[p_{it}\exp[-\zeta_{it}T_t]\right] Y^{it}(e_{it}).$$

If $e_{it} - R_{it} > 0$, then the first-order condition for e_{it} is

$$\nu_t = E_{it}\left[p_{it}\exp[-\zeta_{it}T_t]\right] L_{it} Y^{it\prime}(e_{it}),$$

and the first-order condition for a solution with $R_{it} > 0$ is

$$\nu_t = p_t^R.$$

Substitute for p_{it} and p_t^R in the first-order conditions:

$$w_{it} = p_t \kappa_i \frac{C_t}{L_{it}}, \tag{A-9}$$

$$\nu_t = p_t \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})} C_t, \tag{A-10}$$

$$\nu_t = p_t \frac{c_t'(R_t)}{1 - c_t(R_t)} C_t. \tag{A-11}$$

At a corner solution with $e_{it} = 0$, the second condition's equality would become a $\geq$, and at a corner solution with $R_t = 0$, the third condition's equality would become a $\leq$.

Household maximization implies $p_t = u'(C_t)$. Therefore, using $p_0 = 1$,

$$p_t = \frac{u'(C_t)}{u'(C_0)} = \frac{C_0}{C_t}.$$

Households' budget constraint $\sum_{i=1}^N w_{it} L_{it} = p_t C_t$ and the first-order condition imply $1 = \sum_{i=1}^N \kappa_i$, which does hold. Substitute $p_t C_t = C_0$ in (A-9) through (A-11):

$$w_{it} = \kappa_i \frac{1}{L_{it}} C_0, \tag{A-12}$$

$$\nu_t = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})} C_0, \tag{A-13}$$

$$\nu_t = \frac{c_t'(R_t)}{1 - c_t(R_t)} C_0. \tag{A-14}$$

The wage must be equal in sectors with nonzero employment, so $w_{it} = w_t$ for some $w_t > 0$. Equation (A-12) becomes:

$$L_{it} = \kappa_i \frac{1}{w_t} C_0.$$

From the budget constraint, $w_t = C_0$ and thus $L_{it} = \kappa_i$. Therefore equilibrium L_{it} is independent of e_{it}, ν_t, and T_t.

Conjecture that ν_t/C_0 is independent of T_t (we will confirm this conjecture when studying the regulator's problem). Then from (A-13) and (A-14), the time t market equilibrium is also independent of T_t. It is also independent of the random shocks ϵ_{it} and the unknown damage parameters ζ_{it} and thus is not stochastic.

As ν_t increases, e_{it} strictly decreases while $e_{it} > 0$. Aggregating over all i, $\sum_{i=1}^N e_{it}$ strictly decreases in ν_t and R_t weakly increases in ν_t. There exists some $\bar{\nu}_t > 0$ such that $\sum_{i=1}^N e_{it} - R_t = 0$ if $\nu_t = \bar{\nu}_t$ and $\sum_{i=1}^N e_{it} - R_t > 0$ if $\nu_t < \bar{\nu}_t$.

The regulator solves the following Bellman equation:

$$\tilde{W}(T_t, \tilde{\mu}_t, \tilde{\Omega}_t) = \max_{\nu_t} \tilde{E}_t \Bigg[u\Bigg((1 - c_t(R_t^*)) \prod_{i=1}^N [\exp[-\zeta_{it} T_t] L_{it}^* Y^{it}(e_{it}^*)]^{\kappa_i} \Bigg) + \frac{1}{1+r} \tilde{W}(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}) \Bigg],$$

where $\tilde{E}_t$ denotes expectations over the regulator's information set at the beginning of period t and where a star represents market equilibrium. Market outcomes will be insensitive to ν_t for $\nu_t > \bar{\nu}_t$ and thus the regulator's objective will be constant in ν_t for $\nu_t > \bar{\nu}_t$. Maximized welfare is therefore equivalent for a regulator who solves the following problem in which ν_t is constrained to be less than or equal to $\bar{\nu}_t$:

$$\tilde{W}(T_t, \tilde{\mu}_t, \tilde{\Omega}_t) = \max_{\nu_t \le \bar{\nu}_t} \tilde{E}_t \Bigg[u\Bigg((1 - c_t(R_t^*)) \prod_{i=1}^N [\exp[-\zeta_{it} T_t] L_{it}^* Y^{it}(e_{it}^*)]^{\kappa_i} \Bigg) + \frac{1}{1+r} \tilde{W}(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}) \Bigg].$$

At an interior solution, the regulator's first-order condition is

$$0 = \sum_{i=1}^N \frac{\kappa_i Y_e^{i\prime}(e_{it}^*)}{Y^{it}(e_{it}^*)} \frac{\partial e_{it}^*}{\partial \nu_t} + \frac{1}{1+r} \tilde{E}_t \Big[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}) \Big] \alpha \sum_{i=1}^N \frac{\partial e_{it}^*}{\partial \nu_t} - \frac{c_t'(R_t^*)}{1 - c_t(R_t^*)} \frac{\partial R_t^*}{\partial \nu_t} - \frac{1}{1+r} \tilde{E}_t \Big[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}) \Big] \alpha \frac{\partial R_t^*}{\partial \nu_t}. \tag{A-15}$$

Substitute from (A-13) and (A-14):[40]

$$
\begin{aligned}
0 =& \frac{\nu_t}{C_0^*} \sum_{i=1}^{N} \frac{\partial e_{it}^*}{\partial \nu_t} + \frac{1}{1+r} \tilde{E}_t\left[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1})\right] \alpha \sum_{i=1}^{N} \frac{\partial e_{it}^*}{\partial \nu_t} \\
&- \frac{\nu_t}{C_0^*} \frac{\partial R_t^*}{\partial \nu_t} - \frac{1}{1+r} \tilde{E}_t\left[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1})\right] \alpha \frac{\partial R_t^*}{\partial \nu_t}.
\end{aligned}
$$

Factor out common terms:

$$
0 = \left(\frac{\nu_t}{C_0} + \frac{1}{1+r} \alpha \tilde{E}_t\left[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1})\right] \right) \left(\sum_{i=1}^{N} \frac{\partial e_{it}^*}{\partial \nu_t} - \frac{\partial R_t^*}{\partial \nu_t} \right).
$$

Applying the Implicit Function Theorem to equations (A-13) and (A-14), we find:

$$
\frac{\partial e_{it}^*}{\partial \nu_t} = \frac{-1}{\frac{-Y_{ee}^{it}}{Y^{it\prime}} + \frac{Y^{it\prime}}{Y^{it}}} \frac{1}{\nu_t} < 0 \quad \text{if } e_{it}^* > 0, \tag{A-16}
$$

$$
\frac{\partial R_t^*}{\partial \nu_t} = \frac{1}{\frac{c_t''(R_t^*)}{1-c_t(R_t^*)} C_0^* + \frac{[c_t'(R_t^*)]^2}{[1-c_t(R_t^*)]^2} C_0^*} > 0 \quad \text{if } R_{it}^* > 0. \tag{A-17}
$$

Therefore, if some firm is at an interior solution for either emissions or removal:

$$
0 = \frac{\nu_t}{C_0^*} + \frac{1}{1+r} \alpha \tilde{E}_t\left[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1})\right]. \tag{A-18}
$$

From the envelope theorem:

$$
\tilde{W}_T(T_t, \tilde{\mu}_t, \tilde{\Omega}_t) = -\tilde{\zeta}_t - \tilde{\lambda}_t + \frac{1}{1+r} \tilde{E}_t[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1})],
$$

where we recognize that the market equilibrium is independent of temperature under the conjecture that ν_t/C_0 is independent of temperature. Recursively substitute into (A-18) and rearrange:

$$
\nu_t = C_0^* \, \alpha \frac{1}{r} \left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \tilde{\mu}_t \right].
$$

We have confirmed the conjecture that ν_t/C_0^* is independent of temperature when $\sum_{i=1}^{N} e_{it} - R_t > 0$. We have established Proposition 2.

[40]At a corner with either $e_{it} = 0$ or $R_{it} = 0$, we would have $\partial e_{it}^*/\partial \nu_t = 0$ or $\partial R_t^*/\partial \nu_t = 0$, respectively, and so would end up at the same optimal tax as below as long as at least one is interior.

F Proof of Corollary 2

Let G_t be cumulative revenue collected, which is invested with rate of return r:

$$G_{t+1} = (1+r)\left\{G_t + \nu_t\left[\sum_{i=1}^{N} e_{it} - R_t\right]\right\},$$

with $G_0 \geq 0$. The constraint in time t is now

$$G_t + \nu_t\left[\sum_{i=1}^{N} e_{it} - R_t\right] \geq 0.$$

This constraint binds only if $\sum_{i=1}^{N} e_{it} - R_t \leq 0$. Denote the smallest ν_t at which the constraint binds as $\bar{\nu}_t$. By the implicit function theorem,

$$\frac{\mathrm{d}\bar{\nu}_t}{\mathrm{d}G_t} = -\frac{1}{\sum_{i=1}^{N} e^*_{it}(\bar{\nu}_t) - R^*_t(\bar{\nu}_t) + \sum_{i=1}^{N} \frac{\partial e^*_{it}}{\partial \nu_t}\Big|_{\bar{\nu}_t} - \frac{\partial R^*_t}{\partial \nu_t}\Big|_{\bar{\nu}_t}} > 0,$$

where a star indicates market equilibrium. The sign follows from (A-16), (A-17), and emissions being net negative when the constraint binds.

