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Low Risk Sharing with Many Assets*

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Abstract

Classical contributions in international macroeconomics reconcile low international risk sharing by generating a non-traded component to exchange rates. However, when there is cross-border trade in just one domestic and one foreign-currency-denominated risk-free asset, such price movements are ruled out by no-arbitrage restrictions. Allowing for within-country heterogeneity in stochastic discount factors, we recover low risk-sharing even with cross-border trade in two risk-free assets, as long as heterogeneity increases when exchange rates depreciate.

Keywords: international risk sharing, incomplete financial markets, exchange rates, heterogeneous agents.

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1. Introduction

We revisit the Backus-Smith condition (Kollmann, 1991; Backus and Smith, 1993), which describes the sharing of risk across countries in terms of co-movement of consumption and relative prices (Obstfeld and Rogoff, 2000). While a large class of macroeconomic models can reconcile the cyclicality of exchange rates in the data, these models fail once we account for no-arbitrage restrictions arising from trade in multiple assets. We first explore why this incongruence between macroeconomic mechanisms and no-arbitrage restrictions arises. We then propose a generalization of these models to allow for within-country heterogeneity in stochastic discount factors (SDFs) and illustrate how this extension reconciles low risk sharing with cross-border trade in many assets.

When international financial markets are complete, a large class of models admits the following relationship:

$$\left(\frac{C_{t+1}}{C^*_{t+1}} / \frac{C_t}{C^*_t}\right)^s = \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \tag{1}$$

where C_t is Home aggregate consumption, C^*_t is Foreign aggregate consumption, $\mathcal{E}_t$ is the real exchange rate where an increase signifies a depreciation of Home currency, and s is the inverse of the inter-temporal elasticity of substitution. Risk-sharing implies ex-post redistribution, which, with trade in risk-free assets, must occur entirely through exchange rate movements. For example, following a positive productivity shock in the Home country, the Home currency depreciates leading to higher consumption abroad. Exchange rates thus move to reallocate wealth from Home to Foreign and so are "risky" from the perspective of a Home investor.[1] However, in the data, exchange rates often appreciate when Home consumption rises —which we define as low risk sharing— and constitutes the Backus-Smith *puzzle*.

When markets are incomplete, the condition above needs to hold only in expectation

[1]This implication echoes closed economy models with complete markets which imply a correlation of -1 between the representative agent SDF and the market portfolio, see, e.g. Lettau (2002).

and so may fail ex-post. Classical contributions in international macroeconomics such as Corsetti, Dedola and Leduc (2008); Benigno and Thoenissen (2008) show that non-traded risk, which arises from consumption or production complementarities, enables incomplete market models to generate plausible patterns of international risk sharing. We refer to these economic mechanisms as goods market mechanisms (as opposed to financial). However, these models only allow cross-border trade in a single risk-free asset (denominated in either currency). Lustig and Verdelhan (2019) show that, no-arbitrage restrictions from cross-border trade in at least one Home and one Foreign risk-free asset imply prices always co-move negatively with consumption – so incomplete market models cannot resolve the puzzle of excessive risk-sharing *regardless* of goods market frictions and other economy specifics.[2] Since, in practice, the number of assets traded across borders is very high, this result has far-reaching implications for both theory and practice.

Our first contribution is to highlight how goods markets mechanisms can reconcile the Backus-Smith puzzle. We show that any model that can reconcile the comovement of exchange rates and consumption growth (or SDFs) must generate a non-traded component of relative prices which is "safe" from a domestic investor perspective. We then show that moving from frictionless cross-border trade in a single risk-free asset to cross-border trade in just one Home and one Foreign risk-free asset inhibits *any* goods-market mechanisms from reconciling the patterns of risk-sharing observed in the data since the additional no-arbitrage restriction rules out such safe exchange rate movements (see also Chernov, Haddad and Itskhoki, 2024), regardless of how far the economy is from complete financial markets. Instead, we show that trade in risky assets, such as long-term or nominal bonds, is not generally sufficient to rule out safe fluctuations in the non-traded component of exchange rates. Within a simple two-country, two-good model, with one internationally traded risk-free bond, we confirm that fluctuations in endowments lead to such safe movements in the non-traded

[2] Benigno and Küçük (2012) also find negative comovement in a quantitative international macro model with frictionless cross-border trade in two *nominal* bonds.

component of exchange rates as long as the trade elasticity is sufficiently low.

Our second contribution is to propose generalizations of the representative investor, two-country model to allow the co-existence of multiple SDFs (heterogeneous investors) within the Home country. Domestic heterogeneity implies the presence of non-traded risk within countries which, if correlated with exchange rate movements, is not insured away by additional trade in Home and Foreign risk-free assets. We consider two complementary models which imply similar conditions under which low risk sharing can be obtained, but differ in their interpretation. Both models achieve low risk sharing, even with trade in many assets, without compromising the ability to match the volatility of exchange rates or introducing a predictable component to exchange rate movements.

We first consider a model where a *measure zero* of agents participates in foreign markets–i.e. an intermediary or a George Soros. We model two investors in the Home economy: both price the domestic currency-denominated risk-free asset while only the measure zero investor prices the Foreign currency risk-free asset. We show the model features low risk sharing as long as the exposure of the investor participating in Foreign markets to depreciations is sufficiently high. The threshold is given by the ratio of volatilities of exchange rates and domestic risk. In line with Brandt, Cochrane and Santa-Clara (2006), who argue that exchange rates are smooth from a finance perspective, our results go through if domestic risk is volatile enough to price assets such as equities. Using good-deal bounds (Cochrane and Saá-Requejo, 2000) and portfolio data, showing that investors participating in Foreign markets earn excess an Sharpe ratio over domestic assets, we verify that domestic risk is more volatile than the SDF pricing equities.

Our second model considers heterogeneous consumers in the Home economy facing idiosyncratic consumption risk, all trading in fully *integrated* international markets. We build on a large literature investigating whether idiosyncratic risk has aggregate pricing consequences, most closely Constantinides and Duffie (1996), and Krueger and Lustig

(2010). The presence of idiosyncratic risk implies an Euler inequality for aggregate consumption which, in turn, shapes the properties of exchange rates. With trade in a single risk-free asset internationally, we show that the volatility of idiosyncratic risk implies the model can more easily achieve the low risk-sharing benchmark. Even when there is trade in two nominally risk-free assets, the Backus-Smith puzzle can be resolved as long as idiosyncratic consumption risk increases sufficiently with exchange rate depreciation. This condition implies that the Foreign bonds are a poor hedge for consumption risk, consistent with the diversification puzzle in Heathcote and Perri (2013). Back of the envelope calculations using the average exposure of agents to exchange rate movements from Verner and Gyöngyösi (2020) suggest that empirically plausible heterogeneity can deliver low risk sharing within our framework.

Related Literature Most closely related papers to ours are Benigno and Küçük (2012), Lustig and Verdelhan (2019), Chernov, Haddad and Itskhoki (2024), and Jiang, Krishnamurthy and Lustig (2023). Lustig and Verdelhan (2019) consider multiplicative incomplete market wedges as in Backus, Foresi and Telmer (2001). Benigno and Küçük (2012) and Lustig and Verdelhan (2019) show that introducing a second internationally traded bond breaks the ability of international macro models to reconcile the Backus-Smith puzzle. We extend their frameworks beyond the representative agent assumption, generalizing some of their results, and show how within-country heterogeneity in SDFs can generate low risk sharing as we increase the number of internationally traded assets.

Chernov et al. (2024) investigate how different financial market structures and the mix of locally, globally traded, and unspanned risks contribute to different exchange rate puzzles. Our first model naturally relate to models of segmentation (Alvarez, Atkeson and Kehoe, 2002; Chien, Lustig and Naknoi, 2020) or intermediation (Gabaix and Maggiori, 2015; Itskhoki and Mukhin, 2021). Our second model of incomplete but integrated markets illustrates that segmentation per se is not required to theoretically reconcile the puzzles we investigate. In both models, domestic incompleteness breaks global risks into local risks.

Using a wedge accounting framework, Itskhoki and Mukhin (2023) find that financial shocks can reconcile exchange rate puzzles. Jiang, Krishnamurthy and Lustig (2023) show that Euler equation wedges are necessary to resolve Backus-Smith puzzle. We complement their analysis by structuring these wedges and allowing for multiple within-country SDFs connected by risk-sharing.[3]

From a finance perspective, Bakshi, Cerrato and Crosby (2018) also allow for multiple SDFs considering additive wedges, but their focus is on isolating the spanned and unspanned components and generalizing the results in Brandt et al. (2006). Sandulescu, Trojani and Vedolin (2021) extract minimum variance and minimum entropy SDFs and show that the Backus-Smith condition holds with their model-free SDFs. Orłowski, Tahbaz-Salehi, Trojani and Vedolin (2023) extend the result of Lustig and Verdelhan (2019) to allow for varying degrees of financial integration and different market structures with no-arbitrage pricing.

When we turn to a model with heterogeneous consumers, we tie to a large literature investigating whether idiosyncratic risk affects the aggregate Euler, see, e.g. Mankiw (1986); Weil (1992); Constantinides and Duffie (1996); Guvenen (2009); Kocherlakota and Pistaferri (2009); Krueger and Lustig (2010); Heathcote, Storesletten and Violante (2014); Werning (2015); Kaplan, Moll and Violante (2018); Di Tella, Hébert and Kurlat (2024). Acharya and Pesenti (2024) investigate the role of precautionary savings in generating monetary policy spillovers in a two-country open economy model.[4]

The rest of the paper is organised as follows. Section 2 provides a characterization

[3] Verdelhan (2010), Colacito and Croce (2013), Hassan (2013), and Farhi and Gabaix (2016) investigate the role of habits, long-run risk, country-size, and rare disasters respectively in generating observed correlations between cross country asset returns and exchange rate returns. For a treatment of optimal portfolio choice and international risk sharing, see Devereux and Sutherland (2011) and Heathcote and Perri (2013) amongst others.

[4] In the open economy macro literature, for papers on heterogeneous SDFs arising out of consumption or preference heterogeneity see Ghironi (2006); Kocherlakota and Pistaferri (2007); Farhi and Werning (2016); Fornaro (2018); De Ferra, Mitman and Romei (2020); Auclert, Rognlie, Souchier and Straub (2021); Kekre and Lenel (2024*b*); Guo, Ottonello and Perez (2023).

of how goods markets drive exchange rate cyclicality and how this mechanism works with trade in risk-free and/or risky assets. Section 3 proposes our generalization of incomplete markets with SDF heterogeneity and derives the minimum bounds necessary for generating empirical risk-sharing patterns with different financial structures. Section 4 concludes.

2. Two-country, representative agent, incomplete markets

Consider a two-country model where M_{t+1} denotes the Home representative household's SDF and M^*_{t+1} denotes the Foreign representative household's SDF. Home and Foreign households each trade their respective domestic risk-free real bonds with returns R_t and R^*_t respectively frictionlessly (i.e. no borrowing constraints). No-arbitrage pricing implies:[5]

$$\mathbb{E}_t[M_{t+1}] = 1/R_{t+1}, \tag{2}$$

$$\mathbb{E}_t[M^*_{t+1}] = 1/R^*_{t+1} \tag{3}$$

If the Home (Foreign) households also trade the Foreign (home) bond, and the real exchange rate at time t is denoted with $\mathcal{E}_t$, then we obtain the following two Euler conditions:

$$\mathbb{E}_t\left[M^*_{t+1}\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}\right] = 1/R_{t+1}, \tag{4}$$

$$\mathbb{E}_t\left[M_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = 1/R^*_{t+1}. \tag{5}$$

We assume SDFs, allocations and prices are jointly log-normal.[6] To close the

[5]E.g. in the case of time-separable CRRA utility $M_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-s}$. Alternatively, in the tradition of Hansen and Jagannathan (1991), the SDFs are simply a risk-return operator implied by the traded assets. See Section 4.

