FEDERAL RESERVE BANK OF SAN FRANCISCO

WORKING PAPER SERIES

Methods for Robust Control

Richard Dennis
Federal Reserve Bank of San Francisco

Kai Leitemo
Norwegian School of Management (BI)

Ulf Söderström
Bocconi University

March 2009

Working Paper 2006-10
http://www.frbsf.org/publications/economics/papers/2006/wp06-10bk.pdf

Methods for Robust Control ⋆

Richard Dennis [a,*], Kai Leitemo [b], Ulf Söderström [c]

[a] *Federal Reserve Bank of San Francisco*

[b] *Norwegian School of Management (BI)*

[c] *Sveriges Riksbank and CEPR*

Abstract

Robust control allows policymakers to formulate policies that guard against model misspecification. The principal tools used to solve robust control problems are state-space methods (see Hansen and Sargent, 2008, and Giordani and Söderlind, 2004). In this paper we show that the structural-form methods developed by Dennis (2007) to solve control problems with rational expectations can also be applied to robust control problems, with the advantage that they bypass the task, often onerous, of having to express the reference model in state-space form. In addition, we show how to implement two different timing assumptions with distinct implications for the robust policy and the economy. We apply our methods to a New Keynesian Dynamic Stochastic General Equilibrium model and find that robustness has important effects on policy and the economy.

Key words: Robust control, Misspecification, Optimal policy
JEL classification: C61, E52, E58

⋆ We are grateful for comments from Paolo Giordani and Paul Söderlind, and from participants at the Society for Economic Dynamics Annual Meeting, Vancouver, 2006, and the Centre for Applied Macroeconomic Analysis workshop on "New Methods of Macroeconomic Modelling," Australian National University, 2005. The second author thanks the Norwegian Financial Market Fund for financial support. The views expressed in this paper are those of the authors and do not necessarily reflect the views of the Federal Reserve Bank of San Francisco, the Federal Reserve System, or the Executive Board of Sveriges Riksbank.

∗ Corresponding author. Address: Economic Research Department, Mail Stop 1130, Federal Reserve Bank of San Francisco, 101 Market Street, San Francisco, CA 94105, USA, Phone No: +1 415 974 2235, Fax No: +1 415 974 2168

Email addresses: richard.dennis@sf.frb.org, kai.leitemo@bi.no, ulf.soderstrom@riksbank.se (Ulf Söderström).

1. Introduction

The precision with which economic models can be expressed mathematically belies the fact that they cannot claim to be anything more than approximations to an unknown, and possibly unknowable, data-generating process. This unfortunate reality means that economic decisions are inevitably made in situations where important aspects of the environment are cloaked, hidden behind a cloud of uncertainty. While such uncertainty is hardly welcome, it need not render decisionmakers powerless, as its effects can in principle be mitigated through the application of robust control methods. Robust control provides a set of tools to assist decisionmakers confronting uncertainty who are either unable or unwilling to specify a probability distribution over possible specification errors. The theory establishing that robust control methods can be applied to economic problems has been developed largely in a series of contributions by Hansen and Sargent, contributions that are well summarized in Hansen and Sargent (2008). Among other things, Hansen and Sargent show how to set up and solve discounted robust control problems, and they develop methods to solve for robust policies in backward-looking models and in forward-looking models with commitment. Giordani and Söderlind (2004) extend these methods to forward-looking models with discretion and to simple rules.

A critical component in the application of robust control is the reference model. A reference model is a structural model, possibly arrived at through some (non-modeled) learning process, that is thought to be a good approximation to the underlying data-generating process. The methods described in Hansen and Sargent (2008) and Giordani and Söderlind (2004) require that this reference model be written in a state-space form, following the literature on traditional (non-robust) optimal control. As discussed in Dennis (2007), while state-space methods allow models to be expressed in a form that contains only first-order dynamics, they also have drawbacks. In particular, many models cannot be expressed easily in a state-space form, especially medium- to large-scale models for which the necessary manipulations are often prohibitive.

In this paper we develop an alternative set of tools to solve robust control problems under commitment, tools based on the solution methods developed by Dennis (2007) that have the advantage that they do not require that the reference model be written in a state-space form. Instead, they allow the reference model to be written in structural form, which is more flexible and generally much easier to attain. We also discuss robust policy under two different timing assumptions. Under the first assumption, policy is set after observing the current realizations of the shocks, so only the conditional means of the shocks are distorted. This assumption coincides with that typically used in the state-space approach. Under the second assumption, policy is instead set before observing the current shocks, capturing the notion that the policymaker may have doubts not only about the reference model but also about the current state of the economy. This assumption implies that both the conditional means and the conditional covariances of the shocks are distorted. While the two timing assumptions give identical results under the non-robust policy, they can have important implications for robust decision problems.

To illustrate how the structural-form solution methods work, we study robust monetary policy in a simple New Keynesian Dynamic Stochastic General Equilibrium (DSGE) model often used for monetary policy analysis. We show that robustness has important implications for monetary policy and the economy, and that there are sometimes large differences between the two timing assumptions.

We begin in Section 2 by describing the standard state-space method to applying robust control and documenting the properties of the resulting equilibria. We then show in Section 3 how robust control problems can be formulated and solved when the model is kept in a structural form rather than expressed in a state-space form. In Section 4 we discuss how detection-error probabilities can be calculated while allowing for distortions to both the conditional means and covariances of the shocks, in order to determine the size of distortions taken into consideration by the robust policymaker. In Section 5 we apply our methods to the example economy, before concluding in Section 6.

2. Robust policymaking using state-space methods

Hansen and Sargent (2008, Ch. 16) characterize the decision problem facing a robust Stackelberg leader who sets policy at some initial date while taking into account the behavior of private-agent followers who make decisions sequentially. In the context of an economy where a monopoly producer facing a competitive fringe has doubts about its model, Hansen and Sargent describe a proposed solution. In this section, we explain how their solution method works and, drawing on Dennis (2008), generalize it to the standard stochastic linear-quadratic framework widely used to analyze non-robust decision problems. As Hansen and Sargent discuss, the key to analyzing robust Stackelberg problems is to cast them in a form whereby they can be solved using the same methods used to solve decision problems involving rational expectations.

We begin by documenting how the leader's concern for robustness, that is, its desire to guard against model misspecification, changes its decision problem from the standard non-robust one. Next, we show how this robust decision problem can be solved to obtain the "worst case" and "approximating" equilibria. With the leader guarding against the fear that its model may be misspecified, the approximating equilibrium for a robust decision problem describes outcomes when the robust policy is implemented, but the reference model is actually not misspecified; it represents the analog of the rational expectations equilibrium for a non-robust decision problem. For its part, the worst-case equilibrium, which describes outcomes according to the worst-case fears of the robust decision maker, can usefully be viewed as a vehicle for obtaining the approximating equilibrium.

2.1. *The reference model and policy objectives*

The framework that we consider contains models that can be written in the form

$$\begin{bmatrix} \mathbf{x}_{t+1} \\ \mathrm{E}_t\mathbf{y}_{t+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \end{bmatrix} + \mathbf{B}\mathbf{u}_t + \begin{bmatrix} \mathbf{C}_\mathbf{x} \\ \mathbf{0} \end{bmatrix} \varepsilon_{\mathbf{x}t+1} \tag{1}$$

where $\mathbf{x}_t$ is an $n_\mathbf{x} \times 1$ vector of predetermined variables, $\mathbf{y}_t$ is an $n_\mathbf{y} \times 1$ vector of non-predetermined variables, $\mathbf{u}_t$ is an $n_\mathbf{u} \times 1$ vector of policy control variables, $\varepsilon_{\mathbf{x}t} \sim i.i.d.\,[\mathbf{0}, \mathbf{I}]$ is an $n_\varepsilon \times 1$ ($n_\varepsilon \leq n_\mathbf{x}$) vector of white-noise innovations, and E_t is the mathematical expectations operator conditional upon period t information. Equation (1) describes the reference model, which is the model that the policymaker and private agents believe best describes the data-generating process.

Absent a fear of misspecification, the problem for the policymaker is to choose the sequence of control variables $\{\mathbf{u}_t\}_0^\infty$ to minimize

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\mathbf{z}_t'\mathbf{W}\mathbf{z}_t + 2\mathbf{z}_t'\mathbf{U}\mathbf{u}_t + \mathbf{u}_t'\mathbf{R}\mathbf{u}_t\right], \tag{2}$$

where $\beta \in (0,1)$ is the policymaker's discount factor and $\mathbf{z}_t \equiv \begin{bmatrix} \mathbf{x}_t' & \mathbf{y}_t' \end{bmatrix}'$, subject to equation (1). The symmetric weighting matrices $\mathbf{W}$ and $\mathbf{R}$ are assumed to be positive semidefinite and positive definite, respectively (Anderson, Hansen, McGrattan, and Sargent, 1996).