The regulator solves the following Bellman equation:

$$\begin{aligned}\tilde{W}(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t) = \max_{\nu_t} \tilde{E}_t\Bigg[u\Bigg((1 - c_t(R^*_t)) \prod_{i=1}^{N} [\exp[-\zeta_{it} T_t] L^*_{it} Y^{it}(e^*_{it})]^{\kappa_i}\Bigg) \\ + \frac{1}{1+r}\tilde{W}(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1})\Bigg],\end{aligned}$$

where $\tilde{E}_t$ denotes expectations over the regulator's information set at the beginning of period t. Market outcomes will be insensitive to ν_t for $\nu_t > \bar{\nu}_t$, so the regulator's objective will be constant in ν_t for $\nu_t > \bar{\nu}_t$. Maximized welfare is therefore equivalent for a regulator who solves the following problem in which ν_t is constrained to be less than or equal to $\bar{\nu}_t$:

$$\begin{aligned}\tilde{W}(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t) = \max_{\nu_t \leq \bar{\nu}_t} \tilde{E}_t\Bigg[u\Bigg((1 - c_t(R^*_t)) \prod_{i=1}^{N} [\exp[-\zeta_{it} T_t] L^*_{it} Y^{it}(e^*_{it})]^{\kappa_i}\Bigg) \\ + \frac{1}{1+r}\tilde{W}(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1})\Bigg].\end{aligned}$$

At an interior solution, the regulator's first-order condition is

$$\begin{aligned}0 =& \sum_{i=1}^{N} \frac{\kappa_i Y_e^{i\prime}(e_{it}^*)}{Y^{it}(e_{it}^*)} \frac{\partial e_{it}^*}{\partial \nu_t} + \frac{1}{1+r} \tilde{E}_t \left[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1}) \right] \alpha \sum_{i=1}^{N} \frac{\partial e_{it}^*}{\partial \nu_t} \\ &- \frac{c_t'(R_t^*)}{1 - c_t(R_t^*)} \frac{\partial R_t^*}{\partial \nu_t} - \frac{1}{1+r} \tilde{E}_t \left[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1}) \right] \alpha \frac{\partial R_t^*}{\partial \nu_t} \\ &+ \tilde{E}_t \left[\tilde{W}_G(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1}) \right] \underbrace{\left\{ \sum_{i=1}^{N} e_{it}^* - R_t^* + \nu_t \left[\sum_{i=1}^{N} \frac{\partial e_{it}^*}{\partial \nu_t} - \frac{\partial R_t^*}{\partial \nu_t} \right] \right\}}_{\text{Marginal effect of emission tax on revenue}}. \end{aligned} \tag{A-19}$$

From the envelope theorem,

$$\tilde{W}_G(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t) = \tilde{E}_t \left[\tilde{W}_G(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1}) \right]$$

if $\nu_t < \bar{\nu}_t$ and

$$\tilde{W}_G(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t) = \frac{\mathrm{d}\bar{\nu}_t}{\mathrm{d}G_t} \frac{\partial \tilde{W}(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t)}{\partial \bar{\nu}_t}$$

if $\nu_t = \bar{\nu}_t$. Advancing these to later timesteps, we find

$$\begin{aligned}\tilde{W}_G(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t) =& \sum_{s=1}^{\infty} Pr(\nu_{t+j} < \bar{\nu}_{t+j} \, \forall j \in \{1, ..., s-1\}) Pr(\nu_{t+s} = \bar{\nu}_{t+s}) \\ & \frac{\mathrm{d}\bar{\nu}_{t+s}}{\mathrm{d}G_{t+s}} \frac{\partial \tilde{W}(T_{t+s}, \tilde{\mu}_{t+s}, \tilde{\Omega}_{t+s}, G_{t+s})}{\partial \bar{\nu}_{t+s}}\end{aligned}$$

if $\nu_t < \bar{\nu}_t$. Because increasing $\bar{\nu}_{t+s}$ loosens a constraint, the final derivative is strictly positive. We saw above that the other derivative on the final line is strictly positive. And the probabilities on the first line are strictly positive because they depend on $\tilde{\mu}_{t+j}$ and $\tilde{\mu}_{t+s}$, which in turn depend on draws from normally distributed variables that have infinite support. Therefore $\tilde{W}_G > 0$.

Use ν_t^{NoLB} to denote the optimal ν_t implied by (A-15) and ν_t^{LB} to denote the optimal ν_t implied by (A-19). ν_t^{NoLB} is the tax described in Proposition 2 when emissions are strictly positive. If we evaluate (A-19) around ν_t^{NoLB}, then it reduces to its final line. Since we have established that $\tilde{W}_G > 0$, the sign of that final line of (A-19) matches the sign of the term in curly braces, which is the change in revenue due to a marginal change in the tax. If that change is positive, then this term increases the first-order condition and makes $\nu_t^{LB} > \nu_t^{NoLB}$ by concavity around a maximum. And if that change is negative, then this term decreases the first-order condition and makes $\nu_t^{LB} < \nu_t^{NoLB}$ by concavity around a maximum. We have established the first part of the corollary.

The curly braces in the final line of (A-19) are weakly (strictly) negative when net emissions are weakly (strictly) negative. Because net emissions weakly (strictly) decrease in ν_t when net emissions are weakly (strictly) negative, we have established the second part of the corollary in the case that the dynamic revenue constraint does not bind. And the second part of the corollary holds trivially when the dynamic revenue constraint does bind.

G Proof of Lemma 1

Conjecture that the value of the carbon share depends linearly on each $\hat{E}_t[d_{t+j}]$ and $\hat{E}_t[\Delta_{t+j}]$ for $j \geq 0$:

$$\hat{q}_t = \sum_{j=0}^{\infty} \Lambda_{t,t+j} \hat{E}_t[d_{t+j}] + \sum_{j=0}^{\infty} \Psi_{t,t+j} \hat{E}_t[\Delta_{t+j}],$$

with $\{\Lambda_{t,t+j}\}_{j=0}^{\infty}$ and $\{\Psi_{t,t+j}\}_{j=0}^{\infty}$ sequences to be determined.

First consider a case in which $R_t = 0$. The value of a carbon share in period t is $\hat{E}_t[d_t] + \frac{1}{1+r}\hat{E}_t[\hat{q}_{t+1}]$.

Next consider a case in which $R_t > 0$. The payoff of a shareholder who removes carbon in period t is

$$(1+r)D - \hat{E}_t[\Delta_t] - p_t^R,$$

and the payoff of a shareholder who does not remove carbon in period t is

$$\hat{E}_t[d_t] + \frac{1}{1+r}\hat{E}_t[\hat{q}_{t+1}].$$

In a competitive equilibrium with abundant carbon shares, shareholders must be indifferent between the two options, implying that

$$p_t^R = (1+r)D - \frac{1}{1+r}\hat{E}_t[\hat{q}_{t+1}] - \hat{E}_t[d_t] - \hat{E}_t[\Delta_t]. \tag{A-20}$$

Equilibrium payoffs are then identical whether $R_t = 0$ or $R_t > 0$. By absence of arbitrage, the value of the carbon share is:

$$\hat{q}_t = (1+r)D - \hat{E}_t[\Delta_t] - p_t^R.$$

Substitute for $\hat{q}_{t+1}$ from the guess:

$$\hat{q}_t = \hat{E}_t[d_t] + \frac{1}{1+r}\sum_{j=1}^{\infty} \Lambda_{t+1,t+j} \hat{E}_t[d_{t+j}] + \frac{1}{1+r}\sum_{j=1}^{\infty} \Psi_{t+1,t+j} \hat{E}_t[\Delta_{t+j}].$$

Matching coefficients, $\Lambda_{t,t} = 1$ and $\Psi_{t,t} = 0$. Advancing the analysis by one timestep, we find $\Lambda_{t+1,t+1} = 1$ and $\Psi_{t+1,t+1} = 0$. Therefore $\Lambda_{t,t+1} = 1/(1+r)$ and $\Psi_{t,t+1} = 0$. The lemma follows from repeating these steps for subsequent periods, deriving $\Lambda_{t+j,t+j}$ and $\Psi_{t+j,t+j}$, eventually $\Lambda_{t+1,t+j}$ and $\Psi_{t+1,t+j}$, and finally $\Lambda_{t,t+j}$ and $\Psi_{t,t+j}$.

H Proof of Proposition 4

The cost of emitting in period t is $D - (\hat{q}_t - \hat{E}[d_t])$. From (8),

$$D - (\hat{q}_t - \hat{E}[d_t]) = D - \sum_{j=1}^{\infty} \frac{1}{(1+r)^j} \hat{E}_t[d_{t+j}],$$

from which (7) implies

$$\begin{aligned} D - (\hat{q}_t - \hat{E}[d_t]) =& D - \sum_{j=1}^{\infty} \frac{1}{(1+r)^j} \hat{E}_t[r\, D - \Delta_{t+j}] \\ =& \sum_{j=1}^{\infty} \frac{1}{(1+r)^j} \hat{E}_t[\Delta_{t+j}]. \end{aligned}$$

Using (6) and the properties of truncated normal distributions,

$$D - (\hat{q}_t - \hat{E}[d_t]) = C_0\, \alpha \left[\frac{1}{r} \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \frac{1}{r} \hat{\mu}_t - \sum_{j=1}^{\infty} \frac{1}{(1+r)^j} \chi_{t,t+j} \right], \tag{A-21}$$

where

$$\chi_{t,t+j} \triangleq \frac{\phi\left(\frac{\bar{\mu} - \hat{\mu}_t}{\Sigma_{t,t+j}}\right)}{\Phi\left(\frac{\bar{\mu} - \hat{\mu}_t}{\Sigma_{t,t+j}}\right)} \Sigma_{t,t+j} \geq 0$$

is the adjustment to the mean of a normal distribution (for time j random variables, using the time t information set) for the upper truncation point (from the deposit's definition in (5)), $\phi(\cdot)$ and $\Phi(\cdot)$ are the probability density function and cumulative density function for the standard normal distribution, and $\Sigma_{t,t+j} \triangleq \left(\widehat{Var}_t(\hat{\mu}_{t+j})\right)^{1/2}$ is independent of $\bar{\mu}$. Setting $\nu_t = D - \hat{q}_t$ in (A-13), time t emissions are as in (1) as $\chi_{t,t+j} \to 0$ for all $j > 0$.

Assumption 1 ensures that some shares are outstanding. Applying the foregoing analysis to equation (A-20), we find that, if some shares are exercised,

$$p_t^R = \frac{1}{1+r} \sum_{j=1}^{\infty} \frac{1}{(1+r)^{j-1}} \hat{E}_t[\Delta_{t+j}].$$

Using in equation (A-8), we find that

$$\begin{aligned} \frac{c_t'(R_t)}{1 - c_t(R_t)} =& \frac{1}{1+r} \sum_{j=1}^{\infty} \frac{1}{(1+r)^{j-1}} \hat{E}_t[\Delta_{t+j}] \\ =& C_0\, \alpha \left[\frac{1}{r} \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \frac{1}{r} \hat{\mu}_t - \frac{1}{1+r} \sum_{j=1}^{\infty} \frac{1}{(1+r)^{j-1}} \chi_{t,t+j} \right], \end{aligned} \tag{A-22}$$

with $\chi_{t,t+j}$ as above. Time t removal of carbon emitted in all previous periods is as in (2) as $\chi_{t,t+j} \to 0$ for all $s \in \{1, ..., t\}$ and all $j > 0$.

$\chi_{t,t+j}$ decreases in $\bar{\mu}$ and goes to 0 as $\bar{\mu}$ goes to ∞ (i.e., the mean of a truncated-normal distribution increases in the upper bound and approaches the mean of the untruncated normal distribution as the truncation point goes to infinity). By the foregoing analysis, time t emissions and removal as as in (1) and (2) as $\bar{\mu} \to \infty$. So $\breve{L}_t \to 0$ as $\bar{\mu} \to \infty$. We have proved the proposition.