[6]Lustig and Verdelhan (2019) show this assumption can be relaxed to non log-normal settings

model without explicitly specifying goods markets, an exchange rate process is needed, which is consistent with equations (2)–(5) above. This problem reduces to finding an exchange rate process that satisfies:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(\Delta e_{t+1}) \tag{6}$$

where $x = \log(X)$. Naturally, the process corresponding to complete markets ($\Delta e_{t+1} = m^*_{t+1} - m_{t+1}$) is one candidate. More generally, as shown in Backus, Foresi and Telmer (2001), the following process also satisfies equation (6):

$$\Delta e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1} \tag{7}$$

where η_{t+1} is an incomplete markets wedge which must satisfy certain conditions imposed by asset trade. The Backus-Smith condition (1) restricts the covariance between relative SDFs and exchange rate growth $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$, i.e. the cyclicality of exchange rates, to be positive. In this case, the model implies a high degree of risk sharing. Conversely, if the covariance is negative, the model generates *low* risk sharing.

Combining (2) and (4), with (7) – which implies that the Home bond is internationally traded – yields:

$$\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2} var_t(\eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1}) \tag{8}$$

Combining (3) and (5), with (7) – which implies that the Foreign bond is internationally traded – yields:

$$-\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2} var_t(\eta_{t+1}) - cov_t(m^*_{t+1}, -\eta_{t+1}) \tag{9}$$

These two conditions mirror those studied by Lustig and Verdelhan (2019) and bound the joint dynamics of the incomplete market wedge and the SDFs, carrying strong

using entropy expansions. Co-movement puzzles are more difficult to handle in non-normal settings (see also Chabi-Yo and Colacito, 2019), but their results hold in specific examples such as the case of distributions where moments higher than the skewness are approximately zero.

implications for the macro side of the model. The conditions reflect the return, for a Home and a Foreign investor respectively, from the non-traded component of exchange rates which must compensate for both volatility of returns and the riskiness of the exchange rate.

While the results we derive are preference-free, to relate back to international macro models, consider the case where agents have time-separable, CRRA preferences over consumption. Then, the incomplete markets wedge is related to:[7]

$$\eta_{t+1} = \log \underbrace{\left(\frac{P_{t+1}}{P_t} \frac{P_t^*}{P_{t+1}^*} \right)}_{\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}} - \log \underbrace{\left(\frac{C_t}{C_{t+1}} \frac{C_{t+1}^*}{C_t^*} \right)^s}_{\frac{M_{t+1}}{M_{t+1}^*}}$$

where P_t is the Home price level, C_t is aggregate consumption, s^{-1} is the inter-temporal elasticity of substitution, and terms with asterisks denote the corresponding Foreign objects. The wedge, η, is often interpreted as the non-traded component of exchange rate movements or the wealth gap, see e.g. Corsetti, Dedola and Leduc (2023*a*).

2.1. International Risk-Sharing with Trade in Risk-free Assets

Having now specified our framework, we illustrate the mechanism through which goods market frictions in incomplete market models can help reconcile the pattern of international risk-sharing.

Proposition 1 (One Int'l Traded Asset, Representative Agent No-Arbitrage).
When only Foreign bonds are internationally traded such that equations (2), (3) *and* (5) *hold, then* $cov_t(m_{t+1}^* - m_{t+1}, \Delta e_{t+1}) \leq 0$ *if and only if*

$$cov_t(m_{t+1}, \eta_{t+1}) + \log \mathbb{E}_t[e^{\eta_{t+1}}] \geq var_t(m_{t+1}^* - m_{t+1}) \tag{10}$$

where,

$$cov_t(m_{t+1}, \eta_{t+1}) = cov_t(m_{t+1}, \Delta e_{t+1}) - cov_t(m_{t+1}, m_{t+1}^*) + var_t(m_{t+1}) \tag{11}$$

[7]For a closed form solution of the incomplete markets wedge in a two-country open economy model, see Pavlova and Rigobon (2007).

Proof. See Appendix A.2. □

The RHS of condition (10) is equal to the volatility of the exchange rate growth under complete markets and is strictly positive. The condition is satisfied if either the non-traded component η_{t+1} leads to relative price fluctuations which are safe from the perspective of a Home investor, as captured by $cov_t(m_{t+1}, \eta_{t+1}) > 0$ or that the volatility of the non-traded component is high. Equation (11) shows that non-traded risk results in relative price fluctuations which are particularly *safe* when the Home SDF is very volatile or when international comovement in SDFs is low relative to the comovement of exchange rates and the Home SDF.[8] In Section 2.3, we specify a two-country model with multiple goods and investigate the parametrizations consistent with this condition.

The condition (10) provides a general characterization for goods markets mechanisms developed to resolve the Backus-Smith puzzle in models with consumption or production complementarities (Corsetti et al., 2008; Benigno and Thoenissen, 2008), costly consumer search (Bai and Ríos-Rull, 2015), or global value chain fragmentation (Corsetti, D'Aguanno, Dogan, Lloyd and Sajedi, 2023*b*), amongst others. A limitation of Proposition 1, and the models which satisfy it, is that it may exacerbate other exchange rate puzzles– in particular, that of excess volatility of exchange rates. The RHS of equation (10) is equal to the volatility of exchange rates under complete markets, and models with a low volatility will generally fare better in resolving the cyclicality puzzle, which rationalizes why models such as Corsetti et al. (2008) display very low risk premia (Lustig and Verdelhan, 2019, pp 2241).

Cross-border trade in a second risk-free asset prevents the model from reconciling the Backus-Smith puzzle because this constrains the *safety* of relative price movements due to non-traded risk. From Equation (8), when the non-traded component becomes safe for Home investors, lower returns on their Foreign bond holdings ($\mathbb{E}_t[\eta_{t+1}]$) translate to higher returns on Home bonds for Foreign investors ($\mathbb{E}_t[-\eta_{t+1}]$). This is

[8] Brandt et al. (2006) show that SDFs must comove very strongly to explain the relatively low exchange rate volatility in the data.

only consistent with no arbitrage if the non-traded component becomes sufficiently risky for Foreign investors which would restore the Backus-Smith correlation.

Corollary 1 (Two Int'l Traded Asset, Representative Agent No-Arbitrage).
As the Home bond also becomes internationally traded without arbitrage, $cov_t(m_{t+1}, \eta_{t+1}) \rightarrow$ -$\mathbb{E}_t[\eta_{t+1}] + \frac{1}{2} var_t(\eta_{t+1})$ *as in equation* (8)*, then condition* (10) *implies* $var_t(\Delta e_{t+1}) \leq 0$.

Proof. See Appendix A.2. □

2.2. International Risk Sharing with Trade in Risky Assets

If instead of allowing for trade in both risk-free assets, we allow for trade in Home and Foreign risky assets, then trade in multiple assets does not necessarily restrict the cyclicality of exchange rates.[9] In practice, few assets traded across borders are risk-free in real terms, so this case is likely to be a better approximation of reality. Risky assets could include equity or long maturity bonds. In this case, equations (2)–(5) are replaced by:

$$\mathbb{E}_t[M_{t+1} \tilde{R}_{t+1}] = 1, \tag{12}$$

$$\mathbb{E}_t[M_{t+1} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \tilde{R}^*_{t+1}] = 1, \tag{13}$$

$$\mathbb{E}_t[M^*_{t+1} \tilde{R}^*_{t+1}] = 1, \tag{14}$$

$$\mathbb{E}_t[M^*_{t+1} \left(\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \right)^{-1} \tilde{R}_{t+1}] = 1, \tag{15}$$

where $\tilde{R}$ and $\tilde{R}^*$ are returns on risky Home and Foreign assets respectively.

Proposition 2 (Risky Assets, Representative Agent No-Arbitrage).
When only risky Home and Foreign assets are internationally traded such that equations (12) - (15) *hold, then* $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) \leq 0$ *if and only if*

$$var_t(\Delta e_{t+1}) + cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \leq 0 \tag{16}$$

[9] Lustig and Verdelhan (2019) derive the restrictions imposed by trade in risky assets in addition to two-risk free bonds. In their environment, trade in risky assets can therefore not break patterns of risk sharing.

Proof: See Appendix B.1. □

Trade in risky assets is not sufficient to restrict the cyclicality of exchange rates, unless the risky returns are uncorrelated with domestic non-traded risk. Adding cross-border trade in the Home risk free bond will reimpose equation (8) and cross-border trade in the Foreign risk free bond will reimpose equation (9). Consider the environment in Proposition 1, where only Foreign risk-free bonds are internationally traded. Introducing trade in Home risky assets does not necessarily recover the strong risk-sharing implications that arise when international trade in a second risk-free asset is allowed, as detailed in Corollary 2.

Corollary 2

When Foreign risk-free bonds are internationally traded such that equations (2), (3) and (5) hold, as well as a Home risky asset is internationally traded such that equations (12) and (15) hold, then $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ if and only if

$$var_t(\Delta e_{t+1}) - cov_t(\eta_{t+1}, \tilde{r}_{t+1}) \leq 0 \tag{17}$$

Proof. *Additionally* imposing equation (9) implies $cov_t(\eta_{t+1}, \tilde{r}^*_{t+1}) = 0$. The result to Corollary 2 then follows from Proposition 2. See Appendix B.1 for additional steps. □

As enough assets become traded and markets approach completeness, $\sigma_\eta \to 0$, so Corollary 2 collapses to the impossibility result (Corollary 1). However, even if asset markets are far from the complete markets benchmark, trade in just two nominally risk-free bonds is sufficient to ensure $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) > 0$ as we detail in Appendix B.2.

2.3. An Equilibrium Model

To relate to the existing macroeconomics literature, we specify a parsimonious model and use it as a basis for constructing the investor SDFs.[10] Within this framework, we show that a sufficiently low elasticity of substitution between the goods varieties is enough to generate safe movements in the non-traded component of exchange rates as long as at most one risk-free bond is internationally traded. The representative agent derives per-period utility from consumption:

$$u(C_t) = \beta(C_{t-1})\frac{C_t^{1-s}}{1-s} \tag{18}$$

where $\beta(C_{t-1}) = \omega C_{t-1}^{-u}$, with $\omega \in (0,1)$, is the discount factor used as a stationarity-inducing device following Bodenstein (2011), and the Home consumption bundle is given by:

$$C_t = \left[\alpha^{\frac{1}{\phi}} C_{H,t}^{\frac{\phi-1}{\phi}} + (1-\alpha)^{\frac{1}{\phi}} C_{F,t}^{\frac{\phi-1}{\phi}}\right]^{\frac{\phi}{\phi-1}} \tag{19}$$

where ϕ is the trade elasticity and α is the measure of home-bias. The corresponding price index is given by:

$$P_t = \left[\alpha P_{H,t}^{\frac{\phi-1}{\phi}} + (1-\alpha) P_{F,t}^{\frac{\phi-1}{\phi}}\right]^{\frac{\phi}{\phi-1}} \tag{20}$$

C^* and P^* are defined symmetrically. The real exchange rate is given by $\mathcal{E} = P^*/P$.The domestic budget constraint can be written as:

$$C_t + \tfrac{P_{H,t}}{P_t} Y_{H,t} \leq R_t B_{t-1} - B_t + \mathcal{E}_t R_t^* B_t^* - \mathcal{E}_t B_{t-1}^* \tag{21}$$

and we define $\frac{P_{H,t}}{P_t} Y_{H,t} = Y_t$, $B_t^{(*)}$ is the position in Home (Foreign) risk-free bonds and $R_t^{(*)}$ is the corresponding returns. These bonds are risk-free claims on the respective country's consumption bundle hence these are nominally risk-free assets. Foreign

[10]Relative to Corsetti et al. (2008), we abstract from production and capital accumulation. As a result our mechanism relies heavily on the persistence of the endowment process.

agents face an analogous maximization. Finally, goods market clearing requires:

$$C_{H,t} + C^*_{H,t} = Y_{H,t}; \quad C_{F,t} + C^*_{F,t} = Y^*_{F,t}$$

where endowment process are given by $Y_{H,t} = \rho Y_{H,t-1} + (1-\rho)Y_H + \epsilon_t$, $Y^*_{F,t} = \rho Y^*_{F,t-1} + (1-\rho)Y^*_F + \epsilon^*_t$ and ϵ_t and ϵ^*_t are iid $N(0, \sigma_\epsilon)$ endowment shocks.