Denoting by $\varepsilon_{\mathbf{y}t+1}$ the expectational error $\varepsilon_{\mathbf{y}t+1} \equiv \mathbf{y}_{t+1} - \mathrm{E}_t\mathbf{y}_{t+1}$, and recognizing that in equilibrium these expectation errors will be a linear function of the innovations, $\varepsilon_{\mathbf{y}t} = \mathbf{C}_\mathbf{y}\varepsilon_{\mathbf{x}t}$, where $\mathbf{C}_\mathbf{y}$ has yet to be determined, the reference model can be written in terms of realized values as

$$\begin{bmatrix} \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \end{bmatrix} + \mathbf{B}\mathbf{u}_t + \begin{bmatrix} \mathbf{C}_\mathbf{x} \\ \mathbf{C}_\mathbf{y} \end{bmatrix} \varepsilon_{\mathbf{x}t+1}, \tag{3}$$

or, more compactly, as

$$\mathbf{z}_{t+1} = \mathbf{A}\mathbf{z}_t + \mathbf{B}\mathbf{u}_t + \mathbf{C}\varepsilon_{\mathbf{x}t+1}. \tag{4}$$

2.2. *The distorted model*

The policymaker has doubts about the reference model's adequacy as a description of the data generating process. Specifically, the policymaker fears that the reference model may be misspecified and that a policy optimized to perform well in the reference model might actually produce unintended and unwelcome outcomes. To acknowledge its doubts the policymaker deliberately introduces specification errors, $\mathbf{v}_{t+1}$, which are clocked by the innovations, and surrounds the reference model with a class of models of the form

$$\mathbf{z}_{t+1} = \mathbf{A}\mathbf{z}_t + \mathbf{B}\mathbf{u}_t + \mathbf{C}\left(\mathbf{v}_{t+1} + \varepsilon_{\mathbf{x}t+1}\right). \tag{5}$$

Note that the specification errors, $\mathbf{v}_{t+1}$, which are dated period $t+1$ because they affect outcomes in period $t+1$, are premultiplied by the matrix $\mathbf{C}$, which contains the standard deviations of the innovations. All else equal, therefore, shocks that are more volatile provide greater room for misspecification.

The sequence of specification errors, $\{\mathbf{v}_{t+1}\}_0^\infty$, is constrained to satisfy the boundedness condition

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^{t+1}\mathbf{v}_{t+1}'\mathbf{v}_{t+1} \leq \eta, \tag{6}$$

where $\eta \in [0, \overline{\eta})$ is a robustness parameter that summarizes the policymaker's confidence in the reference model. In the special case that $\eta = 0$, the policymaker is assumed to have complete confidence in the reference model and the non-robust decision problem is restored.

2.3. *The robust commitment problem*

To guard against the specification errors that it fears, the policymaker formulates policy subject to the distorted model with the mind-set that the specification errors will be as damaging as possible, a view that is operationalized through the metaphor that $\{\mathbf{v}_{t+1}\}_0^\infty$ is chosen by a fictitious evil agent whose objectives are diametrically opposed to those of the policymaker. Hansen and Sargent (2001) show that the constraint problem, in which equation (2) is minimized with respect to $\{\mathbf{u}_t\}_0^\infty$ and maximized with respect to $\{\mathbf{v}_{t+1}\}_0^\infty$, subject to equations (5) and (6), can be replaced with an equivalent multiplier problem, in which

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\mathbf{z}_t'\mathbf{W}\mathbf{z}_t + 2\mathbf{z}_t'\mathbf{U}\mathbf{u}_t + \mathbf{u}_t'\mathbf{R}\mathbf{u}_t - \beta\theta\mathbf{v}_{t+1}'\mathbf{v}_{t+1}\right], \tag{7}$$

$\theta \in (\underline{\theta}, \infty)$, is minimized with respect to $\{\mathbf{u}_t\}_0^\infty$ and maximized with respect to $\{\mathbf{v}_{t+1}\}_0^\infty$, subject to equation (5). The multiplier, or robustness parameter, θ, represents the shadow price of a marginal relaxation of the boundedness condition (6). Larger values for θ, which correspond to smaller values of η, signify greater confidence in the adequacy of the reference model. In Section 4, we discuss an entropy-based method for determining θ, following Hansen, Sargent, and Wang (2002) and Anderson, Hansen, and Sargent (2003).

Given a conjecture of $\mathbf{C}_\mathbf{y}$, the Lagrangian for the policymaker's robust decision problem is

$$\begin{aligned} L = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \big[&\mathbf{z}_t'\mathbf{W}\mathbf{z}_t + 2\mathbf{z}_t'\mathbf{U}\mathbf{u}_t + \mathbf{u}_t'\mathbf{R}\mathbf{u}_t - \beta\theta\mathbf{v}_{t+1}'\mathbf{v}_{t+1} \\ &+ 2\lambda_{t+1}'\left(\mathbf{A}\mathbf{z}_t + \mathbf{B}\mathbf{u}_t + \mathbf{C}\mathbf{v}_{t+1} + \mathbf{C}\varepsilon_{\mathbf{x}t+1} - \mathbf{z}_{t+1}\right)\big], \end{aligned} \tag{8}$$

where λ_{t+1} is the vector of Lagrange multipliers associated with equation (5). Differentiating equation (8) with respect to λ_{t+1}, $\mathbf{z}_t$, $\mathbf{u}_t$, and $\mathbf{v}_{t+1}$, the first-order conditions for an optimum, expressed in terms of $\mathbf{x}_t$, $\mathbf{y}_t$ and the Lagrange multipliers $\lambda_{\mathbf{x}t}$ and $\lambda_{\mathbf{y}t}$ are

$$\frac{\partial L}{\partial \lambda_{t+1}} : \mathbf{A}\begin{bmatrix}\mathbf{x}_t \\ \mathbf{y}_t\end{bmatrix} + \mathbf{B}\mathbf{u}_t + \mathbf{C}\mathbf{v}_{t+1} + \begin{bmatrix}\mathbf{C}_\mathbf{x} \\ \mathbf{0}\end{bmatrix}\varepsilon_{\mathbf{x}t+1} - \begin{bmatrix}\mathbf{x}_{t+1} \\ \mathrm{E}_t\mathbf{y}_{t+1}\end{bmatrix} = \mathbf{0}, \tag{9}$$

$$\frac{\partial L}{\partial \mathbf{z}_t} : \mathbf{W}\begin{bmatrix}\mathbf{x}_t \\ \mathbf{y}_t\end{bmatrix} + \mathbf{U}\mathbf{u}_t + \mathbf{A}'\mathrm{E}_t\begin{bmatrix}\lambda_{\mathbf{x}t+1} \\ \lambda_{\mathbf{y}t+1}\end{bmatrix} - \beta^{-1}\begin{bmatrix}\lambda_{\mathbf{x}t} \\ \lambda_{\mathbf{y}t}\end{bmatrix} = \mathbf{0}, \tag{10}$$

$$\frac{\partial L}{\partial \mathbf{u}_t} : \mathbf{U}'\begin{bmatrix}\mathbf{x}_t \\ \mathbf{y}_t\end{bmatrix} + \mathbf{R}\mathbf{u}_t + \mathbf{B}'\mathrm{E}_t\begin{bmatrix}\lambda_{\mathbf{x}t+1} \\ \lambda_{\mathbf{y}t+1}\end{bmatrix} = \mathbf{0}, \tag{11}$$

$$\frac{\partial L}{\partial \mathbf{v}_{t+1}} : -\beta\theta\mathbf{v}_{t+1} + \mathbf{C}'\mathrm{E}_t\begin{bmatrix}\lambda_{\mathbf{x}t+1} \\ \lambda_{\mathbf{y}t+1}\end{bmatrix} = \mathbf{0}, \tag{12}$$

which hold for all $t \geq 0$, with the initial conditions $\mathbf{x}_0$ known and $\lambda_{\mathbf{y}0} = \mathbf{0}$. As Kydland and Prescott (1980) and Currie and Levine (1985, 1993) showed, since $\mathbf{y}_t$ is nonpredetermined, the Lagrange multipliers $\lambda_{\mathbf{y}t}$ which are predetermined, enter the solution as auxiliary state variables. These Lagrange multipliers encode the policy's history dependence, a history dependence arising from the policymaker's commitment to its robust policy.