I Preliminaries for Proofs of Propositions 5 and 6

Let $\breve{\mu}_t$ and $\breve{\Omega}_t$ indicate the indicate the mean and variance for $\sum_{k=1}^{N} \kappa_k \zeta_k$ formed after observing $\tilde{\zeta}_s + \tilde{\lambda}_s$ and q_s for all $s < t$, with the corresponding variance of each ζ_i labeled τ_t^2 (note that this variance will be indepndent of i). I show below how to define τ_t^2 from Bayesian updating for $t > 0$.

Demand for carbon shares and market-clearing price

Conjecture that $\breve{q}_t$ is a linear function of the $\zeta_{it} + \lambda_{it}$ and that q_{t+1} is a linear function of $\tilde{\zeta}_t + \tilde{\lambda}_t$ and $\breve{q}_t$. In this case, $\breve{q}_t$ and q_{t+1} are normally distributed, and by standard properties of normal random variables and $D \to \infty$, the time t maximization problem for traders of type i is equivalent to:

$$\begin{aligned}
&\max_{X_{it}} - \exp\Bigg\{ - A(1+r)(w_{it} - X_{it}\breve{q}_t) - A(y_{it} + X_{it})\breve{E}_t[q_{t+1} + (1+r)d_t | \zeta_{it} + \lambda_{it}, \breve{q}_t] \\
&\qquad\qquad + \frac{1}{2}A^2(y_{it} + X_{it})^2 \breve{Var}_t\left[q_{t+1} - (1+r)C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t] | \zeta_{it} + \lambda_{it}, \breve{q}_t\right] \Bigg\} \\
=&\max_{X_{it}} - \exp\Bigg\{ - A(1+r)(w_{it} - X_{it}\breve{q}_t) + \frac{1}{2}A^2(y_{it} + X_{it})^2 \breve{Var}_t\left[q_{t+1} - (1+r)C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t] | \zeta_{it} + \lambda_{it}, \breve{q}_t\right] \\
&\qquad\qquad - A(y_{it} + X_{it})\left(\breve{E}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t] + (1+r)\left[rD - C_0\alpha\breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \breve{q}_t]\right]\right) \Bigg\},
\end{aligned}$$

where $\breve{E}_t$ and $\breve{Var}_t$ indicate the expectation and variance at the common time t beginning-of-period information set. The first-order condition for a maximum is

$$X_{it} = \frac{h_{it}}{A}\left(\breve{E}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t] + (1+r)\left(rD - C_0\alpha\breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \breve{q}_t]\right) - (1+r)\breve{q}_t\right) - y_{it}, \tag{A-23}$$

where

$$h_{it} \triangleq \left(\breve{Var}_t\left[q_{t+1} - (1+r)C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t]|\zeta_{it} + \lambda_{it}, \breve{q}_t\right]\right)^{-1}. \tag{A-24}$$

The h_{it} are deterministic by standard properties of normal-normal updating. Aggregate net demand for carbon shares is

$$X_t = \sum_{i=1}^{N} \frac{h_{it}}{A}\left(\breve{E}_t[q_{t+1}|\zeta_{it} + \lambda_{it}, \breve{q}_t] + (1+r)\left(rD - C_0\alpha\breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t|\zeta_{it} + \lambda_{it}, \breve{q}_t]\right) - (1+r)\breve{q}_t\right) - \sum_{i=1}^{N} y_{it}.$$

Market-clearing requires

$$X_t = 0.$$

Rearranging, the equilibrium price is:

$$\begin{aligned}\breve{q}_t^* =& \frac{1}{(1+r)\sum_{i=1}^N h_{it}}\left[\sum_{i=1}^{N} h_{it}\left(\breve{E}_t[q_{t+1}|\zeta_{it} + \lambda_{it}, q_t] + (1+r)\left(rD - C_0\alpha\breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t|\zeta_{it} + \lambda_{it}, \breve{q}_t]\right)\right)\right.\\ &\left. - A\sum_{i=1}^{N} y_{it}\right].\end{aligned}$$

Define

$$\breve{h}_{it} \triangleq \frac{h_{it}}{\sum_{i=1}^N h_{it}}. \tag{A-25}$$

We have:

$$\begin{aligned}\breve{q}_t^* =& \frac{1}{1+r}\sum_{i=1}^{N} \breve{h}_{it}\left(\breve{E}_t[q_{t+1}|\zeta_{it} + \lambda_{it}, \breve{q}_t] + (1+r)\left(rD - C_0\alpha\breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t|\zeta_{it} + \lambda_{it}, \breve{q}_t]\right)\right)\\ &- \frac{1}{(1+r)\sum_{k=1}^N h_{kt}} A\sum_{i=1}^{N} y_{it}.\end{aligned}$$

Observe that $\sum_{i=1}^N y_{it} = M_t - M_0$. Therefore

$$\begin{aligned}\breve{q}_t^* =& \frac{1}{1+r}\sum_{i=1}^{N} \breve{h}_{it}\left(\breve{E}_t[q_{t+1}|\zeta_{it} + \lambda_{it}, \breve{q}_t] + (1+r)\left(rD - C_0\alpha\breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t|\zeta_{it} + \lambda_{it}, \breve{q}_t]\right)\right)\\ &- A\frac{M_t - M_0}{(1+r)\sum_{k=1}^N h_{kt}}.\end{aligned} \tag{A-26}$$

Analyzing q_{t+1}

q_{t+1} is the equilibrium price determined by time $t+1$ agents who have common beliefs, so that Lemma 1 applies, modulo the information set. As $D \to \infty$,

$$q_{t+1} = \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} \left[rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k - C_0 \alpha \breve{\mu}_{t+1} \right]$$
$$= \frac{1+r}{r} \left[rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k - C_0 \alpha \breve{\mu}_{t+1} \right]. \tag{A-27}$$

Define $\tilde{q}_t$ as the signal of $\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k$ extracted from $\breve{q}_t$, which implies

$$\breve{E}_t[\tilde{q}_t] = \breve{\mu}_t. \tag{A-28}$$

Conjecture that

$$\breve{\mu}_{t+1} = a'_t \breve{\mu}_t + b'_t \left(\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right) + B'_t \tilde{q}_t, \tag{A-29}$$

where a'_t, b'_t, and B'_t are constants to be determined. Taking the expectation of each side under the information set at the beginning of time t and using (A-28), we find that $1 = a'_t + b'_t + B'_t$. Under the conjecture,

$$\breve{E}_t[\breve{\mu}_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t] = a'_t \breve{\mu}_t + b'_t \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k | \zeta_{it} + \lambda_{it}, \breve{q}_t \right] + B'_t \tilde{q}_t.$$

Using this with (A-27),

$$\breve{E}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t] = \frac{1+r}{r} \Bigg[rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k - C_0 \alpha \left(a'_t \breve{\mu}_t + b'_t \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k | \zeta_{it} + \lambda_{it}, \breve{q}_t \right] + B'_t \tilde{q}_t \right) \Bigg].$$

From (A-26),

$$\breve{q}^*_t = \frac{1+r}{r} \left[rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right] - \frac{1}{r} C_0 \alpha \left(a'_t \breve{\mu}_t + B'_t \tilde{q}_t \right) - \frac{b'_t + r}{r} C_0 \alpha \sum_{i=1}^{N} \breve{h}_{it} \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k | \zeta_{it} + \lambda_{it}, \breve{q}_t \right] - A \frac{M_t - M_0}{(1+r) \sum_{k=1}^{N} h_{kt}}. \tag{A-30}$$

Deriving $\tilde{q}_t$

From (A-30),

$$\begin{aligned}&\sum_{i=1}^{N} \breve{h}_{it} \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k | \zeta_{it} + \lambda_{it}, \breve{q}_t \right] \\ =&\frac{1}{C_0 \alpha (b'_t + r)} \left[(1+r) \left(rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right) - C_0 \alpha (a'_t \breve{\mu}_t + B'_t \tilde{q}_t) - r \breve{q}^*_t - rA \frac{M_t - M_0}{(1+r) \sum_{k=1}^{N} h_{kt}} \right].\end{aligned} \tag{A-31}$$

If we set,

$$\tilde{q}_t = \frac{1}{C_0 \alpha (B'_t + b'_t + r)} \left[(1+r) \left(rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right) - C_0 a'_t \breve{\mu}_t - r \breve{q}_t - rA \frac{M_t - M_0}{(1+r) \sum_{k=1}^{N} h_{kt}} \right], \tag{A-32}$$

then from (A-31) and the definition $\breve{q}_t \triangleq \breve{q}^*_t + \theta_t$,

$$\tilde{q}_t = \sum_{i=1}^{N} \breve{h}_{it} \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k | \zeta_{it} + \lambda_{it}, \breve{q}_t \right] - \frac{r}{C_0 \alpha (B'_t + b'_t + r)} \theta_t. \tag{A-33}$$

Recalling that $\breve{h}_{it} \in (0,1)$ and $\sum_{i=1}^{N} \breve{h}_{it} = 1$, this $\tilde{q}_t$ satisfies the earlier definition of $\tilde{q}_t$ as the signal of $\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k$ extracted from $\breve{q}_t$.

J Proof of Proposition 5

Section I contains preliminaries. A rational expectations equilibrium is fully revealing if and only if $\breve{\mu}_t = \hat{\mu}_{t+1}$ and, from (A-33), $\tilde{q}_t = \hat{\mu}_{t+1}$. From (A-32) and taking $\Theta^2 \to 0$, that $\tilde{q}_t$ clears the carbon share market if and only if

$$r \breve{q}^*_t = (1+r) \left(rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right) - C_0 \alpha a'_t \hat{\mu}_{t+1} - C_0 \alpha (B'_t + b'_t + r) \hat{\mu}_{t+1} - rA \frac{M_t - M_0}{(1+r) \sum_{k=1}^{N} h_{kt}}.$$

Using $a'_t + b'_t + B'_t = 1$,

$$\breve{q}^*_t = \frac{1+r}{r} \left[\left(rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right) - C_0 \alpha \hat{\mu}_{t+1} - rA \frac{M_t - M_0}{(1+r) \sum_{k=1}^{N} h_{kt}} \right].$$

As $\Theta^2 \to 0$, $\breve{q}_t \to \breve{q}_t^*$, in which case traders can back out $\hat{\mu}_{t+1}$ from the observed price $\breve{q}_t$, so $\breve{\mu}_t = \hat{\mu}_{t+1}$. In addition, the conjectured form of $\breve{\mu}_{t+1}$ in (A-29) holds, with $B_t' = 1$, and τ_t^2 can be defined from the posterior variance of informationally efficient beliefs. We have described a rational expectations equilibrium that is fully revealing.

Observe that $\breve{q}_t$ is identical to $\hat{q}_t$ from Lemma 1. Assumption 1 and Proposition 4 then imply that $\breve{L} \to 0$ as $D \to \infty$.

K Proof of Proposition 6

Section I contains preliminaries.