Lemma 1 describes the autarky limit for prices and allocations, attained at the limit of zero liquidity (Corsetti et al., 2008) and full home-bias limit (Itskhoki and Mukhin, 2021, 2023).

Lemma 1 (Autarky Limit).
In the autarky limit $\alpha \to 1$, $B, B^* \to 0$, the model is summarized by the following equations:

$$\begin{aligned} m_{t+1} &= -s g_{y_{H,t+1}}, \\ m^*_{t+1} &= -s g_{y_{F,t+1}}, \\ \Delta e_{t+1} &= \frac{1}{1-2(1-\phi)}(g_{y_{H,t+1}} - g_{y_{F,t+1}}), \\ \eta_{t+1} &= (g_{y_{H,t+1}} - g_{y_{F,t+1}})\frac{1-s}{1-2(1-\phi)} \end{aligned}$$

where $g_{y_{t+1}} = y_{t+1} - y_t$. Then, $m_{t+1}, m^*_{t+1}, \eta_{t+1}$ and Δe_{t+1} are jointly normally distributed.

Proof. See Appendix B.3 □

Our approximation technique relies on taking the autarky limit for real quantities. There are two reasons why this limit is attractive. First, we prove the joint log normality of SDFs and the exchange rate at the full home-bias limit as $\alpha \to 1$, whereas this is not the case for $\alpha < 1$. Second, the allocations and prices are invariant to the number of assets traded.[11] Then, whereas we calculate the covariance in the financial autarky case by direct computation, we back out the implied covariance at the autarky

[11] While the invariance and the log-normality properties also hold at the complete markets allocation or Cole and Obstfeld (1991) limit, these cannot by construction break the Backus-Smith condition.

limit, denoted $cov_t^{\to FA}(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$, between SDFs and depreciations when there is trade in assets by imposing $cov_t(m^{(*)}_{t+1}, \eta_{t+1})$ consistent with (8)-(9).

Proposition 3 (Representative agent, Autarky Limit).
The two-country model at the autarky limit can deliver $cov_t^{\to FA}(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) \leq 0$ *conditional on shocks to* $y_{H,t}$ *in the following cases:*

i. *when no assets traded*:

$$\frac{-s}{1 - 2(1-\phi)} \leq 0 \tag{22}$$

ii. *with trade in one risk-free asset*:

$$\frac{-s(1-s)}{1 - 2(1-\phi)} \geq s^2 - \frac{1}{2}\left[\frac{1-s}{1-2(1-\phi)}\right]^2 \tag{23}$$

iii. with two assets need $\phi \to \infty$ such that $var_t(\Delta e_{t+1}) \to 0$.

Proof. See Appendix A.2. □

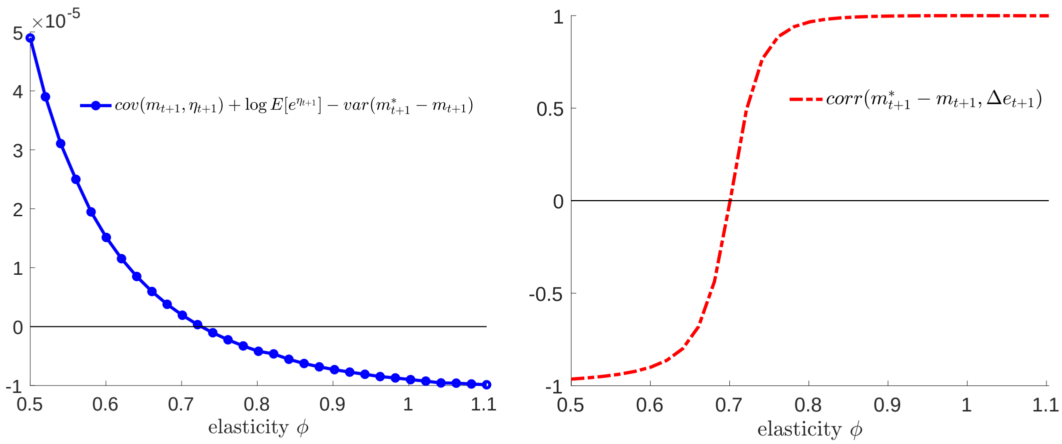
Allowing for a sufficiently low trade elasticity implies that following an increase in Home productivity, demand for Home goods rises so much, that prices must adjust to constrain Foreign consumption of the Home good for markets to clear, as in Corsetti et al. (2008). Under autarky the ability of the model to match risk sharing depends only on ϕ given that $s > 0$. When there is trade in a risk-free asset, case (ii), condition (10) additionally depends on the inter-temporal elasticity of substitution, s.[12] With trade in two nominally risk-free bonds, case (iii), a zero covariance between relative SDFs and the exchange rate arises in the limit where $var_t(\Delta e_{t+1})$ approaches zero, but a negative covariance can never be achieved.

Figure 1a illustrates Proposition 1 away from the autarky limit, i.e. $\alpha < 1$. Specifically, condition (10) is satisfied for values over which the plotted line is positive. Figure 1b shows that the sign on the simulated Backus-Smith correlation tracks condition (10). The correlation is negative for low values of ϕ, positive for higher

[12]Figure 2 in Appendix B.3 shows the range of values for s and ϕ for which condition (23) is satisfied.

Figure 1

(a) *Prop. 1 Statistic* **(b)** *Backus-Smith Correlation*



Calibration: $\omega = 0.96, u = 0.005, s = 1, \phi \in [0.5, 1.1], \alpha = 0.6, \alpha^* = 1 - \alpha, \rho = 0.96, Y_H = 1$. Unconditional moments calculated from second-order simulation with one million draws.

values and intersects 0 at approximately the same value for ϕ as Panel (a), even though, away from the autarky limit, the SDFs and exchange rates in the equilibrium model are not exactly log-normal.

3. Models with heterogeneous SDFs

Next, we show that models with heterogeneous investors can reconcile the Backus-Smith anomaly even when two nominally risk-free assets are internationally traded. We consider two distinct models, reflecting different forms of heterogeneity, but which imply similar results.

The first model features two SDFs – one SDF corresponds to a marginal investor who prices only the Home risk-free bond, and the second SDF to a marginal investor who prices both the Home and the Foreign risk-free bonds and is of *measure zero*. This resembles a model where foreign exposure is very concentrated amongst few agents, such as a model with intermediation (Gabaix and Maggiori, 2015) or where only a George Soros participates in foreign markets. The second model features a continuum of SDFs where all investors frictionlessly participate in both bond markets (i.e. no borrowing constraints) but face uninsurable idiosyncratic consumption risk.

3.1. A model with George Soros

Consider now the case where Home (domestic) financial markets are incomplete. The Foreign economy has a representative investor who can frictionlessly buy Home and Foreign risk-free bonds. The Home economy has two investors characterized by SDFs, M and $\hat{M}$. They both participate frictionlessly in the Home risk-free bond market, but only one of the two Home investors participates in the Foreign risk-free bond market. We assume that the investors who participate in Foreign risk-free bonds are measure zero. This model is characterized by equations (2), (3), (4) and

$$\mathbb{E}_t\left[\hat{M}_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = 1/R^*_{t+1}, \tag{24}$$

and we define:

$$\hat{M}_{t+1} = M_{t+1}D_{t+1} \tag{25}$$

where $D_{t+1} \neq 1$ for some t, captures the degree of heterogeneity within the Home country.

We assume that only the exchange rate markets are segmented so we allow all Home investors to frictionlessly trade a Home risk-free bond such that their marginal utility growth will be equated in expectation:

$$\mathbb{E}_t[M_{t+1}] = \mathbb{E}_t[\hat{M}_{t+1}] \tag{26}$$

Since $\hat{M}_{t+1}$ prices both Home and Foreign bonds, $\hat{M}_{t+1}$ satisfies all conditions in Lustig and Verdelhan (2019) and will comove with exchange rates according to the following analogue of equation (1):

$$\mathbb{E}_t\left[\hat{M}_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = \mathbb{E}_t\left[M^*_{t+1}\right] \tag{27}$$

Frictionless trade in domestic bonds (25)-(26) imposes the following constraint on heterogeneity:

$$\mathbb{E}_t[d_{t+1}] + \frac{1}{2}var_t(d_{t+1}) + cov_t(m_{t+1}, d_{t+1}) = 0 \tag{28}$$

Critically, equation (28) implies that d_{t+1} cannot be an asset specific discounting factor for the marginal investor – i.e. a convenience yield on specific bonds. Heterogeneity is therefore strictly on the investor, as opposed to the asset side. Additionally, d_{t+1} is non-traded risk since d_{t+1} does not affect domestic asset prices. Allowing agents to additionally trade in risky assets further restricts heterogeneity, as expected. In particular, building on Corollary 2, we can show if m and $\hat{m}$ trade in $\tilde{r}$, then $cov(d, \tilde{r}) = 0$.[13]

The extended model admits the same process for exchange rates but a different set of equilibrium restrictions apply to the wedge η_{t+1}.[14] Specifically, equation (8) is unchanged because the Home bond continues to be traded frictionlessly across markets, but market segmentation with respect to the Foreign bond implies the equation (9) is replaced by:

$$\begin{aligned}\mathbb{E}_t[d_{t+1}] + \frac{1}{2}var_t(d_{t+1}) + cov_t(m^*_{t+1}, d_{t+1}) + \mathbb{E}_t[\eta_{t+1}] + \frac{1}{2}var_t(\eta_{t+1})\\ \cdots + cov_t(m^*_{t+1} + d_{t+1}, \eta_{t+1}) = 0\end{aligned} \tag{29}$$

We next derive restrictions on the dynamics of investor heterogeneity and exchange rates, required to match the patterns of international risk sharing observed in the data.

Proposition 4 (Two Int'l Traded Assets, Heterogeneous Marginal Investors).
The two-country model with two internationally traded bonds and heterogeneous marginal investors in the Home country can deliver $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ *if*

[13] By analogy to Corollary 2, if domestic agents trade in a complete set of securities, $\sigma_d \to 0$.