Equations (9) through (12) can be solved in a variety of ways (see Anderson, Hansen, McGrattan, and Sargent, 1996). However they are solved, on the stable manifold, the laws of motion for the state variables have the form

$$\lambda_{\mathbf{y}t+1} = \mathbf{M}^W_{\lambda\lambda}\lambda_{\mathbf{y}t} + \mathbf{M}^W_{\lambda\mathbf{x}}\mathbf{x}_t + \mathbf{N}^W_{\lambda}\varepsilon_{\mathbf{x}t+1}, \tag{13}$$

$$\mathbf{x}_{t+1} = \mathbf{M}^W_{\mathbf{x}\lambda}\lambda_{\mathbf{y}t} + \mathbf{M}^W_{\mathbf{xx}}\mathbf{x}_t + \mathbf{N}^W_{\mathbf{x}}\varepsilon_{\mathbf{x}t+1}, \tag{14}$$

while the decision rules are given by

$$\mathbf{y}_t = \mathbf{H}^W_{\lambda}\lambda_{\mathbf{y}t} + \mathbf{H}^W_{\mathbf{x}}\mathbf{x}_t, \tag{15}$$

$$\mathbf{u}_t = \mathbf{F}^W_{\lambda}\lambda_{\mathbf{y}t} + \mathbf{F}^W_{\mathbf{x}}\mathbf{x}_t, \tag{16}$$

$$\mathbf{v}_{t+1} = \mathbf{K}^W_{\lambda}\lambda_{\mathbf{y}t} + \mathbf{K}^W_{\mathbf{x}}\mathbf{x}_t. \tag{17}$$

In light of equations (13) through (15), the conjecture of $\mathbf{C_y}$ can be revised according to $\mathbf{C_y} \leftarrow \left(\mathbf{H}^W_{\lambda}\mathbf{N}^W_{\lambda} + \mathbf{H}^W_{\mathbf{x}}\mathbf{N}^W_{\mathbf{x}}\right)$, providing the basis for an iterative procedure that, upon convergence, yields the worst-case equilibrium. From this worst-case equilibrium, the approximating equilibrium can be obtained; it is described by equations (13), (15), and (16) with the law of motion for the predetermined variables given by

$$\begin{aligned}\mathbf{x}_{t+1} &= \left(\mathbf{A_{xy}}\mathbf{H}^W_{\lambda} + \mathbf{B_x}\mathbf{F}^W_{\lambda}\right)\lambda_{\mathbf{y}t} + \left(\mathbf{A_{xx}} + \mathbf{A_{xy}}\mathbf{H}^W_{\mathbf{x}} + \mathbf{B_x}\mathbf{F}^W_{\mathbf{x}}\right)\mathbf{x}_t + \mathbf{C_x}\varepsilon_{\mathbf{x}t+1}, \\ &\equiv \mathbf{M}^A_{\mathbf{x}\lambda}\lambda_{\mathbf{y}t} + \mathbf{M}^A_{\mathbf{xx}}\mathbf{x}_t + \mathbf{N}^A_{\mathbf{x}}\varepsilon_{\mathbf{x}t+1},\end{aligned} \tag{18}$$

where $\mathbf{A}$ and $\mathbf{B}$ have been partitioned conformably with $\mathbf{x}_t$, $\mathbf{y}_t$, and $\mathbf{u}_t$.

Once obtained, the approximating equilibrium can be used to construct impulse responses, to perform a variance decomposition, or to build up a likelihood function to be used for estimation and inference, analogous to a rational expectations equilibrium.

3. Robust policymaking using structural-form methods

The solution procedure described above requires that the reference model be written in a state-space form. For many models, however, obtaining a state-space representation can be a lengthy and complicated process, one that opens the door to error. The difficulties are compounded in the context of robust decisionmaking because some transformations involving the shocks and others involving the expectations operator cannot be employed, as illustrated in Section 5. In this section, we reconsider the problem facing a robust policymaker who sets policy with commitment while fearing model misspecification. Like the previous section, we consider models that are linear and objectives that are quadratic. Unlike the previous section, we follow Dennis (2007) and allow the model to be written in a second-order structural form rather than in a state-space form. Since it is often difficult to manipulate even medium scale models into a state-space form, the techniques we describe enable robust decisionmaking to be applied to larger, more sophisticated, models.

We consider robust decisionmaking under two distinct decisionmaking environments. In the first environment the policymaker and private agents make decisions and determine how they will respond to shocks *after* observing the shocks. This environment is consistent with Hansen and Sargent (2008, Ch. 16) and the analysis in Section 2. As an alternative, we also consider an environment in which the policymaker and private agents make decisions *prior* to observing the shocks; here the shocks can be thought of as latent variables that are observed with a one-period delay. [1] This alternative environment is motivated by the idea that, in addition to doubts about their reference model, agents can have doubts about their knowledge of the state variables, doubts justified by the fact that data are often observed with a lag and/or get revised. Although unimportant for (linear-quadratic) non-robust decision problems, this timing assumption does have implications for robust decision problems. Specifically, where the specification errors distort just the conditional means of the shocks when decisions are made *after* observing the shocks, they distort both the conditional means and the conditional covariances of the shocks when decisions are made *prior* to observing the shocks.

3.1. *The reference model and policy objectives*

Let the reference model be described by

$$\mathbf{A}_0\mathbf{y}_t = \mathbf{A}_1\mathbf{y}_{t-1} + \mathbf{A}_2\mathrm{E}_t\mathbf{y}_{t+1} + \mathbf{A}_3\mathbf{u}_t + \mathbf{A}_4\varepsilon_t + \mathbf{A}_5\varepsilon_{t+1}, \tag{19}$$

where $\mathbf{y}_t$ is an $n \times 1$ vector of endogenous variables, $\mathbf{u}_t$ is a $n_\mathbf{u} \times 1$ vector of policy instruments, and ε_t is an $n_\varepsilon \times 1$ ($n_\varepsilon \leq n$) vector of innovations. [2] The matrices $\mathbf{A}_0$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$, and $\mathbf{A}_5$ have dimensions conformable with $\mathbf{y}_t$, $\mathbf{u}_t$, and ε_t, as necessary, and the matrix $\mathbf{A}_0$ is assumed to be nonsingular. We assume that the shocks, denoted $\mathbf{s}_t$, reside at the top of the system and that their evolution is governed by the process

$$\mathbf{s}_t = \Phi\mathbf{s}_{t-1} + \Omega\varepsilon_t, \tag{20}$$

where $|\Phi| < 1$ and the innovations are distributed according to $\varepsilon_t \sim i.i.d.\,[\mathbf{0}, \mathbf{I}]$.

Under the assumption that agents make their decisions after observing $\mathbf{s}_t$, the timing of equation (20) is advanced by one period and $\mathbf{s}_t$ is included within $\mathbf{y}_{t-1}$; then $\mathbf{A}_4 = \mathbf{0}$ and $\mathbf{A}_5 = \begin{bmatrix} \Omega' & \mathbf{0}' \end{bmatrix}'$. Alternatively, under the assumption that agents make their decisions prior to observing $\mathbf{s}_t$, the timing of equation (20) is left unchanged and $\mathbf{s}_{t-1}$ is included within $\mathbf{y}_{t-1}$; then $\mathbf{A}_4 = \begin{bmatrix} \Omega' & \mathbf{0}' \end{bmatrix}'$ and $\mathbf{A}_5 = \mathbf{0}$.

The policy objective function is taken to be

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\mathbf{y}_t'\mathbf{W}\mathbf{y}_t + \mathbf{u}_t'\mathbf{R}\mathbf{u}_t\right], \tag{21}$$

---

[1] Hansen, Sargent, and Wang (2002) consider robust decisionmaking in an environment where the state is partially unobserved. In their framework, however, the full state is never observed.

[2] We recycle some notation used in Section 2 (and also across the two distinct decisionmaking environments) where no confusion is likely to occur.

where $\mathbf{W}$ and $\mathbf{R}$ are matrices containing policy weights that, as earlier, are symmetric positive semidefinite, and symmetric positive definite, respectively. [3]

3.2. *The distorted model*

The policymaker fears that the reference model in equation (19) may be misspecified, distorted by specification errors. Thus, rather than residing in the reference model, the policymaker fears that itself and private agents actually reside within the distorted model. To obtain the distorted model, we first introduce the expectational errors, $\varepsilon_{\mathbf{y}t+1} \equiv \mathbf{y}_{t+1} - \mathrm{E}_t \mathbf{y}_{t+1}$, which will be a linear function of the innovations in equilibrium, $\varepsilon_{\mathbf{y}t+1} = \mathbf{C}\varepsilon_{t+1}$, and write equation (19) in terms of realizations as

$$\mathbf{A}_0 \mathbf{y}_t = \mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{A}_2 \mathbf{y}_{t+1} + \mathbf{A}_3 \mathbf{u}_t + \mathbf{A}_4 \varepsilon_t + (\mathbf{A}_5 - \mathbf{A}_2 \mathbf{C})\, \varepsilon_{t+1}, \tag{22}$$

where the matrix $\mathbf{C}$ has yet to be determined. Next, reflecting the policymaker's concern for misspecification, we surround equation (22) with a class of distorted models of the form

$$\mathbf{A}_0 \mathbf{y}_t = \mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{A}_2 \mathbf{y}_{t+1} + \mathbf{A}_3 \mathbf{u}_t + \mathbf{A}_4 (\mathbf{v}_t + \varepsilon_t) + (\mathbf{A}_5 - \mathbf{A}_2 \mathbf{C})(\mathbf{v}_{t+1} + \varepsilon_{t+1}), \tag{23}$$

where $\mathbf{v}_t$ is an $n_\varepsilon \times 1$ vector of specification errors that is intertemporally constrained to satisfy

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbf{v}_t' \mathbf{v}_t \leq \eta, \tag{24}$$

where $\eta \in [0, \overline{\eta})$. As earlier, smaller values for η imply greater confidence in the reference model.