Expected q_{t+1} as a function of time t information

The combination of normal random variables and an affine information structure implies that the posterior mean is a linear function of the prior and the signals:

$$\breve{E}_t\left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^N \kappa_k \bar{\zeta}_k | \zeta_{it} + \lambda_{it}, \breve{q}_t\right] = a_{it}\breve{\mu}_t + b_{it}(\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) + B_{it}\tilde{q}_t \tag{A-34}$$

for yet-to-be-determined coefficients a_{it}, b_{it}, and B_{it}. Substituting into (A-33), we find:

$$\tilde{q}_t = \sum_{i=1}^N \breve{h}_{it}\left[a_{it}\breve{\mu}_t + b_{it}(\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) + B_{it}\tilde{q}_t\right] - \frac{r}{C_0\alpha(B_t' + b_t' + r)}\theta_t.$$

Solving for $\tilde{q}_t$ yields:

$$\tilde{q}_t = \frac{1}{1 - \sum_{i=1}^N \breve{h}_{it} B_{it}}\left(\sum_{i=1}^N \breve{h}_{it} a_{it}\breve{\mu}_t + \sum_{i=1}^N \breve{h}_{it} b_{it}(\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) - \frac{r}{C_0\alpha(B_t' + b_t' + r)}\theta_t\right).$$

Taking the expectation under the information set at the beginning of time t and setting it equal to (A-28), we find:

$$\breve{\mu}_t = \frac{\breve{\mu}_t}{1 - \sum_{i=1}^N \breve{h}_{it} B_{it}} \sum_{i=1}^N \breve{h}_{it}(a_{it} + b_{it}).$$

This holds if and only if $\sum_{i=1}^N \breve{h}_{it} B_{it} = 1 - \sum_{i=1}^N \breve{h}_{it}(a_{it} + b_{it})$. Define

$$\chi_t \triangleq 1 - \sum_{i=1}^N \breve{h}_{it} B_{it}. \tag{A-35}$$

Then:

$$\tilde{q}_t = \frac{1}{\chi_t}\left(\sum_{i=1}^N \breve{h}_{it} a_{it}\breve{\mu}_t + \sum_{i=1}^N \breve{h}_{it} b_{it}(\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) - \frac{r}{C_0\alpha(B_t' + b_t' + r)}\theta_t\right). \tag{A-36}$$

Deriving $\breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^N \kappa_k \bar{\zeta}_k | \zeta_{it} + \lambda_{it}, \breve{q}_t]$

In time t but prior to observing $\zeta_{it} + \lambda_{it}$ and q_t, the 3×1 random vector

$$\breve{S}_{it} = \begin{bmatrix} \tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^N \kappa_k \bar{\zeta}_k \\ \zeta_{it} + \lambda_{it} - \bar{\zeta}_i \\ \tilde{q}_t \end{bmatrix} \tag{A-37}$$

is jointly normal with unconditional mean

$$\breve{E}_{it}[\breve{S}_{it}] = \begin{bmatrix} \breve{\mu}_t \\ \breve{\mu}_t \\ \breve{\mu}_t \end{bmatrix}$$

and covariance matrix

$$Cov_{it}(\breve{S}_{it}) =$$

$$\begin{bmatrix} \sum_{j=1}^N \kappa_j^2[\tau_t^2+\sigma^2]+2\Gamma\tau_t^2 \sum_{j=1}^N \sum_{k=j+1}^N \kappa_j\kappa_k+\tilde{\omega}^2 & \kappa_i\tau_t^2+(1-\kappa_i)\Gamma\tau_t^2+\kappa_i\sigma^2 & \sum_{k=1}^N(\kappa_k\tau_t^2+(1-\kappa_k)\Gamma\tau_t^2+\kappa_k\sigma^2)\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ \kappa_i\tau_t^2+(1-\kappa_i)\Gamma\tau_t^2+\kappa_i\sigma^2 & \tau_t^2+\sigma^2+\omega^2 & (\tau_t^2+\sigma^2+\omega^2)\frac{\breve{h}_{it}b_{it}}{\chi_t}+\Gamma\tau_t^2\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ \sum_{k=1}^N(\kappa_k\tau_t^2+(1-\kappa_k)\Gamma\tau_t^2+\kappa_k\sigma^2)\frac{\breve{h}_{kt}b_{kt}}{\chi_t} & (\tau_t^2+\sigma^2+\omega^2)\frac{\breve{h}_{it}b_{it}}{\chi_t}+\Gamma\tau_t^2\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} & \begin{matrix}\sum_{k=1}^N\left(\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right)^2\left(\tau_t^2+\sigma^2+\omega^2\right) \\ +2\Gamma\tau_t^2\sum_{j=1}^N\sum_{k=j+1}^N\frac{\breve{h}_{jt}b_{jt}}{\chi_t}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ +\left(\frac{r}{C_0\alpha(B'_t+b'_t+r)}\frac{\Theta}{\chi_t}\right)^2\end{matrix} \end{bmatrix}. \tag{A-38}$$

From the projection theorem,

$$\breve{E}_t\left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^N \kappa_k\bar{\zeta}_k | \zeta_{it}+\lambda_{it}, \breve{q}_t\right]$$

$$= \breve{\mu}_t + \begin{bmatrix} \kappa_i\tau_t^2+(1-\kappa_i)\Gamma\tau_t^2+\kappa_i\sigma^2 \\ \sum_{k=1}^N(\kappa_k\tau_t^2+(1-\kappa_k)\Gamma\tau_t^2+\kappa_k\sigma^2)\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \tau_t^2+\sigma^2+\omega^2 & (\tau_t^2+\sigma^2+\omega^2)\frac{\breve{h}_{it}b_{it}}{\chi_t}+\Gamma\tau_t^2\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ (\tau_t^2+\sigma^2+\omega^2)\frac{\breve{h}_{it}b_{it}}{\chi_t}+\Gamma\tau_t^2\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} & \begin{matrix}\sum_{k=1}^N\left(\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right)^2\left(\tau_t^2+\sigma^2+\omega^2\right) \\ +2\Gamma\tau_t^2\sum_{j=1}^N\sum_{k=j+1}^N\frac{\breve{h}_{jt}b_{jt}}{\chi_t}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ +\left(\frac{r}{C_0\alpha(B'_t+b'_t+r)}\frac{\Theta}{\chi_t}\right)^2\end{matrix} \end{bmatrix}^{-1}$$

$$\begin{bmatrix} \zeta_{it}+\lambda_{it}-\bar{\zeta}_i-\breve{\mu}_t \\ \tilde{q}_t-\breve{\mu}_t \end{bmatrix}.$$

Working through the matrix algebra and matching coefficients to (A-34), we find:

$$\begin{aligned} b_{it} =& \frac{1}{det_{it}} \Bigg\{ \left(\kappa_i \frac{\tau_t^2 + \sigma^2}{\tau_t^2 + \sigma^2 + \omega^2} + (1-\kappa_i) \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2} \right) \\ & \left[\sum_{k=1}^{N} \left(\frac{\breve{h}_{kt} b_{kt}}{\chi_t} \right)^2 + 2 \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2} \sum_{j=1}^{N} \sum_{k=j+1}^{N} \frac{\breve{h}_{jt} b_{jt}}{\chi_t} \frac{\breve{h}_{kt} b_{kt}}{\chi_t} \right. \\ & \left. + \frac{1}{\tau_t^2 + \sigma^2 + \omega^2} \left(\frac{r}{C_0 \alpha (B_t' + b_t' + r)} \frac{\Theta}{\chi_t} \right)^2 \right] \\ & - \left(\sum_{k=1}^{N} \left(\kappa_k \frac{\tau_t^2 + \sigma^2}{\tau_t^2 + \sigma^2 + \omega^2} + (1-\kappa_k) \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2} \right) \frac{\breve{h}_{kt} b_{kt}}{\chi_t} \right) \\ & \left(\frac{\breve{h}_{it} b_{it}}{\chi_t} + \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2} \sum_{k \neq i} \frac{\breve{h}_{kt} b_{kt}}{\chi_t} \right) \Bigg\}, \end{aligned} \tag{A-39}$$

$$\begin{aligned} B_{it} =& \frac{1}{det_{it}} \Bigg\{ \sum_{k \neq i} \Bigg(\left[\kappa_k - \kappa_i \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2} \right] \frac{\tau_t^2 + \sigma^2}{\tau_t^2 + \sigma^2 + \omega^2} \\ & + \left[(1-\kappa_k) - (1-\kappa_i) \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2} \right] \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2} \Bigg) \frac{\breve{h}_{kt} b_{kt}}{\chi_t} \Bigg\}, \end{aligned} \tag{A-40}$$

and

$$a_{it} = 1 - b_{it} - B_{it}, \tag{A-41}$$

where

$$\begin{aligned} det_{it} \triangleq & \sum_{k \neq i} \left(\frac{\breve{h}_{kt} b_{kt}}{\chi_t} \right)^2 + 2 \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2} \sum_{j=1, \neq i}^{N} \sum_{k=j+1, \neq i}^{N} \frac{\breve{h}_{jt} b_{jt}}{\chi_t} \frac{\breve{h}_{kt} b_{kt}}{\chi_t} \\ & + \frac{1}{\tau_t^2 + \sigma^2 + \omega^2} \left(\frac{r}{C_0 \alpha (B_t' + b_t' + r)} \frac{\Theta}{\chi_t} \right)^2 - \left(\frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2} \right)^2 \left(\sum_{k \neq i} \frac{\breve{h}_{kt} b_{kt}}{\chi_t} \right)^2. \end{aligned}$$

(A-41) implies that $\frac{1}{N} \sum_{i=1}^{N} B_{it} = 1 - \sum_{i=1}^{N} (a_{it} + b_{it})$, as required above.