[14] Note that potentially an exchange rate process with η_{t+1} replace by $d_{t+1} + \tilde{\eta}_{t+1}$ could be used. Our results would be unchanged.

and only if

$$1 \geq \rho_{d_{t+1},-\Delta e_{t+1}} \geq \frac{\sigma_t(\Delta e_{t+1})}{\sigma_t(d_{t+1})} \tag{30}$$

where $\rho_{d_{t+1},-\Delta e_{t+1}} \equiv \dfrac{cov_t(d_{t+1}, -\Delta e_{t+1})}{\sigma_t(\Delta e_{t+1})\sigma_t(d_{t+1})}$.

Proof: See Appendix A.2. □

The inequality in Proposition 4 describes the joint dynamics of exchange rates and domestic heterogeneity required to break the covariance between SDFs and exchange rates implied by equation (1), when there is trade in both Home and Foreign bonds. First, a necessary condition is that $\frac{\sigma_t(\Delta e_{t+1})}{\sigma_t(d_{t+1})} < 1$– i.e. there is sufficient domestic heterogeneity relative to the volatility of exchange rates. Since this condition is a critical component for our theory, we evaluate its empirical plausibility in Section 3.1.2.

Second, Proposition 4 bounds the sign of the correlation of SDF heterogeneity (non-traded risk) and exchange rate appreciation to be positive– as should be expected in theory. Consider the Backus-Smith condition (1) where the Home SDF is replaced by $\hat{M}_{t+1}$. Periods of depreciation $\mathcal{E}_{t+1} > \mathcal{E}_t$ are associated with $\hat{M}_{t+1}$ falling (relatively high $\hat{C}_{t+1}$ is associated with low P_{t+1} due to risk sharing). For relatively stable M_{t+1}, D_{t+1} must fall– signifying C_{t+1} is low relative to $\hat{C}_{t+1}$, ceteris paribus. The sufficient condition is that the marginal Home investor in Foreign bond (George Soros) demands a high risk premium for participating in the Foreign bond markets, and thus insures the Home household through the Home asset markets.[15] We provide an illustrative example with portfolio choice in Appendix C.

Corollary 3

As $\sigma_t(d_{t+1}) \to 0$, the model collapses to a representative agent economy and condition

[15] This is reminiscent of Guvenen (2009) who constructs a closed-economy model where non-stockholder's labor income risk gets concentrated on a small set of stockholders through bond markets, and the stockholders therefore demand premia for bearing the aggregate equity risk. See also Chien et al. (2020) for a segmented markets model that explains real exchange rate volatility, low correlation of consumption growth, and high correlation of pricing kernels across countries.

(30) *is violated.*

3.1.1 Confronting volatility and predictability puzzles

Resolving the cyclicality puzzle in isolation would be unsatisfying if the model fit worsens along other dimensions. Representative agent models with trade in a single risk-free bond can reconcile the cyclicality puzzle, but only at the cost of constraining the volatility of exchange rates (Proposition 1). With trade in two nominally risk-free bonds, Lustig and Verdelhan (2019) show that introducing incompleteness cannot reconcile the cyclicality puzzle and introduces a trade-off between making exchange rates sufficiently volatile and keeping them unpredictable. In this section, we show that our proposed resolution does not lead to such trade-offs.

First, our model does not require the volatility of exchange rates to be low. Proposition 4 bounds $\frac{\sigma_{\Delta e_{t+1}}}{\sigma_{d_{t+1}}}$ to be sufficiently small as opposed to $\sigma_{\Delta e_{t+1}}$. In line with Brandt et al. (2006), who argue that exchange rates are smooth from a finance perspective, the bound is easier to satisfy as long as $\sigma_{d_{t+1}}$ is volatile enough to price other assets (e.g. equity $\approx \sigma_{m_{t+1}}$). In the next section, leveraging differences in equity returns across countries, we find evidence that $\sigma_{d_{t+1}} > \sigma_{m_{t+1}}$.

Second, a large literature demonstrates that exchange rate movements are not predictable, so any resolution to cyclicality that introduces predictability is problematic. In our model, heterogeneity as summarised by d_{t+1} is a driver of exchange rate movements which helps reconcile the cyclicality puzzle. Proposition 5 summarizes the implications for predictability of exchange rates by interest rates.

Proposition 5 (No Predictability Puzzle).
In the two-country model with two internationally traded bonds and heterogeneous marginal investors in the Home country, fluctuations in heterogeneity d_{t+1} do not appear in the domestic interest rates.

Proof: The domestic interest rate r_{t+1} is determined by (2). Conditions (26) and (28) imply moments of d_{t+1} must cancel out and do not appear in r_{t+1}. □

Because both agents participate in the domestic market for one period risk-free bonds, d_{t+1} is not spanned by the risk-free rate.[16] In particular, equation (28) implies that the conditional mean and variance of the George Soros' SDF move to perfectly offset one another in response to movement in d_{t+1}, so that r_{t+1} does not change. Therefore, domestic market incompleteness does not contribute to the predictability of exchange rates. Other asset prices such as long term bonds (Chernov and Creal, 2023; Lloyd and Marin, 2023) or equities (Chernov et al., 2024) may span d_{t+1}, depending on model specifics.

Moreover, we contrast to a complimentary explanation that taste shocks can correct the unconditional correlation between real exchange rates and relative consumption (Stockman and Tesar, 1995). Both taste shocks and heterogeneity (d_{t+1}) enter as demand shifters, therefore the latter could be considered a microfoundation for taste shocks. However, unlike taste shocks, d_{t+1} is subject to restrictions imposed by financial market structure (for example, the number of assets traded). While the de facto ability of heterogeneity to generate low risk sharing is restricted, unlike taste shocks, movements in d_{t+1} do not introduce a correlation between exchange rates and the short interest rate differentials.[17]

3.1.2 How much heterogeneity?

We now make a first pass at evaluating the plausibility of the conditions under which our 2 SDF framework can reproduce a correct pattern of risk-sharing. We begin by estimating $\hat{M}_{t+1}$ and M_{t+1} in the spirit of Hansen and Jagannathan (1991).

Consider a Home investor trading in equities:

$$\mathbb{E}_t[M_{t+1}R^e_{t+1}] = 1 \tag{31}$$

[16] In contrast, if there is non-participation in domestic bond markets, interest rates would have predictive power on exchange rates conditional on d_{t+1}.

[17] In a similar vein, Kekre and Lenel (2024*a*) show very persistent discount factor shocks do not contribute to predictability, echoing Chernov and Creal (2023)'s random walk shocks to the SDF, while resolving the unconditional Backus Smith puzzle.

where R^e_{t+1} is the return on equity. Then, we use the Hansen and Jagannathan (1991) bounds to back out a measure for $var_t(m_{t+1})$.[18] We do not measure $var_t(\hat{m}_{t+1})$ directly. Rather, we leverage the concept of "Good-Deal Bounds" of Cochrane and Saá-Requejo (2000), and ask: what additional Sharpe ratio can the domestic investor earn, by participating in the Foreign markets (like $\hat{m}$)?

Lemma 2 (Domestic Market Incompleteness and No Good Deals).
Assume now that there are no good-deals, such that $var_t(\hat{m}_{t+1}) \leq K var_t(m_{t+1})$. Then:

$$-2\mathbb{E}_t[d_{t+1}] \leq (K-1)var_t(m_{t+1}), \tag{32}$$

$$var_t(d_{t+1}) \leq var_t(m_{t+1})\left[1 + K(1 - \frac{2}{\sqrt{K}}\rho_t(\hat{m}_{t+1}, m_{t+1}))\right] \tag{33}$$

Proof. See Appendix A.2. □

The case of $K \leq 1$ corresponds to a world where the maximal Sharpe ratio available to the investor who can access Foreign markets ($\hat{m}_{t+1}$) is no higher than that of the domestic asset investor (m_{t+1}). Then, risk-sharing within the Home economy (28) and no good deals imply a limit on differences in mean SDF growth ($\mathbb{E}_t[d_{t+1}]$). In turn, the volatility of d_{t+1} is bounded by both deviations from arbitrage (K) and the SDF volatility ($var_t(m_{t+1})$), and is decreasing in the correlation between the two SDFs of domestic investors.

We evaluate the above expression using estimates from the literature.[19] A sufficiently high volatility of d_{t+1} makes it more plausible that our generalized model resolves the Backus-Smith puzzle even when there is trade in two risk-free assets.

[18] Assuming $\mathbb{E}_t[M_{t+1}] = 1$ and rearranging yields:

$$var_t(M_{t+1}) \geq sup\left(\underbrace{\frac{\mathbb{E}_t[R^e_{t+1}] - R_{t+1}}{\sqrt{var_t(R^e_{t+1})}}}_{\mathbb{E}_t[SR_{t+1}]}\right)^2$$

The right hand side (RHS) of the condition above is the squared Sharpe ratio. To maximize the RHS, we choose a high return to variance domestically-traded asset such as equity.

[19] Throughout, we proxy conditional moments in our model with unconditional estimates from the literature.

First, we take a Sharpe ratio of 0.5 annually implying $var_t(m_{t+1}) = 0.5$ as in Lustig and Verdelhan (2019). This is on the conservative side, since the gross Sharpe ratio on the S&P 500 is just above 0.6 from time-series momentum strategies (Babu, Levine, Ooi, Pedersen and Stamelos, 2020). Second, $\rho_t(\hat{m}_{t+1}, m_{t+1})$ is the correlation between the two SDFs of domestic investors. Zhang (2021) measures correlations between SDFs of various agents (domestic and foreign) and find $\rho_t(\hat{m}_{t+1}, m_{t+1}) \in [0.21, 0.5]$.[20] A lower correlation would provide a better fit for our model, so, to be conservative, we set the the correlation between the two SDFs of domestic investors to the higher value of 0.5. In the data, $var_t(\Delta e_{t+1}) = 0.11$, see e.g. Lustig and Verdelhan (2019), Lloyd and Marin (2023).

Finally, we leverage the finding in Barroso and Santa-Clara (2015) that carry trade exposure can double the Sharpe ratio of a diversified stock-bond portfolio, i.e. $K \leq 2$.[21] This would imply $var_t(d_{t+1}) = 0.89$ and therefore the threshold correlation between exchange rate growth and SDF heterogeneity is now:

$$\rho^{K=2}_{d_{t+1}, -\Delta e_{t+1}} \geq \frac{0.33}{0.89} \approx 0.37 \tag{34}$$

While this exercise is useful to calibrate the lower bound required by the mechanism, we do not have reliable estimates of the correlation between d and the exchange rate absent portfolio data for a model where measure 0 of agents participate in foreign markets. In section 3.2.1 we calibrate the average exposure of agents in the economy.

[20] They find a value of 0.5 for the correlation between the domestic and the foreign stockholder SDFs, and a value of 0.21 for within country correlation between the stockholders' and the non stockholders' SDFs.

[21] There is substantial variation in the maximum annualized Sharpe ratio documented in the literature. Jordà and Taylor (2012), Asness, Moskowitz and Pedersen (2013), and Burnside, Cerrato and Zhang (2020) find strategies with Sharpe ratio as high as 2.42, 1.59, and 3.73 respectively. Lustig, Roussanov and Verdelhan (2011), Menkhoff, Sarno, Schmeling and Schrimpf (2012), and Hassan and Mano (2019) find currency trade strategies with Sharpe ratio of 0.99, 0.95, and 0.69 respectively. Guvenen (2009) finds a standard deviation of the consumption growth of stockholders is twice as high as that of non-stockholders for the U.S.

3.2. A model with heterogeneous consumers

The starting point of our analysis is the consumption capital asset pricing model (CAPM) building on Rubinstein (1974); Lucas (1978); Breeden (1979), and the two-country model in Lucas (1982), as formulated in Lustig and Verdelhan (2019). We propose a parsimonious extension of this model to allow for heterogeneous SDFs within countries in the spirit of Constantinides and Duffie (1996).