3.3. *The robust commitment problem*

To guard against the specification errors that it fears, the policymaker formulates policy subject to the distorted model with the view that the specification errors will be as damaging as possible. Thus, the policymaker's robust decision problem is for it to choose $\{\mathbf{u}_t\}_0^\infty$ to minimize and for a fictitious evil agent to choose $\{\mathbf{v}_t\}_0^\infty$ to maximize

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\mathbf{y}_t' \mathbf{W} \mathbf{y}_t + \mathbf{u}_t' \mathbf{R} \mathbf{u}_t - \theta \mathbf{v}_t' \mathbf{v}_t\right], \tag{25}$$

subject to the distorted model in equation (23). As in Section 2, the multiplier $\theta \in (\underline{\theta}, \infty)$ is inversely related to η, and in the limit as $\theta \uparrow \infty$, the specification errors become increasingly constrained and the robust decision problem converges to the non-robust decision problem.

Given a conjecture of $\mathbf{C}$, the Lagrangian for the robust decision problem is given by

[3] Penalty terms on the interaction between $\mathbf{y}_t$ and $\mathbf{u}_t$ could be included, but are unnecessary because such terms can be accommodated through a suitable construction of $\mathbf{y}_t$.

$$
\begin{aligned}
L = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \mathbf{y}_t' \mathbf{W} \mathbf{y}_t + \mathbf{u}_t' \mathbf{R} \mathbf{u}_t - \theta \mathbf{v}_t' \mathbf{v}_t \\
& + 2\lambda_t \Big[\mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{A}_2 \mathbf{y}_{t+1} + \mathbf{A}_3 \mathbf{u}_t + \mathbf{A}_4 \left(\mathbf{v}_t + \varepsilon_t \right) \\
& + \left(\mathbf{A}_5 - \mathbf{A}_2 \mathbf{C} \right) \left(\mathbf{v}_{t+1} + \varepsilon_{t+1} \right) - \mathbf{A}_0 \mathbf{y}_t \Big] \Bigg\},
\end{aligned} \tag{26}
$$

where the vector λ_t contains the Lagrange multipliers on equation (23).[4]

3.4. *Shocks observed → decisionmaking → actions*

We first focus on the case where the shocks, $\mathbf{s}_t$, are observed prior to decisions being made. In this case, $\mathbf{s}_t$ enters $\mathbf{y}_{t-1}$, $\mathbf{A}_4 = \mathbf{0}$, and the policymaker and private agents have full confidence in their knowledge of the current state variables. The first order conditions of the Lagrangian (26) with respect to λ_t, $\mathbf{y}_t$, $\mathbf{u}_t$, and $\mathbf{v}_{t+1}$ are

$$
\frac{\partial L}{\partial \lambda_t} : \mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{A}_2 \mathrm{E}_t \mathbf{y}_{t+1} + \mathbf{A}_3 \mathbf{u}_t + \left(\mathbf{A}_5 - \mathbf{A}_2 \mathbf{C} \right) \mathbf{v}_{t+1} + \mathbf{A}_5 \varepsilon_{t+1} - \mathbf{A}_0 \mathbf{y}_t = \mathbf{0}, \tag{27}
$$

$$
\frac{\partial L}{\partial \mathbf{y}_t} : \mathbf{W} \mathbf{y}_t + \beta \mathbf{A}_1' \mathrm{E}_t \lambda_{t+1} + \beta^{-1} \mathbf{A}_2' \lambda_{t-1} - \mathbf{A}_0' \lambda_t = \mathbf{0}, \tag{28}
$$

$$
\frac{\partial L}{\partial \mathbf{u}_t} : \mathbf{R} \mathbf{u}_t + \mathbf{A}_3' \lambda_t = \mathbf{0}, \tag{29}
$$

$$
\frac{\partial L}{\partial \mathbf{v}_{t+1}} : -\beta \theta \mathbf{v}_{t+1} + \left(\mathbf{A}_5 - \mathbf{A}_2 \mathbf{C} \right)' \lambda_t = \mathbf{0}. \tag{30}
$$

Solving equations (27) through (30) returns the solution

$$
\lambda_t = \mathbf{M}_{\lambda\lambda}^W \lambda_{t-1} + \mathbf{M}_{\lambda \mathbf{y}}^W \mathbf{y}_{t-1}, \tag{31}
$$

$$
\mathbf{y}_t = \mathbf{M}_{\mathbf{y}\lambda}^W \lambda_{t-1} + \mathbf{M}_{\mathbf{yy}}^W \mathbf{y}_{t-1} + \mathbf{N}_{\mathbf{y}}^W \varepsilon_{t+1}, \tag{32}
$$

$$
\mathbf{u}_t = \mathbf{F}_\lambda^W \lambda_{t-1} + \mathbf{F}_{\mathbf{y}}^W \mathbf{y}_{t-1}, \tag{33}
$$

$$
\mathbf{v}_{t+1} = \mathbf{K}_\lambda^W \lambda_{t-1} + \mathbf{K}_{\mathbf{y}}^W \mathbf{y}_{t-1}. \tag{34}
$$

To obtain the worst-case equilibrium, we update $\mathbf{C}$ according to $\mathbf{C} \leftarrow \mathbf{M}_{\mathbf{yy}}^W \mathbf{S}$, where $\mathbf{S}$ is the $n \times n_\varepsilon$ selection matrix that picks out the columns of $\mathbf{M}_{\mathbf{yy}}^W$ associated with the shocks (the first n_ε columns when the shocks are ordered at the top of $\mathbf{y}_t$), iterating over equations (27) through (34) until a fix-point is reached. Note that since $\mathbf{s}_t$ is included in $\mathbf{y}_{t-1}$ (which is why $\mathbf{y}_t$ depends on ε_{t+1}) all of the variables in equations (31) through (34) respond to $\mathbf{s}_t$ and hence to ε_t.

Given the worst-case equilibrium, which we can write as

[4] Although the reference model (19) contains both ε_t and ε_{t+1}, we view it as an encompassing specification where in any given application either $\mathbf{A}_4$ or $\mathbf{A}_5$ will equal $\mathbf{0}$. The evil agent will thus choose $\mathbf{v}_{t+1}$ under the first timing assumption and $\mathbf{v}_t$ under the second, but in each case the specification errors are chosen at t.

$$\mathbf{z}_t = \mathbf{M}^W \mathbf{z}_{t-1} + \mathbf{N}^W \varepsilon_{t+1}, \tag{35}$$
$$\mathbf{u}_t = \mathbf{F}_{\mathbf{z}} \mathbf{z}_{t-1}, \tag{36}$$
$$\mathbf{v}_{t+1} = \mathbf{K}_{\mathbf{z}} \mathbf{z}_{t-1}, \tag{37}$$

where $\mathbf{z}_t \equiv \begin{bmatrix} \lambda_t' & \mathbf{y}_t' \end{bmatrix}'$, the approximating equilibrium can be obtained by solving equation (19) (with $\mathbf{A}_4 = \mathbf{0}$) jointly with equations (31) and (33), and can be written as

$$\mathbf{z}_t = \mathbf{M}^A \mathbf{z}_{t-1} + \mathbf{N}^A \varepsilon_{t+1}, \tag{38}$$
$$\mathbf{u}_t = \mathbf{F}_{\mathbf{z}} \mathbf{z}_{t-1}. \tag{39}$$

Importantly, since the innovations ε_{t+1} are neither observed nor realized in period t it holds that $\mathbf{N}^A = \mathbf{N}^W = \begin{bmatrix} \mathbf{0}' & \left(\mathbf{A}_0^{-1}\mathbf{A}_5\right)' \end{bmatrix}'$. As a consequence, while distorting their conditional means, the worst-case specification errors do not distort the conditional covariances of the shocks.