Simplifying, (A-39) becomes:

$$
\begin{aligned}
b_{it} =& \Bigg\{ \sum_{k\neq i} \breve{h}_{kt}^2 b_{kt}^2 + 2\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2} \sum_{j=1,\neq i}^{N} \sum_{k=j+1,\neq i}^{N} \breve{h}_{jt} b_{jt} \breve{h}_{kt} b_{kt} + \left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\right)^2 \frac{\Theta^2}{\tau_t^2+\sigma^2+\omega^2} \\
&\quad - \left(\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right)^2 \left(\sum_{k\neq i} \breve{h}_{kt} b_{kt}\right)^2 \Bigg\}^{-1} \\
&\Bigg\{ \left(\kappa_i \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_i)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) \sum_{k\neq i} \breve{h}_{kt}^2 b_{kt}^2 \\
&\quad + \left(\kappa_i \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_i)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) \left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\right)^2 \frac{\Theta^2}{\tau_t^2+\sigma^2+\omega^2} \\
&\quad + \left(\kappa_i \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_i)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) 2\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2} \sum_{j=1,\neq i}^{N} \sum_{k=j+1,\neq i}^{N} \breve{h}_{jt} b_{jt} \breve{h}_{kt} b_{kt} \\
&\quad + \left(\kappa_i \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_i)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) \frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2} \breve{h}_{it} b_{it} \sum_{k\neq i} \breve{h}_{kt} b_{kt} \\
&\quad - \breve{h}_{it} b_{it} \sum_{k\neq i} \left(\kappa_k \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_k)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) \breve{h}_{kt} b_{kt} \\
&\quad - \frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2} \left(\sum_{k\neq i} \left(\kappa_k \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_k)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) \breve{h}_{kt} b_{kt}\right) \sum_{k\neq i} \breve{h}_{kt} b_{kt} \Bigg\}.
\end{aligned}
$$

Solve for b_{it}:

$$
\begin{aligned}
b_{it} = &\left(\kappa_i \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_i)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) \\
&\Bigg\{ \left(\tau_t^2+\sigma^2+\omega^2-\Gamma\tau_t^2\right)\left(\tau_t^2+\sigma^2+\omega^2+\Gamma\tau_t^2\right)\sum_{k\neq i}\breve{h}_{kt}^2 b_{kt}^2 \\
&\quad + 2\Gamma\tau_t^2\left[\tau_t^2+\sigma^2+\omega^2-\Gamma\tau_t^2\right]\sum_{j=1,\neq i}^{N}\sum_{k=j+1,\neq i}^{N}\breve{h}_{jt}b_{jt}\breve{h}_{kt}b_{kt} \\
&\quad + \left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\right)^2\Theta^2(\tau_t^2+\sigma^2+\omega^2) \\
&\quad - \Bigg[\sum_{k\neq i}\Bigg(\Bigg[\kappa_k(\tau_t^2+\sigma^2+\omega^2)-\kappa_i\Gamma\tau_t^2\Bigg](\tau_t^2+\sigma^2) \\
&\qquad\qquad + \left[(1-\kappa_k)(\tau_t^2+\sigma^2+\omega^2)-(1-\kappa_i)\Gamma\tau_t^2\right]\Gamma\tau_t^2\Bigg)\breve{h}_{kt}b_{kt}\Bigg] \\
&\qquad \sum_{k\neq i}\frac{\Gamma\tau_t^2}{\kappa_i(\tau_t^2+\sigma^2)+(1-\kappa_i)\Gamma\tau_t^2}\,\breve{h}_{kt}b_{kt}\Bigg\} \\
&\Bigg\{ \left(\tau_t^2+\sigma^2+\omega^2-\Gamma\tau_t^2\right)\left(\tau_t^2+\sigma^2+\omega^2+\Gamma\tau_t^2\right)\sum_{k\neq i}\breve{h}_{kt}^2 b_{kt}^2 \\
&\quad + 2\Gamma\tau_t^2\left[\tau_t^2+\sigma^2+\omega^2-\Gamma\tau_t^2\right]\sum_{j=1,\neq i}^{N}\sum_{k=j+1,\neq i}^{N}\breve{h}_{jt}b_{jt}\breve{h}_{kt}b_{kt} \\
&\quad + \left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\right)^2\Theta^2(\tau_t^2+\sigma^2+\omega^2) \\
&\quad + \breve{h}_{it}\sum_{k\neq i}\Bigg(\left[\kappa_k(\tau_t^2+\sigma^2+\omega^2)-\kappa_i\Gamma\tau_t^2\right](\tau_t^2+\sigma^2) \\
&\qquad\qquad + \left[(1-\kappa_k)(\tau_t^2+\sigma^2+\omega^2)-(1-\kappa_i)\Gamma\tau_t^2\right]\Gamma\tau_t^2\Bigg)\breve{h}_{kt}b_{kt}\Bigg\}^{-1}.
\end{aligned}
\tag{A-42}
$$

Observe that

$$
\begin{aligned}
&\left[\kappa_k(\tau_t^2+\sigma^2+\omega^2)-\kappa_i\Gamma\tau_t^2\right](\tau_t^2+\sigma^2)+\left[(1-\kappa_k)(\tau_t^2+\sigma^2+\omega^2)-(1-\kappa_i)\Gamma\tau_t^2\right]\Gamma\tau_t^2 \\
=&\left[(1-\Gamma)\tau_t^2+\sigma^2+\omega^2\right]\left[\kappa_k(\tau_t^2+\sigma^2)+(1-\kappa_i)\Gamma\tau_t^2\right]
\end{aligned}
$$

decreases in κ_i and is strictly positive as $\kappa_i \to 1$, implying that it is strictly positive for all relevant κ_i. Because that expression is positive,

$$b_{it} < \kappa_i \frac{\tau_t^2 + \sigma^2}{\tau_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_i) \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2},$$

and because this last inequality holds for arbitrary i, inspection of (A-42) shows that $b_{it} > 0$. Therefore the set of functions defined by (A-42) for each $i \in \{1, ..., N\}$ maps a vector from

$$\bigtimes_{i=1}^{N} \left[0, \kappa_i \frac{\tau_t^2 + \sigma^2}{\tau_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_i) \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2}\right]$$

into itself. By Brouwer's fixed-point theorem, there exists a fixed point in that space. By inspection, the fixed point does not have any b_{kt} on the boundary. Therefore, for each $i \in \{1, ..., N\}$,

$$b_{it} = \left(\kappa_i \frac{\tau_t^2 + \sigma^2}{\tau_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_i) \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2}\right) \breve{Z}_{it} \tag{A-43}$$

for some $\breve{Z}_{it} \in (0, 1)$. Observe that $\breve{Z}_{it}$ and b_{it} are deterministic because each $\breve{h}_{kt}$ is deterministic.

Deriving $\breve{\mu}_{t+1}$

Substituting (A-43) into (A-36) and using $\sum_{i=1}^N \breve{h}_{it} a_{it} = \chi_t - \sum_{i=1}^N \breve{h}_{it} b_{it}$ from (A-35) and (A-41),

$$\begin{aligned}\tilde{q}_t =& \frac{\chi_t - \sum_{i=1}^N \breve{h}_{it} b_{it}}{\chi_t} \breve{\mu}_t + \sum_{i=1}^{N} \left(\kappa_i \frac{\tau_t^2 + \sigma^2}{\tau_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_i) \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2}\right) \frac{\breve{h}_{it} \breve{Z}_{it}}{\chi_t} (\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) \\ &- \frac{r}{C_0 \alpha (B_t' + b_t' + r)} \frac{\theta_t}{\chi_t}.\end{aligned} \tag{A-44}$$

Consider jointly updating about

$$\begin{bmatrix} \sum_{i=1}^N \kappa_i \zeta_i \\ \sum_{i=1}^N \left(\kappa_i \frac{\tau_t^2 + \sigma^2}{\tau_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_i) \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2}\right) \frac{\breve{h}_{it} \breve{Z}_{it}}{\chi_t} \zeta_i \end{bmatrix}. \tag{A-45}$$

Define the time t signal of this vector as

$$\breve{s}_t \triangleq \begin{bmatrix} \tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^N \kappa_k \bar{\zeta}_k \\ \tilde{q}_t - \frac{\chi_t - \sum_{i=1}^N \breve{h}_{it} b_{it}}{\chi_t} \breve{\mu}_t \end{bmatrix}. \tag{A-46}$$

Let $\Upsilon_{\mu,t}$ indicate the 2×2 precision matrix of the time t prior and $\Upsilon_{s,t}$ indicate the 2×2 conditional precision matrix of the time t signal. Apply the conventional normal-normal updating formula, recursively substitute, and recognize that the time 0 mean is a zero vector:

$$E\left[\left[\begin{matrix}\sum_{i=1}^N \kappa_i \zeta_i \\ \sum_{i=1}^N \left(\kappa_i \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_i)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) \frac{\breve{h}_{it}\breve{Z}_{it}}{\chi_t}\zeta_i\end{matrix}\right]\middle| \{\tilde{\zeta}_j + \tilde{\lambda}_j, \tilde{q}_j\}_{j=0}^t\right] = \sum_{k=0}^t \left[\prod_{j=k+1}^t \pi_j\right](I - \pi_k)\breve{s}_k,$$

where $\pi_j \triangleq (\Upsilon_{\mu,j} + \Upsilon_{s,j})^{-1}\Upsilon_{\mu,j}$ and I is the 2×2 identity matrix. Define $\tilde{\pi}_k$ as element $(1,1)$ of $\left[\prod_{j=k+1}^t \pi_j\right][I - \pi_k]$ and $\breve{\pi}_k$ as element $(1,2)$ of $\left[\prod_{j=k+1}^t \pi_j\right][I - \pi_k]$. Using (A-46) and the definition of $\breve{\mu}_{t+1}$,

$$\breve{\mu}_{t+1} = \sum_{k=0}^t \left[\tilde{\pi}_k\left(\tilde{\zeta}_k + \tilde{\lambda}_k - \sum_{j=1}^N \kappa_j\bar{\zeta}_j\right) + \breve{\pi}_k\left(\tilde{q}_k - \frac{\chi_k - \sum_{i=1}^N \breve{h}_{ik}b_{ik}}{\chi_k}\breve{\mu}_k\right)\right].$$

Substituting for $\breve{\mu}_t$, we have confirmed the conjecture in (A-29) that $\breve{\mu}_{t+1}$ is a linear function of $\breve{\mu}_t$, $\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^N \kappa_k\bar{\zeta}_k$, and $\tilde{q}_t$. Matching coefficients yields a_t', b_t', and B_t'.