We restrict attention to a stochastic discount factor based on a time-separable constant relative risk aversion utility function:

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-s} \tag{35}$$

where β is the discount factor, s is the inverse of the inter-temporal elasticity of substitution, and consumption growth $\frac{C_{t+1}}{C_t} \equiv \exp\left(\Delta c_{t+1}\right)$ is a random variable drawn from a lognormal distribution given by:

$$\Delta c_{t+1}^{(*)} \sim \mathcal{N}\left(\mu_{C_t^{(*)}}, \sigma^2_{C_t^{(*)}}\right)$$

where $\mu_{C_t^{(*)}}$ and $\sigma^2_{C_t^{(*)}}$ are conditional mean and variance of this process, and the asterisks denote the corresponding setup for the Foreign economy.

We extend this representative agent environment to allow for a continuum of consumers indexed by i in the Home economy. Each of these consumers faces uninsurable idiosyncratic consumption risk and trades frictionlessly in Home and Foreign risk-free bonds. Critically, building on a large literature assessing whether idiosyncratic risk matters for aggregate pricing, we assume that an individual consumption draw relates to the aggregate consumption draw with the following log-normal heteroskedastic process:

$$\Delta c_{t+1}^{i} = \log \left(\frac{\delta_{t+1}^{i}}{\delta_t^{i}} \frac{C_{t+1}}{C_t} \right) \sim \mathcal{N}(\mu_{c_t^i}, \sigma^2_{i,t}) \tag{36}$$

where the individual consumption exposures satisfy:

$$\int_i \delta_t^i \, di = 1, \quad \log\left(\frac{\delta_{t+1}^i}{\delta_t^i}\right) \sim \mathcal{N}(\mu_{\delta_t}, \sigma_{\delta_t}^2), \tag{37}$$

$\forall t$. The conditional means and variances for the corresponding normal distributions are noted in the parentheses above.[22] Together, Equations (36) and (37) thus imply $\mu_{c_t^i} = \mu_t^\delta + \mu_{c_t^i}$ and $\sigma_{i,t}^2 = \sigma_{\delta,t}^2 + \sigma_{C,t}^2 + 2\sigma_{\delta_t,C_t}$, where $\sigma_{\delta,t}^2 \equiv var_t\left(\log\left(\frac{\delta_{t+1}^i}{\delta_t^i}\right)\right)$, and $\sigma_{\delta_t,C_t} \equiv cov_t\left(\log\left(\frac{\delta_{t+1}^i}{\delta_t^i}\right), \log\left(\frac{C_{t+1}}{C_t}\right)\right)$. We maintain that the Foreign country has a representative agent.

Several functional forms for δ_t^i can satisfy our requirements (in particular, δ_t^i sum to one by the law of large numbers). One such process, proposed by Constantinides and Duffie (1996), is given by:

$$\frac{\delta_{t+1}^i}{\delta_t^i} = \exp(z_{t+1}^i x_{t+1} - x_{t+1}^2/2) \tag{38}$$

where z_{t+1}^i is distributed as standard normal for all i and t and is independently distributed from the aggregate state x_{t+1}.[23] This process satisfies (37) implying $\log(\delta_{t+1}^i/\delta_t^i) \sim \mathcal{N}(-\frac{x_{t+1}^2}{2}, x_{t+1}^2)$, where x_{t+1}^2 is assumed to be a linear function of aggregate consumption growth, therefore drawing a tight link between idiosyncratic risk and the aggregate state.

Consider the problem faced by an individual Home household i. No-arbitrage pricing requires that the Euler equations for household i investing in Home bonds and Foreign bonds be satisfied:

$$\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-s}\right] = \mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s}\left(\frac{\delta_{t+1}^i}{\delta_t^i}\right)^{-s}\right] = \frac{1}{R_t}, \tag{39}$$

$$\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-s}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = \mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s}\left(\frac{\delta_{t+1}^i}{\delta_t^i}\right)^{-s}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = \frac{1}{R_t^*} \tag{40}$$

[22] We repeatedly use the result that if $X \sim \mathcal{N}(\mu_X, \sigma_X)$, $e^X \sim \log\mathcal{N}(\mu_X, \sigma_X)$ and if $Y \sim \mathcal{N}(\mu_Y, \sigma_Y)$, then $Z = XY \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X + \sigma_Y + 2\sigma_{XY})$. From this it follows that $e^Z \sim \log\mathcal{N}(\mu_X + \mu_Y, \sigma_X + \sigma_Y + 2\sigma_{XY})$.

[23] The law of large numbers follows from properties of the normal distribution for z^i. Treating x_{t+1} as a constant, and using the moment generating function $M_z(h) = e^{hz}$ for $h \in \mathbb{R}$, $\mathbb{E}[e^{z^i x - x^2/2}] = M_z(x)e^{-x^2/2} = e^0 = 1$.

Taking a log expansion and using the SDF definition (35) the Home Euler equation for the Home bond can be written as:

$$-r_t = \mathbb{E}_t[m_{t+1}] + \frac{1}{2}[var_t(m_{t+1}) + s^2 var_t(\log(\delta^i_{t+1})) + 2s^2 cov_t(\log(\delta^i_{t+1}), c_{t+1}))] \quad (41)$$

where the moments of the SDF based on aggregate consumption are given by $\mathbb{E}_t[m_{t+1}] = -s\mu_{C_t}$ and $var_t[m_{t+1}] = s^2\sigma^2_{C_t}$. Moreover, the mean SDF of any two individuals i and j are equated $\mathbb{E}_t[M^i_{t+1}] = \mathbb{E}_t[M^j_{t+1}]$ by risk-sharing. Equation (39) is the analogue of Equation (2) but the RHS now has extra terms which reflect a precautionary motive from facing idiosyncratic consumption risk.

Turning to the Euler for a Home household i investing in Foreign bonds (the analogous equation to equation (5)), and assuming Δe_{t+1} is also jointly normally distributed, we get:

$$-r^*_t = \mathbb{E}_t[m_{t+1}] + \frac{1}{2}[var_t(m_{t+1}) + s^2 var_t(\log(\delta^i_{t+1})) + 2s^2 cov_t(\log(\delta^i_{t+1}), c_{t+1})] + \\ \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2} var_t(\Delta e_{t+1}) + cov_t(m^i_{t+1}, \Delta e_{t+1}) \quad (42)$$

where,

$$cov_t(m^i_{t+1}, \Delta e_{t+1}) = cov_t(m_{t+1}, \Delta e_{t+1}) - s\ cov_t(\log(\delta^i_{t+1}), \Delta e_{t+1})$$

which follows from $m^i_{t+1} = -s\Delta c_{t+1} + \log(\delta^i_{t+1}/\delta^i_t)$ and $m_{t+1} = -s\Delta c_{t+1}$.[24]

This extended economy again admits the exchange rate process described by Equation (7), but with new restrictions imposed on the incomplete markets wedge η_{t+1}.[25] Cross-border trade in Home and Foreign bonds imply:

$$\mathbb{E}_t\left[M_{t+1}e^{-\eta_{t+1}}\right] = \mathbb{E}_t\left[M^i_{t+1}\right],$$

[24] Lemma 2 provides one specific model (the autarky limit) which connects aggregate shocks with depreciation Δe_{t+1}, giving rise to a covariance between idiosyncratic risk and exchange rate movements. In general, idiosyncratic risk can be related to exchange rates through a number of mechanisms.

[25] This can be seen because (3),(4), (39), and (40) imply:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) + cov_t(\delta^i_{t+1}, \Delta e_{t+1})$$

$$\mathbb{E}_t\left[M^i_{t+1}\frac{M^*_{t+1}}{M_{t+1}}e^{\eta_{t+1}}\right] = \mathbb{E}_t\left[M^*_{t+1}\right]$$

The resulting restrictions on the incomplete market wedge are given by:

$$\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2}var_t(\eta_{t+1}) - \log\mathbb{E}_t\left[\left(\frac{\delta^i_{t+1}}{\delta^i_t}\right)^{-s}\right] - cov_t\left(m_{t+1}, \eta_{t+1} - s\log\left(\delta^i_{t+1}\right)\right), \quad (43)$$

$$-\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2}var_t(\eta_{t+1}) + \log\mathbb{E}_t\left[\left(\frac{\delta^i_{t+1}}{\delta^i_t}\right)^{-s}\right] - cov_t\left(m^*_{t+1}, -\eta_{t+1} + s\log\left(\delta^i_{t+1}\right)\right) \quad (44)$$

The non-traded component of exchange rates now reflects idiosyncratic risk, since a Home investor taking a foreign position must be compensated for this exposure.

The next proposition details how the model with heterogeneous consumers can deliver low risk sharing if only the Foreign bond is internationally traded.

Proposition 6 (One Int'l Traded Asset, Heterogeneous Consumers).
When only Foreign bonds are internationally traded and there exist a continuum of heterogeneous Home consumers such that Equations (3), (39), and (40) hold, then $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ *if and only if:*

$$\begin{aligned} cov_t(m_{t+1}, \eta_{t+1}) + \log\mathbb{E}_t[e^{\eta_{t+1}}] \geq \quad & var(m^*_{t+1} - m_{t+1}) \\ & - \log\mathbb{E}_t\left[\frac{\delta^i_{t+1}}{\delta^i_t}^{-s}\right] + s\; cov_t(\log(\delta^i_{t+1}), m^*_{t+1} + \eta_{t+1}) \end{aligned}$$

Proof. Follows from the Proof of Proposition 1 and Equations (44) and (43). □

The model can thus deliver low risk-sharing if the non-traded component is sufficiently safe, generalizing Proposition 1 to an environment with idiosyncratic risk. The threshold for this requirement is now endogenously a function of the idiosyncratic risk. Indeed, without relying on the covariance term (i.e. focusing on the case where idiosyncratic and aggregate risk are uncorrelated), the condition is relaxed by the variance of the idiosyncratic consumption risk faced by the agents.[26] However, the

[26]Furthermore, we can see that low risk sharing can also be attained when the consumption risk is countercyclical, namely $cov_t(\log(\delta^i_{t+1}), c_t) < 0$. In macroeconomic models (Werning, 2015; Bilbiie, 2024), it has been shown that countercyclical consumption risk offers amplification of economic mechanisms. Auclert et al. (2021) generalize Werning (2015)'s as-if complete markets representative agent result to an open economy setting.

variance term cannot help reconcile low risk-sharing when two nominally risk-free bonds are internationally traded.

Proposition 7 (Two Int'l Traded Assets, Heterogeneous Consumers).
The two-country model with two internationally traded bonds and a continuum of heterogeneous Home consumers characterized by Equations (3), (4), (39), and (40), can deliver $cov_t(m^*_{t+1} - \log(\int_i e^{\Delta c^i_t} di)^{-s}, \Delta e_{t+1}) < 0$ *if and only if*

$$1 \geq \rho_{-\delta^i_{t+1}, -\Delta e_{t+1}} \geq \frac{\sigma_t(\Delta e_{t+1})}{s\sigma_t(\log(\delta^i_{t+1}))} \tag{45}$$

where $\rho_{-\log(\delta^i_{t+1}), -\Delta e_{t+1}} \equiv \dfrac{cov_t(-s\delta^i_{t+1}, -\Delta e_{t+1})}{\sigma_t(\Delta e_{t+1})\sigma_t(s\log(\delta^i_{t+1}))}$.