3.5. *Decisionmaking → shocks observed → actions*

We now turn to the environment where decisionmaking occurs prior to the shocks being observed. As discussed earlier, when decisions are made prior to the shocks being observed the policymaker's fears of model misspecification also manifest themselves in the form of uncertainty about the current state of the economy. With this timing assumption, $\mathbf{s}_{t-1}$ enters $\mathbf{y}_{t-1}$, $\mathbf{A}_5 = \mathbf{0}$, and the first order conditions of the Lagrangian (26) with respect to λ_t, $\mathbf{y}_t$, $\mathbf{u}_t$, and $\mathbf{v}_t$ are

$$\frac{\partial L}{\partial \lambda_t} : \mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{A}_2 \mathrm{E}_t \mathbf{y}_{t+1} + \mathbf{A}_3 \mathbf{u}_t + \mathbf{A}_4 \left(\mathbf{v}_t + \varepsilon_t\right) - \mathbf{A}_2 \mathbf{C} \mathrm{E}_t \mathbf{v}_{t+1} - \mathbf{A}_0 \mathbf{y}_t = \mathbf{0}, \tag{40}$$
$$\frac{\partial L}{\partial \mathbf{y}_t} : \mathbf{W}\mathbf{y}_t + \beta \mathbf{A}_1' \mathrm{E}_t \lambda_{t+1} + \beta^{-1} \mathbf{A}_2' \lambda_{t-1} - \mathbf{A}_0' \lambda_t = \mathbf{0}, \tag{41}$$
$$\frac{\partial L}{\partial \mathbf{u}_t} : \mathbf{R}\mathbf{u}_t + \mathbf{A}_3' \lambda_t = \mathbf{0}, \tag{42}$$
$$\frac{\partial L}{\partial \mathbf{v}_t} : -\theta \mathbf{v}_t + \mathbf{A}_4' \lambda_t - \beta^{-1} \left(\mathbf{A}_2 \mathbf{C}\right)' \lambda_{t-1} = \mathbf{0}. \tag{43}$$

Solving equations (40) through (43) yields the solution

$$\lambda_t = \mathbf{M}^W_{\lambda\lambda} \lambda_{t-1} + \mathbf{M}^W_{\lambda\mathbf{y}} \mathbf{y}_{t-1} + \mathbf{N}^W_{\lambda} \varepsilon_t, \tag{44}$$
$$\mathbf{y}_t = \mathbf{M}^W_{\mathbf{y}\lambda} \lambda_{t-1} + \mathbf{M}^W_{\mathbf{y}\mathbf{y}} \mathbf{y}_{t-1} + \mathbf{N}^W_{\mathbf{y}} \varepsilon_t, \tag{45}$$
$$\mathbf{u}_t = \mathbf{F}^W_{\lambda} \lambda_{t-1} + \mathbf{F}^W_{\mathbf{y}} \mathbf{y}_{t-1} + \mathbf{F}^W_{\varepsilon} \varepsilon_t, \tag{46}$$
$$\mathbf{v}_t = \mathbf{K}^W_{\lambda} \lambda_{t-1} + \mathbf{K}^W_{\mathbf{y}} \mathbf{y}_{t-1} + \mathbf{K}^W_{\varepsilon} \varepsilon_t. \tag{47}$$

In this environment, to obtain the worst-case equilibrium we update $\mathbf{C}$ according to $\mathbf{C} \leftarrow \mathbf{N}^W_{\mathbf{y}}$ and iterate over equations (40) through (47) until a fix-point is reached. The worst-case equilibrium can be written as

$$\mathbf{z}_t = \mathbf{M}^W \mathbf{z}_{t-1} + \mathbf{N}^W \varepsilon_t, \tag{48}$$

$$\mathbf{u}_t = \mathbf{F}_{\mathbf{z}}\mathbf{z}_{t-1} + \mathbf{F}_{\varepsilon}\varepsilon_t, \tag{49}$$

$$\mathbf{v}_t = \mathbf{K}_{\mathbf{z}}\mathbf{z}_{t-1} + \mathbf{K}_{\varepsilon}\varepsilon_t. \tag{50}$$

As earlier, the approximating equilibrium, which has the form,

$$\mathbf{z}_t = \mathbf{M}^A\mathbf{z}_{t-1} + \mathbf{N}^A\varepsilon_t, \tag{51}$$

$$\mathbf{u}_t = \mathbf{F}_{\mathbf{z}}\mathbf{z}_{t-1} + \mathbf{F}_{\varepsilon}\varepsilon_t, \tag{52}$$

can be obtained by solving equation (19) (with $\mathbf{A}_5 = \mathbf{0}$) jointly with equations (43) and (46). Unlike in Section 3.4, here $\mathbf{N}^W$ need not equal $\mathbf{N}^A$, implying that the policymaker's fears can distort both the conditional means and the conditional covariances of the shocks.

4. Detection-error probabilities

Anderson, Hansen, and Sargent (2003) describe the concept of a detection-error probability and introduce it as a tool for calibrating θ, the multiplier on the misspecification constraint, which would otherwise be a free parameter. A detection-error probability is the probability that an econometrician observing equilibrium outcomes would make an incorrect inference about whether the approximating equilibrium or the worst-case equilibrium generated the data. The intuitive connection between θ and the probability of making a detection error is that when θ is small, greater differences between the distorted model and the reference model (more severe misspecifications) can arise, which are more easily detected. In this section, we extend the detection-error approach to calibrating θ to the case where the specification errors distort both the conditional means and the conditional covariances of the shocks.

Let A and W denote two models. With a prior that assigns equal weight to each model, Hansen, Sargent, and Wang (2002) show that detection-error probabilities are calculated according to

$$p(\theta) = \frac{\text{prob}(A|W) + \text{prob}(W|A)}{2}, \tag{53}$$

where prob$(A|W)$ (prob$(W|A)$) represents the probability that the econometrician erroneously chooses model A (model W) when in fact model W (model A) generated the data. Let model A denote the approximating model and model W denote the worst-case model, then any sequence of specification errors that satisfies the boundedness condition in equation (24) will be at least as difficult to distinguish from the approximating model as is a sequence that satisfies equation (24) with equality. As such, $p(\theta)$ represents a lower bound on the probability of making a detection error.

To calculate a detection-error probability we require a description of how the econometrician goes about choosing one model over another. Hansen, Sargent, and Wang (2002) assume that this model selection is based on the likelihood ratio principle. Let $\{\mathbf{z}_t^W\}_1^T$ denote a finite sequence of economic outcomes generated according to the worst-case equilibrium, model W, and let L_{AW} and L_{WW} denote the likelihood associated with models A and W, respectively. Then the econometrician chooses model A over model W if $\log(L_{WW}/L_{AW}) < 0$. Generating M independent sequences $\{\mathbf{z}_t^W\}_1^T$, prob$(A|W)$ can be calculated according to

$$\text{prob}\,(A|W) \approx \frac{1}{M}\sum_{m=1}^{M} \text{I}\left[\log\left(\frac{L_{WW}^{m}}{L_{AW}^{m}}\right) < 0\right], \tag{54}$$

where $\text{I}[\log(L_{WW}^{m}/L_{AW}^{m}) < 0]$ is the indicator function that equals one when its argument is satisfied and equals zero otherwise; prob$(W|A)$ is calculated analogously using draws generated from the approximating model. The likelihood function that is generally used to calculate prob$(A|W)$ and prob$(W|A)$ assumes that the innovations are normally distributed.

Although the theory of detection does not require that the evil agent distort only the conditional means of the innovations, and not the conditional covariances, existing methods to calculate detection-error probabilities do (see Hansen, Sargent, and Wang, 2002, for example). To calculate detection-error probabilities while accounting for distortions to both the conditional means and the conditional covariances of the shocks, let

$$\mathbf{z}_t^A = \mathbf{M}^A\mathbf{z}_{t-1}^A + \mathbf{N}^A\varepsilon_t, \tag{55}$$
$$\mathbf{z}_t^W = \mathbf{M}^W\mathbf{z}_{t-1}^W + \mathbf{N}^W\varepsilon_t, \tag{56}$$

govern equilibrium outcomes under the approximating equilibrium and the worst-case equilibrium, respectively. When $\mathbf{N}^A \neq \mathbf{N}^W$, to calculate $p(\theta)$ we must first allow for the stochastic singularity that generally characterizes equilibrium, and second account appropriately for the Jacobian of transformation that enters the likelihood function. Using the QR decomposition we decompose $\mathbf{N}^A$ according to $\mathbf{N}^A = \mathbf{Q}_A\mathbf{R}_A$ and $\mathbf{N}^W$ according to $\mathbf{N}^W = \mathbf{Q}_W\mathbf{R}_W$. By construction, $\mathbf{Q}_A$ and $\mathbf{Q}_W$ are orthogonal matrices and $\mathbf{R}_A$ and $\mathbf{R}_W$ are upper triangular. Let

$$\widehat{\varepsilon}_t^{i|j} = \mathbf{R}_i^{-1}\mathbf{Q}_i'\left(\mathbf{z}_t^j - \mathbf{M}^i\mathbf{z}_{t-1}^j\right), \quad \{i,j\} \in \{A, W\}, \tag{57}$$

represent the inferred innovations in period t when model i is fitted to data $\{\mathbf{z}_t^j\}_1^T$ that are generated according to model j and let $\widehat{\Sigma}^{i|j}$ be the associated estimates of the innovation variance-covariance matrices. Then

$$\log\left(\frac{L_{AA}}{L_{WA}}\right) = \log\left|\mathbf{R}_A^{-1}\right| - \log\left|\mathbf{R}_W^{-1}\right| + \frac{1}{2}\text{tr}\left(\widehat{\Sigma}^{W|A} - \widehat{\Sigma}^{A|A}\right), \tag{58}$$
$$\log\left(\frac{L_{WW}}{L_{AW}}\right) = \log\left|\mathbf{R}_W^{-1}\right| - \log\left|\mathbf{R}_A^{-1}\right| + \frac{1}{2}\text{tr}\left(\widehat{\Sigma}^{A|W} - \widehat{\Sigma}^{W|W}\right), \tag{59}$$

where "tr" is the trace operator.