Using (A-44),

$$\begin{aligned}\breve{\mu}_{t+1} = \sum_{k=0}^t \Bigg[&\tilde{\pi}_k\left(\tilde{\zeta}_k + \tilde{\lambda}_k - \sum_{j=1}^N \kappa_j\bar{\zeta}_j\right) \\ &+ \breve{\pi}_k \sum_{j=1}^N \left(\kappa_j \frac{\tau_k^2+\sigma^2}{\tau_k^2+\sigma^2+\omega^2} + (1-\kappa_j)\frac{\Gamma\tau_k^2}{\tau_k^2+\sigma^2+\omega^2}\right)\frac{\breve{h}_{jk}\breve{Z}_{jk}}{\chi_k}(\zeta_{jk} + \lambda_{jk} - \bar{\zeta}_j) \\ &- \breve{\pi}_k \frac{r}{C_0\alpha(B_k' + b_k' + r)}\frac{\theta_k}{\chi_k}\Bigg].\end{aligned}$$

Define

$$\breve{\kappa}_{is} \triangleq \breve{h}_{is}\breve{Z}_{is}. \tag{A-47}$$

$\breve{\kappa}_{is} \in (0,1)$, because both components are. Also define

$$\breve{\chi}_s \triangleq B_s' + b_s' < 1.$$

Then:

$$
\begin{aligned}
\breve{\mu}_{t+1} =& \sum_{k=0}^{t} \tilde{\pi}_k \left(\tilde{\zeta}_k + \tilde{\lambda}_k - \sum_{j=1}^{N} \kappa_j \bar{\zeta}_j \right) \\
&+ \sum_{k=0}^{t} \frac{\breve{\pi}_k}{\chi_k} \left[\frac{(1-\Gamma)\tau_k^2 + \sigma^2}{\tau_k^2 + \sigma^2 + \omega^2} \sum_{i=1}^{N} \breve{\kappa}_{ik} \kappa_i \left(\zeta_{ik} + \lambda_{ik} - \bar{\zeta}_i \right) \right. \\
&\qquad\qquad \left. + \frac{\Gamma \tau_k^2}{\tau_k^2 + \sigma^2 + \omega^2} \sum_{i=1}^{N} \breve{\kappa}_{ik} (\zeta_{ik} + \lambda_{ik} - \bar{\zeta}_i) \right] \\
&- \sum_{k=0}^{t} \frac{\breve{\pi}_k}{\chi_k} \frac{r}{C_0 \alpha (\breve{\chi}_k + r)} \theta_k. \qquad \text{(A-48)}
\end{aligned}
$$

Letting $\tilde{\Upsilon}_t$ indicate element $(1,1)$ of $(\Upsilon_{\mu,t} + \Upsilon_{s,t})^{-1}$, the conventional normal-normal updating formula for the precision and the definition of τ_t^2 together imply:

$$
\tau_{t+1}^2 = \frac{\tilde{\Upsilon}_t}{(1-\Gamma) \sum_{k=1}^{N} \kappa_k^2 + \Gamma}.
$$

Finally, the following lemma establishes properties of the $\tilde{\pi}$ and the $\breve{\pi}$:

Lemma 2. *If Γ, ω^2, $\tilde{\omega}^2$, and Θ^2 are sufficiently small, then $\tilde{\pi}_k \in (0,1)$ and $\lim_{\Gamma, \tilde{\omega}^2 \to 0} \breve{\pi}_k$ is arbitarily close to zero.*

Proof. See Appendix M. □

Emissions and carbon removal

Adapting (A-21) and (A-22) to the current informational environment and applying the conditions of the proposition, time t firms equate both the marginal cost of emission reductions and the marginal cost of carbon removal to $D - (q_t - \breve{E}_t[d_t])$. Using (A-27),

$$
\begin{aligned}
D - (q_t - \breve{E}_t[d_t]) =& D - \left(\frac{1+r}{r} - 1 \right) \left[rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k - C_0 \alpha \breve{\mu}_t \right] \\
=& \frac{1}{r} C_0 \alpha \left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \breve{\mu}_t \right].
\end{aligned}
$$

The proposition follows from using (A-48) to define $\breve{\mu}_t$.

L Proof of Corollary 4

By inspection, equation (A-42) holds if $b_{it} = 0$ for all $i \in \{1, ..., N\}$ with $\Theta^2 = 0$, and because the denominator contains a term that is linear in the b_{kt} whereas the numerator contains only products of the b, equation (A-42) holds if each b_{it} is arbitrarily small with Θ^2 arbitrarily small. The first part of the corollary follows from these results, $\breve{h}_{kt} \in (0, 1)$, and the definition (A-47).

Now analyze h_{it}. From the definition (A-24),

$$1/h_{it} = \breve{Var}_t \left[q_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t\right] + \breve{Var}_t \left[(1+r)C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t] | \zeta_{it} + \lambda_{it}, \breve{q}_t\right]. \tag{A-49}$$

From (A-29),

$$\breve{Var}_t[\mu_{t+1} | \zeta_{it} + \lambda_{it}, \tilde{q}_t] = (b'_t)^2 \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t].$$

Using this in (A-27),

$$\breve{Var}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, \tilde{q}_t] = \left(\frac{1+r}{r} C_0 \alpha b'_t\right)^2 \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t],$$

$$\breve{Cov}_t \left[q_{t+1}, -(1+r)C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t] | \zeta_{it} + \lambda_{it}, \breve{q}_t\right] = [(1+r)C_0\alpha]^2 \frac{b'_t}{r} \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t].$$

Substituting into (A-49),

$$1/h_{it} = \left[\frac{1+r}{r} C_0 \alpha\right]^2 (b'_t + r)^2 \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t]. \tag{A-50}$$

Apply the projection theorem to (A-37), via (A-38):

$$\begin{aligned}
&\breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t] \\
=&[\tau_t^2 + \sigma^2] \sum_{j=1}^N \kappa_j^2 + 2\Gamma\tau_t^2 \sum_{j=1}^N \sum_{k=j+1}^N \kappa_j \kappa_k + \tilde{\omega}^2 \\
&- \begin{bmatrix} \kappa_i\tau_t^2 + (1-\kappa_i)\Gamma\tau_t^2 + \kappa_i\sigma^2 \\ \sum_{k=1}^N (\kappa_k\tau_t^2 + (1-\kappa_k)\Gamma\tau_t^2 + \kappa_k\sigma^2)\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \end{bmatrix}^\intercal \begin{bmatrix} \tau_t^2+\sigma^2+\omega^2 & (\tau_t^2+\sigma^2+\omega^2)\frac{\breve{h}_{it}b_{it}}{\chi_t} + \Gamma\tau_t^2 \sum_{k\neq i} \frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ (\tau_t^2+\sigma^2+\omega^2)\frac{\breve{h}_{it}b_{it}}{\chi_t} + \Gamma\tau_t^2 \sum_{k\neq i} \frac{\breve{h}_{kt}b_{kt}}{\chi_t} & \begin{matrix} \sum_{k=1}^N \left(\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right)^2 (\tau_t^2+\sigma^2+\omega^2) \\ +2\Gamma\tau_t^2 \sum_{j=1}^N \sum_{k=j+1}^N \frac{\breve{h}_{jt}b_{jt}}{\chi_t}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ + \left(\frac{r}{C_0\alpha(B'_t+b'_t+r)}\frac{\Theta}{\chi_t}\right)^2 \end{matrix} \end{bmatrix}^{-1} \\
&\begin{bmatrix} \kappa_i\tau_t^2 + (1-\kappa_i)\Gamma\tau_t^2 + \kappa_i\sigma^2 \\ \sum_{k=1}^N (\kappa_k\tau_t^2 + (1-\kappa_k)\Gamma\tau_t^2 + \kappa_k\sigma^2)\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \end{bmatrix}
\end{aligned}$$

Using the b_{it} and B_{it} from (A-39) and (A-40), this becomes

$$\breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t] = [(1-\Gamma)\tau_t^2 + \sigma^2] \sum_{j=1}^N \kappa_j^2 + \Gamma\tau_t^2 + \tilde{\omega}^2 - b_{it}\left[\kappa_i[(1-\Gamma)\tau_t^2+\sigma^2] + \Gamma\tau_t^2\right] - B_{it}\left[\sum_{k=1}^N (\kappa_k[(1-\Gamma)\tau_t^2+\sigma^2] + \Gamma\tau_t^2)\frac{\breve{h}_{kt} b_{kt}}{\chi_t}\right].$$

Substituting from (A-43),

$$\breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t] = [(1-\Gamma)\tau_t^2 + \sigma^2] \sum_{j=1}^N \kappa_j^2 + \Gamma\tau_t^2 + \tilde{\omega}^2 - \frac{\left[\kappa_i[(1-\Gamma)\tau_t^2+\sigma^2] + \Gamma\tau_t^2\right]^2}{\tau_t^2+\sigma^2+\omega^2}\breve{Z}_{it} - B_{it}\left[\sum_{k=1}^N \frac{(\kappa_k[(1-\Gamma)\tau_t^2+\sigma^2] + \Gamma\tau_t^2)^2}{\tau_t^2+\sigma^2+\omega^2}\frac{\breve{h}_{kt}\breve{Z}_{kt}}{\chi_t}\right].$$

From (A-42) and (A-43), $\lim_{\Theta^2\to\infty} \breve{Z}_{it} = 1$. And from (A-40), $\lim_{\Theta^2\to\infty} B_{it} = 0$. Therefore,

$$\lim_{\Theta^2\to\infty} \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t] = [(1-\Gamma)\tau_t^2 + \sigma^2] \sum_{j=1}^N \kappa_j^2 + \Gamma\tau_t^2 + \tilde{\omega}^2 - \frac{\left[\kappa_i[(1-\Gamma)\tau_t^2+\sigma^2] + \Gamma\tau_t^2\right]^2}{\tau_t^2+\sigma^2+\omega^2}.$$

Simplifying, we find:

$$\lim_{\Theta^2\to\infty} \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t] = \tilde{\omega}^2 + [(1-\Gamma)\tau_t^2 + \sigma^2]\left(\sum_{j\neq i}\kappa_j^2 + \frac{\omega^2}{\tau_t^2+\sigma^2+\omega^2}\kappa_i^2\right) + \frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\left((1-\kappa_i)^2[(1-\Gamma)\tau_t^2+\sigma^2] + \omega^2\right).$$

Now consider the second part of the corollary. If $\kappa_i = 1/N$ for all $i \in \{1,..,N\}$, then the foregoing implies that $\lim_{\Theta^2\to\infty} \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t]$ is independent of i. From that result, (A-50), and the definition (A-25), $\lim_{\Theta^2\to\infty} \breve{h}_{kt} = 1/N$. And that result with $\lim_{\Theta^2\to\infty} \breve{Z}_{it} = 1$ and the definition (A-47) implies $\lim_{\Theta^2\to\infty} \breve{\kappa}_{kt} = 1/N$. Further, if $\kappa_i = 1/N$ for all $i \in \{1,..,N\}$, then b_{it} equals some constant b_t which, from (A-42), solves

$$\left(\tau_t^2+\sigma^2+\omega^2+(N-1)\Gamma\tau_t^2\right)b_t^3 + \left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\right)^2\Theta^2\frac{\tau_t^2+\sigma^2+\omega^2}{\tau_t^2+\sigma^2+\omega^2-\Gamma\tau_t^2}\frac{N^2}{N-1}b_t$$
$$= \left(\frac{1}{N}\frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + \frac{N-1}{N}\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right)\left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\right)^2\Theta^2\frac{\tau_t^2+\sigma^2+\omega^2}{\tau_t^2+\sigma^2+\omega^2-\Gamma\tau_t^2}\frac{N^2}{N-1}.$$

Substitute from (A-43), writing $\breve{Z}_t$ for $\breve{Z}_{it}$:

$$\frac{1}{N^2}\left((1-\Gamma)\tau_t^2+\sigma^2+\omega^2+N\Gamma\tau_t^2\right)\left(\frac{(1-\Gamma)\tau_t^2+\sigma^2+N\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right)^2\breve{Z}_t^3$$
$$=(1-\breve{Z}_t)\left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\right)^2\Theta^2\frac{\tau_t^2+\sigma^2+\omega^2}{(1-\Gamma)\tau_t^2+\sigma^2+\omega^2}\frac{N^2}{N-1}.$$

Evaluating at $\breve{Z}_t=0$ and $\breve{Z}_t=1$ shows that there exists at least one root in the interval $(0,1)$. For $\breve{Z}_t\in(0,1)$, the left-hand side monotonically increases in $\breve{Z}_t$ and the right-hand side monotonically decreases in $\breve{Z}_t$. Therefore the root in the interval $\breve{Z}_t\in(0,1)$ is unique. This root monotonically increases in Θ^2 for $\Theta^2\in(0,\infty)$. Using $\breve{h}_{kt}=1/N$ and the definition (A-47), we have established the second part of the corollary.