Proof. See Appendix A.2. □

Whereas in a representative agent economy, exchange rate risk becomes traded (is spanned) when households trade in both Home and Foreign real bonds across borders, introducing idiosyncratic consumption risk —which co-moves with the exchange rate— recovers a non-traded component and can deliver low risk-sharing.[27] If idiosyncratic consumption risk rises with depreciations, Foreign bonds are a poor hedge for domestic consumption and low risk-sharing persists even with trade in two nominally risk-free bonds, consistent with Heathcote and Perri (2013). While the model presented in this section has distinct mechanisms from the Soros model in Section 3.1, the conditions under which both models reconcile low risk sharing both rely on the covariance of uninsurable (unspanned) domestic risk with the exchange rate risk.

Similar with the measure zero model, the model with heterogeneous domestic consumers can resolve the cyclicality puzzle without constraining the volatility of exchange rates or introducing predictability through the risk-free rate. Building on

[27] Note that we adjust the risk-sharing measure to consider the co-movement of aggregate consumption and exchange rates, in SDF space, consistent with the original definition of the Backus-Smith covariance. Kocherlakota and Pistaferri (2007) consider domestic market incompleteness but internationally complete markets and derive a link between real exchange rate growth and, potentially, higher order moments of the cross-sectional consumption distribution.

our discussion in Section 3.1.1, we see from Proposition 7 that our model, again, does not contribute to the volatility puzzle. As in the measure zero model, there are parametrizations for which the risk-free rate (41) does not span moments of δ^i_{t+1}. Specifically, this would require $cov_t(\log(\delta^i_{t+1}), c_{t+1}) = -\frac{1}{2}var_t(\log(\delta^i_{t+1}))$ (i.e. a particular cyclicality of consumption risk with respect to domestic endowment), but there are no equilibrium conditions imposing this restriction.

Note that ex-post heterogeneity alone is enough to obtain low risk sharing. Building on the contributions of Weil (1992) and Krueger and Lustig (2010), consider the special case of a two period environment $t = \{0, 1\}$. Agents are identical at date 0 but face idiosyncratic risk drawn from a common distribution at date 1 $c^i_1 = \log C^i_1 \sim \mathcal{N}(\mu_c, \sigma^i_c)$. Idiosyncratic risk can be correlated to aggregate risk and exchange rate depreciation. As before, it is useful to denote the distribution of individual consumption $c^i_1 = \log(\delta^i_1 C_1) \sim \mathcal{N}(\mu_c, \sigma^2_c + \sigma^2_\delta + 2\sigma_{c\delta})$. Conditions (39) and (40) go through unchanged, therefore Propositions 6 and 7 can be recovered in this environment. In the limit with zero liquidity in Foreign bonds, this two-period model can reflect a no-trade equilibrium where agents consume their endowment in every period.

3.2.1 How much heterogeneity, revisited

Finally, we do a back of the envelope calculation to ascertain the empirical plausibility of our mechanism in the model with heterogeneous consumers. We start with the estimates of the amount of idiosyncratic risk from US data. Constantinides (2021) estimates a cross-sectional standard deviation of consumption growth of 0.4, which we use to calibrate σ_δ.[28] We set the inter-temporal elasticity of substitution to 0.1 based on the evidence in Best, Cloyne, Ilzetzki and Kleven (2020), which implies $s = 10$.

[28]This is consistent with estimates for the standard deviation of (annual) idiosyncratic earnings risk of 0.9 in Floden and Lindé (2001) used in Auclert, Rognlie and Straub (2023). Similarly, Acharya, Challe and Dogra (2023) recover an estimate of 0.5 using the evidence in Guvenen, Ozkan and Song (2014). Note that our estimate for σ_δ is lower than for σ_d which is obtained using the asset-price based approach in Section 3.1.2. Therefore, exposure at the country level can be lower than for the individual marginal investor.

Together, these values imply that to achieve low risk-sharing we require a correlation coefficient that is at least as high as :

$$\rho_{-\delta^i_{t+1},-\Delta e_{t+1}} \geq \frac{0.33}{4} = 0.0825 \tag{46}$$

Alongside the process for $\delta^i_{t+1}/\delta^i_t$ given by (38), we further assume two reduced-form macroeconomics relationships. First, as in Constantinides and Duffie (1996), we tie the volatility of idiosyncratic risk to the aggregate state $x^2_{t+1} = \sigma^2_\delta + \phi\Delta c_{t+1}$. We calibrate $\phi = -5.76$ borrowing from Acharya et al. (2023) estimate for the cyclicality of income risk. Second, we assume a reduced form relationship between aggregate consumption and exchange rates $\Delta c_t = \alpha\Delta e_t + \nu_t$, where ν_t is defined as an orthogonal component.[29] Verner and Gyöngyösi (2020) document that an approximately 30% nominal depreciation of Hungarian forint (against the euro) was associated with increase in debt of 10% of disposable income.[30] An average marginal propensity to consume of 0.22 (measured as the percentage points decline in non-durable consumption from a percentage point increase in debt) implies a value of $\alpha = -0.073$. Then,

$$cov_t(-\delta^i_{t+1}, -\Delta e_{t+1}) = -\alpha\frac{\phi}{2}\sigma^2_t(\Delta e_{t+1}) = 0.073 \times 2.88 \times 0.11 = 0.0231 \tag{47}$$

and hence $\rho_{-\delta^i_{t+1},-\Delta e_{t+1}} = 0.175$, significantly higher than required.[31]

4. Conclusion

A classical strand of the literature in international macroeconomics has focused on formulating goods-market mechanisms which generate a negative relationship between consumption growth and depreciation– the opposite sign to that implied by the

[29]This is consistent with a macroeconomic factor model, e.g. Chernov and Creal (2023), who allow exchange rates to feed back into SDFs. A linear relationship between c_t and Δe_t also emerges at the autarky limit of our equilibrium model in Section 2.3.

[30]This corresponds roughly in real terms since inflation in Hungary was about 3% and 0.5% in Europe.

[31]This derivation uses the properties of z^i_{t+1} which imply the law of large numbers of $\delta^i_{t+1}/\delta^i_t$. Specifically $\mathbb{E}_t[z^i_{t+1}x_{t+1}\Delta e_{t+1}] - \mathbb{E}_t[z^i_{t+1}x_{t+1}]\mathbb{E}_t[\Delta e_{t+1}] = 0$, therefore $cov_t(-\delta^i_{t+1}, -\Delta e_{t+1}) = cov_t(-\frac{1}{2}x^2_{t+1}, \Delta e_{t+1})$.

Backus-Smith condition– as long as financial markets are incomplete. We show that any model which achieves this resolution must rely on a non-traded component to relative prices which is "safe" from a domestic investor perspective. However, Lustig and Verdelhan (2019) determine that any two-country model with a representative agent and frictionless trade in Home and Foreign currency denominated risk-free bonds recovers the exchange rate cyclicality implied by complete markets.

We propose generalizations of the model beyond the representative agent no-arbitrage benchmark– first a model with two SDFs featuring limited asset market participation and then a model of heterogeneous consumers trading in integrated markets but facing uninsurable idiosyncratic risk. Introducing heterogeneity in SDFs generates a non-traded component to exchange rate movements, resulting in low risk-sharing even with cross-border trade in both Home and Foreign bonds. Back-of-the-envelope calculations, based on the asset prices and micro estimates, suggest heterogeneity in SDFs is a plausible mechanism to reconcile patterns of cross-country risk sharing.

Our paper highlights the need to extend a workhorse international macroeconomic model to allow not only for heterogeneous agents but to specifically account for portfolio choice with both idiosyncratic and aggregate risk. This is a challenging endeavour that goes beyond the existing literature which either abstracts from heterogeneity (Benigno and Küçük, 2012; Heathcote and Perri, 2013) or delegates the portfolio choice problem to a mutual fund (Auclert et al., 2021).[32]

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A. Appendix

A.1. Additional Derivations for Section 2.

To find the admissible set of processes, consider the log expansions of the above conditions, assuming joint log normality:

$$\mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) = -r_{t+1}, \tag{48}$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) = -r^*_{t+1} \tag{49}$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) - \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m^*_{t+1}, -\Delta e_{t+1}) = -r_{t+1}, \tag{50}$$

$$\mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) + \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m_{t+1}, \Delta e_{t+1}) = -r^*_{t+1}, \tag{51}$$

where lower case levels denote logs, e.g. $\log(M_{t+1}) = m_{t+1}$ and $\Delta e_{t+1} = e_{t+1} - e_t$. Using (48) and (51), and (49) and (50) respectively, yields:

$$\mathbb{E}_t[\Delta e_{t+1}] + r^*_{t+1} - r_{t+1} = -cov_t(m_{t+1}, \Delta e_{t+1}) - \frac{1}{2}var_t(\Delta e_{t+1}), \tag{52}$$

$$\mathbb{E}_t[\Delta e_{t+1}] + r^*_{t+1} - r_{t+1} = cov_t(m^*_{t+1}, -\Delta e_{t+1}) + \frac{1}{2}var_t(\Delta e_{t+1}) \tag{53}$$

A.2. Proofs to Propositions

Proof to Proposition 1 The Backus-Smith condition is related to the covariance $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$ which can be rewritten as:

$$cov_t(m^*_{t+1} - m_{t+1}, m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) \tag{54}$$

$$= var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1}, \eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1}) \tag{55}$$

Imposing (9) (international trade in the Foreign asset), but not (8) (international trade in the Home asset) as is done in Lustig and Verdelhan (2019), assuming $\mathbb{E}_t[\eta_{t+1}] = 0$, and rearranging yields the result. □

Proof to Corollary 1 The volatility of the exchange rate is given by:

$$var_t(\Delta e_{t+1}) = var(m^*_{t+1} - m_{t+1}) + var_t(\eta_{t+1}) + 2cov_t(m^*_{t+1}, \eta_{t+1}) - 2cov_t(m_{t+1}, \eta_{t+1})$$

Imposing (8) and (9):

$$var_t(\Delta e_{t+1}) = var(m^*_{t+1} - m_{t+1}) - var_t(\eta_{t+1})$$

Taking the limit $cov_t(m_{t+1}, \eta_{t+1}) \to$ (10) would imply $var_t(\Delta e_{t+1}) < 0$ which cannot be an equilibrium. □

Proof to Proposition 2: In Section B.1 below, we show that (12)-(15) imply:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) - cov_t(\Delta e_{t+1} - m^*_{t+1} + m_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \tag{56}$$

Using (86) and imposing $\Delta e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1}$:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) - cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \tag{57}$$

In that case, $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ if and only if $var_t(\Delta e_{t+1}) + cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) < 0$. □

Proof to Corollary 2: Next, suppose we reintroduce trade in risk-free bonds. Then (8) and (9) hold. In particular, introducing a Home internationally traded risk-free bond implies:

$$cov_t(\tilde{r}_{t+1}, \eta_{t+1}) = 0 \tag{58}$$

Introducing a Foreign internationally traded risk-free bond implies:

$$cov_t(\tilde{r}^*_{t+1}, \eta_{t+1}) = 0 \tag{59}$$

$\square$

Proof to Proposition 3 From Section B.3 below:

$$cov_t(m_{t+1}, \eta) = -\frac{s(1-s)}{1-2(1-\phi)} var_t(g_{y_{H,t+1}}), \tag{60}$$

$$cov_t(m^*_{t+1}, \eta) = -\frac{s(1-s)}{1-2(1-\phi)} var_t(g_{y_{F,t+1}}), \tag{61}$$

$$var_t(m_{t+1} - m^*_{t+1}) = s^2 var_t(g_{y_{H,t+1}} - g_{y_{F,t+1}}), \tag{62}$$

$$\frac{1}{2} var_t(\eta_{t+1}) = \frac{1}{2}\left[\frac{1-s}{1-2(1-\phi)}\right]^2 var_t(g_{y_{H,t+1}} - g_{y_{F,t+1}}) \tag{63}$$

Assuming $var_t(g_{y_{H,t+1}} - g_{y_{F,t+1}}) = var_t(g_{y_{H,t+1}})$ (i.e no covariance and conditioning on H shocks), under financial autarky, the result is found by computing $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$, where Δe_{t+1} is detailed in Lemma 1.