When $\mathbf{N}^A = \mathbf{N}^W$ (which is the case when the distortions affect only the conditional means of the shocks) it follows that $\mathbf{R}_A = \mathbf{R}_W$ and the Jacobian of transformations associated with the various likelihoods cancel and play no role in the calculations. Consequently, equations (58) and (59) simplify to

$$\log\left(\frac{L_{AA}}{L_{WA}}\right) = \frac{1}{2}\text{tr}\left(\widehat{\Sigma}^{W|A} - \widehat{\Sigma}^{A|A}\right), \tag{60}$$
$$\log\left(\frac{L_{WW}}{L_{AW}}\right) = \frac{1}{2}\text{tr}\left(\widehat{\Sigma}^{A|W} - \widehat{\Sigma}^{W|W}\right), \tag{61}$$

which are equivalent to the expressions that Hansen, Sargent, and Wang (2002) and Hansen and Sargent (2008, Ch. 9) employ. Given equations (58) and (59), equation

(54) is used to estimate prob$(A|W)$ and prob$(W|A)$, which are needed to construct the detection-error probability, as per equation (53). The multiplier θ is then determined by selecting a detection-error probability (or at least its lower bound) and inverting equation (53). Generally this inversion is performed numerically by constructing the mapping between θ and the detection-error probability for a given sample size.

5. Robust monetary policy: An application

We now study the effects of robustness from the perspective of a central bank that has doubts about its model. We analyze a New Keynesian Dynamic Stochastic General Equilibrium (DSGE) model of the type commonly used for modern monetary policy analysis. This model model is difficult to analyze using state-space methods, as consumption habits introduce expectations of future shocks. The model is however easily analyzed using the structural-form solution methods described above.

The model includes three types of agents: firms, households, and a central bank. Firms produce differentiated goods in a monopolistically competitive environment using labor as the only production factor and set prices to maximize profits subject to a downward-sloping demand curve. Following Calvo (1983), firms set prices in a staggered fashion, so only a subset of firms set their price optimally in every period, and as in, for instance, Smets and Wouters (2003), the remaining firms index their price to past inflation. Let $\widehat{\pi}_t$ denote the one-period rate of inflation, mc_t denote real marginal cost, $\beta \in (0,1)$ denote the subjective discount factor, $1-\xi \in (0,1)$ denote the probability for a firm to reoptimize its price in a given period, $\omega \in [0,1]$ denote the degree of indexation, and $\varepsilon_{p,t}$ denote an exogenous markup shock, that is, a disturbance to the elasticity of substitution across different varieties of goods. Log-linearizing the definition for the aggregate price level and the first order condition for optimal price setting around a steady state with zero inflation yields the Phillips curve relationship

$$\widehat{\pi}_t = \frac{\omega}{1+\omega\beta}\widehat{\pi}_{t-1} + \frac{\beta}{1+\omega\beta}\mathrm{E}_t\widehat{\pi}_{t+1} + \frac{(1-\beta\xi)(1-\xi)}{(1+\omega\beta)\xi}mc_t + \varepsilon_{p,t}. \tag{62}$$

Households choose consumption, labor supply, and holdings of a one-period nominal bond to maximize the expected present value of a utility function that is additively separable in consumption and leisure. In order to capture inertia in consumption, the utility function allows for internal habit formation, so households value consumption relative to their past consumption. Let y_t denote aggregate output, $\widehat{\imath}_t$ denote the one-period nominal interest rate, $\sigma > 0$ denote the coefficient of relative risk aversion, $\gamma \in [0,1)$ quantify the importance of habits, and $\varepsilon_{b,t}$ denote an exogenous preference shock. The optimal intertemporal consumption decision coupled with the resource constraint (which equates output to consumption) then implies that aggregate output follows the Euler equation

$$\begin{aligned} y_t = {} & \frac{\gamma}{1+\gamma+\gamma^2\beta}y_{t-1} + \frac{1+\gamma\beta+\gamma^2\beta}{1+\gamma+\gamma^2\beta}\mathrm{E}_t y_{t+1} - \frac{\gamma\beta}{1+\gamma+\gamma^2\beta}\mathrm{E}_t y_{t+2} \\ & - \frac{1-\gamma}{\sigma(1+\gamma+\gamma^2\beta)}\left[(1-\gamma\beta)(\widehat{\imath}_t - \mathrm{E}_t\widehat{\pi}_{t+1}) - \varepsilon_{b,t} + (1+\gamma\beta)\mathrm{E}_t\varepsilon_{b,t+1} - \gamma\beta\mathrm{E}_t\varepsilon_{b,t+2}\right], \end{aligned} \tag{63}$$

see Levin, Onatski, Williams, and Williams (2005).

To a first-order log approximation, firms' real marginal cost, mc_t, obeys

$$mc_t = \left[\chi + \frac{\sigma\left(1+\beta\gamma^2\right)}{(1-\gamma)(1-\beta\gamma)}\right] y_t - \frac{\sigma\gamma}{(1-\gamma)(1-\beta\gamma)} y_{t-1} \tag{64}$$
$$- \frac{\sigma\beta\gamma}{(1-\gamma)(1-\beta\gamma)} \mathrm{E}_t y_{t+1} - (1+\chi)\varepsilon_{z,t} - \frac{1}{1-\gamma\beta}\varepsilon_{b,t} + \frac{\beta\gamma}{1-\gamma\beta}\mathrm{E}_t\varepsilon_{b,t+1},$$

where $\chi > 0$ is the inverse of the Frisch elasticity of labor supply and $\varepsilon_{z,t}$ is an exogenous shock to labor productivity.

The three shocks are assumed to follow the stationary autoregressive processes

$$\varepsilon_{j,t} = \rho_j \varepsilon_{j,t-1} + \sigma_j \eta_{j,t}, \quad j = p, b, z, \tag{65}$$

where $\rho_j \in [0, 1)$, $\sigma_j \geq 0$, and $\eta_{j,t}$ are *i.i.d.* innovations with zero mean and unit variance.

We parameterize the model with coefficient values that correspond to a quarterly frequency. For the coefficients in the firm's price-setting problem, we set the discount factor β to 0.99, the Calvo probability ξ to 0.75, and the degree of price indexation ω to 1/3. In regard to household preferences, we set the coefficient of relative risk aversion σ to 2.0, the habit parameter γ to 0.8, and the inverse of the labor supply elasticity χ to 2.5. For simplicity, the three shocks are all assumed to have an autoregressive parameter of 0.5 and an innovation standard error of 1.0.

The model is closed by assuming that the central bank sets the (annualized) one-period nominal interest rate i_t to minimize the expected discounted present value of the quadratic loss function

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\pi_t^2 + \lambda\left(y_t - y_t^f\right)^2 + \nu i_t^2\right], \tag{66}$$

where $\pi_t \equiv 4\widehat{\pi}_t$ and $i_t \equiv 4\widehat{\imath}_t$ are the rate of inflation and the nominal interest rate expressed in annualized terms, y_t^f is the level of output in the flexible-price equilibrium (without markup shocks), and $\lambda, \nu > 0$ summarize the central bank's preferences for stabilizing the output gap, $y_t - y_t^f$, and the interest rate relative to inflation.

We set the central bank preferences for output gap and interest rate stabilization to $\lambda = 0.5$ and $\nu = 0.05$, respectively. The concern for misspecification, θ, is chosen so that the detection-error probability is 0.2, using $1,000$ simulations of a sample of 200 periods. This implies $\theta = 33.3$ when only the conditional means are distorted and $\theta = 20.9$ when both the conditional means and covariances are distorted.