Finally, consider the third part of the corollary. Without loss of generality, order the sectors by increasing κ_i. Then $\lim_{\Theta^2\to\infty}\breve{Var}_t[\tilde{\zeta}_t+\tilde{\lambda}_t|\zeta_{it}+\lambda_{it},\tilde{q}_t]\geq\lim_{\Theta^2\to\infty}\breve{Var}_t[\tilde{\zeta}_t+\tilde{\lambda}_t|\zeta_{jt}+\lambda_{jt},\tilde{q}_t]$ for all $j>i$ if, for all $\{n,n+k\}$ such that $\kappa_n<\kappa_{n+k}$,

$$\sum_{j\neq n}\kappa_j^2+\frac{\omega^2}{\tau_t^2+\sigma^2+\omega^2}\kappa_n^2+\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}(1-\kappa_n)^2$$
$$\geq\sum_{j\neq n+k}\kappa_j^2+\frac{\omega^2}{\tau_t^2+\sigma^2+\omega^2}\kappa_{n+k}^2+\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}(1-\kappa_{n+k})^2.$$

This sufficient condition is equivalent to

$$(\kappa_{n+k}^2-\kappa_n^2)+\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\left[-2(\kappa_n-\kappa_{n+k})+\kappa_n^2-\kappa_{n+k}^2\right]\geq\frac{\omega^2}{\tau_t^2+\sigma^2+\omega^2}(\kappa_{n+k}^2-\kappa_n^2),$$

and thus is equivalent to

$$\frac{(1-\Gamma)\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2}\frac{\kappa_n+\kappa_{n+k}}{2}\geq-\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}.$$

This condition clearly holds and thus so does the sufficient condition for $\lim_{\Theta^2\to\infty}\breve{Var}_t[\tilde{\zeta}_t+\tilde{\lambda}_t|\zeta_{it}+\lambda_{it},\tilde{q}_t]\geq\lim_{\Theta^2\to\infty}\breve{Var}_t[\tilde{\zeta}_t+\tilde{\lambda}_t|\zeta_{jt}+\lambda_{jt},\tilde{q}_t]$ for all $j>i$. Therefore from (A-50), the sequence $\{h_{1t},...,h_{Nt}\}$ is monotone increasing as $\Theta^2\to 0$ when sectors are ordered by increasing κ_i. And from the definition (A-25), the sequence $\{\breve{h}_{1t},...,\breve{h}_{Nt}\}$ is monotone increasing as $\Theta^2\to 0$, with $\breve{h}_{1t}\leq 1/N$ and $\breve{h}_{Nt}\geq 1/N$. The latter inequalities are strict if any $\kappa_i\neq 1/N$, which in turn is equivalent to $\kappa_1<1/N$. The third part of the corollary follows from the foregoing, $\lim_{\Theta^2\to\infty}\breve{Z}_{it}=1$, and the definition (A-47).

M Proof of Lemma 2

Define

$$\breve{w}_{it} \triangleq \left(\kappa_i \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_i)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right)\frac{\breve{h}_{it}\breve{Z}_{it}}{\chi_t}.$$

By definitions of χ_t and $\breve{Z}_{it}$,

$$\breve{w}_{it} = \frac{\breve{h}_{it} b_{it}}{\sum_i \breve{h}_{it}[a_{it}+b_{it}]},$$

and hence $\breve{w}_{it} \in (0,1)$ with $\sum_{i=1}^N \breve{w}_{it} < 1$. Before updating for time t observations, the prior covariance matrix for the vector of unknown parameters in (A-45) is

$$V_{\mu,t} = \begin{bmatrix} (1-\Gamma)\tau_t^2 \sum_{i=1}^N \kappa_i^2 + \Gamma\tau_t^2 & (1-\Gamma)\tau_t^2 \sum_{i=1}^N \kappa_i \breve{w}_{it} + \Gamma\tau_t^2 \sum_{i=1}^N \breve{w}_{it} \\ (1-\Gamma)\tau_t^2 \sum_{i=1}^N \kappa_i \breve{w}_{it} + \Gamma\tau_t^2 \sum_{i=1}^N \breve{w}_{it} & (1-\Gamma)\tau_t^2 \sum_{i=1}^N \breve{w}_{it}^2 + \Gamma\tau_t^2 \left(\sum_{i=1}^N \breve{w}_{it}\right)^2 \end{bmatrix},$$

and the covariance matrix for the vector of signals in (A-46) is

$$V_{s,t} = \begin{bmatrix} \sigma^2 \sum_{i=1}^N \kappa_i^2 + \tilde{\omega}^2 & \sigma^2 \sum_{i=1}^N \kappa_i \breve{w}_{it} \\ \sigma^2 \sum_{i=1}^N \kappa_i \breve{w}_{it} & [\sigma^2+\omega^2]\sum_{i=1}^N \breve{w}_{it}^2 + \left(\frac{r}{C_0\alpha(B_t'+b_t'+r)\chi_t}\right)^2 \Theta^2 \end{bmatrix}.$$

By definition, $\Upsilon_{\mu,j} = V_{\mu,j}^{-1}$ and $\Upsilon_{s,j} = V_{s,j}^{-1}$. Define $det_{\mu,j}$ and $det_{s,j}$ as the determinants of $V_{\mu,j}$ and $V_{s,j}$, respectively. These are weakly positive. Recall that $\pi_j \triangleq (\Upsilon_{\mu,j}+\Upsilon_{s,j})^{-1}\Upsilon_{\mu,j}$. Using superscripts to indicate elements, standard matrix inversion and matrix algebra yield:

$$\pi_j = \frac{1}{det_{\mu,j} + det_{s,j} + V_{\mu,j}^{(1,1)}V_{s,j}^{(2,2)} + V_{s,j}^{(1,1)}V_{\mu,j}^{(2,2)} - 2V_{\mu,j}^{(1,2)}V_{s,j}^{(1,2)}} \begin{bmatrix} det_{s,j} + V_{s,j}^{(1,1)}V_{\mu,j}^{(2,2)} - V_{\mu,j}^{(1,2)}V_{s,j}^{(1,2)} & V_{\mu,j}^{(1,1)}V_{s,j}^{(1,2)} - V_{s,j}^{(1,1)}V_{\mu,j}^{(1,2)} \\ V_{\mu,j}^{(2,2)}V_{s,j}^{(1,2)} - V_{s,j}^{(2,2)}V_{\mu,j}^{(1,2)} & det_{s,j} + V_{\mu,j}^{(1,1)}V_{s,j}^{(2,2)} - V_{\mu,j}^{(1,2)}V_{s,j}^{(1,2)} \end{bmatrix}.$$

The leading fraction is the determinant of $V_{\mu,j}+V_{s,j}$ and thus is weakly positive. Substituting, we find that

$$\begin{aligned} V_{\mu,j}^{(1,1)}V_{s,j}^{(1,2)} - V_{s,j}^{(1,1)}V_{\mu,j}^{(1,2)} =& -\tilde{\omega}^2\left[(1-\Gamma)\tau_j^2\sum_{i=1}^N \kappa_i\breve{w}_{ij} + \Gamma\tau_j^2\sum_{i=1}^N \breve{w}_{ij}\right] \\ & - \Gamma\tau_j^2\sigma^2\left(\left[\sum_{i=1}^N \kappa_i^2\right]\left[\sum_{i=1}^N \breve{w}_{ij}\right] - \sum_{i=1}^N \kappa_i\breve{w}_{ij}\right), \end{aligned} \tag{A-51}$$

which goes to 0 as $\tilde{\omega}^2$ and Γ become small. We also find:

$$\begin{aligned}V_{\mu,j}^{(2,2)}V_{s,j}^{(1,2)} - V_{s,j}^{(2,2)}V_{\mu,j}^{(1,2)} =&\Gamma\tau^2\sigma^2\left(\sum_{i=1}^{N}\breve{w}_{ij}\right)\left[\left(\sum_{i=1}^{N}\breve{w}_{ij}\right)\sum_{i=1}^{N}\kappa_i\breve{w}_{ij} - \sum_{i=1}^{N}\breve{w}_{ij}^2\right]\\ &- \left[\omega^2\sum_{i=1}^{N}\breve{w}_{ij}^2 + \left(\frac{r}{C_0\alpha(B_j' + b_j' + r)\chi_j}\right)^2\Theta^2\right]\\ &\left[(1-\Gamma)\tau_j^2\sum_{i=1}^{N}\kappa_i\breve{w}_{ij} + \Gamma\tau_j^2\sum_{i=1}^{N}\breve{w}_{ij}\right].\end{aligned}$$

This goes to 0 as Γ, ω^2, and Θ^2 become small. So π_j becomes approximately diagonal as Γ, ω^2, Θ^2, and $\tilde{\omega}^2$ become small.