When there is trade in the Foreign bond across borders, (9) implies that $cov_t(m^*_{t+1}, \eta_{t+1}) = -\frac{1}{2} var_t(m_{t+1})$.[33] Imposing this restriction:

$$cov_t^{\to FA}(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + cov_t^{\to FA}(m^*_{t+1}, \eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1})$$

and $cov_t^{\to FA}(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) \leq 0$ if:

$$\frac{-s(1-s)}{1-2(1-\phi)} > s^2 - \frac{1}{2}\left[\frac{1-s}{1-2(1-\phi)}\right]^2 \tag{64}$$

Naturally, $cov_t^{\to FA}(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$ at the autarky limit defined by $ToT_t C_{F,t} = C_{H,t}$.

Analogously, with both Home and Foreign bonds internationally traded, (8) also

[33] In general, the two quantities are not equivalent in the FA limit, but do coincide at $\phi\sigma = \frac{1}{2}$, which is outside of the region of interest.

binds. Then,

$$cov_t^{\to FA}(m^*_{t+1}-m_{t+1},\Delta e_{t+1}) = var_t(m^*_{t+1}-m_{t+1})+cov_t^{\to FA}(m^*_{t+1},\eta_{t+1})-cov_t^{\to FA}(m_{t+1},\eta_{t+1})$$

and $cov_t^{\to FA}(m^*_{t+1}-m_{t+1},\Delta e_{t+1}) \leq 0$ requires:

$$\frac{1}{2}\left[\frac{1}{1-2(1-\phi)}\right]^2 \leq 0$$

which is only satisfied with equality at $\phi \to \infty$ so that $var_t(\Delta e_{t+1}) = 0$. □

Deriving exchange rate process for section 3.1. We begin by deriving the condition that must be satisfied by an exchange rate progress satisfying no-arbitrage in the generalized model. Combining (2), (3), (24), (28) yields:

$$\mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m_{t+1},\Delta e_{t+1}) + cov_t(d_{t+1},\Delta e_{t+1}) + r^*_{t+1} - r_{t+1} = 0, \quad (65)$$

$$-\mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m^*_{t+1},-\Delta e_{t+1}) - r^*_{t+1} + r_{t+1} = 0 \quad (66)$$

Combining the above, the restriction that must be satisfied by any exchange rate process which admits no arbitrage is therefore:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1}-m_{t+1},\Delta e_{t+1}) - cov_t(d_{t+1},\Delta e_{t+1}) \quad (67)$$

Assuming $\Delta e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1}$:

$$var_t(\Delta e_{t+1}) = var_t(m^*_{t+1}-m_{t+1}) + var_t(\eta_{t+1}) + 2cov_t(m^*_{t+1}-m_{t+1},\eta_{t+1}) \quad (68)$$

Using equations (8) and (29), we can express the covariance term as

$$\begin{aligned} cov_t(m^*_{t+1}-m_{t+1},\eta_{t+1}) = {} & -\mathbb{E}_t[d_{t+1}] - \frac{1}{2}var_t(d_{t+1}) - cov_t(m^*_{t+1},d_{t+1}) \\ & -cov_t(d_{t+1},\eta_{t+1}) - var_t(\eta_{t+1}) \end{aligned} \quad (69)$$

Using equations (28) and (69), we can simplify equation (68) :

$$\begin{aligned} var_t(\Delta e_{t+1}) &= var_t(m^*_{t+1}-m_{t+1}) + var_t(\eta_{t+1}) + cov_t(m^*_{t+1}-m_{t+1},\eta_{t+1}) + \cdots \\ &\quad \left\{cov_t(m_{t+1},d_{t+1}) - cov_t(m^*_{t+1},d_{t+1}) - cov_t(d_{t+1},\eta_{t+1}) - var_t(\eta_{t+1})\right\}, \\ var_t(\Delta e_{t+1}) &= var_t(m^*_{t+1}-m_{t+1}) + cov_t(m^*_{t+1}-m_{t+1},\eta_{t+1}) - cov_t(d_{t+1},\Delta e_{t+1}) \end{aligned} \quad (70)$$

so equation (67) is satisfied. $\square$

Proof to Proposition 4 The covariance can be rewritten as:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1}, \eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1}),$$

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) - \cdots$$
$$\left\{\mathbb{E}_t[d_{t+1}] + \frac{1}{2}var_t(d_{t+1}) + cov_t(m^*_{t+1} + \eta, d_{t+1}) + \mathbb{E}_t[\eta_{t+1}] + \frac{1}{2}var_t(\eta_{t+1})\right\} + \cdots$$
$$\left\{\mathbb{E}_t[\eta_{t+1}] - \frac{1}{2}var_t(\eta_{t+1})\right\}$$

Simplifying:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) - \cdots$$
$$\left\{\log \mathbb{E}_t[D_{t+1}] + cov_t(m^*_{t+1} + \eta, d_{t+1})\right\} - var_t(\eta_{t+1}) \tag{71}$$

where $\log \mathbb{E}_t[D_{t+1}] = \mathbb{E}_t[d_{t+1}] + \frac{1}{2}var_t(d_{t+1})$. Using equation (29), we can rewrite this:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(\Delta e_{t+1}) + \log \mathbb{E}_t D_{t+1} + cov_t(m_{t+1} - \eta_{t+1}), d^*_{t+1}) \tag{72}$$

The model can reconcile Backus-Smith if and only if:

$$var_t(\Delta e_{t+1}) + \log \mathbb{E}_t[D_{t+1}] + cov_t(m^*_{t+1} + \eta_{t+1}, d_{t+1}) \leq 0 \tag{73}$$

Additionally, using equation (28) we get:

$$cov_t(d_{t+1}, -\Delta e_{t+1}) \geq var_t(\Delta e_{t+1}) \tag{74}$$

Finally, the Cauchy Schwarz identity implies:

$$cov_t(d_{t+1}, -\Delta e_{t+1}) \leq \sqrt{var_t(d_{t+1}) var_t(\Delta e_{t+1})} \tag{75}$$

Combining the inequalities and dividing by $\sigma_t(d_{t+1})$ yields the result. $\square$

Proof to Corollary 4 With heterogeneous marginal investors in the domestic country, when only the Foreign bond is traded across borders, the relevant Euler equations are (2), (3),(24) and (26). Using (55) but replacing (9) by (29) yields the result. $\square$

Proof to Proposition 7. First, we deal with constructing the correct measure of the Backus-Smith correlation, in SDF space, when there are heterogeneous agents. If log quantities are jointly normally distributed it follows that the log of the Backus-Smith covariance in this economy is equal to the covariance of the logs:[34]

$$\log\left(cov_t\left(\int_i e^{\Delta c^i_{t+1}} di, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)\right) = cov_t\left(\log \underbrace{\int_i e^{\Delta c^i_{t+1}} di}_{C_{t+1}}, \Delta e_{t+1}\right) \implies$$

$$\log\left(cov_t\left(\int_i M^i_{t+1} di, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)\right) = cov_t\left(m_{t+1}, \Delta e_{t+1}\right) \tag{76}$$

Then combining Equations (3), (4), (39), and (40) leads to:

$$var_t(\Delta e_{t+1}) \leq s\ cov_t(-\log(\delta^i_{t+1}), -\Delta e_{t+1}) \tag{77}$$

Using the Cauchy-Schwarz identity as in Proposition 4 gives the result. □

Proof to Lemma 2 The volatility of $\hat{m}_{t+1}$ is given by:

$$var_t(\hat{m}_{t+1}) = var_t(m_{t+1}) + var_t(d_{t+1}) + 2cov_t(m_{t+1}, d_{t+1}) \tag{78}$$

However, since the investors in the Home country share risk, $cov_t(m_{t+1}, d_{t+1})$ is pinned down by equation (28). The result on $\mathbb{E}_t[d_{t+1}]$ follows by substituting $var_t(\hat{m}_{t+1}) = K var_t(m_{t+1})$ in equation (78) and imposing within country risk-sharing equation (28).

For the variance bound, consider:

$$var_t(d_{t+1}) = var(\hat{m}_{t+1}) + var(m_{t+1}) - 2cov_t(\hat{m}_{t+1}, m_{t+1}),$$
$$var_t(d_{t+1}) = var(\hat{m}_{t+1}) + var(m_{t+1}) - 2\rho_t(\hat{m}_{t+1}, m_{t+1})\sigma_t(\hat{m}_{t+1})\sigma_t(m_{t+1})$$

Then,

$$var_t(d_{t+1}) \leq (K+1)var(m_{t+1}) - 2\rho_{\hat{m}_{t+1}, m_{t+1}}\sqrt{K}\sigma^2_t(m_{t+1}), \tag{79}$$

Rearranging yields the result. □

[34]In general $cov_t(X_{t+1}, Y_{t+1}) = \mathbb{E}_t[X_{t+1}]\mathbb{E}_t[Y_{t+1}]\left(e^{cov_t(U,V)} - 1\right)$ when $U, V \sim \mathcal{N}$.

B. ONLINE APPENDIX

B.1. Trade in Risky Assets

Suppose Home and Foreign households trade in Home and Foreign currency denominated risky assets $\tilde{R}_{t+1}$ such that (12)- (15) hold. Assuming joint log normality, the above Euler equations imply:

$$\mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) + cov_t(m_{t+1}, \tilde{r}_{t+1}) = 0, \quad (80)$$

$$\mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \cdots$$
$$cov_t(m_{t+1}, \tilde{r}^*_{t+1}) + cov_t(m_{t+1}, \Delta e_{t+1}) + cov_t(\Delta e_{t+1}, \tilde{r}^*_{t+1}) = 0, \quad (81)$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + cov_t(m^*_{t+1}, \tilde{r}^*_{t+1}) = 0, \quad (82)$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) - \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) + \cdots$$
$$cov_t(m^*_{t+1}, \tilde{r}_{t+1}) + cov_t(m^*_{t+1}, -\Delta e_{t+1}) + cov_t(-\Delta e_{t+1}, \tilde{r}_{t+1}) = 0 \quad (83)$$

Combining (80) and (81):

$$\mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) - \mathbb{E}_t[\tilde{r}_{t+1}] - \frac{1}{2}var_t(\tilde{r}_{t+1}) + \cdots$$
$$cov_t(m_{t+1}, \Delta e_{t+1}) + cov_t(\Delta e_{t+1}, \tilde{r}^*_{t+1}) + cov_t(m_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) = 0 \quad (84)$$

Combining (82) and (83):

$$-\mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) - \mathbb{E}_t[\tilde{r}^*_{t+1}] - \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + \cdots$$
$$cov_t(m^*_{t+1}, -\Delta e_{t+1}) + cov_t(-\Delta e_{t+1}, \tilde{r}_{t+1}) - cov_t(m^*_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) = 0 \quad (85)$$

Together, the above conditions yield:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) - cov_t(\Delta e_{t+1} - m^*_{t+1} + m_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \quad (86)$$

Assuming the exchange rate process is given by $\Delta e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1}$ this condition reduces to:

$$var_t(\Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) - cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \quad (87)$$

Imposing the exchange rate process, we can derive restrictions to the incomplete market wedge analogous to equations (8) and (9). Then, doing a log expansion from combining equations (15), (80), and the exchange rate process, we get:

$$cov_t(m_{t+1}, \eta_{t+1}) = -\mathbb{E}_t[\eta_{t+1}] + \frac{1}{2}var_t(\eta_{t+1}) - cov_t(\tilde{r}_{t+1}, \eta_{t+1}) \tag{88}$$

Additionally, equations (13) and (82) imply:

$$cov_t(m^*_{t+1}, \eta_{t+1}) = -\mathbb{E}_t[\eta_{t+1}] - \frac{1}{2}var_t(\eta_{t+1}) - cov_t(\tilde{r}^*_{t+1}, \eta_{t+1}) \tag{89}$$

The volatility of the exchange rate is given by:

$$\begin{aligned} var_t(\Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + var_t(\eta_{t+1}) + 2cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) = \cdots \\ var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) - cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \end{aligned} \tag{90}$$

which verifies (86), so the exchange rate process is admissible.