Figures 1–3 show the responses of inflation, the output gap, real marginal costs, and the nominal interest rate to one-standard-deviation innovations in the three shocks. Each panel displays the responses according to the rational expectations equilibrium and the approximating equilibria from the two modeling assumptions, where policy is set after observing the shocks (so conditional means are distorted) and where policy is set before observing the shocks (so also conditional covariances are distorted).

Figure 1 presents the responses following a technology shock. The innovation to technology raises the marginal productivity of labor and thereby lowers real marginal costs. The innovation also raises both output and the flexible-price level of output, however, due to sticky prices, the latter rises more than the former, opening up a negative output gap. Through the Phillips curve, the reduction in marginal costs asserts downward pressure on inflation and with inflation and the output gap both below baseline the policy response is to lower the nominal interest rate. Inflation, the output gap, and marginal costs then return to steady state after a period of over-shooting.

——— Insert Figure 1 around here ———

When the central bank has doubts about the model and sets policy after observing the shock, it fears that the technology shock is more persistent than under rational expectations, and that the positive technology shock is associated with negative future shocks to preferences and the price markup. As such developments would lead to a more negative output gap and lower inflation, the robust central bank reduces the interest rate more than with rational expectations. As a consequence, inflation, the output gap, and marginal cost are all higher in future periods with the robust policy. The central bank that sets policy before observing the shock fears that the shock, as well as being more persistent, also has a larger contemporaneous effect on the economy. The robust policy is therefore again to reduce the interest rate more than with rational expectations, but the time profile of the interest rate response is less expansionary than when the central bank sets policy after observing the shocks.

Since the optimal policy is to accommodate the technological innovation, it is not surprising that the responses for the three equilibria are all quite similar. The greatest difference among the three sets of responses resides in the behavior of the nominal interest rate. With the additional distortions hidden by the consumption preference shock representing a serious concern for the robust central bank (see below), in the equilibrium where only conditional means of shocks are distorted the central bank responds to the lower output gap with a looser monetary policy, accepting higher inflation as a consequence.

Figure 2 displays the responses to a one-standard-deviation preference shock. Looking at the results for the rational expectations equilibrium, due to an increase in labor supply real marginal costs first decline following the shock, but then increase. This response leads to a small increase in inflation and a small positive output gap opens up (since output rises by slightly more than flex-price output). The central bank expands monetary policy to counter the increase in inflation, and this policy response reduces marginal costs, inflation and the output gap.

——— Insert Figure 2 around here ———

The robust central bank fears that the preference shock has a larger impact on inflation, output and marginal costs in the short term, and therefore increases interest rates more, in particular when the central bank sets policy after observing the shocks, and so fears distortions to the conditional means only. This result is due to the presence of expected future preference shocks, which are affected by conditional-mean distortions, but not by conditional-variance distortions, and that, due to the interest rate stabilization motive, are not easily offset by the robust central bank. Nevertheless, the responses of inflation, the output gap, and marginal costs to the preference shock are all small relative to the other shocks.

Figure 3 shows the responses to a price markup shock. As expected, the markup shock causes inflation to rise, and the central bank responds by tightening monetary policy. The increase in the interest rate opens up a negative output gap and lowers real marginal costs. For this shock, the greatest difference between the rational expectations responses and the approximating equilibrium responses resides in the behavior of inflation and the nominal interest rate. The robust policies imply higher interest rates and slightly lower inflation than the rational expectations policy. At the same time, for this shock, as for the technology and the preference shock, differences in behavior between the various equilibria are relatively small, and are observed most obviously in the interest rate itself.

——— Insert Figure 3 around here ———

Table 1 reports the unconditional variances of key variables in the model and the value of the loss function in equation (66). As suggested by the impulse responses, the robust policies typically lead to more volatility in the interest rate and inflation, in particular when the central bank sets policy after observing the shocks. Panel (b) shows that under this policy the worst-case and approximating equilibria imply a large deterioration in performance compared with the case of rational expectations, with loss rising by 56 and 49 percent, respectively. In contrast, when the central bank sets policy without knowledge of the current shocks, in panel (c), loss is 21 percent higher in the worst-case equilibrium and just 2.1 percent higher in the approximating equilibrium. Clearly, for the same detection-error probability, distortions to the conditional means of the shocks are of considerable concern while distortions that raise the volatility of the innovations are less so. As explained above, this result is mainly driven by the expectations of future preference shocks that are affected by conditional-mean distortions, but not by conditional-variance distortions. [5]

——— Insert Table 1 around here ———

Turning to the effects that robustness has on macroeconomic volatility in the approximating equilibrium, when the robust decision is made prior to observing the shocks in panel (c), the central bank's desire for robustness raises the variances of all of the macroeconomic variables that we consider, with the exception of inflation. In this respect, the desire for robustness acts similarly to a fall in the relative weight assigned to output gap stabilization, λ. For the case where the robust decision is made subsequent to observing the shocks in panel (b), the variance of output, the output gap, and real marginal costs—the three variables whose equations contain expectations of future shocks—all fall, while the variance of inflation and the nominal interest rate rise. Here the robust central bank designs its policy to guard against distortions hidden by the preference shocks, which have important effects through the shock's expectation structure, accepting greater interest rate and inflation volatility as a consequence.

6. Final comments

In this paper we show how structural-form solution methods can be applied to solve robust control problems, thereby making it easier to analyze complex models using robust control methods. As an additional contribution, we show that, upon departing from rational expectations, different assumptions regarding the timing of decisionmaking relative to the realization of shocks can have a material impact on the robust decision rule. Specifically, if the shocks are realized prior to decisions being made, then the worst-case specification errors distort the conditional means of the shock process distortions whereas if the shocks are realized subsequent to decisions being made, then the worst-case specification errors distort both the conditional means and the conditional covariances of

[5] Excluding the expected future preference shocks from the model makes distortions to the conditional means and variances more costly than those to only the conditional means.

the shocks. To accommodate distortions to the conditional volatility of the shocks, we generalize the existing method for calculating detection-error probabilities.

We illustrate the structural-form solution methods by applying them to a business cycle model of the genre widely used to study monetary policy under rational expectations. A key finding from this exercise is that the strategically designed specification errors will tend to distort the Phillips curve in an effort to make inflation more persistent, and hence harder and more costly to stabilize. The optimal response to these distortions is for the central bank to become more activist in its response to shocks. Finally, with the business cycle model serving as a laboratory, we show that the distortions to the conditional volatility of the shocks have implications for monetary policy and for economic outcomes that are both qualitatively and quantitatively important.

References

Anderson, E.W., Hansen, L.P., McGrattan, E.R., Sargent, T.J., 1996. Mechanics of forming and estimating dynamic linear economies, in: Amman, H.M., Kendrick, D.A., Rust, J. (Eds.), Handbook of Computational Economics, Volume 1. North-Holland.

Anderson, E.W., Hansen, L.P., Sargent, T.J., 2003. A quartet of semigroups for model specification, robustness, prices of risk, and model detection. Journal of the European Economic Association 1 (1), 68–123.

Calvo, G.A., 1983. Staggered prices in a utility-maximizing framework. Journal of Monetary Economics 12 (3), 383–398.

Currie, D., Levine, P., 1985. Optimal feedback rules in an open economy macromodel with rational expectations. European Economic Review 27 (2), 141–163.

Currie, D., Levine, P., 1993. Rules, Reputation and Macroeconomic Policy Coordination. Cambridge University Press.

Dennis, R., 2007. Optimal policy rules in rational-expectations models: New solution algorithms. Macroeconomic Dynamics 11 (1), 31–55.

Dennis, R., 2008. Robust control with commitment: A modification to Hansen-Sargent. Journal of Economic Dynamics and Control 32 (7), 2061–2084.

Giordani, P., Söderlind, P., 2004. Solution of macromodels with Hansen-Sargent robust policies: Some extensions. Journal of Economic Dynamics and Control 28 (12), 2367–2397.

Hansen, L.P., Sargent, T.J., 2001. Robust control and model uncertainty. American Economic Review Papers and Proceedings 91 (2), 60–66.

Hansen, L.P., Sargent, T.J., 2008. Robustness. Princeton University Press.

Hansen, L.P., Sargent, T.J., Wang, N., 2002. Robust permanent income and pricing with filtering. Macroeconomic Dynamics 6 (1), 40–84.

Kydland, F.E., Prescott, E.C., 1980. Dynamic optimal taxation, rational expectations and optimal control. Journal of Economic Dynamics and Control 2 (1), 79–91.

Leitemo, K., Söderström, U., 2008 Robust monetary policy in the New-Keynesian framework. Macroeconomic Dynamics 12 (S1), 126–135.

Levin, A.T., Onatski, A., Williams, J.C., Williams, N, 2005. Monetary policy under uncertainty in micro-founded macroeconometric models, in: Gertler, M., Rogoff, K. (Eds.), NBER Macroeconomics Annual. The MIT Press.