Observe that:

$$\begin{aligned}&det_{s,j} + V_{s,j}^{(1,1)}V_{\mu,j}^{(2,2)} - V_{\mu,j}^{(1,2)}V_{s,j}^{(1,2)}\\ =&\left[\sigma^2\sum_{i=1}^{N}\kappa_i^2 + \tilde{\omega}^2\right]\left[[(1-\Gamma)\tau^2 + \sigma^2 + \omega^2]\sum_{i=1}^{N}\breve{w}_{ij}^2 + \Gamma\tau_j^2\left(\sum_{i=1}^{N}\breve{w}_{ij}\right)^2 + \left(\frac{r}{C_0\alpha(B_j' + b_j' + r)\chi_j}\right)^2\Theta^2\right]\\ &- \left[\sigma^2\sum_{i=1}^{N}\kappa_i\breve{w}_{ij}\right]\left[[(1-\Gamma)\tau_j^2 + \sigma^2]\sum_{i=1}^{N}\kappa_i\breve{w}_{ij} + \Gamma\tau_j^2\sum_{i=1}^{N}\breve{w}_{ij}\right]\\ >&\left[\sigma^2\sum_{i=1}^{N}\kappa_i^2\right]\left[[(1-\Gamma)\tau^2 + \sigma^2]\sum_{i=1}^{N}\breve{w}_{ij}^2 + \Gamma\tau_j^2\left(\sum_{i=1}^{N}\breve{w}_{ij}\right)^2\right]\\ &- \left[\sigma^2\sum_{i=1}^{N}\kappa_i\breve{w}_{ij}\right]\left[[(1-\Gamma)\tau_j^2 + \sigma^2]\sum_{i=1}^{N}\kappa_i\breve{w}_{ij} + \Gamma\tau_j^2\sum_{i=1}^{N}\breve{w}_{ij}\right].\end{aligned}$$

As Γ becomes small, the final two lines go to

$$\begin{aligned}&\sigma^2[\tau_j^2 + \sigma^2]\left[\left(\sum_{i=1}^{N}\kappa_i^2\right)\left(\sum_{i=1}^{N}\breve{w}_{ij}^2\right) - \left(\sum_{i=1}^{N}\kappa_i\breve{w}_{ij}\right)^2\right]\\ =&[\tau_j^2 + \sigma^2]\lim_{\Theta^2,\omega^2,\tilde{\omega}^2\to 0} det_{s,j}.\end{aligned}$$

We also have:

$$\begin{aligned}
&det_{s,j} + V_{\mu,j}^{(1,1)}V_{s,j}^{(2,2)} - V_{\mu,j}^{(1,2)}V_{s,j}^{(1,2)}\\
=&\left[[\sigma^2+\omega^2]\sum_{i=1}^{N}\breve{w}_{ij}^2 + \left(\frac{r}{C_0\alpha(B_j'+b_j'+r)\chi_j}\right)^2\Theta^2\right]\left[[(1-\Gamma)\tau_j^2+\sigma^2]\sum_{i=1}^{N}\kappa_i^2 + \Gamma\tau_j^2 + \tilde{\omega}^2\right]\\
&-\left[\sigma^2\sum_{i=1}^{N}\kappa_i\breve{w}_{ij}\right]\left[[(1-\Gamma)\tau_j^2+\sigma^2]\sum_{i=1}^{N}\kappa_i\breve{w}_{ij} + \Gamma\tau_j^2\sum_{i=1}^{N}\breve{w}_{ij}\right]\\
>&\left[\sigma^2\sum_{i=1}^{N}\breve{w}_{ij}^2\right]\left[[(1-\Gamma)\tau_j^2+\sigma^2]\sum_{i=1}^{N}\kappa_i^2 + \Gamma\tau_j^2\right]\\
&-\left[\sigma^2\sum_{i=1}^{N}\kappa_i\breve{w}_{ij}\right]\left[[(1-\Gamma)\tau_j^2+\sigma^2]\sum_{i=1}^{N}\kappa_i\breve{w}_{ij} + \Gamma\tau_j^2\sum_{i=1}^{N}\breve{w}_{ij}\right].
\end{aligned}$$

As Γ becomes small, the final two lines go to

$$\begin{aligned}
=&\left[\sigma^2\sum_{i=1}^{N}\breve{w}_{ij}^2\right]\left[[\tau_j^2+\sigma^2]\sum_{i=1}^{N}\kappa_i^2\right] - \left[\sigma^2\sum_{i=1}^{N}\kappa_i\breve{w}_{ij}\right]\left[[\tau_j^2+\sigma^2]\sum_{i=1}^{N}\kappa_i\breve{w}_{ij}\right]\\
=&[\tau_j^2+\sigma^2]\lim_{\Theta^2,\omega^2,\tilde{\omega}^2\to 0} det_{s,j}.
\end{aligned}$$

Because $det_{s,j} \geq 0$ by properties of covariance matrices, $det_{s,j} + V_{s,j}^{(1,1)}V_{\mu,j}^{(2,2)} - V_{\mu,j}^{(1,2)}V_{s,j}^{(1,2)} > 0$ and $det_{s,j} + V_{\mu,j}^{(1,1)}V_{s,j}^{(2,2)} - V_{\mu,j}^{(1,2)}V_{s,j}^{(1,2)} > 0$. Therefore elements $(1,1)$ and $(2,2)$ of π_j are each between 0 and 1 for Γ sufficiently small.

These results imply that $\prod_{j=k+1}^{t}\pi_j$ is approximately diagonal (in that its off-diagonal elements are arbitarily small), with diagonal elements between 0 and 1, as Γ, ω^2, Θ^2, and $\tilde{\omega}^2$ become small.

To analyze $\tilde{\pi}_k$ and $\breve{\pi}_k$ under these same conditions, it remains to analyze the first row of $(\Upsilon_{\mu,k}+\Upsilon_{s,k})^{-1}\Upsilon_{s,k}$: the product of element $(1,1)$ of $(\Upsilon_{\mu,k}+\Upsilon_{s,k})^{-1}\Upsilon_{s,k}$ and element $(1,1)$ of $\prod_{j=k+1}^{t}\pi_j$ dominates $\tilde{\pi}_k$ as Γ, ω^2, Θ^2, and $\tilde{\omega}^2$ become small, and the product of element $(1,2)$ of $(\Upsilon_{\mu,k}+\Upsilon_{s,k})^{-1}\Upsilon_{s,k}$ and element $(1,1)$ of $\prod_{j=k+1}^{t}\pi_j$ dominates $\breve{\pi}_k$ as Γ, ω^2, Θ^2, and $\tilde{\omega}^2$ become small.

Matrix algebra yields that element $(1,1)$ of $(\Upsilon_{\mu,k}+\Upsilon_{s,k})^{-1}\Upsilon_{s,k}$ is

$$\frac{det_{\mu,k} + V_{\mu,k}^{(1,1)}V_{s,k}^{(2,2)} - V_{\mu,k}^{(1,2)}V_{s,k}^{(1,2)}}{det_{\mu,k} + V_{\mu,k}^{(1,1)}V_{s,k}^{(2,2)} - V_{\mu,k}^{(1,2)}V_{s,k}^{(1,2)} + det_{s,k} + V_{s,k}^{(1,1)}V_{\mu,k}^{(2,2)} - V_{\mu,k}^{(1,2)}V_{s,k}^{(1,2)}}.$$

We already established that the denominator is strictly positive as Γ becomes small and that the final three terms in the denominator sum to a strictly positive number as Γ becomes small. The numerator is

$$\left[(1-\Gamma)\tau_j^2\sum_{i=1}^N \kappa_i^2+\Gamma\tau_j^2\right]\left[[(1-\Gamma)\tau_j^2+\sigma^2+\omega^2]\sum_{i=1}^N \breve{w}_{it}^2+\Gamma\tau_j^2\left(\sum_{i=1}^N \breve{w}_{it}\right)^2+\left(\frac{r}{C_0\alpha(B_t'+b_t'+r)\chi_t}\right)^2\Theta^2\right]$$
$$-\left[(1-\Gamma)\tau_j^2\sum_{i=1}^N \kappa_i\breve{w}_{it}+\Gamma\tau_j^2\sum_{i=1}^N \breve{w}_{it}\right]\left[[(1-\Gamma)\tau_j^2+\sigma^2]\sum_{i=1}^N \kappa_i\breve{w}_{it}+\Gamma\tau_j^2\sum_{i=1}^N \breve{w}_{it}\right]$$
$$>\left[(1-\Gamma)\tau_j^2\sum_{i=1}^N \kappa_i^2+\Gamma\tau_j^2\right]\left[[(1-\Gamma)\tau_j^2+\sigma^2]\sum_{i=1}^N \breve{w}_{it}^2+\Gamma\tau_j^2\left(\sum_{i=1}^N \breve{w}_{it}\right)^2\right]$$
$$-\left[(1-\Gamma)\tau_j^2\sum_{i=1}^N \kappa_i\breve{w}_{it}+\Gamma\tau_j^2\sum_{i=1}^N \breve{w}_{it}\right]\left[[(1-\Gamma)\tau_j^2+\sigma^2]\sum_{i=1}^N \kappa_i\breve{w}_{it}+\Gamma\tau_j^2\sum_{i=1}^N \breve{w}_{it}\right].$$

As Γ becomes small, the final two lines go to

$$\left[\tau_j^2\sum_{i=1}^N \kappa_i^2\right]\left[[\tau_j^2+\sigma^2]\sum_{i=1}^N \breve{w}_{it}^2\right]-\left[\tau_j^2\sum_{i=1}^N \kappa_i\breve{w}_{it}\right]\left[[\tau_j^2+\sigma^2]\sum_{i=1}^N \kappa_i\breve{w}_{it}\right]$$
$$=\frac{\tau_j^2+\sigma^2}{\sigma^2}\frac{\tau_j^2}{\sigma^2}\lim_{\Theta^2,\omega^2,\tilde{\omega}^2\to 0} det_{s,j}. \tag{A-52}$$

Therefore the numerator is strictly positive, which means that element $(1,1)$ of $(\Upsilon_{\mu,k}+\Upsilon_{s,k})^{-1}\Upsilon_{s,k}$ is $\in(0,1)$ for Γ sufficiently small. Because element $(1,1)$ of $\prod_{j=k+1}^t \pi_j$ is $\in(0,1)$ and element $(1,2)$ of $\prod_{j=k+1}^t \pi_j$ is arbitarily small for Γ, ω^2, Θ^2, and $\tilde{\omega}^2$ small, we have established that $\tilde{\pi}_k\in(0,1)$ for $\Gamma,\omega^2,\tilde{\omega}^2,\Theta^2$ sufficiently small.

Element $(1,2)$ of $(\Upsilon_{\mu,j}+\Upsilon_{s,j})^{-1}\Upsilon_{s,j}$ is:

$$\frac{V_{s,k}^{(1,1)}V_{\mu,k}^{(1,2)}-V_{\mu,k}^{(1,1)}V_{s,k}^{(1,2)}}{det_{\mu,k}+V_{\mu,k}^{(1,1)}V_{s,k}^{(2,2)}-V_{\mu,k}^{(1,2)}V_{s,k}^{(1,2)}+det_{s,k}+V_{s,k}^{(1,1)}V_{\mu,k}^{(2,2)}-V_{\mu,k}^{(1,2)}V_{s,k}^{(1,2)}}.$$

From (A-51), the numerator is 0 if both Γ and $\tilde{\omega}^2$ are zero. Therefore as $\Gamma,\tilde{\omega}^2\to 0$ with ω^2 and Θ^2 not too large, $\breve{\pi}_k$ is equal to zero times element $(1,1)$ of $\prod_{j=k+1}^t \pi_j$ plus a term that scales with element $(1,2)$ of $\prod_{j=k+1}^t \pi_j$ and thus is arbitarily small.

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