B.2. Degree of Market Completeness and Risk-Sharing

To illustrate that two nominally risk-free asssets suffice to span exchange rates, regardless of market completeness, we turn to a framework in the tradition of Lucas (1978). Agents have time-separable, CRRA preferences over an exogenous consumption stream. In particular:

$$\Delta c_{t+1} = \sum_{k=1}^{N} g_{y_{k,t+1}}, \tag{91}$$

$$m_{t+1} = -s\Delta c_{t+1}, \tag{92}$$

where $g_{y_{k,t+1}} = y_{k,t+1} - y_{k,t} \sim i.i.d\ \mathcal{N}(\mu_{y_k}, \sigma_{y_k})$ denotes the growth rate of k-th productive unit that comprises the consumption good. Corresponding variables for the Foreign economy are denoted with an asterisk. Start with the case of $N = 1$ productive units, discussed in (Lustig and Verdelhan, 2019, Sec III.A). Frictionless international trade in Home and Foreign risk-free bonds (8)- (9) and additional trade in a Home and a Foreign risky asset (a claim on $g_{y_{k,t+1}}, g^*_{y_{k,t+1}}$ respectively) will imply that the incomplete markets wedge η_{t+1} is orthogonal to $g_{y_{k,t+1}}, g^*_{y_{k,t+1}}$, and it then follows that the only equilibrium is $\eta_{t+1} = 0$– i.e. markets are complete. When $N > 1$, additional risky claims need to traded to complete the market. However, for any N, frictionless international trade in just the Home and the Foreign real bonds ensures risk-sharing consistent with complete markets (Corollary 1). Relatedly, Chernov et al. (2024) show that incompleteness alone cannot match exchange rate puzzles, unless there is some lack of integration in financial markets (i.e. only Foreign bonds traded as in Corollary 2).

B.3. An equilibrium model

In this appendix we build on the equilibrium two-country, two-good, endowment model in Section 2.3. We derive the autarky limit and express SDFs and prices as functions of exogenous variables. Starting with the static conditions:

$$C = \left[\alpha^{\frac{1}{\phi}} C_H^{\frac{\phi-1}{\phi}} + (1-\alpha)^{\frac{1}{\phi}} C_F^{\frac{1}{\phi}}\right]^{\frac{\phi}{\phi-1}} \tag{93}$$

Relative demand for goods requires:

$$\frac{C_F}{C_H} = \tfrac{1-\alpha}{\alpha} ToT^{-\phi}, \tag{94}$$

$$Y \tfrac{C_F^*}{C_H^*} = \tfrac{1-\alpha^*}{\alpha^*} ToT^{-\phi}, \tag{95}$$

where τ denotes the terms of trade.Market clearing requires:

$$C_H + C_H^* = Y_H \tag{96}$$

$$C_F + C_F^* = Y_F \tag{97}$$

The real exchange rate is given by $\mathcal{E} = \frac{P^*}{P}$ and the terms of trade by $ToT = \dfrac{P_F}{P_H}$. The law of one price holds for each good but not the aggregate basket unless $\alpha = \alpha^*$.

Under financial autarky, $PC = P_H Y_H$ and $P^* C^* = P_F Y_F$. Combining this with relative demand yields:

$$C_H + ToT\ C_F = Y_H, \tag{98}$$

$$C_F = \frac{1-\alpha}{\alpha}\left(\frac{1}{ToT}\right)^{\phi} C_H, \quad C_H = Y_H\left[\frac{\alpha}{\alpha + ToT^{1-\phi}(1-\alpha)}\right] \tag{99}$$

Abroad,

$$C_H^* ToT^{-1} + C_F^* = Y_F, \tag{100}$$

$$C_F^* = \frac{1-\alpha^*}{\alpha^*}\left(\frac{1}{ToT}\right)^{\phi} C_H^*, \quad C_H^* = Y_F\left[\frac{\alpha^*}{\alpha^* ToT^{-1} + ToT^{-\phi}(1-\alpha^*)}\right] \tag{101}$$

Balanced trade, and the law of one price, requires $\tau_t C_F = C_{H^*}$ in every period. Using relative demand:

$$ToT_t = \frac{\alpha^* \frac{P_t^*}{p_{H,t}}^{\phi} C_t^*}{(1-\alpha)\frac{P_t}{p_{F,t}}^{\phi} C_t}$$

Using autarky again:

$$ToT_t = \frac{\alpha^* \frac{P_t^*}{p_{H,t}}^{\phi} \frac{p_{F,t}^*}{P_t^*} Y_{F,t}^*}{(1-\alpha)\frac{P_t}{p_{F,t}}^{\phi} \frac{p_{H,t}}{P_t} Y_{H,t}}$$

Imposing $\alpha^* = (1-\alpha)$:

$$ToT_t = \underbrace{\frac{P_t^*}{P_t}^{\phi-1}}_{\mathcal{E}_t^{\phi-1}} \frac{Y_{F,t}^*}{Y_{H,t}} \underbrace{\frac{p_{F,t}}{p_{H,t}}^{1+\phi}}_{ToT_t^{1+\phi}}$$

So:

$$ToT_t^{-\phi} = \mathcal{E}_t^{\phi-1}\frac{Y_{F,t}^*}{Y_{H,t}} \tag{102}$$

A first order approximation and $e = (2\alpha - 1)\tau$,

$$\tau = \frac{y_H - y_F}{1 - 2\alpha(1-\phi)}, \tag{103}$$

$$\Delta e = (2\alpha - 1) * \Delta\tau \tag{104}$$

We now show that at the limit of $\alpha \to 1$, the model coincides with a two-country, two-good (no trade), CAPM. Home and Foreign consumption is given by:

$$c_t = c_{H,t} = y_{H,t} \tag{105}$$

$$c_t^* = c_{F_t}^* = y_{F,t} \tag{106}$$

Assuming $g_{y_{H,t}}$, $g_{y_{F,t}} \sim \mathcal{N}(\mu, \sigma_y^2)$, then $m_{t+1}^{(*)} \sim \mathcal{N}(-s\mu, s^2\sigma_{y_i}^2)$, $i \in \{H, F\}$. We can then construct:

$$\eta_{t+1} = (g_{y_{H,t+1}} - g_{y_{F,t+1}})\frac{1-s}{1-2\alpha(1-\phi)} \tag{107}$$

using (7).

Returning to the financial side of the model, the Home agents' inter-temporal allocation satisfies (2), (5). The international risk sharing condition in the model is given by:

$$\begin{aligned} \mathbb{E}_t[M_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}] &= \mathbb{E}_t\left[M_{t+1}^*\right] \leftrightarrow \\ \mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] &= \mathbb{E}_t\left[\beta\left(\frac{C_{t+1}^*}{C_t^*}\right)^{-s}\right] \end{aligned} \tag{108}$$

Critically, if the Foreign risk-free bond was also traded then the second risk-sharing condition below would also need to be satisfied:

$$\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s}\right] = \mathbb{E}_t\left[\beta\left(\frac{C_{t+1}^*}{C_t^*}\right)^{-s}\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}\right] \tag{109}$$

Notice that (108) and (109) are the same if approximated to first order, but in general will imply significantly different results.

B.3.1 Numerical Illustration for Proposition 3

The figure below illustrates the range of parameters for which case (ii) is satisfied, over a grid of s and ϕ.

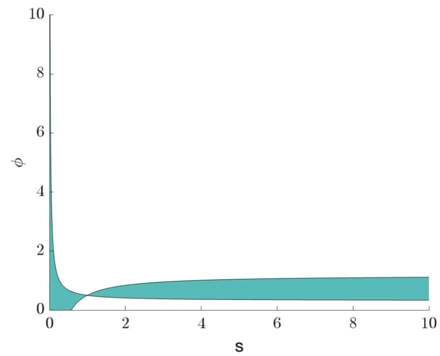

Figure 2: *Shaded region reflects parameters for which the model can satisfy the empirical Backus Smith correlation when there is trade in a risk-free bond, at the limit of financial autarky and full home-bias.*

C. Portfolio Example for Section 3.1

To gain concrete understanding of condition (30), we flesh out the financial market structure in the Home economy. The simplest model of heterogeneity consistent with our framework is one where the investor characterized by m_{t+1} and the investor characterized by $\hat{m}_{t+1}$ are identical except the latter participates in financial markets for Foreign assets. Imposing consumption utility structure on the SDFs, $\hat{m}_{t+1} = \log(u'(y_{t+1} + w^H_{t+1} + w^F_{t+1}))$ and $m_{t+1} = \log(u'(y_{t+1} + w^H_{t+1}))$, where y_{t+1} is the value of the Home country's endowment, w^H_{t+1} is wealth after trade in a set of basis assets (e.g. just the Home bond) and w^F_{t+1} is defined as the residual portfolio wealth after trade in both the set of basis assets and the Foreign bond.[35] As is standard in portfolio choice, we assume exponential utility (CARA) which allows us to break the individual components by abstracting from wealth effects. Specifically, $u(C) = -e^{-sC}$. Assuming for exposition that m^*_{t+1} does not vary a lot and utility is exponential, equation (1) implies:

$$cov_t(-sw^F_{t+1}, \Delta e_{t+1}) \leq 0 \tag{110}$$

[35] w^H is the return on the basis asset portfolio which are freely traded by both investors. Note that the autarky limit is where $cov_t(m_{t+1}, -\Delta e_{t+1}) < 0$, requiring $cov_t(y_{t+1}, -\Delta e_{t+1}) > 0$, consistent with Proposition 2. Moreover, at the autarky limit $m_{t+1} \to \hat{m}_{t+1}$.

In other words, exchange rates are risky for the marginal investor, consistent with redistribution across countries, but this investor does not pass the risk on to the domestic household through domestic asset markets. It is useful to note that the implied comovement of $\hat{m}_{t+1}$ and m_{t+1} in this framework is given by $s^2 var_t(y_{t+1} + w^H_{t+1}) + s^2 cov_t(y_{t+1} + w^H_{t+1}, w^F_{t+1})$, which will depend on how portfolios are formed and the underlying structure of shocks which we have not specified. Corsetti, Dedola and Leduc (2014) discipline portfolios using the data and show there is low-risk sharing when there is trade in one international nominal risk-free bond, and trade in international equities.