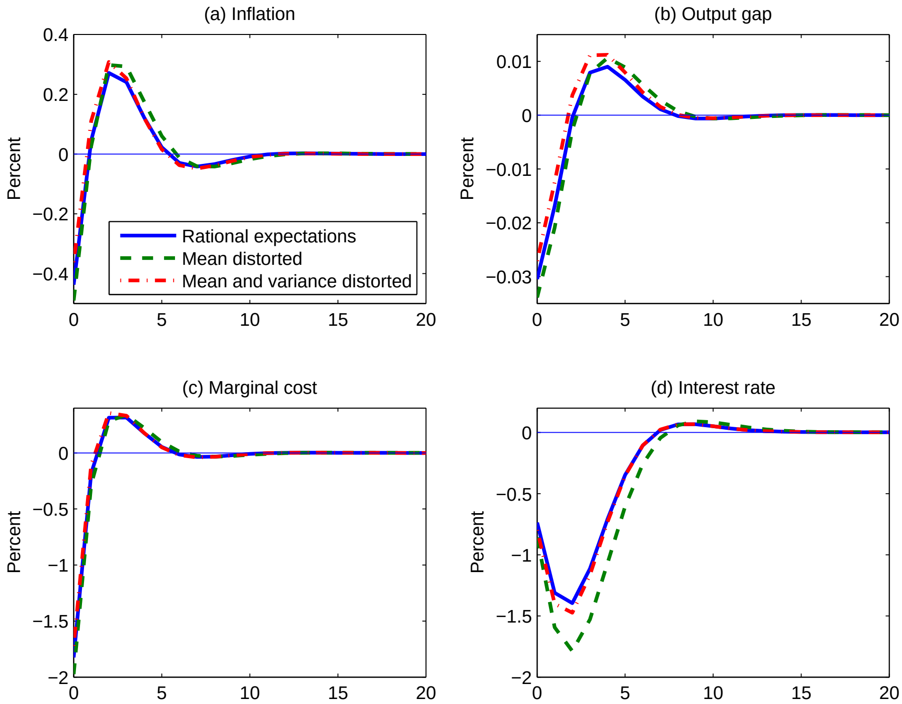
Smets, F., Wouters, R., 2003. An estimated dynamic stochastic general equilibrium model of the Euro area. Journal of the European Economic Association 1 (5), 1123–1175.

Table 1
Unconditional variances and value of loss function in New Keynesian model

	$\text{Var}(\pi_t)$	$\text{Var}(y_t)$	$\text{Var}(y_t - y_t^n)$	$\text{Var}(mc_t)$	$\text{Var}(i_t)$	Loss
(a) *Rational expectations*						
	0.769	0.158	0.066	12.638	13.588	1.444
(b) *Conditional means distorted*						
WO	1.032	0.148	0.068	12.587	25.109	2.255
AP	1.015	0.156	0.065	12.357	23.216	2.147
(c) *Conditional means and covariances distorted*						
WO	0.902	0.192	0.077	13.720	17.695	1.776
AP	0.722	0.166	0.071	13.220	15.185	1.475

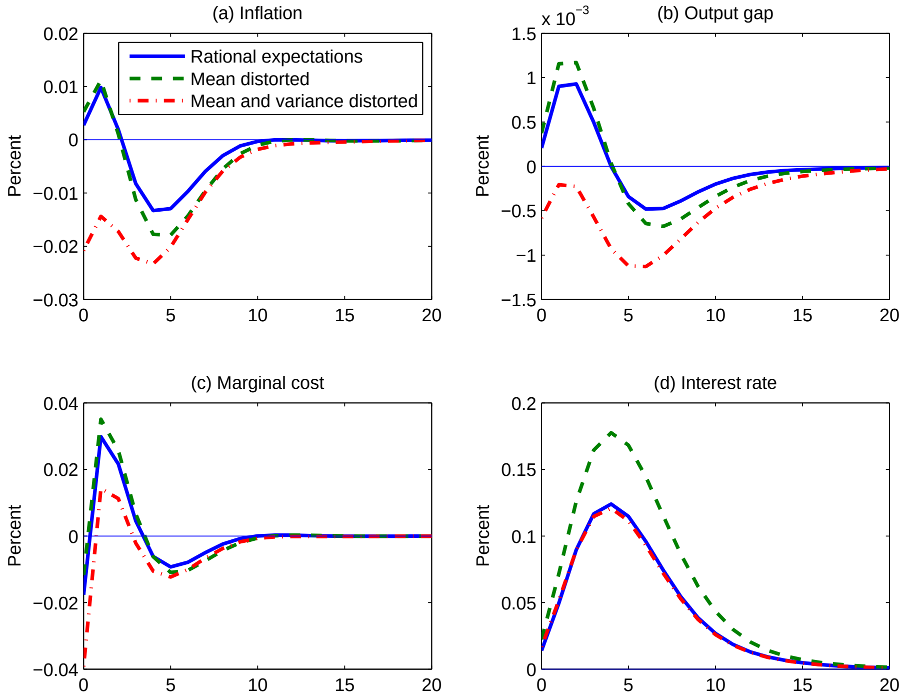
This table reports the unconditional variances and the value of the loss function (66) in the New Keynesian model in (a) the rational expectations equilibrium with the non-robust policy; (b) the worst-case and approximating equilibria when the conditional means of the shocks are distorted (policy set after observing the shocks); and (c) the worst-case and approximating equilibria when the conditional means and covariances of the shocks are distorted (policy set before observing the shocks).

Fig. 1. Impulse responses to technology shock in the New Keynesian model



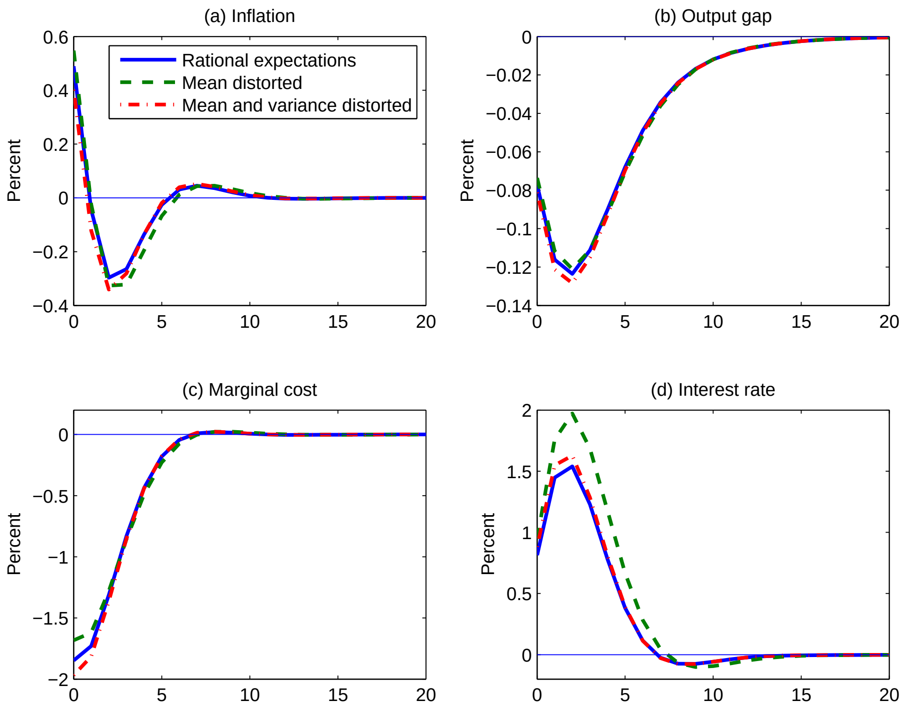
This figure shows the impulse responses to a one-standard-deviation technology shock in the New Keynesian model in (i) the rational expectations equilibrium with the non-robust policy; (ii) the approximating equilibrium when the conditional means of the shocks are distorted (policy set after observing the shocks); and (iii) the approximating equilibrium when the conditional means and covariances of the shocks are distorted (policy set before observing the shocks).

Fig. 2. Impulse responses to preference shock in the New Keynesian model



This figure shows the impulse responses to a one-standard-deviation preference shock in the New Keynesian model in (i) the rational expectations equilibrium with the non-robust policy; (ii) the approximating equilibrium when the conditional means of the shocks are distorted (policy set after observing the shocks); and (iii) the approximating equilibrium when the conditional means and covariances of the shocks are distorted (policy set before observing the shocks).

Fig. 3. Impulse responses to price markup shock in the New Keynesian model



This figure shows the impulse responses to a one-standard-deviation price markup shock in the New Keynesian model in (i) the rational expectations equilibrium with the non-robust policy; (ii) the approximating equilibrium when the conditional means of the shocks are distorted (policy set after observing the shocks); and (iii) the approximating equilibrium when the conditional means and covariances of the shocks are distorted (policy set before observing the shocks).